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Imperfect Bifurcation in the Presence of Symmetry

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Introduction

Consider the familiar principle that typically (or generically) a system of m scalar equations in n variables where $m>n$ has no solutions. This principle can be reformulated geometrically as follows. If S is a submanifold of a manifold X with codimension m (i.e. $m=\dim X - \dim S$) and if $f:\mathbf{R}^n \to X$ is a smooth mapping where $m>n$, then usually – or generically – Image $f \cap S$ is empty. One of the basic tenets in the application of singularity theory is that this principle holds in a general way in function spaces. In the next few paragraphs we shall try to explain this more general situation as well as to explain its relevance to bifurcation problems.

First we describe an example through which these ideas may be understood. Consider the buckling of an Euler column. Let λ denote the applied load and x denote the maximum deflection of the column. After an application of the Lyapunov-Schmidt procedure the potential energy function V for this system may be written as a function of x and λ alone and hence the steady-state configurations of the column may be found by solving

$$(0.1)\quad G(x,\lambda)=\frac{\partial V}{\partial x}(x,\lambda)=0.$$

See for example [6, Sect. 6]. It is shown there that near the buckling point (which we assume to be at $\lambda=0$) we may write

$$(0.2)\quad G(x,\lambda)=x^3-\lambda x+\dots.$$

Moreover, the lowest order terms dominate so that the pitchfork $x^3-\lambda x=0$ describes qualitatively the various steady-state configurations of the column near the buckling point.

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The key word in the last paragraph is "qualitatively". This word is interpreted – in singularity theory – to mean "up to an appropriate change of coordinates". Our earlier paper [6] was devoted to the study of the machinery of singularity theory in a context appropriate to steady state problems in bifurcation theory. In particular, let $\mathscr{E}_{n,m}$ denote the space of germs of C^∞ mappings of $(\mathbf{R}^n, 0) \to \mathbf{R}^m$. Then a bifurcation problem is the solution to

(0.3) $G(x, \lambda) = 0$

where $G \in \mathscr{E}_{n+1,m}$ and $(d_x G)(0) = 0$. Here we assume that either G is obtained directly or – as in the Euler column – as the result of the Lyapunov-Schmidt procedure applied to some non-linear operator. In this context, we suggest that the phrase "appropriate change of coordinates" be interpreted through the following equivalence. Two bifurcation problems G_1 and G_2 are *contact equivalent* if

(0.4) $G_2(x, \lambda) = T(x, \lambda) \cdot G_1(X(x, \lambda), \Lambda(\lambda))$

where $X(0,0) = 0$, $\Lambda(0) = 0$, $\det(d_x X)(0) > 0$, $\dfrac{d\Lambda}{d\lambda}(0) > 0$, and for each (x, λ), $T(x, \lambda)$ is an invertible $m \times m$ matrix.

In [6] we showed that the bifurcation equation for the Euler column is contact equivalent to $x^3 - \lambda x = 0$.

It should be clear from the above example that the set

(0.5) $\mathcal{O}_G = \{H \in \mathscr{E}_{n+1,m} \mid H \text{ is contact equivalent to } G\}$

should play a role in the study of the bifurcation problem G. A crude measure for the complexity of the bifurcation problem G is the codimension of $\mathcal{O}_G$ in $\mathscr{E}_{n+1,m}$. In [6] we gave a prescription for computing codim G. As a simple example, the codimension of $x^3 - \lambda x$ was shown to be two.

Now suppose one actually performs the experiment of buckling columns. To each column one assigns a potential function $\bar{V}$ and a bifurcation problem $\bar{G}$. Since real columns tend to have imperfections one should not expect $\bar{G}$ to be equivalent to the idealized problem G described above. Moreover, the principle ennunciated in the first paragraph agrees with this comment. This principle also suggests that if a pitchfork is to be observed experimentally – that is, a column found for which $\bar{G}$ is in $\mathcal{O}_G$ – then one must vary a two parameter family of columns. For example, one could add arbitrarily a central load and put an arbitrary initial curvature in the column.

In general, if we let $\alpha \in \mathbf{R}^k$ represent the various auxilliary or imperfection parameters then for each α one obtains a bifurcation problem $\bar{G}_\alpha$. The total experiment is then described by a mapping $F: \mathbf{R}^k \to \mathscr{E}_{n+1,m}$ defined by $\alpha \mapsto \bar{G}_\alpha$.

In this context, our stated principle is: the observation of a bifurcation problem of codimension k by an experiment which varies fewer than k auxilliary parameters is non-generic. This discussion suggests that contact codimension measures the difficulty of observing a particular bifurcation problem.

In this paper we discuss several ways in which the contact codimension fails as such a measure. These deficiencies are associated with both the way a particular problem is idealized into a mathematical model and the mathematical assumption that the contact codimension is the relevant number. As the first deficiency is the main topic of this paper, we illustrate it with a simple example.

As noted above the buckling of an Euler column yields a bifurcation problem with contact codimension two; yet, in the idealized model there are no auxilliary parameters. Our question is simple. Why does a mathematical model of codimension two appear when no extra parameters are varied? The answer seems to be in the way the Euler column is idealized. In this model it is assumed that the potential energy associated to a given configuration of the column u is the same as that of $-u$. This means that the bifurcation problem G associated to this model must satisfy

(0.6) $G(-x,\lambda)=-G(x,\lambda)$.

Clearly $G(x,\lambda)=x^3-\lambda x$ satisfies (0.6). We suggest that to apply the stated principle to the idealized problem one must include the $\mathbf{Z}_2$ symmetry which has been imposed in the problem. The appropriate question is "what is the codimension of $\mathcal{O}_G$ in the space of germs satisfying (0.6)?" Care must be taken that the contact equivalences which are used to define $\mathcal{O}_G$ be restricted to those equivalences which preserve the symmetry condition (0.6). This will be formalized in Sect. 1. In Sect. 2 we compute this new codimension for $x^3-\lambda x$ and find that it is zero. So the principle is satisfied when the restrictions imposed by the idealization are present.

Many authors [3, 5, 13–15, 18, 20] have realized the significance of imposing a group of symmetries on the mathematical model for a given physical problem. Our purpose here is to integrate the presence of a symmetry group with our approach to bifurcation problems. The necessary hard mathematics has already been formulated in the case that the symmetry group is compact. We restrict to this case.

With regard to the Euler column, we emphasize that the dichotomy is caused by idealization. Experimentally one rarely observes the pitchfork; one usually sees perturbations which have contact codimension zero. The reader is referred to discussions in [9] and [3].

In the preceding discussion we have concentrated mainly on the framework of singularity theory; now we focus on one of the major theorems, the unfolding theorem. It follows from this theorem (in the contexts where it has been proved) that if a germ f has codimension k then an arbitrary small perturbation of f may – up to an appropriate change of coordinates – be written as

(0.7) $f(x)+a_1p_1(x)+\ldots+a_kp_k(x)$

where the p_i's are fixed perturbation terms and the a_i's are scalars which depend smoothly on the particular perturbation. (0.7) is called a *universal unfolding* of f and the a_i's are called the *unfolding parameters*.

The unfolding parameters themselves fall into two classes, *modal* and *non-modal*. The modal parameters may be characterized by the fact that they parametrize the largest family M of perturbations of f in (0.7) such that no two perturbations in this family are equivalent. To illustrate this point we briefly describe an example which is considered in more detail in Sect. 3. Let

(0.8) $G(x,y,\lambda)=(x^2+y^2+\lambda x, cxy+\lambda y)$.

We note that G satisfies the $\mathbf{Z}_2$ symmetry relation

$$(0.9)\quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot G(x, -y, \lambda) = G(x, y, \lambda).$$

It is easy to check that as c varies none of the problems (0.8) are equivalent by a contact equivalence which preserves the symmetry relation (0.9). We also show that the universal unfolding for (0.8) is

$$(0.10)\quad F(x, y, \lambda, c, \alpha, \beta) = (x^2 + y^2 + \lambda x + \alpha, cxy + (\lambda - 2\beta)y).$$

We claim that c is a modal parameter and that α and β are non-modal parameters. This will be shown in Sect. 3.

Given a bifurcation problem G, its universal unfolding F, and the modal family M define the *module packet* of G to be

$$(0.11)\quad \mathcal{O}_M = \{H \in \mathcal{E}_{n+1,m} \mid H \text{ is contact equivalent to some member of the family } M\}.$$

We follow Arnold [1] and take the view that the module packet $\mathcal{O}_M$ and not $\mathcal{O}_G$ contains all the bifurcation problems which are qualitatively similar to G. (Note: Perturbations of $H \in M$ may be different from perturbations of G as happens when $c = 2$ in (0.8). However these differences occur on a subvariety and do not alter the following discussion.) If this is the case then the codimension of $\mathcal{O}_M$; that is, the number l of non-modal parameters, is the appropriate number to be used with our stated principle. We shall try to make this point more clearly in our discussion of the application of (0.8) to the Brusselator model considered in [16].

There are two justifications for considering l rather than k as the relevant number. First, if one considers an l-parameter family $E : \mathbf{R}^l \to \mathcal{E}_{n+1,m}$, then generically Image $E \cap \mathcal{O}_G = \emptyset$. Yet it is clear that Image $E \cap \mathcal{O}_M$ may be non-empty in an unavoidable manner. To avoid the module packet one must vary fewer than l parameters. Second, the members of the modal family M although not equivalent via smooth contact equivalences are topologically contact equivalent – at least in the cases which have been analyzed. (More accurately, the module packet decomposes into a finite union of semi-algebraic subsets which contain topologically equivalent problems.)

This behavior may be clarified by an analogy with systems of linear differential equations, say

$$(0.12)\quad \dot{x} = Ax.$$

The topological behavior of solutions of (0.12) is determined by the number of eigenvalues of A with negative real parts. Although the eigenvalues of a perturbed matrix $A + \varepsilon B$ will be slightly different, there are large regions of the matrix space where the topological behavior of (0.12) will be unaffected by such changes. In addition, there are only a finite number of regions in this matrix space where the topological behavior is distinct.

While pursuing this work we have benefited from the ideas of Sattinger [13–15]. Sattinger makes the following two observations about bifurcation problems with symmetry. First, the form of the Taylor expansion of such a problem is severely restricted making computations less difficult than the general case. Second, the presence of a symmetry group often forces an eigenvalue of high multiplicity in the bifurcation problem. (Note that in (0.3) n is the multiplicity and, in practice, we only consider examples where $n=m$.) Generally the Taylor expansions of problems with eigenvalues of high multiplicity (like 2 or 3) are difficult to compute; the first observation states that the difficulties are perhaps overstated.

We make a third observation; namely, the presence of a symmetry group forces an artificially high contact codimension. This observation should be compared with Thom's philosophy that only perturbations of problems with low codimension can be completely understood. We shall see that if one considers problems with symmetry then there are many more problems which have low codimension – at least in the symmetry class. For these problems, it is possible to describe qualitatively all of the symmetry preserving perturbations; that is, the unfolding theorem is true in this context. This is the main result of the next section. Of course the complete description of all symmetry breaking perturbations is much more difficult as here one must come to grips with problems of high contact codimension.

The structure of this paper is as follows. Section 1 contains the theory of Γ-equivariant bifurcation problems where Γ is a compact group of symmetries. Certain $\mathbf{Z}_2$-equivariant problems are described in Sects. 2 and 3. The results of Sect. 3 are applied to the bifurcations of a model chemical reaction in [16]. Section 4 contains a $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-equivariant bifurcation problem which appears in a model for the buckling of a rectangular plate. The results of this application will appear in [17]. Here we describe results of Bauer, Keller, and Reiss [2] in terms of our theory. A non-linear eigenvalue problem for the Laplacian on the unit disk in the plane is described in Sect. 5. We view this as a model problem with $\mathbf{0}(2)$ symmetry. In the last section we suggest an analysis for a bifurcation problem associated with the buckling of an annular plate. Here the symmetry group is $\mathbf{0}(2) \times \mathbf{Z}_2$.

1. Γ-Unfolding Theory

Let Γ be a compact group acting orthogonally on $\mathbf{R}^n$ and $\mathbf{R}^m$. We call a germ $f:(\mathbf{R}^n, 0) \to (\mathbf{R}^m, 0)$ *Γ-equivariant* if $f(\gamma x) = \gamma f(x)$ for all $\gamma \in \Gamma$ and denote by $\mathscr{E}^{\Gamma}_{n,m}$ the space of Γ-equivariant germs. A function germ $h:(\mathbf{R}^n, 0) \to \mathbf{R}$ is called *Γ-invariant* if $h(\gamma x) = h(x)$ for all $\gamma \in \Gamma$. The space of Γ-invariant germs is denoted by $\mathscr{E}^{\Gamma}_n$. In what follows it is assumed that Γ acts trivially on any factor except the given $\mathbf{R}^n$ and $\mathbf{R}^m$.

A *bifurcation problem with symmetry group Γ* is a germ G in $\mathscr{E}^{\Gamma}_{n+1,m}$. We modify the notion of contact equivalence given in [6] to include symmetry as follows. Two bifurcation problems, G_1 and G_2, with symmetry group Γ, are *Γ-equivalent* if

(1.1) $G_1(x,\lambda) = T(x,\lambda) \cdot G_2(X(x,\lambda), \Lambda(\lambda))$

where $\det(d_x X)_0 > 0$, $\Lambda'(0) > 0$, and for each (x, λ), $T(x,\lambda)$ is an invertible $m \times m$ matrix. Moreover, we require the symmetry conditions.

(1.2) $X(\gamma x, \lambda) = \gamma X(x, \lambda)$ and

(1.3) $\gamma^{-1} T(\gamma x, \lambda) \gamma = T(x, \lambda)$

hold for all $\gamma \in \Gamma$ where in (1.3) γ is viewed as an orthogonal matrix acting on $\mathbf{R}^m$. It is easy to check that (1.2) and (1.3) imply that if G_2 is in $\mathscr{E}^\Gamma_{n+1,m}$ then so is G_1.

The usual definitions for unfolding theory can now be given in terms of Γ-equivalence. An *l-parameter Γ-unfolding of G* is a germ $F(x,\lambda,\alpha) \in \mathscr{E}^\Gamma_{n+1+l,m}$ where $F(x,\lambda,0) = G(x,\lambda)$. Let $H(x,\lambda,\beta)$ in $\mathscr{E}^\Gamma_{n+1+k,m}$ be a k-parameter Γ-unfolding of G. Then *H factors through F* if for each $\beta \in \mathbf{R}^k$, $H(\cdot,\cdot,\beta)$ is Γ-equivalent to $F(\cdot,\cdot,\alpha)$ for some α with the Γ-equivalence depending smoothly on β. More precisely, we have

(1.4) $\quad H(x,\lambda,\beta) = T(x,\lambda,\beta) \cdot F(X(x,\lambda,\beta), \Lambda(\lambda,\beta), \alpha(\beta))$

with the appropriate equivariance conditions (1.2) and (1.3) holding. Finally, F in $\mathscr{E}^\Gamma_{n+1+l,m}$ is a *universal Γ-unfolding of G* if every Γ-unfolding H of G factors through F.

The motivation, statement, and method of proof of the unfolding theorem for Γ-equivalence are the same as the unfolding theorem for contact equivalence given in [6] with one difference. For the proof of this theorem, which we shall state, one must use a Γ-equivariant version of the Malgrange Preparation Theorem. Such a generalization has been proved by Poenaru [11] in the case where Γ is a compact group with an orthogonal action. It is for this reason that we have imposed restrictions on the type of symmetry group.

To state the unfolding theorem we must identify the tangent space ΓG to $\mathcal{O}_G = \{G_1 \in \mathscr{E}^\Gamma_{n+1,m} \mid G_1$ is Γ-equivalent to $G\}$ at G. It is easy to check that

(1.5) $\quad \Gamma G = \tilde{\Gamma} G + \mathscr{E}_\lambda \cdot \left\{\dfrac{\partial G}{\partial \lambda}\right\}$

where

(1.6) $\quad \tilde{\Gamma} G = \{(d_x G)(X(x,\lambda)) + T(x,\lambda) G\}$.

Here X and T satisfy (1.2) and (1.3) respectively; X is arbitrary in $\mathscr{E}^\Gamma_{n+1,m}$ and T is a (possibly singular) $m \times m$ matrix. It is clear that $\tilde{\Gamma} G$ is a $\mathscr{E}^\Gamma_{n+1}$-module whereas, in general, ΓG is not. This fact is the cause of the inelegance in the following definition.

Definition 1.7. Let G be a bifurcation problem with symmetry group Γ. Then

a) G has *finite codimension* if $\dim \mathscr{E}^\Gamma_{n+1,m}/\tilde{\Gamma} G < \infty$, and

b) the *Γ-codimension* of G is $\dim \mathscr{E}^\Gamma_{n+1,m}/\Gamma G$.

We now state the unfolding theorem.

Theorem 1.8. *Let $G(x,\lambda)$ be a finite codimension bifurcation problem with symmetry group Γ. Let $G_1(x,\lambda), \ldots, G_l(x,\lambda)$ in $\mathscr{E}^\Gamma_{n+1,m}$ project onto a spanning set for $\mathscr{E}^\Gamma_{n+1,m}/\Gamma G$. Then $F(x,\lambda,\alpha) = G(x,\lambda) + \alpha_1 G_1(x,\lambda) + \ldots + \alpha_l G_l(x,\lambda)$ is a universal Γ-unfolding for G.*

The proof of this theorem requires nothing more than checking that the proof of Theorem 2.4 [6] holds in the presence of a symmetry group Γ; thus we leave the details to the reader.

For the remainder of this section we outline how one might actually compute universal Γ-unfoldings for a given G. These computations break naturally into two parts; one involves the group Γ and the other the specific bifurcation problem G. As all of the examples considered in this paper satisfy $m=n$, we restrict to this case. The reader may wish to continue with the examples in the subsequent sections and return to the remainder of this section later.

The point of these calculations is the computation of $\tilde{\Gamma}G$. First we describe a theorem due to Schwarz [19]. Let $\sigma_1, \ldots, \sigma_p$ be a finite set of polynomials which generate – as a ring – the ring of polynomials in $\mathscr{E}_n^\Gamma$. Such a set exists by Hilbert's Basis Theorem. Let $\sigma:(\mathbf{R}^n,0)\to(\mathbf{R}^p,0)$ be defined by

$$(1.9)\quad \sigma(x)=(\sigma_1(x), \ldots, \sigma_p(x)).$$

Schwarz's Theorem states that under the condition that Γ is a compact group acting orthogonally on $\mathbf{R}^n$ any invariant function f in $\mathscr{E}_n^\Gamma$ may be written as $f(x) = h(\sigma(x))$ where $h:(\mathbf{R}^p,0)\to\mathbf{R}$ is a smooth germ. For example an even function $f:(\mathbf{R},0)\to\mathbf{R}$ may be written as $f(x)=h(x^2)$.

Next one considers the $\mathscr{E}_n^\Gamma$-modules $\mathscr{E}_{n,n}^\Gamma$ and $\mathscr{M}_{n,n}^\Gamma$ where

$$(1.10)\quad \mathscr{M}_{n,m}^\Gamma=\{T:(\mathbf{R}^n,0)\to\text{space of } m\times m \text{ matrices} \mid T \text{ satisfies the equivariance condition (1.3)}\}.$$

In the example these modules are finitely generated and the first computation is to find an explicit set of generators $X_1, \ldots, X_q$ for $\mathscr{E}_{n,n}^\Gamma$ and $T_1, \ldots, T_r$ for $\mathscr{M}_{n,n}^\Gamma$. Since Γ acts trivially on the λ-factor of $\mathbf{R}^n\times\mathbf{R}$ these generators also generate $\mathscr{E}_{n+1,n}^\Gamma$ and $\mathscr{M}_{n+1,n}^\Gamma$ as modules over $\mathscr{E}_{n+1}^\Gamma$. One now observes that $\tilde{\Gamma}G$ is generated as a module over $\mathscr{E}_{n+1}^\Gamma$ by the $(q+r)$-generators

$$(1.11)\quad (d_xG)(X_1), \ldots, (d_xG)(X_q);\ T_1\cdot G, \ldots, T_r\cdot G.$$

The next step is to determine whether $\mathscr{E}_{n+1,n}^\Gamma/\tilde{\Gamma}G$ is a finite dimensional vector space and then to find a spanning set for $\mathscr{E}_{n+1,n}^\Gamma/\Gamma G$. We describe an algorithm which – at least in the examples considered in the later sections – gives a method for reducing these computations to Taylor Theorem arguments of the type found in [6] for bifurcation problems without symmetry.

We assume that G is in $\mathscr{E}_{n+1,n}^\Gamma$; hence G has the form

$$G(x,\lambda)=a_1(\sigma(x),\lambda)X_1(x)+\ldots+a_q(\sigma(x),\lambda)X_q(x),$$

since $X_1, \ldots, X_q$ generate $\mathscr{E}_{n+1,n}^\Gamma$ over $\mathscr{E}_n^\Gamma$. Note that the a_i's are uniquely determined if $\mathscr{E}_{n+1,n}^\Gamma$ is a free module over $\mathscr{E}_{n+1}^\Gamma$ with ordered basis $X_1, \ldots, X_q$. This we assume while noting that all of the examples considered in this paper satisfy this hypothesis.

We may now identify $\mathscr{E}_{n+1,n}^\Gamma$ with $\mathscr{E}_{p+1,q}$ as follows: define $\varphi:\mathscr{E}_{n+1,n}^\Gamma\to\mathscr{E}_{p+1,q}$ by $\varphi(G)=(a_1(u,\lambda), \ldots, a_q(u,\lambda))$ where $u=(u_1, \ldots, u_p)$ denotes the coordinates on $\mathbf{R}^p$. It is clear that $\Phi G=\varphi(\tilde{\Gamma}G)$ is an $\mathscr{E}_{p+1}$-submodule of $\mathscr{E}_{p+1,q}$ and that φ induces an isomorphism between $\mathscr{E}_{n+1,n}^\Gamma/\tilde{\Gamma}G$ and $\mathscr{E}_{p+1,q}/\Phi G$. The calculations for finding a spanning set of $\mathscr{E}_{p+1,q}/\Phi G$ are very much the same as the calculations performed in [6]. One remark remains, ΦG is generated as a module over $\mathscr{E}_{p+1}$ by the image under φ of the generators of $\tilde{\Gamma}G$ listed in (1.11). Moreover, the computation of these generators in any given case is perhaps tedious but certainly straightforward.

We now turn to another aspect of the singularity theory approach to applications; namely the notion of determinancy. In most applications one is rarely given the bifurcation problem G explicitly; rather one usually has G defined implicitly along with methods for computing the Taylor expansion of G at the bifurcation point. Thus it would be useful to know when one has enough terms of this expansion – at least qualitatively. This is reflected in the following: the bifurcation problem G with symmetry group Γ is *Γk-determined* if for every $H \in \mathscr{E}^{\Gamma}_{n+1,m}$ where H vanishes up to order $k+1$, $G+H$ is Γ-equivalent to G. We use the method outlined above for finding the Γ-codimension of G to compute the Γ-determinacy of G.

For bifurcation problems $G \in \mathscr{E}_{n+1,m}$ the condition $\mathscr{M}^k \mathscr{E}_{n+1,m} \subset \mathscr{M}\tilde{T}G$ implies that G is k-determined. Here $\mathscr{M}$ is the maximal ideal in $\mathscr{E}_{n+1}$ and $\tilde{T}G$ is the module part of the tangent space to the orbit of G formed by contact equivalence. One major step of this proof is the demonstration that $\tilde{T}G_t = \tilde{T}G$ for $0 \leqq t \leqq 1$ where $G_t = G + tH$. (See the proof of Proposition 3.11 of [6].) The proof for Γk-determinacy works in exactly the same way; the details are left to the reader.

Proposition 1.12. *Let G and H be in $\mathscr{E}^{\Gamma}_{n+1,m}$ and let $G_t = G + tH$. Assume that for all t with $0 \leqq t \leqq 1$, $\tilde{\Gamma}G_t = \tilde{\Gamma}G$. Then $G+H$ is Γ-equivalent to G.*

A necessary condition that the assumption $\tilde{\Gamma}G_t = \tilde{\Gamma}G$ is that the generators (1.11) with G replaced by G_t be in $\tilde{\Gamma}G$. The validity of this condition is more easily computed in $\mathscr{E}_{p+1,s}$. Once it has been determined that $\Phi G_t \subset \Phi G$, one expands the $q+r$ generators $\varphi((d_x G_t)(X_i))$ and $\varphi(T_j \cdot G_t)$ in terms of the same generators for $t=0$. This yields a $(q+r) \times (q+r)$ matrix whose invertibility at $u=0$ guarantees the hypothesis of Proposition 1.12.

It is sometimes useful to use Nakayama's Lemma in these calculations, so we state it here.

Lemma 1.13. *Let B be an $\mathscr{E}_{p+1}$-submodule of $\mathscr{E}_{p+1,q}$ generated by $b_1, \ldots, b_l$. Let $c_1, \ldots, c_l$ be in $\mathscr{M}B$ where $\mathscr{M}$ is the maximal ideal in $\mathscr{E}_{p+1}$. Then $b_1 + c_1, \ldots, b_l + c_l$ form a set of generators for B.*

2. $\mathbf{Z}_2$ Actions on the Line

We consider now several bifurcation problems with $\Gamma = \mathbf{Z}_2 = \{\pm 1\}$ symmetry where $n = m = 1$ and $\mathbf{Z}_2$ acts by multiplication on $\mathbf{R}$. The bifurcation problems along with their universal $\mathbf{Z}_2$ unfoldings are given in Table 1.

Table 1

	$G(x, \lambda)$	Γ-codim	Universal Γ-unfolding	Contact codim
a)	$x^3 - \lambda x$	0	$x^3 - \lambda x$	2
b)	$x^3 - \lambda^2 x$	1	$x^3 - (\lambda^2 + \alpha) x$	5
c)	$x^5 - \lambda x$	1	$x^5 - \alpha x^3 - \lambda x$	4

Note that the contact codimension of these problems were computed in Sect. 4 of [6].

We follow the notation and computational rules described in Sect. 1. The invariant functions $\mathscr{E}_1^\Gamma$ form the space of even functions; thus $p=1$ and $\sigma(x)=x^2$. The equivariant mappings $\mathscr{E}_{1,1}^\Gamma$ consist of odd functions; thus $q=1$ and $X_1(x)=x$. The module $\mathscr{M}_{1,1}^\Gamma$ consists of even functions; thus $r=1$ and $T_1(x)=1$.

Let $G(x,\lambda)$ be in $\mathscr{E}_{2,1}^\Gamma$, then

(2.1) $G(x,\lambda)=a(x^2,\lambda)x$

where $a(u,\lambda)$ is a smooth germ. The generators $(d_xG)(X_1)$ and T_1G for $\tilde{\Gamma}G$ are just

(2.2) $(2a_u(x^2,\lambda)x^2+a(x^2,\lambda))X_1$ and $a(x^2,\lambda)X_1$.

Thus the $\mathscr{E}_2$-module ΦG is generated by

(2.3) ua_u and a.

For the three cases listed in Table 1, the module ΦG is

(2.4) $\mathscr{E}_2\{u,\lambda\}$, $\mathscr{E}_2\{u,\lambda^2\}$, and $\mathscr{E}_2\{u^2,\lambda\}$

respectively. Spanning sets for $\mathscr{E}_{2,1}/\Phi G$ are

(2.5) $\{1\}$, $\{1,\lambda\}$, and $\{1,u\}$

respectively. Therefore spanning sets for $\mathscr{E}_{2,1}^\Gamma/\tilde{\Gamma}G$ are

(2.6) $\{x\}$, $\{x,\lambda x\}$, and $\{x,x^3\}$

respectively. Finally $\frac{\partial G}{\partial \lambda}$ is, in the three cases, $-x$, $-2\lambda x$, $-x$; hence bases for $\mathscr{E}_{2,1}^\Gamma/\Gamma G$ are given by

(2.7) $\emptyset$, $\{x\}$, $\{x^3\}$

respectively. The universal Γ-unfoldings in Table 1 follow from Theorem 1.8.

Lemma 2.8. *Let $G(x,\lambda)$ be one of the bifurcation problems listed in Table 1. Let $H(x,\lambda)$ be in $\mathscr{E}_{2,1}^\Gamma$ consist of higher order terms for G, then $G+H$ is Γ-equivalent to G. Specifically H has the form*

a) $H(x,\lambda)=x^5a(x^2,\lambda)+\lambda^2xb(x^2,\lambda)+\lambda x^3c(x^2,\lambda)$
b) $H(x,\lambda)=x^5a(x^2,\lambda)+\lambda^3xb(x^2,\lambda)+\lambda^2x^3c(x^2,\lambda)$
c) $H(x,\lambda)=x^7a(x^2,\lambda)+\lambda^2xb(x^2,\lambda)+\lambda x^3c(x^2,\lambda)$

in the three cases.

Proof. We proceed as outlined at the end of Sect. 1. Let $G_t=G+tH$. For example in (a),

(2.9) $G_t=(h+t(x^4a+\lambda^2b+\lambda x^2c))X_1$

where $h(x,\lambda)=x^2-\lambda$. Then ΦG_t is generated by

(2.10) $h+t(u^2a+\lambda^2b+\lambda uc)$, $uh_u+t(2u^2a+\lambda uc)$.

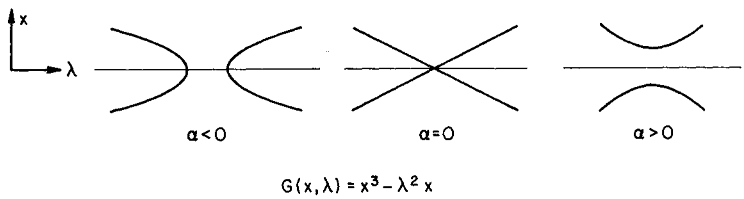

Fig. 1

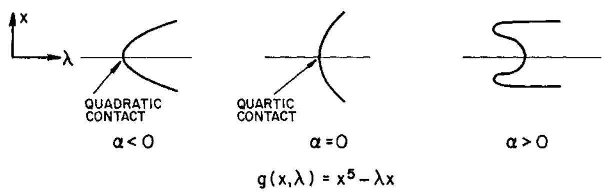

Fig. 2

As ΦG is generated by u and λ (see (2.4)) we have that generators for ΦG_t are

(2.11) $h+m_1$ and uh_u+m_2

where m_1 and m_2 are in $\mathscr{M}\Phi G$. We now apply Nakayama's Lemma to see that $\Phi G_t = \Phi G$ and Proposition 1.12 implies this lemma. The proofs for the other two cases are similar.

We next describe the effects of the symmetric imperfections on the bifurcation diagrams. The pitchfork $x^3-\lambda x=0$ is $\mathbf{Z}_2$-stable. The other two examples have bifurcation diagrams represented in Figs. 1 and 2 respectively.

A few comments are in order. We have now shown – as promised in the Introduction – that the Γ-codimension of $x^3-\lambda x$ is zero. As suggested this may explain why the pitchfork appears in many idealized bifurcation problems whereas the fact that its contact codimension is two explains why it is difficult to observe experimentally. The other two problems offer more evidence for this dichotomy. In Sect. 7 of [6] we described a spring problem considered by Poston and Stewart [12] in which both of the other problems appear. This we found most surprising given the contact codimensions of these problems since only one spring constant was being varied. The computations above indicate that the appearance of these problems was in no way an accident.

3. Quadratic Bifurcation Problems in Two Variables

In [6], Sect. 5, we considered the following class of bifurcation problems

(3.1) $G(x,y,\lambda)=(p(x,y)+\lambda x, q(x,y)+\lambda y)$

where p and q are homogeneous polynomials of degree two. Using an observation of McLeod and Sattinger [10] we defined a class of *non-degenerate* problems as those satisfying

(3.2) (a) p and q have no common factors
(b) the surfaces $p(x,y)+\lambda x=0$ and $q(x,y)+\lambda y=0$ are tangent only at the origin.

A normal form was found for non-degenerate G and the contact codimension was computed to be seven. It was noted that non-degenerate G appear in a reaction-diffusion equation known as the Brussellator model. The application of our theory to this model will be given in [16]; here we note that there is a natural $\mathbf{Z}_2$ symmetry in the Brussellator. We describe this symmetry below as it appears in (3.1). We also note that the number of non-modal parameters in a universal unfolding of a non-degenerate problem (3.1) is five and of those, only three preserve the trivial solution $x=y=0$. We shall show that the corresponding numbers for the $\mathbf{Z}_2$-unfoldings are two and one.

First we describe the $\mathbf{Z}_2$ action on $\mathbf{R}^2$ and then determine the general non-degenerate $\mathbf{Z}_2$-equivariant problem (3.1). In the action of $\mathbf{Z}_2=\{\pm 1\}$ on $\mathbf{R}^2$ under consideration, -1 acts as the matrix $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$; that is, $(x,y)\mapsto(x,-y)$. It is easy to check that the Γ-equivariant mappings $\mathscr{E}^{\Gamma}_{2,2}$ consist of those maps whose first component is even in y and whose second component is odd in y. So the general quadratic Γ-equivariant problem (3.1) has the form

$$(3.3)\quad G(x,y,\lambda)=(ax^2+by^2+\lambda x,\, cxy+\lambda y).$$

Note that if (3.3) satisfies (3.2) (a), then $a\neq 0$, $b\neq 0$, and $c\neq 0$. Also note that $-G(-x,-y,\lambda)$ is both $\mathbf{Z}_2$-equivalent to G, and of the form (3.3). So in (3.3) we may assume that $a>0$. Next by scaling (3.3) we may assume that $a=|b|=1$, yielding the normal form

$$(3.4)\quad G^{\pm}(x,y,\lambda)=(x^2\pm y^2+\lambda x,\, cxy+\lambda y).$$

Finally (3.4) must satisfy (3.2) (b) which implies $c\neq 1$ and $c\neq 0$.

Having determined the non-degenerate $\mathbf{Z}_2$-equivariant problems (3.1) we will compute their universal $\mathbf{Z}_2$-unfoldings to be

$$(3.5)\quad F^{\pm}(x,y,\lambda,\alpha,\beta,c)=(x^2\pm y^2+\lambda x+\alpha,\, cxy+(\lambda-2\beta)y).$$

Note that the computation of (3.4) shows that c is a modal parameter with $c\neq 0$ and $c\neq 1$. Also the trivial solution is maintained when $\alpha=0$.

We use the methods and notations from Sect. 1 to compute (3.5). The module $\mathscr{E}^{\Gamma}_{n,n}$ is generated by $X_1=(1,0)$ and $X_2=(0,y)$ over $\mathscr{E}^{\Gamma}_2$. Note that $\mathscr{E}^{\Gamma}_2$ consists of those germs $f(x,y)$ which are even in y; so $f(x,y)=h(x,y^2)$ for some smooth germ h. A computation shows that the module $\mathscr{M}^{\Gamma}_{2,2}$ – consisting of those matrix valued functions satisfying (1.4) – is generated by

$$T_1=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\quad T_2=\begin{pmatrix}0 & y\\ 0 & 0\end{pmatrix},\quad T_3=\begin{pmatrix}0 & 0\\ y & 0\end{pmatrix},\quad T_4=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}.$$

Let $G(x,y,\lambda)=a(x,y^2,\lambda)X_1+b(x,y^2,\lambda)X_2$ be the general bifurcation problem in $\mathscr{E}^{\Gamma}_{3,2}$. One computes the generators of the module ΦG to be

(3.6) $(a,0)$, $(vb,0)$, $(0,a)$, $(0,b)$, (a_u,b_u), (va_v,vb_v).

Here we view a and b as functions of u, v, and λ.

In the case of interest $a=u^2\pm v+\lambda$ and $b=cu+\lambda$. It is easy to check that (3.6) implies that ΦG is generated by

(3.7) $(u^2+\lambda u,0)$, $(0,u^2+v+\lambda u)$, $(0,cu+\lambda)$, $(2u+\lambda,c)$, $(v,0)$.

Observe that $(0,v)=\frac{v}{c}(2u+\lambda,c)-\frac{2u+\lambda}{c}(v,0)$. Hence an equivalent set of generators is

(3.8) (u^2,cu), $(0,u^2)$, $(0,cu+\lambda)$, $(2u+\lambda,c)$, $(v,0)$, $(0,v)$.

One may use the middle two generators to eliminate λ from the problem of computing $\mathscr{E}_{3,2}/\Phi G$. Since $(u^3,0)$ is in ΦG it is easy to check that

(3.9) $(1,0)$, $(0,1)$, $(u,0)$, $(0,u)$

form a spanning set for $\mathscr{E}_{3,2}/\Phi G$. Hence

(3.10) $(1,0)$, $(0,y)$, $(x,0)$, $(0,xy)$

is a spanning set in $\mathscr{E}^{\Gamma}_{3,2}/\tilde{\Gamma}G$. Now $\frac{\partial G}{\partial\lambda}=(x,y)$, hence

(3.11) $(1,0)$, $(0,y)$, $(0,xy)$

is a spanning set for $\mathscr{E}^{\Gamma}_{3,2}/\Gamma G$ and the universal Γ-unfolding is as indicated in (3.5).

Lemma 3.12. *The non-degenerate* $\mathbf{Z}_2$ *equivariant bifurcation problems* (3.4) *are* $\mathbf{Z}_2$ *2-determined.*

Proof. First consider a special perturbation term

(3.13) $H(x,y,\lambda)=(Q(x,y^2,\lambda)+y^2L(x,y^2,\lambda),\ C(x,y^2,\lambda)y+Ky^3)$

where K is a constant, L is linear, C is cubic, and Q is quartic. We claim that $G+H$ is $\mathbf{Z}_2$-equivalent to G. As in Sect. 1, let $G_t=G+tH$. It is a straightforward computation to show that the generators for ΦG prescribed by (3.6) also generate ΦG_t. Now apply Proposition 1.12.

The general perturbation of third order includes terms of the form

(3.14) $(Ax^3+Bx\lambda^2+Cx^2\lambda,\ Dx^2y+E\lambda^2y+F\lambda xy)$

for constants $A,\dots,F$. To complete the proof observe that if $\bar{H}$ is a general perturbation then a $\mathbf{Z}_2$ equivalence of $G+\bar{H}$ given by

(3.15) $$\begin{pmatrix}1+ax+by & 0\\ 0 & 1+dx+ey\end{pmatrix}(G+\bar{H})(x+fx^2+gx\lambda,y,\lambda)$$

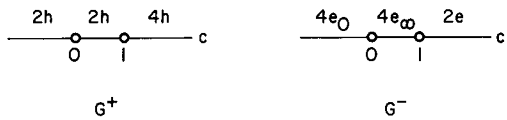

Fig. 3

is of the form $G+H$ where H is as in (3.13) – for appropriate choices of the constants $a, \ldots, g$. In fact, it is not hard to show that the constants $A, \ldots, F$ in (3.14) depend linearly on $a, \ldots, g$ and that the associated 6×6 matrix is invertible.

For the remainder of this section we describe the effects of the unfolding parameters on the bifurcation diagrams. To understand this we must first analyze how the bifurcation diagrams $G^{\pm}=0$ depend on the modal parameter c. The degeneracies $c=0$ and $c=1$ divide the real line into three regions for each of G^+ and G^-. The type of bifurcation is described in Fig. 3; the nomenclature is as in [6, Sect. 5]. Here the numbers 2 and 4 denote the number of solutions for $\lambda=\lambda_0 \neq 0$, the letters e and h denote whether the singularity of $(x, y) \to (x^2 \pm y^2, cxy)$ is elliptic or hyperbolic, and the subscripts 0 and ∞ denote whether or not the trivial solution is in the convex hull of the three non-trivial solutions.

The solution branches are all straight lines given by

(1) $x=y=0$

(2) $x=-\lambda;\ y=0$

(3)–(4) $\lambda=-cx;\ y=\pm\sqrt{|c-1|}\,x$.

Solutions (1) and (2) occur in all cases while (3) and (4) occur for $c>1$ in G^+ and for $c<1$ in G^-. The solutions – as must be the case – are symmetric with respect to the plane $y=0$; the first two actually lie in this plane.

We now consider the effects of the non-modal parameters. Only α effects the first two solutions; i.e. those in the plane $y=0$. The equations defining the perturbed solutions are:

$$(3.16)\qquad y=0;\ x^2+\lambda x+\alpha=0.$$

It is clear that the second equation in (3.16) defines a hyperbola with asymptotes $x=0$ and $x=-\lambda$. The sign of α determines in which quadrants the hyperbola lies.

Observe that the other solutions satisfy the equations

$$(3.17)\qquad \lambda=2\beta-cx\,;\qquad \left(x-\frac{\beta}{c-1}\right)^2-\frac{\varepsilon}{c-1}y^2=\frac{\beta^2+\alpha(c-1)}{(c-1)^2}$$

where $\varepsilon=\pm 1$ according to whether G^+ or G^- is under analysis. So these solutions – if they exist – lie in the plane $\lambda+cx=2\beta$ which is perpendicular to the plane $y=0$. It is also clear that these solutions form a hyperbola if $\varepsilon(c-1)>0$. This hyperbola intersects the plane $y=0$ only if $\beta^2+\alpha(c-1)>0$. If $\varepsilon(c-1)<0$, then the solutions form an ellipse when $\beta^2+\alpha(c-1)>0$ and do not exist otherwise.

To obtain the qualitative nature of the bifurcation diagrams, one should make two further observations. First, when the conic (3.17) intersects the plane $y=0$ bifurcation must occur, as (3.17) with y set to 0 satisfies (3.16). Second, it is clear

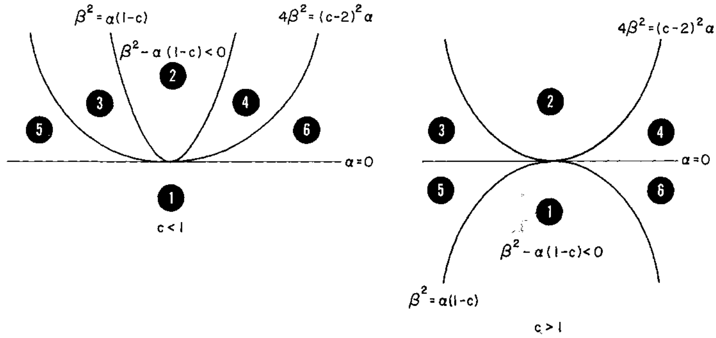

Fig. 4. The division of the $\alpha\beta$-plane by Σ

that when the conic (3.17) intersects $y=0$ it does so at a vertex. However, the intersection point on (3.16) is not necessarily a vertex for that conic. When both conics intersect at their vertices, a hill-top bifurcation occurs (see [6, Sect. 5] or [21]). Points (α, β, c) where such behavior occurs are easily computed as both $F=0$ and $d_x F=0$ must be satisfied. Elimination of x, y, λ yields

$$(3.18) \quad \beta^2=\left(\frac{c}{2}-1\right)^2 \alpha\,; \qquad \alpha \geqq 0.$$

From the above discussion one sees that the qualitative nature of the bifurcation diagrams are found for fixed c by sectioning the $\alpha\beta$-plane with the curves $\alpha=0$, $\beta^2=\alpha(1-c)$, and (3.18). The union of these curves is called Σ and is shown in Fig. 4. Note that the last two equations are identical when $c=0$. The bifurcation diagrams are given in Figs. 5–12. Note also that $c=2$ yields a degeneration of the parabola (3.18) into a ray. This degeneracy is not observable in (3.4) but is observable in the perturbations considered in (3.5). This information is also contained in the figures.

Observe that the parabolas and line defining Σ are all tangent at $\alpha=\beta=0$, this point corresponding to one of the unperturbed problems enumerated by Fig. 3. Let A be a connected component of the complement to Σ in the $\alpha-\beta$ plane. We consider $A_r=\lim_{r\to 0} \text{area}(A\cap B_r)/\text{area}(B_r)$ where B_r is the ball of radius r centered at the origin as a measure of the likelihood of observing bifurcation diagrams represented by parameters in the region A. With this notion of "generic imperfections" the complete description of all the likely bifurcation diagrams is straightforward; namely, only those regions A which include portions of the α-axis are generic.

Finally a degree may be assigned to each solution branch of $F^{\pm}=0$. This degree is just the sign of $\det(d_{xy}F^{\pm})$ and is represented on the figures by $+$'s and $-$'s. In practice, this degree is computed by computing the degrees for the solution branches for the unperturbed problem $G^{\pm}=0$ (these degrees were listed in Fig. 5.6 of [6]), and then matching the degree assigment on the perturbed diagrams at ∞. Since only simple eigenvalues occur in the perturbed diagrams, the principle of "exchange of stability" may be used to complete the assignment of degrees.

4. The Double Cusp with Symmetry

In this section we consider

(4.1) $G(x,y,\lambda)=(p(x,y)-\lambda x,\ q(x,y)-\lambda y)$

where p and q are homogeneous polynomials of degree three. Such bifurcation problems occur in the Von Karman equations model for the buckling of a rectangular plate [4, 7] and in a finite element analogue for this problem considered by Bauer, Keller, and Reiss [2]. As in Sect. 3, we consider only those problems (4.1) which satisfy the non-degeneracy conditions (3.2). As shown in [17] there is a natural $Z_2 \oplus Z_2$ symmetry group for problems of type (4.1) which occur in these physical models. In particular $(-1,0)$ acts as $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ in $\mathbf{R}^2$ and $(0,-1)$ acts as $\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$. We assume that $\Gamma=\mathbf{Z}_2\oplus\mathbf{Z}_2$ acts the same way on both the domain and range of G. The equivariant bifurcation problems (4.1) are those for which p is odd on x and even in y while q is odd in y and even in x. The most general such problem is

(4.2) $G(x,y,\lambda)=(ax^3+bxy^2-\lambda x,\ cx^2y+dy^3-\lambda y)$.

Note that p and q in (4.2) have common factors iff $a=0$, $d=0$, or $ad-bc=0$. Thus the non-degenerate problems (4.2) may be rescaled by a Γ-equivalence so that $|a|=|d|=1$; the rescaling is accomplished by setting $x=\bar{x}/|a|^{1/2}$ and $y=\bar{y}/|d|^{1/2}$ and then multiplying the first coordinate of G by $|a|^{1/2}$ and the second coordinate by $|b|^{1/2}$. If it assumed that the physically interesting problems are those for which the trivial solution $x=y=0$ is the only solution when $\lambda<0$, then we will show that both a and d are positive. In any case we may assume, at the expense of letting $\lambda \mapsto -\lambda$ that $a=1$. So non-degenerate equivariant problems (4.2) have the form

(4.3) $G^{\pm}(x,y,\lambda)=(x^3+bxy^2-\lambda x,\ cx^2y\pm y^3-\lambda y)$

where $bc\neq\varepsilon$. Here $\varepsilon=1$ for G^+ and $\varepsilon=-1$ for G^-. Finally the surfaces $p(x,y)=\lambda x$ and $q(x,y)=\lambda y$ in (4.2) have non-zero points of tangency when $b=d$ or $a=c$. Thus we assume that $c\neq 1$ and $b\neq\varepsilon$ in (4.3). No further simplifications of (4.3) can be made by Γ-equivalences so b and c are modal parameters for these problems.

We shall show that the Γ-codimension of the non-degenerate problems (4.3) is three. This should be compared with the contact codimension which was shown to be 16. (See [6, Sect. 5]). The universal Γ-unfolding is

(4.4) $F^{\pm}(x,y,\lambda,b,c,\alpha)=G^{\pm}(x,y,\lambda,b,c)+(0,\alpha y)$.

The computation of $F^{\pm}$ is similar to those in previous sections; we sketch the results. For the action of $\Gamma=\mathbf{Z}_2\oplus\mathbf{Z}_2$ described above, $\mathscr{E}_2^\Gamma$ consists of function germs $f(x,y)$ which are even in both x and y separately; so $\sigma_1(x,y)=x^2$ and $\sigma_2(x,y)=y^2$. The $\mathscr{E}_2^\Gamma$-module $\mathscr{E}_{2,2}^\Gamma$ was described above; its generators are $X_1=(x,0)$ and $X_2=(0,y)$. The $\mathscr{E}_2^\Gamma$-module $\mathscr{M}_{2,2}^\Gamma$ consists of those matrix-valued functions $T=\begin{pmatrix}t_1 & t_2\\ t_3 & t_4\end{pmatrix}$ where t_1 and t_4 are even in both x and y and t_2 and t_3 are

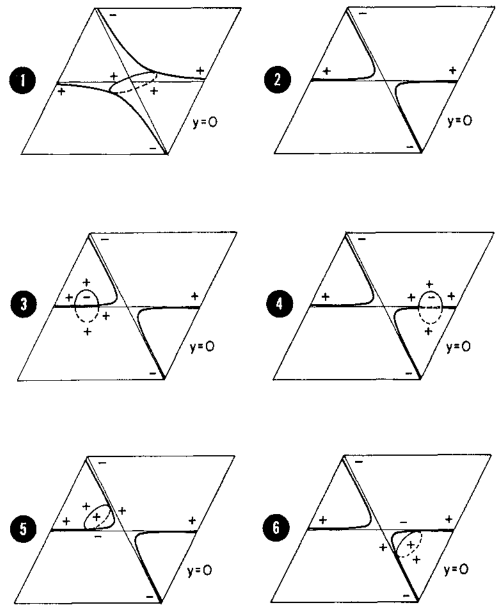

Fig. 5. G^+, $c<0$, $2h$

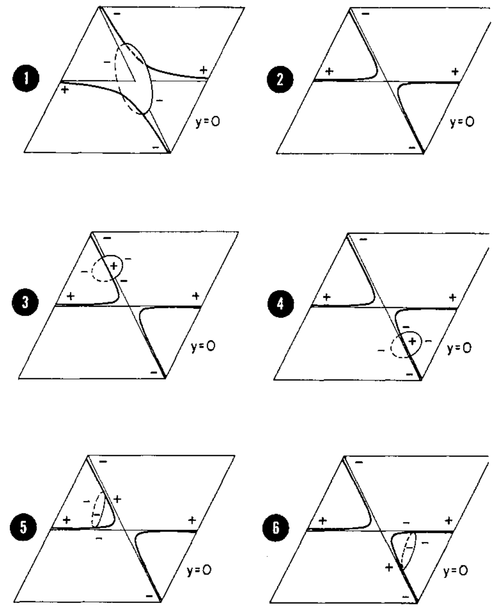

Fig. 6. G^+, $0<c<1$, $2h$

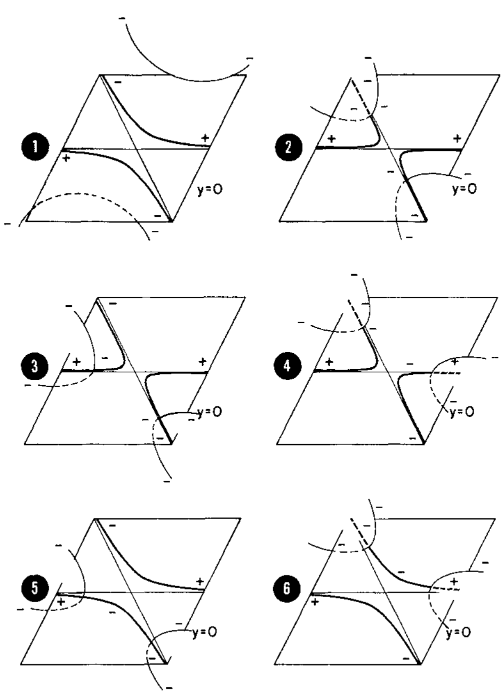

Fig. 7. G^+, $1<c<2$, $4h$

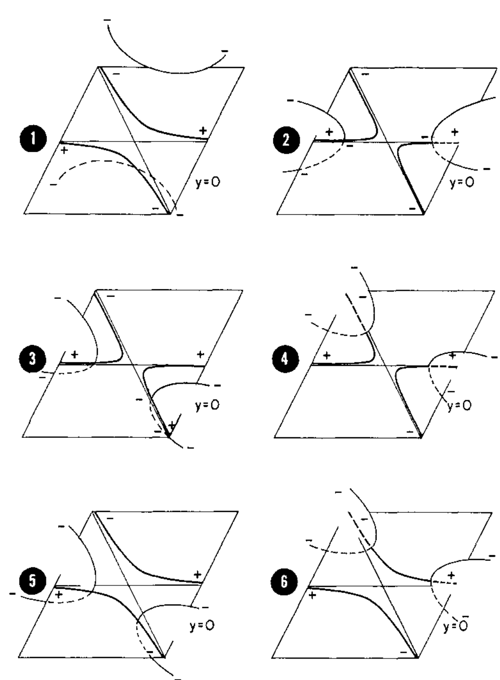

Fig. 8. G^+, $2<c$, $4h$

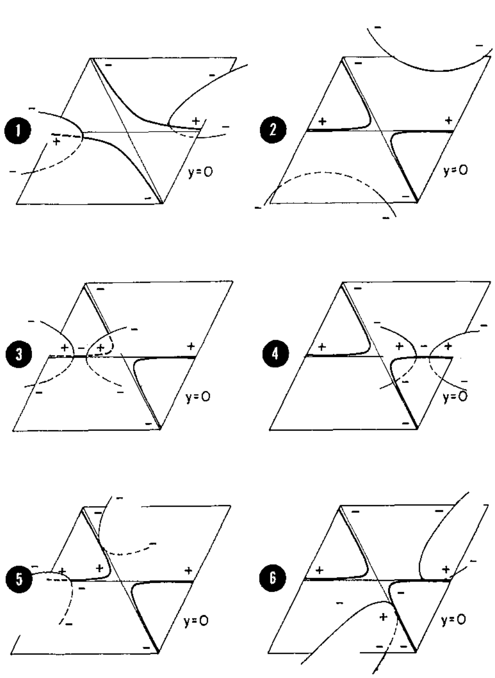

Fig. 9. G^{-}, $c<0$, $4e_0$

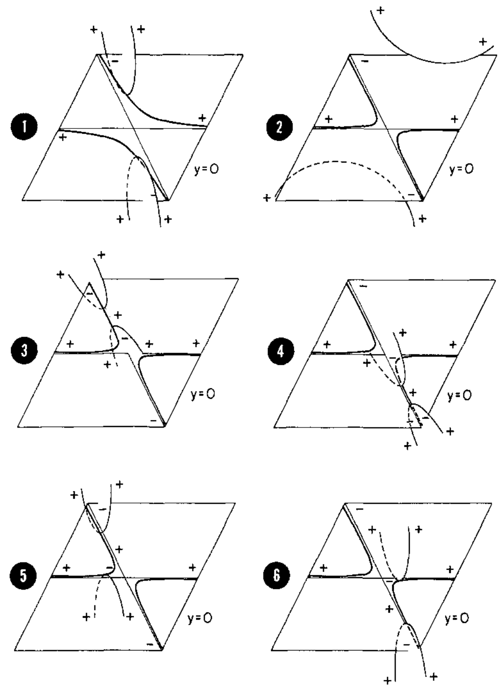

Fig. 10. G^{-}, $0<c<1$, $4e_{\infty}$

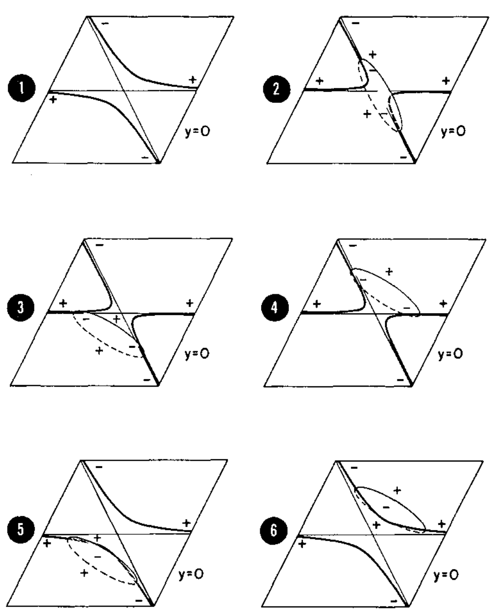

Fig. 11. G^{-}, $1<c<2$, $2e$

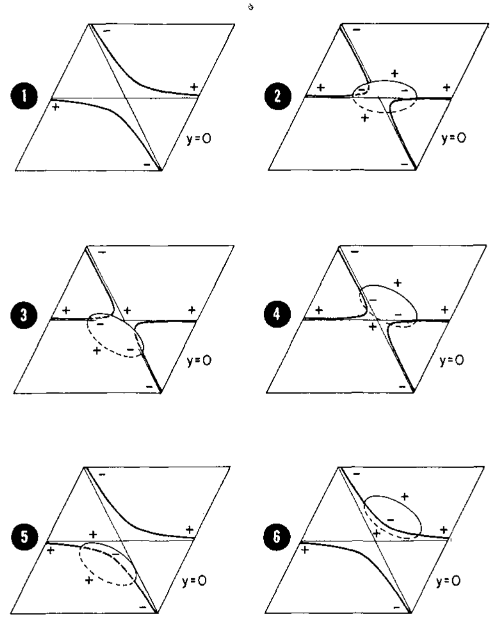

Fig. 12. G^{-}, $2<c$, $2e$

odd in both x and y. Thus $T_1=\begin{pmatrix}1&0\\0&0\end{pmatrix}$, $T_2=\begin{pmatrix}0&xy\\0&0\end{pmatrix}$, $T_3=\begin{pmatrix}0&0\\xy&0\end{pmatrix}$ and $T_4=\begin{pmatrix}0&0\\0&1\end{pmatrix}$ are generators for $\mathcal{M}^{\Gamma}_{2,2}$ over $\mathcal{E}^{\Gamma}_2$.

Let G be in $\mathcal{E}^{\Gamma}_{3,2}$. We may write

(4.5) $G(x,y,\lambda)=(a(x^2,y^2,\lambda)x, b(x^2,y^2,\lambda)y)=aX_1+bX_2$.

A computation shows that $\mathcal{E}^{\Gamma}_{3,2}/\tilde{\Gamma}G\cong\mathcal{E}_{3,2}/\Phi G$ where ΦG is the $\mathcal{E}_3$-module generated by

(4.6) $(a,0),(vb,0),(0,ua),(0,b),(ua_u,ub_u),(va_v,vb_v)$.

Here we denote the coordinates on the domain of $\mathcal{E}_{3,2}$ by u, v, and λ. For the bifurcation problem (4.3), we have that

(4.7) $a(u,v,\lambda)=u+bv-\lambda, b(u,v,\lambda)=cu+\varepsilon v-\lambda$.

The generators for ΦG then reduce to

(4.8) $(u+bv-\lambda,0),(0,cu+\varepsilon v-\lambda),(0,(1-c)u^2+(b-\varepsilon)uv),$
$((c-1)uv-(b-\varepsilon)v^2,0),(u,cu),(bv,\varepsilon v)$.

Using the non-degeneracy assumptions it is not hard to show that

(4.9) $(u+bv-\lambda,0),(0,cu+\varepsilon v-\lambda),(v^2,0),(0,u^2),(u,cu),(\varepsilon bv,v)$

generate ΦG and that all quadratic terms are in ΦG. Thus $\mathcal{E}_{3,2}/\Phi G$ is generated by

(4.10) $(1,0),(0,1),(0,u),(v,0)$.

This implies that

(4.11) $(x,0),(0,y),(0,x^2y),(xy^2,0)$

generate $\mathcal{E}^{\Gamma}_{3,2}/\tilde{\Gamma}G$. Since $\dfrac{\partial G}{\partial\lambda}=-(x,y)$ we see that

(4.12) $(0,y),(0,x^2y),(xy^2,0)$

form a set of generators for $\mathcal{E}^{\Gamma}_{3,2}/\Gamma G$. Thus (4.4) is a universal Γ-unfolding for (4.3).

Lemma 4.13. *The non-degenerate bifurcation problems* (4.3) *are* Γ *3-determined.*

Proof. Let $G_t=G+tH$ where $H(x,y,\lambda)$ is equivariant beginning with terms of order 4. It is clear from (4.9) that $\Phi G_t\subset\Phi G$ for all t. A calculation which involves showing that a 6×6 matrix is invertible shows that $\Phi G_t=\Phi G$. Apply Proposition 1.12.

We next describe the bifurcation diagrams associated to (4.3) as the modal parameters b and c are varied. This information is summarized in Fig. 13. First note the degeneracy conditions $c=1$, $b=\varepsilon$, $bc=\varepsilon$ divide the bc-plane into seven regions. The number of solutions to $G^{\pm}=0$ for fixed λ is constant over each region.

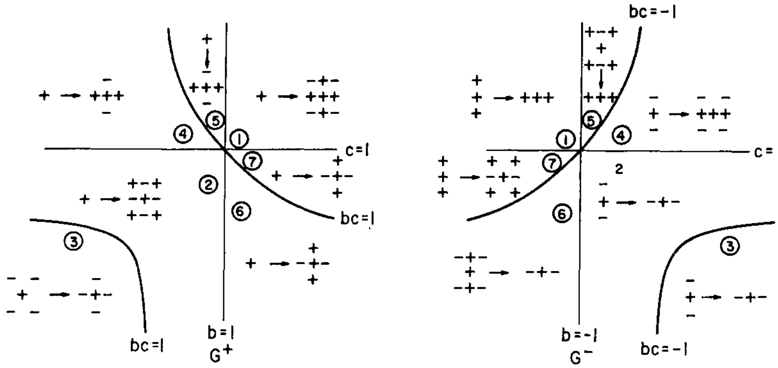

Fig. 13

In fact, the number of solutions depends only on the sign of λ. For $\lambda \neq 0$ a degree may be assigned to each solution that degree being "$-$" when the number of eigenvalues of $d_x G$ with positive real parts is one and "$+$" otherwise. The number of solutions for $\lambda > 0$ and $\lambda < 0$ along with the degrees and relative positions of the solutions in the $\lambda = \pm 1$ planes are shown in Fig. 13.

As mentioned after (4.2) the physically interesting regions occur for G^+ excluding region 3.

The computations to produce Fig. 13 are straightforward. We outline the case G^+. After factoring, the equation $G^+ = 0$ divides into four pairs of equations, grouped vertically

$$(4.14)\quad \begin{matrix} x=0 \\ y=0 \end{matrix}; \quad \begin{matrix} x=0 \\ y^2=\lambda \end{matrix}; \quad \begin{matrix} x^2=\lambda \\ y=0 \end{matrix}; \quad \begin{matrix} x^2+by^2=\lambda \\ cx^2+y^2=\lambda \end{matrix}.$$

The first pair of equations yields the line of trivial solutions. The second and third pairs yield parabolas in the planes $x=0$ and $y=0$ respectively when $\lambda > 0$. On elimination of λ from the last pair of equations in (4.14) we obtain

$$(4.15)\quad (c-1)x^2 = (b-1)y^2.$$

Since $b-1$ and $c-1$ must have the same signs, these solutions can only occur in regions 1, 2, and 3. One then checks that two parabolas of solutions occur when $\lambda > 0$ in regions 1 and 2 for $\lambda < 0$ in region 3. The degree assignments are obtained by computing $d_x G^+$ when $G^+ = 0$. For the four cases defined by (4.14) one has the following formulas for $d_x G^+$:

$$(4.16)\quad \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix} \begin{pmatrix} (b-1)y^2 & 0 \\ 0 & 2y^2 \end{pmatrix} \begin{pmatrix} 2x^2 & 0 \\ 0 & (c-1)x^2 \end{pmatrix} \begin{pmatrix} 2x^2 & 2bxy \\ 2cxy & 2y^2 \end{pmatrix}.$$

The degrees are obvious for the first three cases. For the last case note that $\det(d_x G^+) = 4x^2y^2(1-bc)$. This information is sufficient to complete Fig. 13. Finally note that the modal parameters have no effect on the line of trivial solutions and the parabolas which are in the planes $x=0$ and $y=0$. The modal parameters do affect the planes and focal lengths of the other two parabolas. The effect on the planes is given by (4.15).

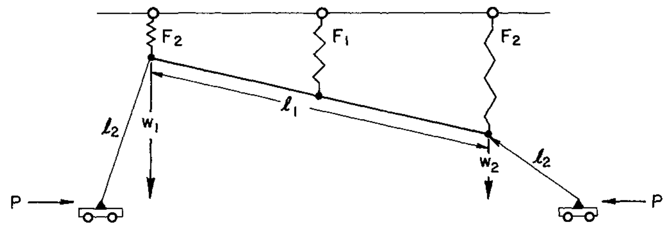

Fig. 14

For each of the fourteen regions in Fig. 13, the effect of the unfolding parameter α in (4.4) may be computed. Some of these effects were explored in the paper of Bauer, Keller, and Reiss [2]. They observed that secondary bifurcations occur in the bifurcation diagrams associated to regions 1, 7, and 6 if α is non-zero. We add to this observation by noting that secondary bifurcation may be understood just from the degree considerations alone. Note that the equations $F^+=0$ divide into four cases upon factoring

$$(4.17)\qquad \begin{matrix} x=0 \\ y=0 \end{matrix};\quad \begin{matrix} x=0 \\ y^2=\lambda-\alpha \end{matrix};\quad \begin{matrix} x^2=\lambda \\ y=0 \end{matrix};\quad \begin{matrix} x^2+by^2=\lambda \\ cx^2+y^2=\lambda-\alpha \end{matrix}.$$

The degree assignments are given for the first three cases by the eigenvalues of the matrices d_xF^+.

$$(4.18)\qquad \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix} \begin{pmatrix} (b-1)\lambda-b\alpha & 0 \\ 0 & 2(\lambda-\alpha) \end{pmatrix} \begin{pmatrix} 2\lambda & 0 \\ 0 & (c-1)\lambda+\alpha \end{pmatrix}.$$

Examining for example region 1, given by $b>1$ and $c>1$, we see that when $\alpha<0$ the degree of the third case is "$-$" when λ is small and positive. For λ large the degree of that branch is "$+$"; the change in degree is effected by a secondary bifurcation. Similarly if $\alpha>0$ then the second pair of equations yields a solution if $\lambda>\alpha$. However for $\lambda<\frac{b}{b-1}\alpha$ that solution has degree "$-$". Again for λ large that degree is "$+$", so a secondary bifurcation occurs. We also point out that secondary bifurcation fails to occur when symmetry breaking perturbations are added to this system.

For the remainder of this section we describe a rod and spring problem considered by Bauer, Keller, and Reiss. Aside from providing an example for our theory this analysis will add two points. First the fact that (4.4) is a universal Γ-unfolding implies that $B-K-R$ had indeed found all of the relevant symmetry preserving parameters. Second, the determinacy result (4.13) proves that nothing qualitative would be gained by analyzing higher order terms – at least in the interior of the regions of the modal plane (Fig. 13).

The configuration of the rod and spring system is given in Fig. 14. The springs F_1 and F_2 are non-linear with restoring forces $F_i(w)=\alpha_i w+\beta_i w^3$ $(i=1,2)$ for a displacement w. In the figure, P is an applied force, l_1 and l_2 are rod lengths, and w_1 and w_2 describe the states of the system. Spring F_1 is attached at the midpoint of rod l_1. It is assumed that with $P=0$ the state $w_1=w_2=0$ is the rest state of the system.

We note that there are two natural symmetries in the problem; namely, if (w_1, w_2) is a solution for a given load P then so are (w_2, w_1) and $(-w_1, -w_2)$.

The analysis in [2] leads to the following bifurcation problem – after neglecting higher order terms –

$$(4.19)\quad H(u,v,\mu)=\left(\tau^2u^3+3auv^2+(\tau^2-\mu\tau)u, v^3+\frac{2\tau}{\sigma a}u^2v+(1-\mu\tau)v\right)$$

where μ is proportional to P, u is proportional to w_1+w_2, v is proportional to w_1-w_2, a is $1+2l_2/l_1$, σ is $a(2+\beta_1/\beta_2)/6$, and τ is a composite variable depending on all of the parameters in the problem except P.

The symmetry described above translates in the (u,v) coordinates to the following; if (u,v) is a solution to $H=0$, then so are $(u,-v)$ and $(-u,-v)$. Thus H is a bifurcation problem with the symmetry group $\Gamma=\mathbf{Z}_2\oplus\mathbf{Z}_2$.

We next scale H to the form of the unfolding (4.4) by setting $x=u$, $y=v/\tau$, $\lambda=\frac{\mu}{\tau}-1$, $\alpha=1-1/\tau^2$, and premultiplying the resulting equations by the matrix $\begin{pmatrix}\tau^2 & 0\\ 0 & \tau^3\end{pmatrix}^{-1}$ thus obtaining

$$(4.20)\quad F(x,y,\lambda)=\left(x^3+3axy^2-\lambda x, \frac{2}{\sigma a}x^2y+y^3-(\lambda+\alpha)y\right).$$

Note that this is a bifurcation problem of type G^+; moreover, the modal parameter $b=3a$ is always greater than 1, so that only regions 1, 7, and 6 in Fig. 13 are accessible. $B-K-R$ make the observation that the qualitative nature of the problem is independent of a – which is confirmed by our theory – and then they set $a=2$. Thus the modal parameters are $b=6$ and $c=1/\sigma$. As was noted by $B-K-R$ the problems for $\sigma<1$, $1<\sigma<6$, and $6<\sigma$ are qualitatively different. These restrictions correspond to problems from regions 1, 7, and 6. Qualitatively only the sign of α matters. This corresponds to the regions $\tau>1$, $\tau=1$, and $\tau<1$ studied by $B-K-R$.

We shall apply the results of this section in [17] to give a mathematical explanation for the phenomenon of mode jumping observed in the buckling of a rectangular plate near a double eigenvalue.

5. An Example with O(2) Symmetry

We consider as a model the bifurcation problem

$$(5.1)\quad \begin{aligned} N_\lambda u=\Delta u+(\lambda+\lambda_0)h(u)=0 &\quad \text{on}\quad D\\ u=0 &\quad \text{on}\quad \partial D\end{aligned}$$

where D is the unit disk in the plane, Δ is the Laplace operator, $h(u)=u+\bar{h}(u)$ where $\bar{h}(u)=au^3+\dots$ with $a\neq 0$, and λ_0 is an eigenvalue for the linearized problem

$$(5.2)\quad \begin{aligned} L\varphi \equiv \Delta\varphi+\lambda_0\varphi=0 &\quad \text{on} \quad D\\ \varphi=0 &\quad \text{on} \quad \partial D \end{aligned}$$

at $\lambda=0$. For definiteness we assume that $N_\lambda: \mathscr{H}\to L^2(D)$ where $\mathscr{H}$ is the Hilbert space $\mathscr{H}_2(D)\cap\mathring{\mathscr{H}}_1(D)$ consisting of those functions which vanish on ∂D and whose derivatives of order $\leqq 2$ are all in $L^2(D)$. We consider $L^2(D)$ with the standard inner product $\langle u,v\rangle=\int_D uv$.

Note that the eigenvalues and eigenfunctions for Δ with Dirichlet boundary conditions on D may be computed by separation of variables in polar coordinates. The eigenvalues are $\lambda_{m,n}>0$ where $\sqrt{\lambda_{m,n}}$ is the m-th root of the Bessel function $J_n(r)$; the corresponding eigenfunctions are

$$(5.3)\quad \{J_n(\sqrt{\lambda_{m,n}}\,r)\cos(n\theta), J_n(\sqrt{\lambda_{m,n}}\,r)\sin(n\theta)\}.$$

Note that $\lambda_{m,n}$ is a double eigenvalue for Δ when $n\geqq 1$. As a simple example of our techniques, we analyze the bifurcation of (5.1) from the trivial solution $u\equiv 0$, near $\lambda=0$, when $\lambda_0=\lambda_{m,n}$ for some m, n with $n\geqq 1$.

It is clear that (5.1) is invariant under the group of rotations in the plane $\mathbf{SO}(2)$ and, in fact, invariant under $O(2)$ – the group of 2×2 orthogonal matrices. As observed by Sattinger [14] the Lyapunov-Schmidt reduction of (5.1) yields a bifurcation problem $G:(\mathbf{R}^2\times\mathbf{R},0)\to(\mathbf{R}^2,0)$ of the form

$$(5.4)\quad G(x,y,\lambda)=0$$

where G is equivariant with respect to $\mathbf{O}(2)$. We shall produce this reduction, but first we discuss the algebraic properties of $\mathbf{O}(2)$.

Let $\Gamma=\mathbf{O}(2)$; the action of Γ on $\mathbf{R}^2$ is the standard one given by matrix multiplication. The ring of invariant functions $\mathscr{E}_2^\Gamma$ consists of smooth germs $f(x^2+y^2)$ where f is arbitrary.

Lemma 5.5. *The module $\mathscr{E}_{2,2}^\Gamma$ is generated by the one element (x,y) over $\mathscr{E}_2^\Gamma$.*

Proof. Let g be a polynomial mapping in $\mathscr{E}_{2,2}^\Gamma$. We write g in the complex coordinates z, $\bar{z}$ as

$$(5.6)\quad g(z,\bar{z})=\sum(a_{jk}+ib_{jk})z^j\bar{z}^k$$

where $a_{j,k}$ and $b_{j,k}$ are real. The action of $\mathbf{SO}(2)$ on (5.6) is easy to describe, being given by complex multiplication by $e^{i\theta}$. The equivariance of g may be written as

$$(5.7)\quad e^{-i\theta}g(e^{i\theta}z,e^{-i\theta}\bar{z})\equiv g(z,\bar{z}).$$

The identity (5.7) holds for (5.6) only if the summation is over pairs (j,k) with $j=k+1$. So (5.6) may be written as

$$(5.8)\quad g(z,\bar{z})=\sum_k(a_k+ib_k)z(z\bar{z})^k.$$

As $z\bar{z}=x^2+y^2$, we have that g is in the module generated by z and iz over $\mathscr{E}_2^\Gamma$; that is, the module generated by (x,y) and $(-y,x)$. A quick check shows that $(-y,x)$ is not invariant under $\mathbf{O}(2)$; thus proving the lemma for polynomials. We now apply Schwarz's Theorem [19] described in Sect. 1.

In the language of Sect. 1, we have that $X_1=(x,y)$. We now compute a set of generators for $\mathscr{M}_{2,2}^\Gamma$.

Lemma 5.9. *$\mathscr{M}_{2,2}^\Gamma$ is generated by* $T_1=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, $T_2=\begin{pmatrix}x^2-y^2 & 2xy\\ 2xy & y^2-x^2\end{pmatrix}$, $T_3=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$, *and* $T_4=\begin{pmatrix}-2xy & x^2-y^2\\ x^2-y^2 & 2xy\end{pmatrix}$ *as a module over $\mathscr{E}_2^\Gamma$ when $\Gamma=\mathbf{SO}(2)$. Moreover, T_1 and T_2 generate $\mathscr{M}_{2,2}^\Gamma$ when $\Gamma=\mathbf{O}(2)$.*

Proof. The last statement follows from the first as T_3 and T_4 do not satisfy the equivariance condition (1.3) when $\gamma=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$, while T_1 and T_2 do satisfy this condition. The proof of the first statement again uses complex notation.

Let $T(z,\bar{z})$ be in $\mathscr{M}_{2,2}^\Gamma$. Then $T(z,\bar{z})$ is a linear mapping on $\mathbf{R}^2$ satisfying (1.3). We write T in terms of the complex coordinates $w,\bar{w}$. So

$$(5.10)\quad T(z,\bar{z},w,\bar{w})=t_1(z,\bar{z})w+t_2(z,\bar{z})\bar{w} + i(t_3(z,\bar{z})w+t_4(z,\bar{z})\bar{w})$$

where each t_i is real-valued. Rotation through angle θ is obtained by multiplying by $e^{i\theta}$. So condition (1.3) becomes

$$(5.11)\quad e^{-i\theta}T(e^{i\theta}z,\, e^{-i\theta}\bar{z},\, e^{i\theta}w,\, e^{-i\theta}\bar{w})=T(z,\bar{z},w,\bar{w}).$$

Applying (5.11) to the form of T given in (5.10) yields

$$(5.12)\quad \begin{aligned} &t_1(e^{i\theta}z,e^{-i\theta}\bar{z})=t_1(z,\bar{z}); && t_3(e^{i\theta}z,e^{-i\theta}\bar{z})=t_3(z,\bar{z});\\ &t_2(e^{i\theta}z,e^{-i\theta}\bar{z})=e^{2i\theta}t_2(z,\bar{z}); && t_4(e^{i\theta}z,e^{-i\theta}\bar{z})=e^{2i\theta}t_4(z,\bar{z}).\end{aligned}$$

Hence $\mathscr{M}_{2,2}^\Gamma$ when $\Gamma=\mathbf{SO}(2)$ is generated by w, iw, $z^2\bar{w}$, and $iz^2\bar{w}$ as a module over $\mathscr{E}_2^\Gamma$. Translation of these generators into matrix form proves the lemma.

From Lemma 5.5 we observe that any bifurcation problem (5.4) with symmetry group $\mathbf{O}(2)$ has the form

$$(5.13)\quad G(x,y,\lambda)=(g(x^2+y^2,\lambda)(x,y)$$

where $g(z,\lambda)$ is a smooth germ.

Using the methods of Sect. 1, one computes that

$$(5.14)\quad \mathscr{E}_{3,2}^\Gamma/\tilde{T}G\cong\mathscr{E}_{2,1}/\Phi G \quad \text{where} \quad \Phi G=\langle g, zg_z\rangle.$$

Remark. Any bifurcation problem (5.13) has infinite Γ-codimension when $\Gamma=\mathbf{SO}(2)$. Note that in the proof of Lemma 5.5, $X_1=(x,y)$ and $X_2=(-y,x)$ were shown to be generators for $\mathscr{E}_{2,2}^\Gamma$ when $\Gamma=\mathbf{SO}(2)$. It follows that

$$(5.15)\quad \mathscr{E}_{3,2}^\Gamma/\tilde{T}G\cong\mathscr{E}_{2,2}/\langle (g,0),(zg_z,0),(0,g)\rangle.$$

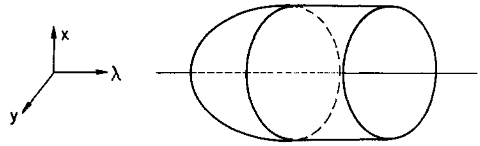

Fig. 15

So G has infinite codimension whenever $g(0)=0$ since at least four generators are needed for ΦG if G were to have finite codimension. For universal Γ-unfoldings the difference between using $\mathbf{O}(2)$ and $\mathbf{SO}(2)$ symmetry is dramatic.

This dichotomy can be observed on the diagram level. The bifurcation diagram $G=0$ for (5.13) has two parts. The first is the trivial solution $x=y=0$ and the second is the hypersurface $g(x^2+y^2,\lambda)=0$. Consider the perturbed problem

$$(5.16)\quad G_\varepsilon(x,y,\lambda)=g(x^2+y^2,\lambda)(x,y)+\varepsilon(-y,x).$$

Note that the symmetry group for G_ε is $\mathbf{SO}(2)$ not $\mathbf{O}(2)$. It is easy to check that only the trivial solution remains as a solution to $G_\varepsilon=0$. This problem has been considered from another point of view by Dancer [5].

Next consider the sample problem

$$(5.17)\quad G(x,y,\lambda)=(x^2+y^2-\lambda)(x,y).$$

From (5.14) we see that $\dim \mathscr{E}^\Gamma_{3,2}/\tilde{\Gamma}G=1$. (We are now using $\Gamma=\mathbf{O}(2)$.) In fact, $\mathscr{E}^\Gamma_{3,2}/\tilde{\Gamma}G$ is spanned by $\{(x,y)\}$. Next note that $\frac{\partial G}{\partial\lambda}=-(x,y)$ so that Γ-codim $G=0$. Thus (5.17) is a Γ-(infinitesimally) stable bifurcation problem whose universal Γ-unfolding is itself. The bifurcation diagram associated with (5.17) is given in Fig. 15.

Note that (5.17) is a degenerate double cusp problem (4.3) which has infinite contact codimension.

Lemma 5.18. *Let $H(x,y,\lambda)$ be a bifurcation problem with symmetry group* $\mathbf{O}(2)$. *Let $H(x,y,\lambda)=h(x^2+y^2,\lambda)(x,y)$ where $h(z,\lambda)=az-b\lambda+\dots$ with $a>0$ and $b>0$. Then H is* $\mathbf{O}(2)$*-equivalent to* (5.17).

Proof. First rescale z and λ so that $a=b=1$. Observe that now H is a perturbation of G and that the perturbation term $H-G$ when viewed in $\mathscr{E}_{2,1}/\Phi G$ via (5.14) is actually in $\mathscr{M}\Phi G$. Apply Lemma 1.13 and Proposition 1.12.

We are now in a position to perform the Lyapunov-Schmidt reduction of (5.1).

Proposition 5.19. *The reduced bifurcation equations associated to* (5.1) *are* $\mathbf{O}(2)$*-equivalent to* (5.17).

The Lyapunov-Schmidt procedure is an elementary application of the Implicit Function Theorem. Let φ_1 and φ_2 be an orthonormal basis for $\operatorname{Ker} L$ relative to the inner product $\langle f,g\rangle=\int_D fg$. Thus φ_1 and φ_2 are the eigenfunctions (5.3) scaled to have norm 1. Define $E: L^2(D) \to \operatorname{Ker} L$ by

(5.20) $\quad E(\varphi) = \langle \varphi, \varphi_1 \rangle \varphi_1 + \langle \varphi, \varphi_2 \rangle \varphi_2$.

Let V be the orthogonal complement to $\operatorname{Ker} L$ in $\mathscr{H}$. Define $W: \operatorname{Ker} L \times \mathbf{R} \to V$ implicitly by

(5.21) $\quad E' N_\lambda(l + W(l, \lambda)) = 0\,; \qquad W(0) = 0$

for all $l \in \operatorname{Ker} L$ where $E' = id - E$. Since $L|V$ is non-singular the Implicit Function Theorem guarantees the existence of W. Observe that

(5.22) $\quad N_\lambda(l + W(l, \lambda)) = 0 \quad \text{iff} \quad E N_\lambda(l + W(l, \lambda)) = 0$

by (5.21) and define

(5.23) $\quad H(x, y, \lambda) = E N_\lambda(x\varphi_1 + y\varphi_2 + W(x, y, \lambda))$

where $W(x, y, \lambda) = W(x\varphi_1 + y\varphi_2, \lambda)$ Expand N_λ by its definition (5.1) and use φ_1, φ_2 as coordinates on the range of H to obtain

(5.24) $\quad H(x, y, \lambda) = \lambda(x, y) + (\lambda + \lambda_0)$
$$\cdot(\langle \bar{h}(x\varphi_1 + y\varphi_2 + W, \varphi_1 \rangle), \langle \bar{h}(x\varphi_1 + y\varphi_2 + W, \varphi_2 \rangle)).$$

Since $\bar{h}(0) = \bar{h}'(0) = \bar{h}''(0) = 0$ we have that

(5.25) $\quad H(x, y, \lambda) = \lambda(x, y) + \text{terms of order } 3 + \ldots$.

Since H must be equivariant with respect to $\mathbf{O}(2)$ there is only one possible term of order 3; namely,

(5.26) $\quad K(x^2 + y^2)(x, y)$

for some constant K. If we can show that $K \neq 0$ then we may apply Lemma 5.18 and prove the proposition. To compute K we need only compute the first coordinate of $\frac{\partial^3 H}{\partial x^3}(0)$. From (5.24) we see that this coordinate is

(5.27) $\quad \lambda_0 \bar{h}'''(0) \langle (\varphi_1 + W_x(0))^3, \varphi_1 \rangle$.

We claim that

(5.28) $\quad W_x(0) = 0$.

Assuming this, it is clear that $\langle \varphi_1^3, \varphi_1 \rangle = \int_D \varphi_1^4 > 0$ which along with the assumption that $\bar{h}'''(0) \neq 0$ proves the proposition. To prove (5.28) we use (5.21) in the form

(5.29) $\quad H(x, y, \lambda) = N_\lambda(x\varphi_1 + y\varphi_2 + W)$

which implies

(5.30) $\quad LW = (\lambda + \lambda_0)\,[\langle \bar{h}, \varphi_1 \rangle \varphi_1 + \langle \bar{h}, \varphi_2 \rangle \varphi_2 - \bar{h}]$

where $\bar{h}=\bar{h}(x\varphi_1+y\varphi_2+W)$. Using the fact that $\bar{h}'(0)=0$, we have that $LW_x(0)=0$. Since the range of W is in V so is $W_x(0)$. Since $L|V$ is an isomorphism we see that (5.28) holds.

6. Buckling of an Annular Plate

It is generally known that when a clamped circular plate is subjected to a uniform compressive force the plate will first buckle into a radially symmetric mode. In [8] Majumdar observes that this is not necessarily the case when the plate is annular rather than circular, the inner edge being free. The appropriate parameter is the ratio, τ, of the inner to the outer radius of the annulus. When τ is small the annular plate behaves like a circular plate. For $\tau \approx 0.42$ both a radially symmetric and a radially non-symmetric mode occur simultaneously as the first buckling mode. This is clearly a bifurcation problem with a multiple eigenvalue.

Before describing the model for this problem note that any idealization will include an $\mathbf{O}(2)\times\mathbf{Z}_2$ symmetry group. The action of $\mathbf{O}(2)$ comes from the circular symmetry of the problem while the action of $\mathbf{Z}_2$ is induced by the fact that buckling up is assumed to be the same as buckling down. The inclusion of $\mathbf{O}(2)$ symmetry implies that the non-radially symmetric mode must itself be a double eigenvalue similar to the problem considered in the last section. In fact, the simplest situation for τ set at this critical ratio is that the resulting eigenvalue is triple. This is in fact what occurs and we analyze this situation first in the abstract.

Let $\Gamma=\mathbf{O}(2)\times\mathbf{Z}_2$ act on $\mathbf{R}^2\times\mathbf{R}$ in the following way: $\mathbf{O}(2)$ acts by matrix multiplication on the first factor and $\mathbf{Z}_2$ acts by scalar multiplication on the second factor. The invariant functions $f(x,y,z)\in\mathscr{E}_3^\Gamma$ are clearly of the form $h(x^2+y^2,z^2)$. In the notation of Sect. 1 we let $\sigma_1(x,y,z)=x^2+y^2$ and $\sigma_2=z^2$. The $\mathscr{E}_3^\Gamma$- module $\mathscr{E}_{3,3}^\Gamma$ is generated by $X_1=(x,y,0)$ and $X_2=(0,0,z)$. The computation of a set of generators for $\mathscr{M}_{3,3}^\Gamma$ is similar to the computations for $\mathscr{M}_{2,2}^\Gamma$ in Sect. 5. There are five generators; namely,

$$(6.1)\quad T_1=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}\quad T_2=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\quad T_3=\begin{pmatrix}x^2-y^2&2xy&0\\2xy&y^2-x^2&0\\0&0&0\end{pmatrix}$$

$$T_4=\begin{pmatrix}0&0&xz\\0&0&yz\\0&0&0\end{pmatrix}\quad T_5=\begin{pmatrix}0&0&0\\0&0&0\\xz&yz&0\end{pmatrix}.$$

The general Γ-equivariant bifurcation problem in $\mathscr{E}_{4,3}^\Gamma$ has the form

$$(6.2)\quad G(x,y,z,\lambda)=a_1(\sigma_1,\sigma_2,\lambda)X_1+a_2(\sigma_1,\sigma_2,\lambda)X_2.$$

A straightforward calculation shows that the $\mathscr{E}_3^\Gamma$ module ΦG is generated by

$$(6.3)\quad (a,0),\ (0,ua),\ (vb,0),\ (0,b),\ (ua_u,ub_u),\ (va_v,vb_v)$$

where a and b are functions of u, v, and λ. The similarity of this result with (4.6) is apparant.

As the buckling of the annular plate has only one extra parameter, τ, we should expect – if the philosophy outlined in the Introduction is to be satisfied – that the analysis of the bifurcation equations of the plate should yield a singularity with codimension (excluding modal parameters) equal to one. Moreover, the role of τ is to split the multiple eigenvalue into simple eigenvalues. The simplest such case is the problem with two modal parameters considered in Sect. 4. In particular

(6.4) $a(u, v, \lambda) = u + bv - \lambda$; $b(u, v, \lambda) = cu + \varepsilon v - \lambda$.

In terms of x, y, z, λ we have

(6.5) $a = x^2 + y^2 + bz^2 - \lambda$; $b = c(x^2 + y^2) + \varepsilon z^2 - \lambda$.

Again, for a physically motivated problem, ε should equal $+1$. A good guess for the form of the universal unfolding for the plate problem is

(6.6) $$F(x, y, z, \lambda, \tau) = (x^3 + xy^2 + bxz^2 - \lambda x,\ x^2 y + y^3 + byz^2 - \lambda y,\ c(x^2 + y^2)z + z^3 - (\lambda - \tau)z)$$

where b and c are modal parameters depending on the various constants in the problem. As mentioned in Sect. 4, the qualitative nature of the perturbed bifurcation equations depends on the specific values of the modal parameters. We hope to treat in the future the Lyapunov-Schmidt reduction, either explicitly or numerically, to determine the specific values of a and b for this problem, as we have done for the rectangular plate in [17].

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