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ABSTRACT: The continuum damage mechanics-based elasto-plastic damage theory, that extends the total form of Hayakawa and Murakami equations, is developed. Weak elastic-plastic dissipation coupling is assumed by the use of two dissipation potentials, plastic and damage, where only isotropic plasticity and damage hardening is included, whereas kinematic hardening is not accounted for. Unilateral damage condition, based on the concept of generalized projection operators, accounts for a partial damage deactivation, which allows for an influence of negative principal components of the stress tensor on damage evolution. The incremental representation of the elastic-damage constitutive equations is derived. Both elastic-damage and plastic-damage compliance matrices are developed for plane stress condition, and implemented to ABAQUS finite element code by the user-supplied procedure for non-standard material properties. Effective computation algorithm for plastic and damage loading/unloading conditions based on the doubly passive predictor and plastic-damage corrector approach is proposed. Numerical examples are presented by applying the model calibration by Hayakawa and Murakami for the spheroidized graphite cast iron FCD400. The examples illustrate the capability of the model to describe elastic-plastic damage evolution under monotonic loading. Under reverse loading conditions a partial elastic stiffness recovery was demonstrated on the consecutive increasing strain-controlled loading cycles and some limitation of the model was shown.

KEY WORDS: elasto-plastic damage, damage-induced anisotropy, damage deactivation, stiffness recovery, doubly passive predictor–corrector approach.

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INTRODUCTION

IN MOST ENGINEERING materials the nonlinear material softening may be explained by two distinct degradation phenomena: plasticity due to dislocation along slip planes and damage due to microcrack nucleation and growth. This is why double-dissipative coupled models based on thermodynamic frameworks have been developed during the last two decades, e.g., Hansen and Shreyer (1994); Zhu and Cescotto (1995); Chaboche (1997); Hayakawa and Murakami (1997); Abu Al-Rub and Voyiadjis (2003). These models are usually highly complicated, so that they can prescribe and predict complex physical phenomena, including initial or damage anisotropy, damage deactivation, to mention only the most important of them. A consistent thermodynamic formalism is well established and developed; however, calibration of the complicated constitutive equations and methods of their implementation to finite element codes are still difficult tasks.

Recently, Kuna-Ciskał and Skrzypek (2004) have adopted and extended the model of elastic-damage material, initiated by Murakami and Kamiya (1997), to describe damage anisotropy in concrete. The effective elastic-damage stiffness matrix has explicitly been derived and implemented to the finite element ABAQUS code. By adopting loss of the positive definiteness of the tangent stiffness matrix as the failure criterion, the model has effectively been used to simulate and predict different crack growth mechanisms in a plane-stress concrete specimen under tension or compression.

A motivation for the present article is to include the second dissipation mechanism due to plastic flow to the model explored so far, and to implement the coupled elastic-plastic-damage constitutive equations to a commercial FEM code. The use of the theory developed by Hayakawa and Murakami (1997) is the simplest way to achieve this goal. Although the present formulation is based on the Gibbs thermodynamic potential, contrary to the Murakami and Kamiya model where the Helmholtz free energy was used as the state potential, in what follows the elastic-damage compliance matrix has the analogous representation to the previously derived elastic-damage stiffness matrix (Kuna-Ciskał and Skrzypek, 2004). However, the numerical algorithm developed and implemented by the present work is much more complicated than the previously used one, but it can capture various damage/plasticity dissipation mechanisms: passive/passive, passive/active, active/passive, and active/active, and ensure stability and convergence. The theory formulated by Hayakawa and Murakami (1997, 1998) was proved by the series of experiments, which calibrated it for the spheroidized graphite cast iron. Accounting for an additional plastic softening, the recent investigations show the capability of the method developed, for capturing complex plane stress loading conditions, without

the necessity to introduce additional failure criterion (Skrzypek et al., 2004).

Implementation of the model to the commercial FE ABAQUS code has considerable meaning in engineering application of damage mechanics.

COUPLED ELASTO-PLASTIC DAMAGE THEORIES – STATE OF THE ART

A thermodynamically consistent framework for description of elasto-(visco)-plasticity coupled with damage meets two general questions:

(i) how to include damage to the Helmholtz free energy or the Gibbs thermodynamic potential (state coupling) and
(ii) how to couple dissipation potential(s) with damage (dissipation coupling).

State Coupling

Three cases can be distinguished (Chaboche, 1999):

NO STATE COUPLING

When the simplest approach is used, the state potential term Ψ^{p} associated with plastic hardening α_j is not coupled with damage; in other words damage affects only the elastic term Ψ^{e} through the damage variable D, e.g., Chaboche (1977)

$$\Psi = \Psi^{\mathrm{e}}(\varepsilon^{\mathrm{e}}, D) + \Psi^{\mathrm{p}}\left(\alpha_j\right) \tag{1}$$

or Mou and Han (1996)

$$\Psi = \Psi^{\mathrm{e}}(\varepsilon^{\mathrm{e}}, D, T) + \Psi^{\mathrm{p}}_{\mathrm{iso}}(r, T) + \Psi^{\mathrm{p}}_{\mathrm{kin}}(\alpha, T) \tag{2}$$

where both isotropic r and kinematic hardening α variables are not coupled with damage.

STATE COUPLING THROUGH THE ADDITIONAL TERM OF THE STATE POTENTIAL Ψ^{d}

In this concept new state variable β, that is a scalar measure of cumulative damage, is introduced by the analogy with isotropic hardening variable r in plasticity. This approach was introduced by Cordebois and Sidoroff (1982)

$$\Psi = \Psi^{\mathrm{e}}(\varepsilon^{\mathrm{e}}, \boldsymbol{D}, T) + \Psi^{\mathrm{p}}\left(\alpha_j\right) + \Psi^{\mathrm{d}}(\beta) \tag{3}$$

where $\boldsymbol{D}$ is the second-rank damage tensor, and α_j denotes the set of hardening variables for plasticity. A similar concept was used by Zhu and Cescotto (1995), where only isotropic plastic hardening is considered. A more general formulation was recently published by Voyiadjis and Deliktas (2000) and Abu Al-Rub and Voyiadjis (2003), where kinematic and isotropic hardening was admitted for both plasticity (α, p) and damage $(\boldsymbol{\Gamma}, r)$ terms

$$\Psi = \Psi^{\mathrm{E}}\left(\varepsilon^{\mathrm{E}}, \boldsymbol{\Phi}\right) + \Psi^{\mathrm{p}}(\alpha, p) + \Psi^{\mathrm{d}}(\boldsymbol{\Gamma}, r) \tag{4}$$

whereas $\varepsilon^{\mathrm{E}} = \varepsilon^{\mathrm{e}} + \varepsilon^{\mathrm{ed}}$ denotes a sum of the ordinary elastic strain and the elastic-damage strain (reversible) and $\boldsymbol{\Phi}$ denotes the damage tensor.

A slightly different approach was proposed by Hayakawa and Murakami (1997), who introduced the Gibbs thermodynamic potential, instead of the usually applied Helmholtz free energy, $\Gamma(\sigma, r, \boldsymbol{D}, \beta) = \sigma : \varepsilon^{\mathrm{e}} - \Psi$

$$\Gamma = \Gamma^{\mathrm{e}}(\sigma, \boldsymbol{D}) + \Gamma^{\mathrm{p}}(r) + \Gamma^{\mathrm{d}}(\beta) \tag{5}$$

However, for the sake of simplicity only isotropic plasticity r and damage β hardening was admitted. This model is applied as the basis for further extension in this article.

DIRECT HARDENING-DAMAGE COUPLING

A more general approach consists in a hardening-damage coupling independent of variable β (through the damage variable D). Such an approach is used e.g., in Saanouni et al. (1994), where classical variables $\varepsilon^{\mathrm{e}}, \alpha$, and r are replaced by the effective variables $\widetilde{\varepsilon}^{\mathrm{e}}(D), \widetilde{\boldsymbol{\alpha}}(D)$, and $\widetilde{r}(D)$ by the use of total energy equivalence principle, to yield the following Helmholtz free energy representation

$$\Psi = \Psi^{\mathrm{e}}(\widetilde{\varepsilon}^{\mathrm{e}}, T) + \Psi^{\mathrm{an}}(\widetilde{\boldsymbol{\alpha}}, \widetilde{r}). \tag{6}$$

A more general formulation was given by Hansen and Schreyer (1994), who proposed a separable form of the free energy

$$\Psi = W(\varepsilon, \varepsilon^{\mathrm{p}}, \boldsymbol{D}) + H\left(\varepsilon^{\mathrm{S}}, \varepsilon^{\mathrm{H}}, \boldsymbol{D}^{\mathrm{S}}, \boldsymbol{D}^{\mathrm{H}}\right) + G_{\mathrm{D}}(\boldsymbol{D}), \tag{7}$$

where W is the stored (elastic) term, H the hardening contribution, and G_{D} the surface energy term, whereas $\varepsilon^{\mathrm{S}}, \varepsilon^{\mathrm{H}}$ and $\boldsymbol{D}^{\mathrm{S}}, \boldsymbol{D}^{\mathrm{H}}$ are kinematic, isotropic plastic, and damage hardening variables, respectively.

Dissipation Coupling

Coupling between plastic and damage dissipation, in general, can be done in two ways: either (a) a single plastic dissipation potential coupled with damage is used (strong coupling), or (b) two (or more) dissipation potential functions: plastic dissipation potential and damage dissipation potential are assumed and independently defined (weak coupling).

STRONG DISSIPATION COUPLING

This frequently used approach imposes limitations that are too restrictive because it does not allow for damage evolution without simultaneous plastic dissipation, since both mechanisms are governed by the single plastic multiplier. Hence, this approach is not applicable for a brittle-like damage in elastic moderately plastic materials, but only for a ductile-like damage in elastic-plastic materials for which damage in the elastic range (without plasticity) is negligible. In spite of this the strong coupling is frequently used by many authors for its simplicity and easy model calibration. Among them, we mention the following articles: Cordebois and Sidoroff (1982); Lemaitre and Chaboche (1985); Lemaitre (1992); Saanouni et al. (1994); Chaboche (1999); Chow and Tai (2000); Pederson and Tvergaard (2000); Hesebeck (2001); Nesnas and Saanouni (2002); Olsson and Ristnmaa (2003). A discussion of different approaches to damage effect on the elastic limit criterion was given by Chaboche (1999). Three possibilities are discussed in this article:

1. No hardening-damage coupling (only the effective stress concept $\widetilde{\sigma}$ is used), e.g., Benallal (1989) and Lemaitre (1992).
2. Partly coupled hardening-damage (through effective kinematic hardening $\widetilde{\boldsymbol{X}}$), e.g., Lemaitre and Chaboche (1985).
3. Fully coupled hardening-damage (through both effective hardening variables $\widetilde{\boldsymbol{X}}$ and $\widetilde{R}$), e.g., Chaboche (1977) and Saanouni (1988).

In the usual case of J_2 plasticity models, all three approaches are summarized as follows:

$$\begin{aligned} f_1 &= f(\widetilde{\sigma}, \boldsymbol{X}, R) = J_2(\widetilde{\sigma} - \boldsymbol{X}) - R - \sigma_y \leq 0 \\ f_2 &= f(\widetilde{\sigma}, \widetilde{\boldsymbol{X}}, R) = J_2(\widetilde{\sigma} - \widetilde{\boldsymbol{X}}) - R - \sigma_y \leq 0 \\ f_3 &= f(\widetilde{\sigma}, \widetilde{\boldsymbol{X}}, \widetilde{R}) = J_2(\widetilde{\sigma} - \widetilde{\boldsymbol{X}}) - \widetilde{R} - \sigma_y \leq 0 \end{aligned} \tag{8}$$

WEAK DISSIPATION COUPLING

Increasing demands for higher performance materials, especially elastic-moderate plastic materials, composites, and others, in which strong

plasticity-damage dissipation coupling models are not applicable, present an increasing necessity to develop more physically justified approaches based on the weak coupling concept. Among the numerous articles based on the weak coupling theories, let us mention the following: Chow and Wang (1987); Simo and Ju (1987, 1989); Lemaitre and Chaboche (1990); Stevens and Liu (1992); Hansen and Schreyer (1994); Zhu and Cescotto (1995); Chaboche (1997); Hayakawa and Murakami (1997, 1998); Voyiadjis and Park (1997, 1999); Voyiadjis and Deliktas (2000); Voyiadjis et al. (2001); Abu Al-Rub and Voyiadjis (2003). Chow and Wang (1987) introduced two dissipation functions that describe the plastic dissipation surface F^{p} and damage dissipation surface F^{d} as follows:

$$\begin{aligned} F^{\mathrm{p}} &= f(\sigma, \boldsymbol{D}, R) = \widetilde{\sigma}^{\mathrm{p}} - [R_0 + R(p)] = 0 \quad \widetilde{\sigma}^{\mathrm{p}} = \left\{ \frac{1}{2} \sigma^{\mathrm{T}} : \widetilde{\boldsymbol{H}} : \sigma \right\}^{1/2} \\ F^{\mathrm{d}} &= f(\sigma, \boldsymbol{D}, B) = \widetilde{\sigma}^{\mathrm{d}} - [B_0 + B(\beta)] = 0 \quad \widetilde{\sigma}^{\mathrm{d}} = \left\{ \frac{1}{2} \sigma^{\mathrm{T}} : \widetilde{\boldsymbol{D}} : \sigma \right\}^{1/2} \end{aligned} \tag{9}$$

Then, using a thermodynamic formulation for anisotropic plasticity ($\widetilde{\boldsymbol{H}}$ is Hill's effective plastic characteristic tensor) and anisotropic damage ($\widetilde{\boldsymbol{D}}$ denotes effective damage characteristic tensor), the plastic evolution and damage evolution equations are derived for variables $\dot{\varepsilon}^{\mathrm{p}}$, $\dot{p}$ and $\dot{\boldsymbol{D}}$, $\dot{\beta}$.

In general, all these theories are based on the concept of the existence of two (or even more) dissipation surfaces, and the maximum dissipation principle: 'Actual state of the thermodynamic forces is that which maximizes the dissipation function over all other possible admissible states'. A generalized normality rule is usually applied to obtain evolution rules. In the previously mentioned article by Hansen and Schreyer (1994), the Lagrangian functional is built with two independent multipliers $\dot{\lambda}^{\mathrm{p}}$ and $\dot{\lambda}^{\mathrm{d}}$

$$L = -\gamma + \dot{\lambda}^{\mathrm{p}} \Phi^{\mathrm{p}} + \dot{\lambda}^{\mathrm{d}} \Phi^{\mathrm{d}}, \tag{10}$$

where γ denotes the dissipation due to plasticity and damage

$$\gamma = \sigma : \dot{\varepsilon}^{\mathrm{p}} + \sigma^{\mathrm{S}} : \dot{\varepsilon}^{\mathrm{S}} + \sigma^{\mathrm{H}} : \dot{\varepsilon}^{\mathrm{H}} + \boldsymbol{Y} : \dot{\boldsymbol{D}} + \boldsymbol{Y}^{\mathrm{S}} : \dot{\boldsymbol{D}}^{\mathrm{S}} + \boldsymbol{Y}^{\mathrm{H}} : \dot{\boldsymbol{D}}^{\mathrm{H}}, \tag{11}$$

whereas two constraints $\Phi^{\mathrm{p}} = 0$ and $\Phi^{\mathrm{d}} = 0$ represent the yield and damage dissipation potentials

$$\begin{aligned} \Phi^{\mathrm{p}}\left(\widehat{\sigma}, \widehat{\sigma}^{\mathrm{S}}, \widehat{\sigma}^{\mathrm{H}}\right) &= \Phi^{\mathrm{p}1}\left(\widehat{\sigma} - \widehat{\sigma}^{\mathrm{S}}\right) - \left[\Phi^{\mathrm{p}2}\left(\widehat{\sigma}^{\mathrm{H}}\right) + \sigma_y\right] \leq 0 \\ \Phi^{\mathrm{d}}\left(\boldsymbol{Y}, \boldsymbol{Y}^{\mathrm{S}}, \boldsymbol{Y}^{\mathrm{H}}\right) &= \Phi^{\mathrm{d}1}\left(\boldsymbol{Y} - \boldsymbol{Y}^{\mathrm{S}}\right) - \left[\Phi^{\mathrm{d}2}\left(\boldsymbol{Y}^{\mathrm{H}}\right) + \omega_0\right] \leq 0. \end{aligned} \tag{12}$$

In the notation used above $\widehat{\sigma}^{\mathrm{S}}$, $\widehat{\sigma}^{\mathrm{H}}$ and $\boldsymbol{Y}^{\mathrm{S}}$, $\boldsymbol{Y}^{\mathrm{H}}$ denote the conjugate forces for ε^{S}, ε^{H} and $\boldsymbol{D}^{\mathrm{S}}$, $\boldsymbol{D}^{\mathrm{H}}$, respectively, whereas the symbol $(\widehat{\ })$ stands for effective variables e.g., $\widehat{\sigma} = \boldsymbol{M}(\boldsymbol{D}) : \sigma$, $\widehat{\varepsilon}^{\mathrm{p}} = \boldsymbol{M}^{-\mathrm{T}}(\boldsymbol{D}) : \varepsilon^{\mathrm{p}}$, etc.

The evolution rules of $\dot{\widehat{\varepsilon}}^{\mathrm{p}}$, $\dot{\widehat{\varepsilon}}^{\mathrm{S}}$, $\dot{\widehat{\varepsilon}}^{\mathrm{H}}$ and $\dot{\boldsymbol{D}}, \dot{\boldsymbol{D}}^{\mathrm{S}}, \dot{\boldsymbol{D}}^{\mathrm{H}}$, for the actual values of thermodynamic forces that maximize entropy, are

$$\begin{aligned}
&\frac{\partial L}{\partial \sigma} = 0 \rightarrow \dot{\widehat{\varepsilon}}^{\mathrm{p}} = \dot{\lambda}^{\mathrm{p}} \frac{\partial \Phi^{\mathrm{p1}}}{\partial \sigma} \qquad \frac{\partial L}{\partial \boldsymbol{Y}} = 0 \rightarrow \dot{\boldsymbol{D}} = \dot{\lambda}^{\mathrm{d}} \frac{\partial \Phi^{\mathrm{d1}}}{\partial \boldsymbol{Y}} \\
&\frac{\partial L}{\partial \sigma^{\mathrm{S}}} = 0 \rightarrow \dot{\widehat{\varepsilon}}^{\mathrm{S}} = \dot{\lambda}^{\mathrm{p}} \frac{\partial \Phi^{\mathrm{p1}}}{\partial \sigma^{\mathrm{S}}} \qquad \frac{\partial L}{\partial \boldsymbol{Y}^{\mathrm{S}}} = 0 \rightarrow \dot{\boldsymbol{D}}^{\mathrm{S}} = \dot{\lambda}^{\mathrm{d}} \frac{\partial \Phi^{\mathrm{d1}}}{\partial \boldsymbol{Y}^{\mathrm{S}}} \\
&\frac{\partial L}{\partial \sigma^{\mathrm{H}}} = 0 \rightarrow \dot{\widehat{\varepsilon}}^{\mathrm{H}} = \dot{\lambda}^{\mathrm{p}} \frac{\partial \Phi^{\mathrm{p2}}}{\partial \sigma^{\mathrm{H}}} \qquad \frac{\partial L}{\partial \boldsymbol{Y}^{\mathrm{H}}} = 0 \rightarrow \dot{\boldsymbol{D}}^{\mathrm{H}} = \dot{\lambda}^{\mathrm{d}} \frac{\partial \Phi^{\mathrm{d2}}}{\partial \boldsymbol{Y}^{\mathrm{H}}}
\end{aligned} \tag{13}$$

and the Khun-Tucker loading/unloading conditions for both independent dissipation mechanisms, plasticity and damage, hold

$$\dot{\lambda}^{\mathrm{p}} \geq 0,\ \Phi^{\mathrm{p}} \leq 0,\ \dot{\lambda}^{\mathrm{p}} \Phi^{\mathrm{p}} = 0 \ \text{ and } \ \dot{\lambda}^{\mathrm{d}} \geq 0,\ \Phi^{\mathrm{d}} \leq 0,\ \dot{\lambda}^{\mathrm{d}} \Phi^{\mathrm{d}} = 0. \tag{14}$$

Similar thermodynamic formulation as described above was also used by Zhu and Cescotto (1995), where the dissipation due to coupled plasticity and damage contains only four terms, hence

$$\Phi = \sigma : \dot{\varepsilon}^{\mathrm{p}} - R\dot{\alpha} - \boldsymbol{Y} : \dot{\boldsymbol{D}} - B\dot{\beta} \ \text{ and } \ \overline{\Phi} = \Phi - \dot{\lambda}^{\mathrm{p}} F^{\mathrm{p}} - \dot{\lambda}^{\mathrm{d}} F^{\mathrm{d}} \tag{15}$$

However, anisotropic elasticity, anisotropic plasticity, and anisotropic damage are included in this formulation. To this end the effective orthotropic elastic stiffness tensor $\widehat{\boldsymbol{C}}^{\mathrm{e}}$, the effective Hill's plastic characteristic tensor $\widehat{\boldsymbol{H}}$, and the damage characteristic tensor $\boldsymbol{J}$ are used. Additionally, an anisotropic microcrack opening/closing mechanism is accounted for by the use of a concept of spectral projection tensors and a modified damage energy release rate for ductile-like or brittle-like damage conditions.

A possibility to define a multiple dissipation potential was discussed by Chaboche (1997), who introduced three independent potentials expressed in the space of thermodynamic forces $(A_k^{\mathrm{p}}, A_j^{\mathrm{d}})$ associated with the plastcity and damage hardening variables $(V_k^{\mathrm{p}}, V_j^{\mathrm{d}})$. The potentials stand for three dissipation mechanisms: Ω^{p} – plastic flow and hardening process, Ω^{s} – microstructure evolution, and Ω^{d} – damage process.

Finally, introducing three Lagrange multipliers $\dot{\lambda}^{\mathrm{p}}, \dot{\lambda}^{\mathrm{s}}$, and $\dot{\lambda}^{\mathrm{d}}$ the generalized normality rule holds

$$\begin{aligned} \dot{\varepsilon}^{\mathrm{p}} &= \dot{\lambda}^{\mathrm{p}} \frac{\partial \Omega^{\mathrm{p}}}{\partial \sigma} \\ -\dot{V}_k^{\mathrm{p}} &= \dot{\lambda}^{\mathrm{p}} \frac{\partial \Omega^{\mathrm{p}}}{\partial A_k^{\mathrm{p}}} + \dot{\lambda}^{\mathrm{s}} \frac{\partial \Omega^{\mathrm{s}}}{\partial A_k^{\mathrm{p}}} \\ -\dot{V}_j^{\mathrm{d}} &= \dot{\lambda}^{\mathrm{d}} \frac{\partial \Omega^{\mathrm{d}}}{\partial A_j^{\mathrm{d}}} \end{aligned} \tag{16}$$

All the discussed models of weak elastic-plastic dissipation coupling assume the existence of both plastic and damage potentials. In general, the assumption of existence of damage potential, related to the similar concept of plastic potential, should be validated. Hence Hayakawa and Murakami (1997) performed an experimental verification of the existence of a damage potential and the corresponding normality rule for the spheroidized graphite cast iron. Experiments proved that both existence and normality hold in the space of damage conjugate forces, which are easily related to stresses $F^{\mathrm{d}}[\boldsymbol{Y}(\sigma)]$.

UNILATERAL DAMAGE CONCEPTS

When the material is subjected to reverse tension–compression cycles, the crucial question that arises is how to properly describe a phenomenon of the unilateral damage, also called the damage deactivation or the crack closure/opening effect. To this end a decomposition of the stress or strain tensors into the positive or negative projections is usually introduced using the fourth-rank projection operators written in terms of principal directions of the strain or stress tensors $\boldsymbol{n}_{\varepsilon}^{(i)}$ or $\boldsymbol{n}_{\sigma}^{(i)}$ as follows (Lubarda et al., 1994; Hansen and Schreyer, 1995; Krajcinovic, 1996):

$$\begin{aligned} \boldsymbol{P}_{\varepsilon}^{+} &= \sum_{i=1}^{3} \langle\langle \varepsilon^{(i)} \rangle\rangle \boldsymbol{n}_{\varepsilon}^{(i)} \otimes \boldsymbol{n}_{\varepsilon}^{(i)} \otimes \boldsymbol{n}_{\varepsilon}^{(i)} \otimes \boldsymbol{n}_{\varepsilon}^{(i)}, \qquad \boldsymbol{P}_{\varepsilon}^{-} = \boldsymbol{I} - \boldsymbol{P}_{\varepsilon}^{+} \\ \boldsymbol{P}_{\sigma}^{+} &= \sum_{i=1}^{3} \langle\langle \sigma^{(i)} \rangle\rangle \boldsymbol{n}_{\sigma}^{(i)} \otimes \boldsymbol{n}_{\sigma}^{(i)} \otimes \boldsymbol{n}_{\sigma}^{(i)} \otimes \boldsymbol{n}_{\sigma}^{(i)}, \qquad \boldsymbol{P}_{\sigma}^{-} = \boldsymbol{I} - \boldsymbol{P}_{\sigma}^{+} \end{aligned} \tag{17}$$

where the double angular bracket is defined as $\langle\langle a \rangle\rangle = 1$ for $a \geq 0$ or 0 for $a < 0$, and $\varepsilon^{(i)}$, $\sigma^{(i)}$ are the principal strain or stress components. Hence

$$\begin{aligned} \varepsilon^{+} &= \boldsymbol{P}_{\varepsilon}^{+} : \varepsilon, \quad \varepsilon^{-} = \boldsymbol{P}_{\varepsilon}^{-} : \varepsilon \\ \sigma^{+} &= \boldsymbol{P}_{\sigma}^{+} : \sigma, \quad \sigma^{-} = \boldsymbol{P}_{\sigma}^{-} : \sigma \end{aligned} \tag{18}$$

where only the positive or the negative components of the strain or stress tensors are extracted. It is usually assumed that the negative components of the strain or stress tensors remain inactive as long as the loading conditions again render them active. A more realistic description of damage deactivation allows for an influence of negative principal components of strain and stress tensors for damage evolution, as observed in brittle materials (Hayakawa and Murakami, 1997; Murakami and Kamiya, 1997). These concepts reduce to introduction of the generalized projection operators (Ganczarski et al., 2003) and the modified strain or stress tensors as follows:

$$\overline{\boldsymbol{P}}_{\varepsilon} = \boldsymbol{P}_{\varepsilon}^{+} + \zeta_{\varepsilon}\boldsymbol{P}_{\varepsilon}^{-} \quad \text{or} \quad \overline{\boldsymbol{P}}_{\sigma} = \boldsymbol{P}_{\sigma}^{+} + \zeta_{\sigma}\boldsymbol{P}_{\sigma}^{-} \tag{19}$$

and

$$\overline{\varepsilon} = \overline{\boldsymbol{P}}_{\varepsilon} : \varepsilon \quad \text{or} \quad \overline{\sigma} = \overline{\boldsymbol{P}}_{\sigma} : \sigma \tag{20}$$

In the case when $\zeta_{\varepsilon} = 1$ or $\zeta_{\sigma} = 1$ then $\overline{\boldsymbol{P}}_{\varepsilon} = \boldsymbol{I}$ or $\overline{\boldsymbol{P}}_{\sigma} = \boldsymbol{I}$, such that the unique mapping holds: $\overline{\varepsilon} = \varepsilon$ or $\overline{\sigma} = \sigma$. In contrast, if $\zeta_{\varepsilon} = 0$ or $\zeta_{\sigma} = 0$, then $\overline{\boldsymbol{P}}_{\varepsilon} = \boldsymbol{P}_{\varepsilon}^{+}$ or $\overline{\boldsymbol{P}}_{\sigma} = \boldsymbol{P}_{\sigma}^{+}$, hence the negative principal components of the strain or stress tensors have no influence on damage evolution, $\overline{\varepsilon} = \varepsilon^{+}$ or $\overline{\sigma} = \sigma^{+}$. When general coordinate systems are used the modified strain or stress tensors are expressed in terms of the actual ones by the following mappings:

$$\overline{\varepsilon}_{ij} = \sum_{I=1}^{3} \xi(\varepsilon_I) n_{iI}^{(\varepsilon)} n_{jI}^{(\varepsilon)} n_{Ik}^{(\varepsilon)} n_{Il}^{(\varepsilon)} \varepsilon_{kl} = B_{ijkl}^{(\varepsilon)} \varepsilon_{kl} \tag{21}$$

or

$$\overline{\sigma}_{ij} = \sum_{I=1}^{3} \xi(\sigma_I) n_{iI}^{(\sigma)} n_{jI}^{(\sigma)} n_{Ik}^{(\sigma)} n_{Il}^{(\sigma)} \sigma_{kl} = B_{ijkl}^{(\sigma)} \sigma_{kl} \tag{22}$$

where the fourth-rank tensors $B_{ijkl}^{(\varepsilon)}$ or $B_{ijkl}^{(\sigma)}$ are built of direction cosines between the principal and the current spatial systems.

Fourth-rank positive projection operators $\boldsymbol{P}_{\varepsilon}^{+}$ that extract the tensile strain components were used by Hansen and Schreyer (1994), whereas Zhu and Cescotto (1995) applied the spectral decomposition of the stress tensor $\sigma^{+} = \boldsymbol{P}_{\sigma}^{+} : \sigma$ in order to account for microcrack opening/closure effect in elastic-plastic-damage materials. These concepts are somewhat similar to that introduced earlier by Ortiz (1985), who considered, however, only brittle materials (concrete). Limitations of the unilateral damage condition

concept applied to continuum damage theories were discussed by Chaboche (1992, 1993); Chaboche et al. (1995) and Halm and Dragon (1996). It was shown that some existing theories (e.g., Krajcinovic and Fonseka, 1981; Ju, 1989) either lead to nonsymmetries of the elastic stiffness or may yield to non-realistic discontinuous stress– strain response under non-proportional loading. It is easy to illustrate that, if unilateral damage condition does influence both diagonal and off-diagonal components of the stiffness or compliance matrix, a stress discontinuity occurs when one of the principal strains changes sign, whereas the others remain unchanged, see Skrzypek and Kuna-Ciskał (2003).

In what follows, the crack closure phenomenon is stress controlled, which means the zeroth closure stress $\sigma_c = 0$ (Figure 1(a)). A continuous description of the stiffness recovery process due to crack closure effect was recently proposed by Ganczarski (2004), where Ylinen's approximation was modified such that only the damage variable that acts on the volumetric

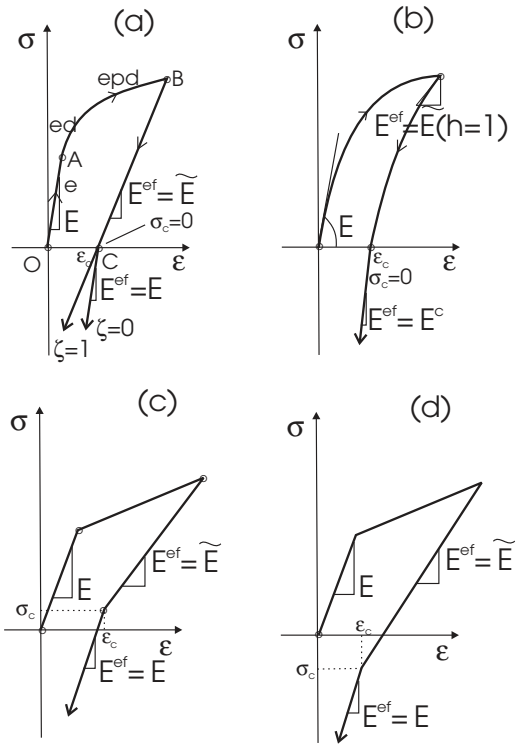

Figure 1. *Tension/compression deactivation diagrams: (a) stiffness recovery controlled by ζ (present model), (b) smooth stiffness recovery model (Ganczarski, 2004), (c), (d) two closure positions $\sigma_c > 0$, $\sigma_c < 0$ (Chaboche et al., 1998).*

stress is subjected to the closure effect, whereas the isotropic ones remain unchanged on compression (Figure 1(b)). Hence, only partial (but continuous) stiffness recovery occurs on the loading cycle when a low cycle fatigue-damage is observed in 316L stainless steel. Instead of the usually used crack closure parameter h_c, a continuous function $h(\sigma)$ is defined such that a smooth stiffness recovery is met. A more general concept of microcrack closure was studied by Chaboche et al. (1998), who considered two cases of positive $\sigma_c > 0$ or negative $\sigma_c < 0$ closure stresses in composite materials (Figure 1(c) and (d)).

The extended Hayakawa and Murakami model is the basis of the present consideration. It is adopted here to the reverse cyclic loading conditions, where the microcrack opening/closure mechanism is significant. The matrix representation of the incremental form of the Hayakawa and Murakami elastic-plastic-damage equations, derived by Bielski et al. (2002) is implemented to the ABAQUS finite element code. Effective algorithm for plastic and damage loading/unloading conditions based on the doubly passive predictor–plastic damage corrector approach is used.

STATE POTENTIAL AND ELASTIC-PLASTIC-DAMAGE CONSTITUTIVE EQUATIONS

The basic set of state and evolution equations for elastic-plastic damage material of a moderate ductility is accepted after Hayakawa and Murakami (1997). It is usually assumed that the Helmholtz free energy is used as the state potential. Such an approach was applied e.g., by Murakami and Kamiya (1997) to describe elastic-brittle damage materials (high strength concrete) by the use of a total stress–strain formulation $\sigma = {}^{s}\widetilde{\mathbf{\Lambda}} : \varepsilon^{e}$, where ${}^{s}\widetilde{\mathbf{\Lambda}}(\boldsymbol{D})$ denotes the effective secant stiffness tensor. Skrzypek and Kuna-Ciskał (2003) and Kuna-Ciskał and Skrzypek (2004) extended this model to the incremental formulation $\mathrm{d}\sigma = {}^{t}\widetilde{\mathbf{\Lambda}} : \mathrm{d}\varepsilon^{e}$, where ${}^{t}\widetilde{\mathbf{\Lambda}}(\varepsilon^{e}, \boldsymbol{D})$ stands for the effective tangent stiffness tensor. A general failure criterion based on the Drucker stability postulate has also been proposed:

$$\frac{\partial^2 \Psi}{\partial \varepsilon^{e}_{ij} \partial \varepsilon^{e}_{kl}} \mathrm{d}\varepsilon^{e}_{ij} \mathrm{d}\varepsilon^{e}_{kl} = H_{ijkl} \mathrm{d}\varepsilon^{e}_{ij} \mathrm{d}\varepsilon^{e}_{kl} > 0 \tag{23}$$

if loss of the positive definiteness of the Hessian matrix [**H**] is accepted as the failure criterion (also Chen and Han, 1995). Note that, when Helmholtz free energy based approach is used, the damage conjugate forces $\boldsymbol{Y}$ are expressed in terms of elastic strains $\boldsymbol{Y}(\varepsilon^{e})$. However, for the sake of simplicity of experimental validation of the model, the damage conjugate

forces should rather be related to the stress tensor $\boldsymbol{Y}(\sigma)$. Such an approach enables consistent definition of both dissipation surfaces, plastic $F^{\mathrm{p}}(\sigma, R)$ and damage $F^{\mathrm{d}}(\boldsymbol{Y}, B)$, in the stress space.

To make the experimental verification straightforward, i.e., to have damage conjugate forces expressed in stresses instead of strains, the Gibbs thermodynamic potential from Equation (5) is applied instead of the Helmholtz one

$$\Gamma(\sigma, r, \boldsymbol{D}, \beta) = \sigma : \varepsilon^{\mathrm{e}} - \Psi(\varepsilon^{\mathrm{e}}, r, \boldsymbol{D}, \beta) \tag{24}$$

where $\Psi(\varepsilon^{\mathrm{e}}, r, \boldsymbol{D}, \beta)$ is the Helmholtz free energy per unit mass.

Following the scheme typical for the irreversible thermodynamics the elastic strain tensor as well as the conjugate forces are obtained as

$$\varepsilon^{\mathrm{e}} = \frac{\partial \Gamma}{\partial \sigma}, \qquad R \equiv \frac{\partial \Gamma}{\partial r}, \qquad \boldsymbol{Y} \equiv \frac{\partial \Gamma}{\partial \boldsymbol{D}}, \qquad B \equiv \frac{\partial \Gamma}{\partial \beta} \tag{25}$$

where σ is the stress tensor, r is the scalar plastic isotropic hardening variable, $\boldsymbol{D}$ is the second rank symmetric damage tensor, and β is the scalar variable to describe an evolution of the damage surface. R, $\boldsymbol{Y}$, and B are the thermodynamic force-conjugates of state variables r, $\boldsymbol{D}$, and β, respectively. Note that kinematic hardening variables for plasticity and damage terms are disregarded in Equation (24).

The form of the Gibbs thermodynamic potential is postulated as a sum of three parts

$$\Gamma(\sigma, r, \boldsymbol{D}, \beta) = \Gamma^{\mathrm{e}}(\sigma, \boldsymbol{D}) + \Gamma^{\mathrm{p}}(r) + \Gamma^{\mathrm{d}}(\beta) \tag{26}$$

where Γ^{e} is the complementary energy due to the elastic deformation, which is assumed to be an isotropic symmetric function depending on the tensors σ and $\boldsymbol{D}$, since the undamaged (initial) material is considered isotropic. Furthermore, due to the postulated isotropic and linear behavior of the undamaged material, Γ^{e} is assumed to be a quadratic with respect to σ. Due to the expected decrease of Γ^{e} as damage develops, the form linear with respect to $\boldsymbol{D}$ is supposed. Eventually, after the crack closure effect (under compressive state) is included, the form for Γ^{e} is expressed as a linear combination of the six basic invariants of the tensors σ and $\boldsymbol{D}$, instead of 10 basic invariants in a general case (Spencer, 1971; Rymarz, 1993):

$$\begin{aligned}\Gamma^{\mathrm{e}}(\sigma, \boldsymbol{D}) = &-\frac{\nu_0}{2E_0}(\operatorname{tr}\sigma)^2 + \frac{1+\nu_0}{2E_0}\operatorname{tr}(\sigma\cdot\sigma) + \vartheta_1 \operatorname{tr}\boldsymbol{D}(\operatorname{tr}\sigma)^2 \\ &+ \vartheta_2 \operatorname{tr}\boldsymbol{D}\operatorname{tr}(\overline{\sigma}\cdot\overline{\sigma}) + \vartheta_3 \operatorname{tr}\sigma\operatorname{tr}(\sigma\cdot\boldsymbol{D}) + \vartheta_4 \operatorname{tr}(\overline{\sigma}\cdot\overline{\sigma}\cdot\boldsymbol{D})\end{aligned} \tag{27}$$

with $\overline{\sigma}$ being the modified stress tensor

$$\begin{aligned}\overline{\sigma}_{ij} &= \sum_{I=1}^{3} \overline{\sigma}_I Q_{Ii} Q_{Ij} = \sum_{I=1}^{3} \xi(\sigma_I)\sigma_I Q_{Ii} Q_{Ij} \\ &= \sum_{I=1}^{3} \xi(\sigma_I)\sigma_{kl} Q_{Ik} Q_{Il} Q_{Ii} Q_{Ij} = \left[\sum_{I=1}^{3} \xi(\sigma_I) Q_{Ii} Q_{Ij} Q_{Ik} Q_{Il}\right]\sigma_{kl} = B^{(\sigma)}_{ijkl}\sigma_{kl}\end{aligned} \tag{28}$$

Q_{Ii} is a direction cosine between the principal stress direction I and the current spatial system direction i, σ_I are the principal values for σ, whereas the modified ones are

$$\begin{aligned}\overline{\sigma}_I &= \langle\sigma_I\rangle - \zeta\langle-\sigma_I\rangle \equiv \xi(\sigma_I)\sigma_I \quad (I = 1, 2, 3, \text{no sum.}) \\ \xi(\sigma_I) &= H(\sigma_I) + \zeta H(-\sigma_I) = \xi_I \\ 0 &\le \zeta \le 1\end{aligned} \tag{29}$$

ν_0, E_0 are the elastic constants for initial undamaged material and ϑ_1, ϑ_2, ϑ_3, ϑ_4, and ζ are elastic-damage material constants. In Equation (27), the modified stress tensor appears only in the fourth and sixth terms of Γ^{e}, since the unilateral response is not observed in the virgin elastic state and the continuity requirements must hold during unloading when $\sigma_I = 0$. Hence, the off-diagonal components of the effective elastic compliance tensor ${}^{\mathrm{s}}\boldsymbol{C}^{\mathrm{e}}(\boldsymbol{D})$ are not affected by the unilateral damage, in a similar fashion as shown by Kuna-Ciskał and Skrzypek (2004) in the case of elastic-brittle damage.

Γ^{p} is the part of the Gibbs potential related to plastic deformation (e.g., distortion energy of crystal lattice related to dislocation structure) and it represents the effect of the isotropic plastic hardening. Note that the influence of other state variables, including damage, is neglected. Hence, we assume

$$\Gamma^{\mathrm{p}}(r) = R_\infty\left[r + \frac{1}{b}\exp(-br)\right] \tag{30}$$

where R_∞ and b are plastic material constants.

Γ^{d} is the damage part of the Gibbs potential (e.g., surface energy due to the nucleation of cavities); here the linear relation between internal variable β and its conjugate force B is postulated, hence

$$\Gamma^{\mathrm{d}}(\beta) = \frac{1}{2}K_{\mathrm{d}}\beta^2 \tag{31}$$

where K_{d} is another damage material constant.

Now, applying Equations (25) and (26) one comes to the relation for elastic strains expressed in stresses and damage tensor in a total form

$$\varepsilon^{\mathrm{e}} = \frac{\partial \Gamma^{\mathrm{e}}}{\partial \sigma} = -\frac{\nu_0}{E_0}(\operatorname{tr}\sigma)\boldsymbol{I} + \frac{1+\nu_0}{E_0}\sigma + 2\vartheta_1(\operatorname{tr}\boldsymbol{D}\operatorname{tr}\sigma)\boldsymbol{I} + 2\vartheta_2(\operatorname{tr}\boldsymbol{D})\overline{\sigma} : \frac{\partial \overline{\sigma}}{\partial \sigma} + \vartheta_3[\operatorname{tr}(\boldsymbol{D})\boldsymbol{I} + (\operatorname{tr}\sigma)\boldsymbol{D}] + \vartheta_4(\overline{\sigma}\cdot\boldsymbol{D} + \boldsymbol{D}\cdot\overline{\sigma}) : \frac{\partial \overline{\sigma}}{\partial \sigma} \tag{32}$$

or

$$\varepsilon^{\mathrm{e}} = {}^{\mathrm{s}}\boldsymbol{C}^{\mathrm{e}}(\boldsymbol{D}) : \sigma \tag{33}$$

The thermodynamic conjugate forces corresponding to the internal state variables $\boldsymbol{D}$, r, and β are

$$\boldsymbol{Y} \equiv \frac{\partial \Gamma^{\mathrm{e}}}{\partial \boldsymbol{D}} = \left[\vartheta_1(\operatorname{tr}\sigma)^2 + \vartheta_2 \operatorname{tr}(\overline{\sigma}\cdot\overline{\sigma})\right]\boldsymbol{I} + \vartheta_3(\operatorname{tr}\sigma)\sigma + \vartheta_4(\overline{\sigma}\cdot\overline{\sigma}) \tag{34}$$

$$R \equiv \frac{\partial \Gamma^{\mathrm{p}}}{\partial r} = R_\infty[1 - \exp(-br)] \tag{35}$$

$$B \equiv \frac{\partial \Gamma^{\mathrm{d}}}{\partial \beta} = K_{\mathrm{d}}\beta \tag{36}$$

${}^{\mathrm{s}}\boldsymbol{C}^{\mathrm{e}}(\boldsymbol{D})$ is the fourth-rank secant elastic compliance tensor with damage tensor $\boldsymbol{D}$ as argument. The details of derivation of the effective compliance tensor ${}^{\mathrm{s}}\boldsymbol{C}^{\mathrm{e}}$, including the derivative of the modified stress tensor with respect to the stress tensor $\partial\overline{\sigma}/\partial\sigma$, are analogous to those in Kuna-Ciskał and Skrzypek (2004), where the effective secant stiffness tensor was derived based on the formulation by Murakami and Kamiya (1997) for elastic damage materials. Hence, when Equation (28) is used, the corresponding derivatives are calculated according to the scheme

$$D^{(\sigma)}_{ijkl} = \frac{\partial \overline{\sigma}_{ij}}{\partial \sigma_{kl}} = B^{(\sigma)}_{ijkl} + \frac{\partial B^{(\sigma)}_{ijpq}}{\partial \sigma_{kl}}\sigma_{pq} \tag{37}$$

This formula leads to a somewhat cumbersome representation of the fourth-rank tensor $D^{(\sigma)}_{ijkl}$ that accounts for unilateral damage if the incremental form of the transformation rule i.e., Equation (28) is used

$$\mathrm{d}\overline{\sigma}_{ij} = D^{(\sigma)}_{ijkl}\mathrm{d}\sigma_{kl} \tag{38}$$

DISSIPATION POTENTIALS AND EVOLUTION RULES FOR PLASTICITY AND DAMAGE

The next step is the assumption that there exists a dissipation potential surface

$$F(X_m;\ r, \boldsymbol{D}, \beta) = 0 \tag{39}$$

for each conjugate force X_m, where ($\boldsymbol{X} = \{\sigma, R, \boldsymbol{Y}, B\}$) and current state variables, together with the normality law

$$J_m = \dot{\lambda}_m \frac{\partial F}{\partial X_m} \tag{40}$$

with $\boldsymbol{J} = \{\dot{\varepsilon}^{\mathrm{p}}, \dot{r}, \dot{\boldsymbol{D}}, \dot{\beta}\}$.

We assume, after Hayakawa and Murakami (1997), that the dissipation potential is composed of two parts, plastic and damage, and the maximum dissipation principle holds (weak dissipation coupling). The plastic dissipation potential is mainly related to the dislocation motion, whereas the damage one is related to the internal energy release in cavities. Thus

$$F\big(\sigma, \boldsymbol{Y}, R, B; \boldsymbol{D}, r, \beta\big) = F^{\mathrm{p}}\big(\sigma, R; \boldsymbol{D}\big) + F^{\mathrm{d}}(\boldsymbol{Y}, B;\ \boldsymbol{D}, r, \beta) \tag{41}$$

Note that only isotropic plasticity and damage hardening are accounted for in the present formulation, such that thermodynamic force conjugates to the fluxes $\boldsymbol{J} = \{\dot{\varepsilon}^{\mathrm{p}}, \dot{r}, \dot{\boldsymbol{D}}, \dot{\beta}\}$ are $\boldsymbol{F} = \{\sigma, R, \boldsymbol{Y}, B\}$. This formulation introduces limitation to relatively simple loading conditions. If more complex loading history is considered, kinematic hardening should also be included to both the state and the dissipation potentials to yield a more general form of the fluxes and thermodynamic force vectors, e.g., $\{\dot{\varepsilon}^{\mathrm{p}}, \dot{\boldsymbol{\alpha}}, \dot{p}, \dot{\boldsymbol{\Gamma}}, \dot{\boldsymbol{\Phi}}, \dot{r}\}$ and $\{\sigma, \boldsymbol{X}, R, \boldsymbol{Y}, \boldsymbol{H}, K\}$, respectively, where the notation used in Abu Al-Rub and Voyiadjis (2003) was used.

Having in mind that damage will, in general, influence a yield surface, and assuming the associate flow rule, the form

$$F^{\mathrm{p}}(\sigma, R;\ \boldsymbol{D}) = \sigma_{\mathrm{eq}} - (R_0 + R) = 0 \tag{42}$$

$$\sigma_{\mathrm{eq}} = \sqrt{\frac{3}{2}\sigma' : \boldsymbol{M}(\boldsymbol{D}) : \sigma'} \tag{43}$$

with R_0 being a material constant (initial yield stress) and σ' being the deviatoric stress tensor, is proposed as the plastic dissipation potential (for relatively simple loading processes).

The tensor function $\boldsymbol{M}(\boldsymbol{D})$ linear in $\boldsymbol{D}$ is specified in order to describe the increased effect of stress due to damage

$$[\boldsymbol{M}(\boldsymbol{D})]_{ijkl} = \frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) + \frac{1}{2}c^{\mathrm{p}}\left(\delta_{ik}D_{jl} + D_{ik}\delta_{jl} + \delta_{il}D_{jk} + D_{il}\delta_{jk}\right) \tag{44}$$

with c^{p} being a material constant. Thus, Equation (42) is an extension of the Huber–Mises–Hencky yield condition for the damage materials.

The damage dissipation potential is built in the force conjugate $\boldsymbol{Y}$ space and is assumed to depend on damage as well as on plastic deformation and hydrostatic stress. The relation is easily transformed into the stress space via Equation (34). Hence

$$F^{\mathrm{d}}(\boldsymbol{Y}, B;\, \boldsymbol{D}, r, \beta) = Y_{\mathrm{eq}} + c^{\mathrm{r}} r \,\mathrm{tr}\,\boldsymbol{D}\,\mathrm{tr}\,\boldsymbol{Y} - (B_0 + B) = 0 \tag{45}$$

with

$$Y_{\mathrm{eq}} = \sqrt{\frac{1}{2}\boldsymbol{Y} : \boldsymbol{L}(\boldsymbol{D}) : \boldsymbol{Y}} \tag{46}$$

tensor function $\boldsymbol{L}(\boldsymbol{D})$ linear in $\boldsymbol{D}$

$$[\boldsymbol{L}(\boldsymbol{D})]_{ijkl} = \frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) + \frac{1}{2}c^{\mathrm{d}}\left(\delta_{ik}D_{jl} + D_{ik}\delta_{jl} + \delta_{il}D_{jk} + D_{il}\delta_{jk}\right) \tag{47}$$

and material constants c^{d}, c^{r}, and B_0. Expressions for $\boldsymbol{M}(\boldsymbol{D})$ and $\boldsymbol{L}(\boldsymbol{D})$ that appear in formulae for σ_{eq} (Equation (43)) and Y_{eq} (Equation (46)) are based on the assumption that in no-damage states both the yield and damage functions are isotropic.

A more general formulation was given by Zhu and Cescotto (1995) where, instead of the simplified formulae (43) and (46), the following expressions were used

$$\sigma_{\mathrm{eq}} = \left\{\frac{1}{2}\sigma^{\mathrm{T}} : \overline{\boldsymbol{H}} : \sigma\right\}^{1/2}, \qquad \overline{\boldsymbol{H}} = \boldsymbol{M}^{\mathrm{ZC}} : \boldsymbol{H} : \boldsymbol{M}^{\mathrm{ZC}}, \tag{48}$$

$$Y_{\mathrm{eq}} = \left\{\frac{1}{2}\boldsymbol{Y}^{\mathrm{T}} : \boldsymbol{J} : \boldsymbol{Y}\right\}^{1/2}, \tag{49}$$

where $\boldsymbol{H}$ is Hill's plastic characteristic tensor, $\boldsymbol{M}^{ZC}(\boldsymbol{D})$ denotes damage effect tensor, and $\boldsymbol{J}$ stands for damage characteristic tensor. Hence, the anisotropy of both the plastic surface and the damage surface is accounted for. It was experimentally proved that for the purpose of modeling spheroidized graphite cast iron the formulae (44) and (47) are justified.

Using Equations (32)–(36) and (40) the Clausius–Duhem inequality becomes

$$\Phi = \sigma : \dot{\varepsilon}^{\mathrm{p}} - R\dot{r} - \boldsymbol{Y} : \dot{\boldsymbol{D}} - B\dot{\beta} \geq 0 \tag{50}$$

where Φ denotes the non-negative dissipation due to plasticity and damage that is subjected to two constraints $F^{\mathrm{p}} = 0$ and $F^{\mathrm{d}} = 0$. Hence, when two Lagrange multipliers $\dot{\lambda}^{\mathrm{p}}$ and $\dot{\lambda}^{\mathrm{d}}$ are introduced, the functional is built

$$\overline{\Phi} = \Phi - \dot{\lambda}^{\mathrm{p}} F^{\mathrm{p}} - \dot{\lambda}^{\mathrm{d}} F^{\mathrm{d}} = \sigma : \dot{\varepsilon}^{\mathrm{p}} - R\dot{r} - \boldsymbol{Y} : \dot{\boldsymbol{D}} - B\dot{\beta} - \dot{\lambda}^{\mathrm{p}} F^{\mathrm{p}} - \dot{\lambda}^{\mathrm{d}} F^{\mathrm{d}} \tag{51}$$

Next, maximizing $\overline{\Phi}$, we arrive at the plasticity and damage constitutive equations:

$$\frac{\partial \overline{\Phi}}{\partial \sigma} = 0 \longrightarrow \dot{\varepsilon}^{\mathrm{p}} = \dot{\lambda}^{\mathrm{p}} \frac{\partial F^{\mathrm{p}}}{\partial \sigma}, \qquad \dot{\varepsilon}^{\mathrm{p}}_{ij} = \frac{3}{2} \dot{\lambda}^{\mathrm{p}} \frac{M_{ijkl} \sigma'_{kl}}{\sigma_{\mathrm{eq}}} = \dot{\lambda}^{\mathrm{p}} m_{ij} \tag{52}$$

$$\frac{\partial \overline{\Phi}}{\partial (-R)} = 0 \longrightarrow \dot{r} = \dot{\lambda}^{\mathrm{p}} \frac{\partial F^{\mathrm{p}}}{\partial (-R)} = \dot{\lambda}^{\mathrm{p}} \tag{53}$$

$$\frac{\partial \overline{\Phi}}{\partial \boldsymbol{Y}} = 0 \longrightarrow \dot{\boldsymbol{D}} = \dot{\lambda}^{\mathrm{d}} \frac{\partial F^{\mathrm{d}}}{\partial \boldsymbol{Y}} = \dot{\lambda}^{\mathrm{d}} \left[\frac{\boldsymbol{L} : \boldsymbol{Y}}{2 Y_{\mathrm{eq}}} + c_{\mathrm{r}} r \, (\mathrm{tr}\, \boldsymbol{D}) \boldsymbol{I} \right] \tag{54}$$

$$\frac{\partial \overline{\Phi}}{\partial (-B)} = 0 \longrightarrow \dot{\beta} = \dot{\lambda}^{\mathrm{d}} \frac{\partial F^{\mathrm{d}}}{\partial (-B)} = \dot{\lambda}^{\mathrm{d}} \tag{55}$$

where, for convenience of further derivation, a new definition of the second-rank tensor $\boldsymbol{m}(\sigma, \boldsymbol{D})$ was used, $\boldsymbol{m} = (3/2)\,(\boldsymbol{M}(\boldsymbol{D}) : \sigma'/\sigma_{\mathrm{eq}})$, with Equation (44) taken into account for the fourth-rank tensor function $\boldsymbol{M}(\boldsymbol{D})$.

The Kuhn–Tucker relations are used in order to specify loading/unloading conditions both for plasticity and damage

$$\dot{\lambda}^{\mathrm{p}} \geq 0, \quad F^{\mathrm{p}} \leq 0, \quad \dot{\lambda}^{\mathrm{p}} F^{\mathrm{p}} = 0 \tag{56}$$

$$\dot{\lambda}^{\mathrm{d}} \geq 0, \quad F^{\mathrm{d}} \leq 0, \quad \dot{\lambda}^{\mathrm{d}} F^{\mathrm{d}} = 0 \tag{57}$$

The values of plastic multiplier $\dot{\lambda}^{\mathrm{p}}$ and damage multiplier $\dot{\lambda}^{\mathrm{d}}$ are found by solving the consistency conditions simultaneously for the yield surface and

the damage surface (e.g., Abu Al-Rub and Voyiadjis, 2003):

$$\dot{F}^{\mathrm{p}} \equiv \frac{\partial F^{\mathrm{p}}}{\partial \sigma} : \dot{\sigma} + \frac{\partial F^{\mathrm{p}}}{\partial R}\dot{R} + \frac{\partial F^{\mathrm{p}}}{\partial \boldsymbol{D}} : \dot{\boldsymbol{D}} = 0 \tag{58}$$

$$\dot{F}^{\mathrm{d}} \equiv \frac{\partial F^{\mathrm{d}}}{\partial \boldsymbol{Y}} : \dot{\boldsymbol{Y}} + \frac{\partial F^{\mathrm{d}}}{\partial B}\dot{B} + \frac{\partial F^{\mathrm{d}}}{\partial \boldsymbol{D}} : \dot{\boldsymbol{D}} = 0 \tag{59}$$

hence, when plasticity and damage evolution rules (Equations (52)–(55)) are substituted for $\dot{r}$, $\dot{\beta}$, and $\dot{\boldsymbol{D}}$, we arrive at

$$\begin{aligned}\dot{\lambda}^{\mathrm{p}} &= \frac{(\partial F^{\mathrm{p}}/\partial \sigma) : \dot{\sigma} + \left((\partial F^{\mathrm{p}}/\partial \boldsymbol{D}) : (\partial F^{\mathrm{d}}/\partial \boldsymbol{Y})\right)\dot{\lambda}^{\mathrm{d}}}{\mathrm{d}R/\mathrm{d}r} \\ &= \frac{(\partial F^{\mathrm{p}}/\partial \sigma) : \dot{\sigma}}{\mathrm{d}R/\mathrm{d}r} + \frac{\left((\partial F^{\mathrm{p}}/\partial \boldsymbol{D}) : (\partial F^{\mathrm{d}}/\partial \boldsymbol{Y})\right)\left((\partial F^{\mathrm{d}}/\partial \boldsymbol{Y}) : \dot{\boldsymbol{Y}}\right)}{(\mathrm{d}R/\mathrm{d}r)((\mathrm{d}B/\mathrm{d}\beta) - (\partial F^{\mathrm{d}}/\partial \boldsymbol{D}) : (\partial F^{\mathrm{d}}/\partial \boldsymbol{Y}))}\end{aligned} \tag{60}$$

and

$$\dot{\lambda}^{\mathrm{d}} = \frac{(\partial F^{\mathrm{d}}/\partial \boldsymbol{Y}) : \dot{\boldsymbol{Y}}}{(\mathrm{d}B/\mathrm{d}\beta) - (\partial F^{\mathrm{d}}/\partial \boldsymbol{D}) : (\partial F^{\mathrm{d}}/\partial \boldsymbol{Y})} \tag{61}$$

Note that the thermodynamic force $\boldsymbol{Y}$ is expressed in terms of σ (see Equation (34)) such that from the rate of it, $\dot{\boldsymbol{Y}}$ may be understood as a product of the fourth-rank tensor $\boldsymbol{Z} = \partial \boldsymbol{Y}/\partial \sigma$ and the stress rate tensor $\dot{\sigma}$

$$\dot{\boldsymbol{Y}} = \boldsymbol{Z} : \dot{\sigma} \tag{62}$$

Finally, both plasticity and damage multipliers $\dot{\lambda}^{\mathrm{p}}$ and $\dot{\lambda}^{\mathrm{d}}$ are given in terms of $\dot{\sigma}$ as

$$\dot{\lambda}^{\mathrm{d}} = \left[\frac{(\partial F^{\mathrm{d}}/\partial \boldsymbol{Y}) : \boldsymbol{Z}}{(\mathrm{d}B/\mathrm{d}\beta) - (\partial F^{\mathrm{d}}/\partial \boldsymbol{D}) : (\partial F^{\mathrm{d}}/\partial \boldsymbol{Y})}\right] : \dot{\sigma} = \boldsymbol{\Lambda}^{\mathrm{d}} : \dot{\sigma} = \left\{\boldsymbol{\Lambda}^{\mathrm{d}}\right\}\{\dot{\sigma}\} \tag{63}$$

$$\begin{aligned}\dot{\lambda}^{\mathrm{p}} &= \left[\frac{\partial F^{\mathrm{p}}/\partial \sigma}{\mathrm{d}R/\mathrm{d}r} + \frac{\left((\partial F^{\mathrm{p}}/\partial \boldsymbol{D}) : (\partial F^{\mathrm{d}}/\partial \boldsymbol{Y})\right)\left((\partial F^{\mathrm{d}}/\partial \boldsymbol{Y}) : \boldsymbol{Z}\right)}{(\mathrm{d}R/\mathrm{d}r)((\mathrm{d}B/\mathrm{d}\beta) - (\partial F^{\mathrm{d}}/\partial \boldsymbol{D}) : (\partial F^{\mathrm{d}}/\partial \boldsymbol{Y}))}\right] : \dot{\sigma} \\ &= \boldsymbol{\Lambda}^{\mathrm{p}} : \dot{\sigma} = \left\{\boldsymbol{\Lambda}^{\mathrm{p}}\right\}\{\dot{\sigma}\}\end{aligned} \tag{64}$$

where second-rank tensors $\boldsymbol{\Lambda}^{\mathrm{d}}$ and $\boldsymbol{\Lambda}^{\mathrm{p}}$ depend on the current material state variables σ, $\boldsymbol{Y}$, $\boldsymbol{D}$, and r.

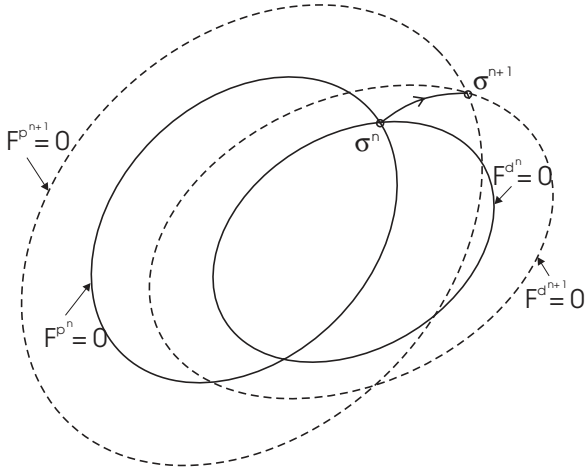

Figure 2. *Coupled dissipation surfaces, plastic and damage, simultaneously corrected at loading step.*

In other words, in the stress space two coupled dissipation surfaces exist, the plastic surface $F^{\mathrm{p}} = 0$ and the damage $F^{\mathrm{d}} = 0$, that should be corrected together on the subsequent loading steps $\Delta\sigma = \sigma^{n+1} - \sigma^{n}$ (Figure 2).

It happens, of course, that when one of the two multipliers $\dot{\lambda}^{\mathrm{p}}$ or $\dot{\lambda}^{\mathrm{d}}$ equals zero, then the system of Equations (63) and (64) reduces to $\dot{\lambda}^{\mathrm{p}} > 0$ or $\dot{\lambda}^{\mathrm{d}} > 0$, for passive damage or plastic processes, respectively. On the other hand if both multipliers $\dot{\lambda}^{\mathrm{p}}$ and $\dot{\lambda}^{\mathrm{d}}$ equal zero a doubly passive plastic and damage process takes place.

FINITE ELEMENT IMPLEMENTATION

Incremental Elastic-Plastic-Damage Constitutive Equations – Effective Tangent Compliance Matrix

When the Hayakawa and Murakami concept was used, elastic-damage constitutive equations are formulated in a total form (Equation (33)) $\varepsilon^{\mathrm{e}} = {}^{\mathrm{s}}\boldsymbol{C}^{\mathrm{e}}(\boldsymbol{D}) : \sigma$, where ${}^{\mathrm{s}}\boldsymbol{C}^{\mathrm{e}}$ stands for the effective secant compliance matrix. Since the plasticity and damage constitutive equations (52)–(55) are given in the incremental form, in order to solve the coupled elastic–plastic-damage process we also need the incremental form of Equation (33).

To derive incremental elastic-damage constitutive equations we follow the scheme described in Kuna-Ciskał and Skrzypek (2004), where the effective tangent stiffness matrix has been determined based on the relevant model of elastic-damage materials developed by Murakami and Kamiya (1997).

It was shown that when deriving explicit general formulae for the effective tangent matrix the main difficulty arises from the unilateral damage effect included, as described by the fourth-rank transformation tensor $D^{\varepsilon}_{ijkl} = \partial\overline{\varepsilon}_{ij}/\partial\varepsilon_{kl}$. In the present article, after Hayakawa and Murakami (1997), the Gibbs formulation is applied for the state potential instead of the Helmholtz free energy, such that elastic-damage constitutive equations are expressed in stresses instead of strains. This makes experimental verification straightforward, in which stress-controlled processes are usually examined.

On the other hand, the implementation of these relations into a standard displacement finite element code requires special treatment since the iteration process is driven by strain increments. Moreover, both the plastic flow and damage evolution are the history-dependent processes, hence the integration of the constitutive equations at each subincremental step must be performed from the last equilibrium state instead of simple cumulation of subincrements of stresses.

For numerical operations by finite element method the vector/matrix representation of all tensor quantities is applied. The Voigt notation is used to perform tensorial multiplication with appropriate contractions. At each integration point where the constitutive relations are to be integrated it is assumed that the state variables $(\{\varepsilon\},\ r, \{\boldsymbol{D}\}, \beta)$ together with the conjugate forces $(\{\sigma\},\ R, \{\boldsymbol{Y}\}, B)$ are known at the beginning of a 'time' increment. In the frame of the small strain theory, the given strain increment $\{\dot{\varepsilon}\}$ is the sum of the elastic and plastic part

$$\{\dot{\varepsilon}\} = \{\dot{\varepsilon}^{\mathrm{e}}\} + \{\dot{\varepsilon}^{\mathrm{p}}\} \tag{65}$$

whereas the stress increment $\{\dot{\sigma}\}$ is sought along with conjugate pairs' increase of $\dot{\boldsymbol{R}}$ and $\dot{r}$, $\{\dot{\boldsymbol{Y}}\}$ and $\{\dot{\boldsymbol{D}}\}$, and $\dot{\boldsymbol{B}}$ and $\dot{\beta}$.

The general incremental form of the elastic-damage constitutive Equation (33) may be expressed as

$$\{\dot{\varepsilon}^{\mathrm{e}}\} = [\boldsymbol{C}^{\mathrm{e}}(\sigma, \boldsymbol{D}, \boldsymbol{Y})]\{\dot{\sigma}\} \tag{66}$$

with the local tangent elastic compliance matrix $\boldsymbol{C}^{\mathrm{e}}$ being

$$\boldsymbol{C}^{\mathrm{e}}(\boldsymbol{D}, \sigma) = {}^{\mathrm{s}}\boldsymbol{C}^{\mathrm{e}}(\boldsymbol{D}) + \frac{\partial^{\mathrm{s}}\boldsymbol{C}^{\mathrm{e}}(\boldsymbol{D})}{\partial \boldsymbol{D}} : \frac{\partial \boldsymbol{D}}{\partial \sigma} : \sigma \tag{67}$$

Taking into account Equation (52) for the plastic-damage strain increment $\dot{\varepsilon}^{\mathrm{p}}$ together with Equation (64) the following is furnished

$$\dot{\varepsilon}^{\mathrm{p}} = \boldsymbol{m}^{\mathrm{p}}\dot{\lambda}^{\mathrm{p}} = \boldsymbol{m}^{\mathrm{p}}(\boldsymbol{\Lambda}^{\mathrm{p}} : \dot{\sigma}) = (\boldsymbol{m}^{\mathrm{p}}\boldsymbol{\Lambda}^{\mathrm{p}}) : \dot{\sigma} \tag{68}$$

or if a matrix form is used

$$\{\dot{\varepsilon}^{\mathrm{p}}\} = \left[\left\{\boldsymbol{m}^{\mathrm{p}}(\sigma, \boldsymbol{D})\right\}\left\{\boldsymbol{\Lambda}^{\mathrm{p}}(\sigma, \boldsymbol{D}, \boldsymbol{Y}, r)\right\}\right]\{\dot{\sigma}\} = [\boldsymbol{C}^{\mathrm{p}}(\sigma, \boldsymbol{D}, \boldsymbol{Y}, r)]\{\dot{\sigma}\} \tag{69}$$

Note that, when Voigt notation is used, in Equation (69) $[\{\boldsymbol{m}^{\mathrm{p}}\}\{\boldsymbol{\Lambda}^{\mathrm{p}}\}]$ stands for the matrix product of the corresponding vectors.

Hence, from Equations (65), (66), and (69), we eventually arrive at the incremental elastic-plastic-damage constitutive equation

$$\{\dot{\varepsilon}\} = [\boldsymbol{C}^{\mathrm{e}}]\{\dot{\sigma}\} + [\boldsymbol{C}^{\mathrm{p}}]\{\dot{\sigma}\} = [\boldsymbol{C}(\sigma, \boldsymbol{D}, \boldsymbol{Y}, r)]\{\dot{\sigma}\} \tag{70}$$

with the local elastic-plastic-damage compliance matrix $\boldsymbol{C} = \boldsymbol{C}^{\mathrm{e}}(\sigma, \boldsymbol{D}, \boldsymbol{Y}) + \boldsymbol{C}^{\mathrm{p}}(\sigma, \boldsymbol{D}, \boldsymbol{Y}, r)$ dependent on the variables given at the starting equilibrium point (see Equations (67) and (69)).

Then taking inverse of the matrix $\boldsymbol{C}$, the increments of stresses are calculated as:

$$\{\dot{\sigma}\} = [\boldsymbol{C}^{-1}]\{\dot{\varepsilon}\} \tag{71}$$

and the elastic-plastic-damage stiffness matrix $\boldsymbol{C}^{-1}$ is accepted as an approximation of the local Jacobian matrix of the constitutive model. The general representation of the elastic-damage compliance matrix $\boldsymbol{C}^{\mathrm{e}}(\boldsymbol{D}, \sigma)$ (Equation (67)) as well as the plastic-damage matrix $\boldsymbol{C}^{\mathrm{p}}(\sigma, \boldsymbol{D}, \boldsymbol{Y}, r)$ (Equation (69)) is somewhat cumbersome for numerical implementation into ABAQUS finite element code via the user-supplied procedure that defines non-standard material properties. Hence, in what follows we confine ourselves to the plane stress, following procedures described in Skrzypek and Kuna-Ciskał (2003).

Matrix Constitutive Equations in Plane Stress State

If the plane stress state $\sigma_{33} = 0$ is considered then in vector/matrix representation we deal with:

$$\sigma = \{\sigma_{11}, \sigma_{22}, \sigma_{12}\}, \qquad \boldsymbol{Y} = \{Y_{11}, Y_{22}, Y_{33}, Y_{12}\}, \qquad \boldsymbol{D} = \{D_{11}, D_{22}, D_{33}, D_{12}\}, \tag{72}$$

$$\boldsymbol{\Lambda}^{\mathrm{p}} = \left\{\Lambda^{\mathrm{p}}_{11}, \Lambda^{\mathrm{p}}_{22}, \Lambda^{\mathrm{p}}_{12}\right\}, \qquad \boldsymbol{\Lambda}^{\mathrm{d}} = \left\{\Lambda^{\mathrm{d}}_{11}, \Lambda^{\mathrm{d}}_{22}, \Lambda^{\mathrm{d}}_{12}\right\}, \tag{73}$$

$$\boldsymbol{Z} = \left[\frac{\partial \boldsymbol{Y}}{\partial \sigma}\right] = \begin{bmatrix} Z_{11} & Z_{12} & Z_{14} \\ Z_{21} & Z_{22} & Z_{24} \\ Z_{31} & Z_{32} & Z_{34} \\ Z_{41} & Z_{42} & Z_{44} \end{bmatrix}$$
$$\left\{\dot{Y}_{11}, \dot{Y}_{22}, \dot{Y}_{33}, \dot{Y}_{12}\right\} = \begin{bmatrix} Z_{11} & Z_{12} & Z_{14} \\ Z_{21} & Z_{22} & Z_{24} \\ Z_{31} & Z_{32} & Z_{34} \\ Z_{41} & Z_{42} & Z_{44} \end{bmatrix} \begin{Bmatrix} \dot{\sigma}_{11} \\ \dot{\sigma}_{22} \\ \dot{\sigma}_{12} \end{Bmatrix} \tag{74}$$

The effective application of Equation (28) to the constitutive equation (32) requires the explicit formula for the derivative $\partial\overline{\sigma}_{ij}/\partial\sigma_{kl}$, therefore the unilateral transformation tensor $D^{(\sigma)}_{ijkl}$ from Equation (38) must explicitly be expressed. In the case of plane stress considered, the transformation reduces to plane rotation by the angle β

$$\beta = \frac{1}{2}\arctan\frac{2\sigma_{12}}{\sigma_{11} - \sigma_{22}} \tag{75}$$

Hence, the derivative $\partial\overline{\sigma}_{ij}/\partial\sigma_{kl}$ must be calculated according to the procedure for the complex function

$$D^{(\sigma)}_{ijkl} = \frac{\partial\overline{\sigma}_{ij}}{\partial\sigma_{kl}} = B^{(\sigma)}_{ijkl} + \frac{\partial B^{(\sigma)}_{ijpq}}{\partial\beta}\frac{\partial\beta}{\partial\sigma_{kl}}\sigma_{pq} \tag{76}$$

The above scheme is cumbersome, therefore, after Kuna-Ciskał and Skrzypek (2004), we reduce Equation (28) to the simplified form

$$\begin{Bmatrix} \overline{\sigma}_{11} \\ \overline{\sigma}_{22} \\ \overline{\sigma}_{12} \end{Bmatrix} = \begin{bmatrix} a & 0 & b \\ 0 & c & b \\ 0.5b & 0.5b & d \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} \tag{77}$$

where $a = (1/2)(\xi_1 + \xi_2) + (1/2)(\xi_1 - \xi_2)\cos 2\beta$, $b = (1/2)(\xi_1 - \xi_2)\sin 2\beta$, $c = (1/2)(\xi_1 + \xi_2) - (1/2)(\xi_1 - \xi_2)\cos 2\beta$, and $d = (1/2)(\xi_1 + \xi_2)$, whereas $\xi_I = \xi(\sigma_I)$ denote coefficients in the stress modification rule (Equation (29)) due to damage deactivation.

In the plane stress conditions, the total form of the matrix constitutive equation (32) has the form

$$\begin{Bmatrix} \varepsilon^{\mathrm{e}}_{11} \\ \varepsilon^{\mathrm{e}}_{22} \\ \gamma^{\mathrm{e}}_{12} \end{Bmatrix} = \begin{bmatrix} {}^{\mathrm{s}}C^{\mathrm{e}}_{11} & {}^{\mathrm{s}}C^{\mathrm{e}}_{12} & {}^{\mathrm{s}}C^{\mathrm{e}}_{14} \\ {}^{\mathrm{s}}C^{\mathrm{e}}_{12} & {}^{\mathrm{s}}C^{\mathrm{e}}_{22} & {}^{\mathrm{s}}C^{\mathrm{e}}_{24} \\ {}^{\mathrm{s}}C^{\mathrm{e}}_{14} & {}^{\mathrm{s}}C^{\mathrm{e}}_{24} & {}^{\mathrm{s}}C^{\mathrm{e}}_{44} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} \tag{78}$$

The effective secant compliance matrix ${}^{\mathrm{s}}\boldsymbol{C}^{\mathrm{e}}(\boldsymbol{D})$ is furnished as follows:

$$
\begin{aligned}
{}^{\mathrm{s}}C^{\mathrm{e}}_{11} &= \frac{1}{E_0} + 2\vartheta_1 \operatorname{tr}\boldsymbol{D} + 2\vartheta_2 \operatorname{tr}\boldsymbol{D}(a^2 + 0.5b^2) + 2\vartheta_3 D_{11} \\
&\quad + \vartheta_4[D_{11}(2a^2 + 0.5b^2) + 0.5D_{22}b^2 + 2abD_{12}] \\
{}^{\mathrm{s}}C^{\mathrm{e}}_{12} &= -\frac{\nu_0}{E_0} + 2\vartheta_1 \operatorname{tr}\boldsymbol{D} + \vartheta_2 b^2 \operatorname{tr}\boldsymbol{D} + \vartheta_3(D_{11} + D_{22}) \\
&\quad + \vartheta_4[0.5b^2(D_{11} + D_{22}) + b(a + c)D_{12}] \\
{}^{\mathrm{s}}C^{\mathrm{e}}_{14} &= 2b(a + d)\vartheta_2 \operatorname{tr}\boldsymbol{D} + 2\vartheta_3 D_{12} \\
&\quad + \vartheta_4[D_{11}b(2a + d) + bdD_{22} + 2D_{12}(b^2 + ad)] \\
{}^{\mathrm{s}}C^{\mathrm{e}}_{22} &= \frac{1}{E_0} + 2\vartheta_1 \operatorname{tr}\boldsymbol{D} + 2\vartheta_2 \operatorname{tr}\boldsymbol{D}(c^2 + 0.5b^2) + 2\vartheta_3 D_{22} \\
&\quad + \vartheta_4[0.5D_{11}b^2 + D_{22}(2c^2 + 0.5b^2) + 2bcD_{12}] \\
{}^{\mathrm{s}}C^{\mathrm{e}}_{24} &= 2b(c + d)\vartheta_2 \operatorname{tr}\boldsymbol{D} + 2\vartheta_3 D_{12} \\
&\quad + \vartheta_4[bdD_{11} + D_{22}b(2c + d) + 2D_{12}(b^2 + cd)] \\
{}^{\mathrm{s}}C^{\mathrm{e}}_{44} &= 2\left\{ \frac{1 + \nu_0}{E_0} + 2\vartheta_2 \operatorname{tr}\boldsymbol{D}(1 + b^2 + d^2) \right. \\
&\quad \left. + \vartheta_4[(b^2 + d^2)(D_{11} + D_{22}) + 4bdD_{12}] \right\}
\end{aligned}
\tag{79}
$$

The thermodynamic conjugate force $\boldsymbol{Y}$ defined by Equation (34) is expressed as:

$$
\begin{Bmatrix} Y_{11} \\ Y_{22} \\ Y_{33} \\ Y_{12} \end{Bmatrix} = \begin{Bmatrix} \vartheta_1(\operatorname{tr}\sigma)^2 + \vartheta_2 \operatorname{tr}(\overline{\sigma}\cdot\overline{\sigma}) + \vartheta_3(\operatorname{tr}\sigma)\sigma_{11} + \vartheta_4\left[(a\sigma_{11} + b\sigma_{12})^2 + (0.5b(\sigma_{11} + \sigma_{22}) + d\sigma_{12})^2\right] \\ \vartheta_1(\operatorname{tr}\sigma)^2 + \vartheta_2 \operatorname{tr}(\overline{\sigma}\cdot\overline{\sigma}) + \vartheta_3(\operatorname{tr}\sigma)\sigma_{22} + \vartheta_4\left[(c\sigma_{22} + b\sigma_{12})^2 + (0.5b(\sigma_{11} + \sigma_{22}) + d\sigma_{12})^2\right] \\ \vartheta_1(\operatorname{tr}\sigma)^2 + \vartheta_2 \operatorname{tr}(\overline{\sigma}\cdot\overline{\sigma}) \\ \vartheta_3(\operatorname{tr}\sigma)\sigma_{12}\vartheta_4[0.5b(\sigma_{11} + \sigma_{22}) + d\sigma_{12}](a\sigma_{11} + 2b\sigma_{12} + c\sigma_{22}) \end{Bmatrix}
\tag{80}
$$

When the general incremental form of the elastic-damage constitutive equation (66) is applied to the plain stress state $\mathrm{d}\sigma_{33} = 0$, the following 3×3 tangent elastic-damage stiffness matrix representation $C^{\mathrm{e}}_{ij}(\sigma, \boldsymbol{D}, \boldsymbol{Y})$ is

furnished in terms of the corresponding secant matrix components ${}^{s}C^{e}_{ij}(\boldsymbol{D})$ Equation (79), according to formula (67)

$$\begin{Bmatrix} \dot{\varepsilon}^{e}_{11} \\ \dot{\varepsilon}^{e}_{22} \\ \dot{\gamma}^{e}_{12} \end{Bmatrix} = \begin{bmatrix} C^{e}_{11} & C^{e}_{12} & C^{e}_{14} \\ C^{e}_{12} & C^{e}_{22} & C^{e}_{24} \\ C^{e}_{14} & C^{e}_{24} & C^{e}_{44} \end{bmatrix} \begin{Bmatrix} \dot{\sigma}_{11} \\ \dot{\sigma}_{22} \\ \dot{\sigma}_{12} \end{Bmatrix}. \tag{81}$$

For instance, as regards the first element of the matrix (81), the following formula holds:

$$\begin{aligned} C^{e}_{11} = {}^{s}C^{e}_{11} &+ \left\{ 2\vartheta_1 \frac{\partial \operatorname{tr} \boldsymbol{D}}{\partial \sigma_{11}} + 2\vartheta_2 \frac{\partial \operatorname{tr} \boldsymbol{D}}{\partial \sigma_{11}} \left(a^2 + 0.5b^2\right) + 2\vartheta_3 \frac{\partial D_{11}}{\partial \sigma_{11}} \right. \\ &\left. + \vartheta_4 \left[\frac{\partial D_{11}}{\partial \sigma_{11}} \left(2a^2 + 0.5b^2\right) + 0.5 \frac{\partial D_{22}}{\partial \sigma_{11}} b^2 + 2ab \frac{\partial D_{12}}{\partial \sigma_{11}} \right] \right\} \sigma_{11} \\ &+ \left\{ 2\vartheta_1 \frac{\partial \operatorname{tr} \boldsymbol{D}}{\partial \sigma_{11}} + \vartheta_2 b^2 \frac{\partial \operatorname{tr} \boldsymbol{D}}{\partial \sigma_{11}} + \vartheta_3 \left(\frac{\partial D_{11}}{\partial \sigma_{11}} + \frac{\partial D_{22}}{\partial \sigma_{11}} \right) \right. \\ &\left. + \vartheta_4 \left[0.5b^2 \left(\frac{\partial D_{11}}{\partial \sigma_{11}} + \frac{\partial D_{22}}{\partial \sigma_{11}} \right) + \frac{\partial D_{12}}{\partial \sigma_{11}} b(a + c) \right] \right\} \sigma_{22} \\ &+ \left\{ 2b(a + d)\vartheta_2 \frac{\partial \operatorname{tr} \boldsymbol{D}}{\partial \sigma_{11}} + 2\vartheta_3 \frac{\partial D_{12}}{\partial \sigma_{11}} \right. \\ &\left. + \vartheta_4 \left[\frac{\partial D_{11}}{\partial \sigma_{11}} b(2a + d) + bd \frac{\partial D_{22}}{\partial \sigma_{11}} + 2 \frac{\partial D_{12}}{\partial \sigma_{11}} \left(b^2 + ad\right) \right] \right\} \sigma_{12} \end{aligned} \tag{82}$$

Computational Algorithm for Elastic-Plastic Damage Material under Unilateral Damage Condition

The iteration of global equilibrium of a system is performed by ABAQUS with the use of Newton–Raphson method. The applied scheme is iteration-independent, which means that all variables are only updated by the end of an increment step after the convergence is achieved. The task to be performed by the user-supplied routine is to integrate physical relations at a point level (Gauss point of an element) when starting from a known equilibrium state and for a total strain increment given in each iteration. The output information are stress and all other state variables updated (integrated) by the end of the iteration increment as well as the local stiffness matrix.

The integration is performed here with an explicit forward Euler scheme. That means the derivatives (stiffness) are known at the starting point and kept constant along the increment. Such an approach is not unconditionally

stable and may be succesfully used only for relatively simple loading history and sufficiently small incremental step.

The particular forms of matrices $\boldsymbol{C}^{\mathrm{e}}$ and $\boldsymbol{C}^{\mathrm{p}}$ not only depend on the state variables and the conjugate forces but also on the kind of deformation process taking place through a strain increment. Namely, they depend on whether the process is active or passive. Here, the terms 'active', 'passive', and 'neutral' are generalized for both damage and plasticity. 'Active' (loading) denotes a process, which implies evolution of the limit surface; 'passive' (unloading) stands for changes inside the limit surface; and 'neutral' denotes a process tangent to the limit surface.

The predictor–corrector approach is applied here since the limit surfaces depend on the state variables and may be verified after the variables are updated at the end of an incremental step. At each iteration step we start with a doubly passive predictor, which means that matrix $\boldsymbol{C}^{\mathrm{e}}$ is determined for $\dot{\lambda}^{\mathrm{d}} = 0$ and matrix $\boldsymbol{C}^{\mathrm{p}}$ is zero because of $\dot{\lambda}^{\mathrm{p}} = 0$, i.e., we assume there are no plastic strain increments and those elastic do not involve the damage parameters evolution. Afterwards, the stress increments $\{\dot{\sigma}\}$ are calculated, then $\{\dot{\boldsymbol{Y}}\}$ from Equation (62) is found, and before the conjugate forces are updated (only $\{\sigma\}$ and $\{\boldsymbol{Y}\}$ are changed) checking is performed. There are, in general, four possible situations at the end of the increment:

$$(1) \qquad F^{\mathrm{d}}\left(\boldsymbol{Y} + \dot{\boldsymbol{Y}}, B, \boldsymbol{D}, r\right) \leq 0 \quad \wedge \quad F^{\mathrm{p}}(\sigma + \dot{\sigma}, R, \boldsymbol{D}) \leq 0 \tag{83}$$

which means that none of the limit surfaces is exceeded, the pure (no damage) elastic increment is accepted and the conjugate forces $\{\sigma\}$ and $\{\boldsymbol{Y}\}$ are updated: $\sigma \longleftarrow \sigma + \dot{\sigma}$, $\boldsymbol{Y} \longleftarrow \boldsymbol{Y}(\sigma)$ (Equation (34));

otherwise – the corrector phase ((2), (3), or (4)) is necessary

$$(2) \qquad F^{\mathrm{d}}\left(\boldsymbol{Y} + \dot{\boldsymbol{Y}}, B, \boldsymbol{D}, r\right) > 0 \quad \wedge \quad F^{\mathrm{p}}(\sigma + \dot{\sigma}, R, \boldsymbol{D}) \leq 0 \tag{84}$$

(only damage surface is exceeded) the matrix $\boldsymbol{C}^{\mathrm{e}}$ is rebuilt with the elastic-damage evolution allowed for (with $\dot{\lambda}^{\mathrm{d}} > 0$), whereas $\dot{\lambda}^{\mathrm{p}} = 0$ is kept constant. New solution for stress increment $\{\dot{\sigma}\}$ is obtained, $\dot{\lambda}^{\mathrm{d}}$ (Equation (63)) and $\{\dot{\boldsymbol{Y}}\}, \{\dot{\boldsymbol{D}}\}$, and $\dot{\beta}$ are determined from Equations (62), (54), and (55), and checking is done again for the plasticity limit:

if

$$F^{\mathrm{p}}\left(\sigma + \dot{\sigma}, R, \boldsymbol{D} + \dot{\boldsymbol{D}}\right) \leq 0 \tag{85}$$

then the increment is accepted, the state variables $\boldsymbol{D}$ and β as well as the conjugate forces σ, $\boldsymbol{Y}$, and B are updated: $\boldsymbol{D} \longleftarrow \boldsymbol{D} + \dot{\boldsymbol{D}}$, $\beta \longleftarrow \beta + \dot{\beta}$, $\sigma \longleftarrow \sigma + \dot{\sigma}$, $\boldsymbol{Y} \longleftarrow \boldsymbol{Y}(\sigma)$ (Equation (34)), $B \longleftarrow B(\beta)$ (Equation (36));

else – Point (4) is performed

(3) $$F^{\mathrm{d}}\left(\boldsymbol{Y}+\dot{\boldsymbol{Y}}, B, \boldsymbol{D}, r\right) \leq 0 \quad \wedge \quad F^{\mathrm{p}}(\sigma+\dot{\sigma}, R, \boldsymbol{D}) > 0 \tag{86}$$

(only plastic surface is exceeded) the matrix $\boldsymbol{C}^{\mathrm{p}}$ is built with $\dot{\lambda}^{\mathrm{p}} > 0$ (plastic surface evolution allowed for) but with $\dot{\lambda}^{\mathrm{d}} = 0$ (no damage evolution), whereas the matrix $\boldsymbol{C}^{\mathrm{e}}$ is kept unchanged as in the predictor phase. After the solution for stress increment $\{\dot{\sigma}\}$ is obtained, the plastic multiplier $\dot{\lambda}^{\mathrm{p}}$ (Equation (64)) is determined as well as $\dot{r}$ (Equation (53)), and $\{\dot{\boldsymbol{Y}}\}$ (Equation (62)) are calculated. Then checking is done for the damage limit:

if

$$F^{\mathrm{d}}\left(\boldsymbol{Y}+\dot{\boldsymbol{Y}}, B, \boldsymbol{D}, r+\dot{r}\right) \leq 0 \tag{87}$$

then the increment is accepted, the state variable r as well as the conjugate forces σ, $\boldsymbol{Y}$, and R are updated: $r \longleftarrow r+\dot{r}$, $R \longleftarrow R(r)$ (Equation (35)), $\sigma \longleftarrow \sigma+\dot{\sigma}$, $\boldsymbol{Y} \longleftarrow \boldsymbol{Y}(\sigma)$ (Equation (34));

else – Point (4) is performed

(4) $$F^{\mathrm{d}}\left(\boldsymbol{Y}+\dot{\boldsymbol{Y}}, B, \boldsymbol{D}, r\right) > 0 \quad \wedge \quad F^{\mathrm{p}}(\sigma+\dot{\sigma}, R, \boldsymbol{D}) > 0 \tag{88}$$

(both limit surfaces are exceeded) the matrices $\boldsymbol{C}^{\mathrm{e}}$ and $\boldsymbol{C}^{\mathrm{p}}$ are built for $\dot{\lambda}^{\mathrm{p}} > 0$ (plastically active process) together with $\dot{\lambda}^{\mathrm{d}} > 0$ (damage evolution). After the stress increments $\{\dot{\sigma}\}$ are calculated, the plastic multiplier $\dot{\lambda}^{\mathrm{p}}$ (Equation (64)) and the damage multiplier $\dot{\lambda}^{\mathrm{d}}$ (Equation (63)) are determined, $\dot{r}$ (Equation (53)), $\{\dot{\boldsymbol{D}}\}$ (Equation (54)), and $\dot{\beta}$ (Equation (55)) are determined, and finally the state variables r, $\boldsymbol{D}$, and β together with the conjugate forces σ, $\boldsymbol{Y}$, R, and B are updated: $r \longleftarrow r+\dot{r}$, $\boldsymbol{D} \longleftarrow \boldsymbol{D}+\dot{\boldsymbol{D}}$, $\beta \longleftarrow \beta+\dot{\beta}$, $\sigma \longleftarrow \sigma+\dot{\sigma}$, $\boldsymbol{Y} \longleftarrow \boldsymbol{Y}(\sigma)$ (Equation (34)), $R \longleftarrow R(r)$ (Equation (35)), $B \longleftarrow B(\beta)$ (Equation (36)).

Eventually, the local stiffness matrix (treated as tangent at the beginning of the step), possibly corrected after predictor–corrector treatment, is accepted for computing the global stiffness of a system.

A more general method is presented in Zhu and Cescotto (1995). There, the elastic predictor is followed by the two-step coupled plastic-damage corrector and the stiffness matrix is built by a numerical perturbation method.

EXAMPLES OF APPLICATION TO MONOTONIC AND CYCLIC LOADINGS

The presented constitutive model and the local iteration method are implemented into the ABAQUS finite element code via the user-supplied

procedure that defines non-standard material properties. The primary goal is to test the efficiency and convergence of the global iteration process for such a model of elastic-plastic-damage material in uniaxial or plane stress state.

Material Data

The following material constants are taken for spheroidized graphite cast iron FCD400 after Hayakawa and Murakami (1999):

$$E_0 = 169 \times 10^3\,(\mathrm{MPa}), \quad \nu_0 = 0.285, \quad \zeta = 0.89,$$
$$\vartheta_1 = -3.95 \times 10^{-7}\,(\mathrm{MPa}^{-1}), \quad \vartheta_2 = 4.00 \times 10^{-6}\,(\mathrm{MPa}^{-1}),$$
$$\vartheta_3 = -4.00 \times 10^{-7}\,(\mathrm{MPa}^{-1}), \quad \vartheta_4 = 2.50 \times 10^{-6}\,(\mathrm{MPa}^{-1}),$$
$$b = 15, \quad R_0 = 293.0\,(\mathrm{MPa}), \quad R_\infty = 250.0\,(\mathrm{MPa}), \quad c^{\mathrm{p}} = 1.0,$$
$$K_{\mathrm{d}} = 1.3\,(\mathrm{MPa}), \quad B_0 = 0.273\,(\mathrm{MPa}), \quad c^{\mathrm{d}} = -15.0, \quad c^{\mathrm{r}} = 50.0$$

Monotonic Loading

Initially, a uniform stress state is analyzed and a plane rectangular shield is used for a structural model.

A strain control is used because, for the stress-controlled processes together with the forward Euler integration scheme and lack of consistent linearization, a divergence might be met at the point-level iteration, especially for the material model with asymptotic hardening.

The results of integration for monotonic strain increase are presented in Figures 3–5. Figure 3 presents the stress–strain relation with initial elastic (doubly passive) range followed by elastic-damage process and concluded with elastic-damage-plastic deformation. The asymptotic nature of plastic hardening is also observed. The effect of acquired anisotropy is confirmed in Figure 4. The diagonal components of the damage tensor, initially of value zero increase in course of deformation but the component D_{11} (connected with the direction of tension) is greater than D_{22} observed in the perpendicular direction. One may also observe that evolution of plastic deformation slows down significantly the evolution of damage process. Finally, Figure 5 presents the deterioration of elastic material properties due to damage. C_{11}^{-1}, which is the diagonal component of elasticity tensor (Equation (71)), is of constant value as long as the process is doubly passive, then rapidly decreases after the elastic-damage process is activated, and still decreases, but to a much lesser extent, after plasticity is coupled with damage. Due to asymptotic plasticity of the material model, the deterioration observed

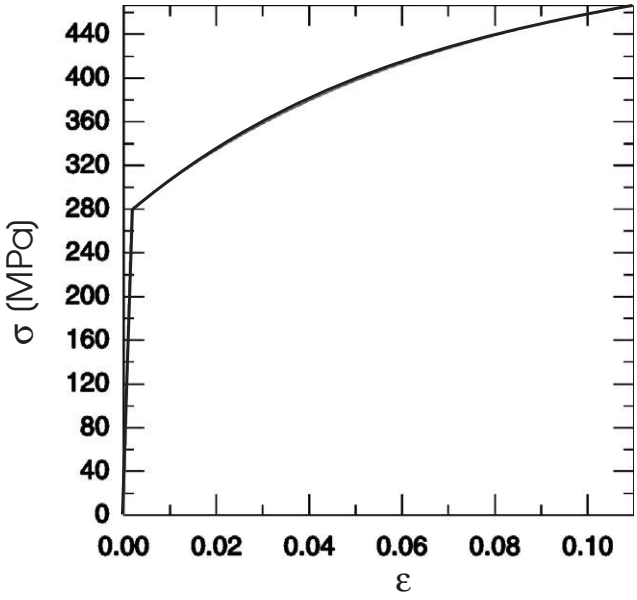

Figure 3. *Strain-controlled process with observed stages of: purely elastic, elastic-damage, and elastic-damage-plastic responses.*

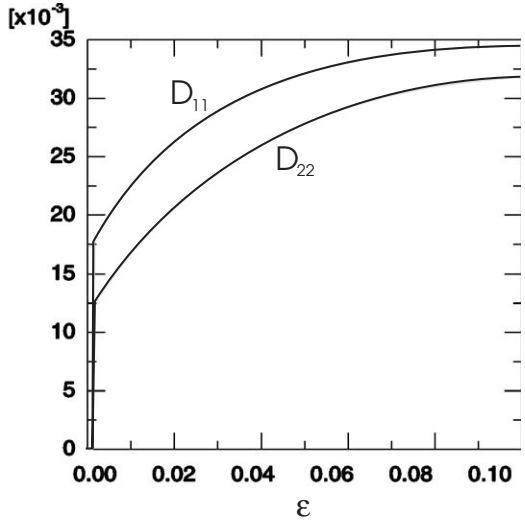

Figure 4. *Damage anisotropy and effect of plastic deformation on damage rate.*

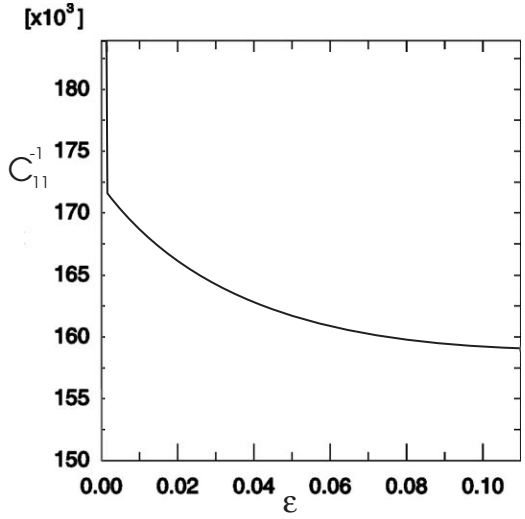

Figure 5. *Stiffness drop due to elastic-damage and elastic-plastic-damage growth.*

is also of asymptotic nature. In the initial state of virgin elastic material and for plane stress conditions $C_{11}^{-1} = E_0/(1 - \nu_0^2)$.

Cyclic Loading

Figures 6–10 present the results of cyclic strain-controlled loading process for relatively small strain value. The cyclic process is assumed with amplitude increasing for each half-cycle (Figure 6) so as to enable activation of both dissipation processes on the tension and compression side. The stress–strain relation is given in Figure 7. The initial fraction of the multi-cycle diagram resembles what is presented above for monotonic load. The evolution of damage tensor in course of cyclic deformation is presented in Figure 8. Not only is the anisotropy of damage effects confirmed but one may also easily recognize whether the process is 'damagely' active or passive. This diagram should be correlated with Figure 9, where the hardening parameters are given and the passive or active stages of the process in a sense of plastic r and damage β hardening are shown. The curve for evolution of the damage hardening parameter β is in exact affinity with those for the evolution of damage tensor components. Clearly, the damage rate is proportional to the rate of β (Equation (54)). The evolution of the plastic hardening parameter r indicates a plastic-active or plastic-passive process. It may be observed that the plastically active

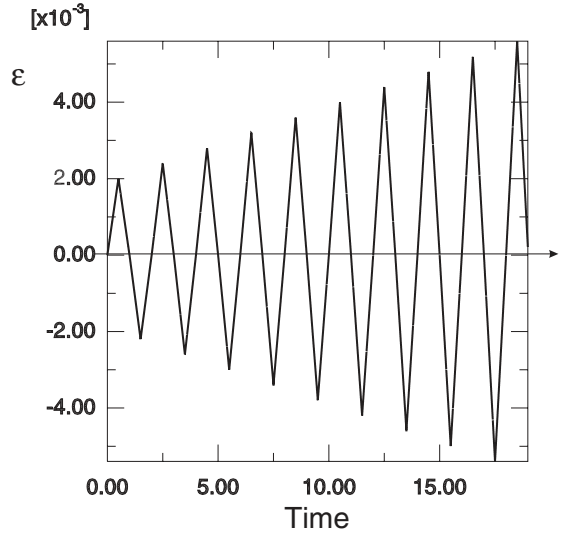

Figure 6. *Increasing strain-controlled deformation cycles.*

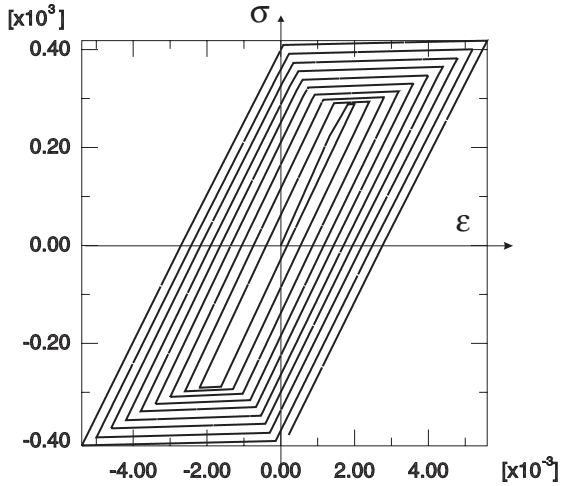

Figure 7. *Stress–strain curves on loading cycles with partial damage deactivation ($\zeta = 0.89$) for spheroidized cast iron.*

deformation evolves both on the tension and compression side, whereas damage evolves almost exclusively on the tension side. This is explained based on the fact that on the compression side both the dissipation surfaces are isotropically extended due to doubly active deformation and, after the

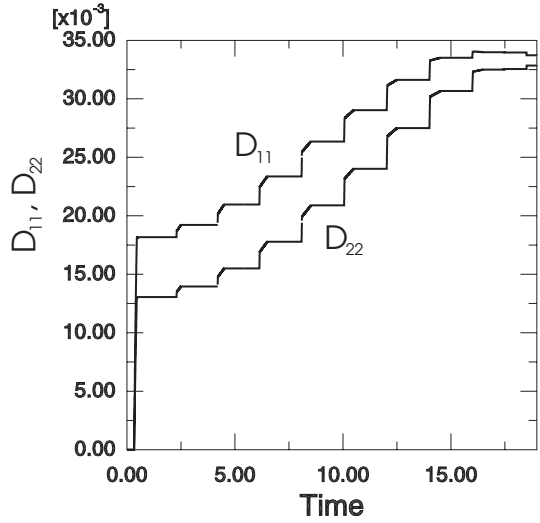

Figure 8. *Evolution of anisotropic damage components in course of cyclic deformation.*

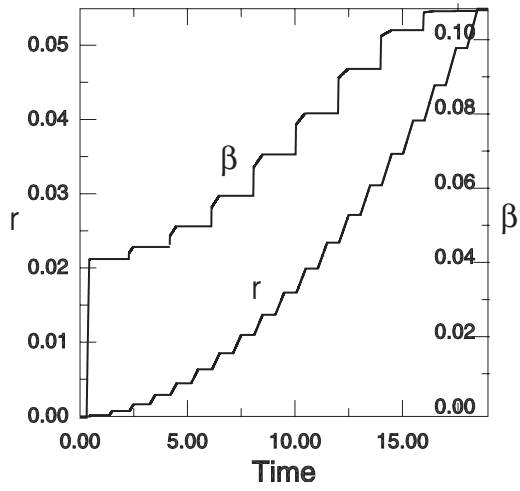

Figure 9. *Evolution of hardening parameters, plastic (r) and damage (β), on loading cycles.*

sign of stress becomes negative, the plastically active process precedes any damage since the effective damage stress is reduced by the unilateral effect. This, however, must not be taken as a rule. Probably for a bigger change in the cycle amplitude on the compression side both the dissipation processes would be active.

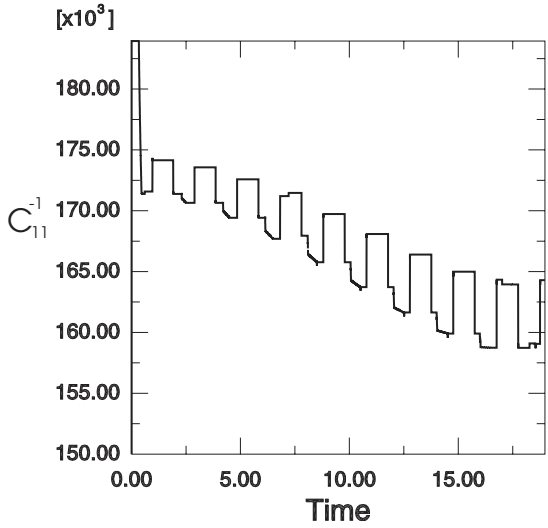

Figure 10. *Consecutive stiffness drop/recovery during tensile/compressive loading half-cycles.*

Figure 10 presents a change of the elastic material properties during the cycle loading. The diagonal component of the elasticity tensor C_{11}^{-1} varies in 'time'. Again, the deterioration of material stiffness is confirmed. One may also observe the partial recovery of the stiffness after each change of stress for negative sign. The deterioration has an asymptotic nature: the decrease of elastic modulus on the tension side becomes slower in course of the process. Also, the recovery of the stiffness on the compression side is asymptotically convergent. The change of the elastic modulus (directly connected with the unilateral effect) is noticeable but not significant for the accepted material constants (note a big scale of the vertical axis in Figure 10). That is why the change of slope in the stress–strain diagram (Figure 7) is hardly visible.

CONCLUSIONS

1. The extended incremental elastic-plastic-damage model is capable of describing both the purely elastic-damage and the plastic-damage behavior.
2. The applied model of unilateral damage by the use of generalized projection operator enables to account for a partial damage deactivation controlled by the additional material constant ζ, which results in a slower damage evolution on a compressive side.

3. For the material data used for model calibration, damage evolution on the compressive side is almost eliminated due to reduction of the effective damage stress by the stiffness recovery. In contrast, the plastic deformation evolves on both the tensile and compressive side.
4. Application of the model developed is limited to relatively simple loading conditions due to isotropic hardening applied to both plastic and damage surfaces. For more complex loading conditions it is necessary to extend the model by the kinematic hardening as well.

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