Implementing individual savings decisions for retirement with bounds on wealth

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Abstract

We present a savings plan for retirement that removes risk by fixing a constraint on a life–long pension so that it has an upper and a lower bound. This corresponds to the ideas of Nobel laureate R.C. Merton whose implementation has never been published. We show with an illustration that our proposed practical algorithm reproduces the theoretical results after a savings period of around thirty years by using daily, monthly, weekly or yearly updates of the investment positions. We calculate the percentiles of the final accumulated wealth distribution for the adjusted implementation. In the simulated illustration we observe that the adjusted values converge to the theoretical values of the percentiles when the frequency of update increases. We conclude that monthly adjustments result in a practical way to implement theoretical results that were obtained under the hypothesis of a continuous process by Donnelly et al. (2015). This method is easy to use in practice by pension savers and fund managers.

1 Introduction and motivation

Nobel laureate Robert C. Merton's vision in his 2014 Harvard Business Review paper (Merton, 2014) was a new communication framework for pensions, where the individual customer has only three parameters to consider: their target pension, their worst case scenario pension and the associated probabilities. Merton suggests that the communication with the pension saver could for example be as follows: if you are interested in a target pension of \$10,000 per annum, then there is 2/3 chance of actually reaching it. If you do not reach your target pension, then \$7,000 per annum is the lowest pension that you will get.

Merton stressed the need for a real income in retirement, and the need to focus the investment strategy around providing a real income. **Merton is involved with the leading global investment manager Dimensional Fund Advisors and one can actually see the structure of a real-life version of Merton's vision when consulting Dimensional's website. However, the technical details behind Dimensional's implementation of Merton's vision are not revealed. It is quite properly considered proprietary material.** Yet pension investment strategies are often aimed at maximising a nominal amount at retirement. Moreover, there is often a mismatch between the hurried, day-to-day approach to investments in many investment departments and the long term goal of providing a real income in retirement.

In our work, we provide a possible implementation of Merton's vision. However, we do not believe that it fully coincides with its implementation by the investment manager, Dimensional Fund Advisors, because it probably does not involve the type of investments that we propose.

The Merton idea can be implemented via incorporating a lower bound into the framework introduced by Donnelly et al. (2015). Donnelly et al. (2015) design strategies to control wealth at retirement, so they focus on the analysis of the stochastic distribution of terminal wealth. These authors propose ways to guarantee the stability of accumulated wealth after a certain horizon. This is easily generalized to the upside and downside terminal wealth constraints, which constitute a way to automatically smooth the feasible retirement annuities, which are proportional to the wealth accumulated during the savings period. The mechanism is based on compensating the elimination of large losses by the elimination of large gains. The pension has a target value, which cannot be exceeded. Giving up the possibility of having a larger pension than the target pays for the control of the downside, meaning that the pension cannot fall below a stated minimum amount.

Since the practical implementation of these new methods was originally not discussed by Donnelly et al. (2015), here we carry out a presentation of the algorithm. The practical implementation of investment strategies for retirement turns out to be straightforward, and similarly to Merton's proposal, the investor establishes an upper and a lower bound on the annuity pension. The results are a generalization of those that were initially proposed by Donnelly et al. (2015). The latter authors solve the portfolio selection problem of an investor with a deterministic savings plan who is constrained to have no more than a target wealth at retirement (an upper bound). Here we extend the results by adding a lower bound to the terminal wealth that can be expressed in terms of constraints in the life annuity. **We remark that while the strategy developed in the paper may be able to ensure that accumulated final wealth lies between the designated target values, we cannot guarantee an annuity income, as annuity prices will depend on mortality rates and, primarily, on interest rates at the retirement date.**

The proposed practical mechanism results in a transparent and automatic pension savings product where the portfolio during the savings phase is regularly rebalanced, so that the accumulated wealth at any moment is constrained by the lower and upper bounds.

We emphasize that this paper is primarily an asset allocation paper. The contribution is precisely to show the practical implementation that allows to compensate the risk of a bad performance of savings by reducing the possibility of large gains. This is exactly what can be understood as a high-strike call or upper bound, versus low-strike-out or lower bound. Our aim is to show that implementing a savings strategy in practice requires reconsideration of the portfolio at a certain frequency and, that weekly updates seem to approximate the optimal solution quite well. We also make it clear that we only

focus on a one-time investment. However, we do not expect to find diverging conclusions for a pension saver that invests an additional sum every month in a scheme similar to Gerrard et al. (2014).

This paper suggests a simplification of the pension saver decision process when deciding how much risk to take. It focuses on strategies designed to buy an annuity, by tailoring Merton's vision. We deliberately leave out inflation. We assume that the control of inflation is carried out by investment experts. Individuals should only focus on a lower and an upper bound for their pension at current prices.[1] People might have an understanding about how much they might need today, which is why they should specify their target in today's terms and leave it to the investment manager to make sure that that will be appropriately translated into values in the future.

The paper is organized as follows. In Section 2 the background is shown. In Section 3 we present the mathematical problem to solve and the practical algorithm is shown in Section 4. In Section 5 we carry out the numerical illustration. Section 6 concludes and discusses the implications.

2 Background

The idea that funds which aim to accumulate savings for retirement should be under some form of control is not new. There are many authors introducing some constraint on the portfolio or the terminal wealth. For instance, Korn and Trautmann (1995) set a constraint on the expected value of the final wealth.

Many contributions point into that direction and analyze the stochastic distribution of retirement wealth for proposing investment strategies. Greninger et al. (2000) conclude that nine-tenths of the experts who participated in their study agreed that families should have achieved 50-60% of their retirement savings goal by age 50 and 85-90% by age 60. Regarding asset allocation, over 60% of experts feel that it is prudent to start moving toward more conservative investments about 3-5 years before retirement. However, as noted by Basu et al. (2011) such deterministic switching rules produce inferior

[1]Many authors have shown that financial literacy enhances peoples' likelihood of contributing to their pension saving (Behrman et al., 2012; Lusardi and Mitchell, 2007).

wealth outcomes for the investor compared to strategies that dynamically alter the allocation between growth and conservative assets based on cumulative portfolio performance relative to a set target.

Grossman and Zhou (1996) impose the constraint that the terminal wealth must be at least some fraction of the initial wealth, and consequently find that risk aversion rises when stock prices fall. Browne (1999) finds the strategy that maximizes the probability of reaching a given wealth level by a given fixed terminal time, for the case where an investor can allocate his wealth at any time between investment opportunities given by a collection of risky stocks, as well as a risk-free asset that has a positive return.

Recently, Donnelly et al. (2015) found that by constraining the final wealth by using an upper bound, the investor increases their chance of attaining the desired target retirement wealth, and even if he fails to reach it, he still has a higher wealth than if he has no such upper bound. Note that Donnelly et al. (2015) proposed a different formulation compared to Dhaene et al. (2005), in which at least the target capital is attained with maximum probability. Donnelly et al. (2015) have also a different approach compared to Browne (1999), as Browne (1999) maximizes directly the probability of reaching the target retirement wealth. Here we consider the same approach as Donnelly et al. (2015) but adding also a lower bound for the final wealth. Note that our approach is also different from Gerrard et al. (2014) who analyze the lowest part of the terminal wealth distribution after savings and consumption.

Here we concentrate on the savings phase (by choosing a saving period of thirty years) and we constrain the terminal wealth by using an upper and a lower bound. Additionally, we show how to implement daily adjustments to investment to obtain results that replicate the theoretical optimal strategies.

Other relevant contributions where some constraint on the terminal wealth is introduced can be found in Van Weert et al. (2010) who generalize portfolio selection problems to the case where a minimal return requirement is imposed. They derive an intuitive formula that can be used in provisioning and terminal wealth problems as a constraint on the admissible investment portfolios, in order to guarantee a minimal annualized return. Bouchard et al. (2010) deal with target constraints on stochastic processes and Gaibh et al. (2009) consider optimal selection of portfolios for utility maximizing investors under joint budget and shortfall risk constraints, where shortfall risk is measured in terms of the expected loss. Hainaut and Devolder (2007) maximize the utility of dividends and of terminal surplus under a budget constraint.

Boyle and Tian (2007) investigate the portfolio selection problem for an investor who desires to outperform some benchmark index with a certain confidence level. The benchmark is chosen to reflect some particular investment objective and it can be either deterministic or stochastic. However, this approach is about short-term investment rather than savings in the long term. Cuoco (1997) examines the intertemporal optimal consumption and investment problem in the presence of a stochastic endowment and constraints on the portfolio choices and, in the same spirit, Zariphopoulou (1994) focuses on consumption-investment models with constraints. Bernard et al. (2014) construct optimal strategies explicitly and show how they outperform traditional diversified strategies under worst-case scenarios.

3 Optimizing annuities with constraints

3.1 The underlying financial market

In this section, we assume investment in a continuous-time financial market model over a finite time horizon $[0, T]$ for an integer $T > 0$. Sometimes we refer to T as the *terminal time*. In our context, time 0 is the start of the savings phase and T is the retirement date.

The market consists of one risky stock and one risk-free bond. At time t, the risk-free bond has price $S_0(t)$ and the risky stock has price $S_1(t)$. Their price dynamics are

$$\mathrm{d}S_0(t) = rS_0(t)\,\mathrm{d}t, \qquad \mathrm{d}S_1(t) = S_1(t)\left(\mu dt + \sigma dW(t)\right), \tag{3.1}$$

in which W is a Brownian motion, $\sigma > 0$, $S_0(0) = 1$ and $S_1(0)$ being a fixed, strictly positive constant. We assume that $\mu > r$, where μ is the mean stock return and r is the risk-free return.

The information $\mathcal{F}_t$ available to investors at time t is the information generated by the Brownian motion up to time t. The market price of risk is $\theta := (\mu - r)/\sigma$.

3.2 Investor

An investor starts with an initial wealth $x_0 > 0$ and plans to make a sequence of known future savings, each of amount $a > 0$. Define $C(t)$ to be the sum from time 0 to time t of the investor's planned discrete savings, with

$$dC(t) = \begin{cases} a & \text{if } t = 1, 2, \ldots, T-1 \\ 0 & \text{otherwise.} \end{cases}$$

In other words, at the end of each unit time period, the investor pays an amount $a > 0$ into their fund.

A portfolio process $\pi = \{\pi(t); t \in [0, T]\}$ is a square-integrable, $\{\mathcal{F}_t\}$-progressively measurable process. The investor follows a self-financed strategy, investing at each instant $t \in [0, T]$ a monetary amount $\pi(t)$ in the stock such that $\pi = \{\pi(t); t \in [0, T]\}$ is a portfolio process.

The wealth process $X^\pi = \{X^\pi(t); t \in [0, T]\}$ corresponding to a portfolio process is the $\{\mathcal{F}_t\}$-adapted process given by the *wealth equation*

$$dX^\pi(t) = (rX^\pi(t) + \pi(t)\sigma\theta)\, dt + \pi(t)\sigma\, dW(t) + dC(t), \quad X^\pi(0) = x_0 \text{ a.s.} \tag{3.2}$$

We define the *savings plan* g of the investor, i.e. the discounted sum of the future savings by the investor by

$$g(t) := \int_t^T e^{-r(s-t)} dC(s), \quad \forall t \in [0, T]. \tag{3.3}$$

Then the set of *admissible portfolios* for the investor's initial wealth $x_0 > 0$ is defined to be

$$\mathcal{A} := \{\pi : \Omega \times [0, T] \to \mathbb{R} : X^\pi(0) = x_0, \text{ a.s. and } X^\pi(t) + g(t) \geq 0, t \in (0, T] \text{ a.s.}\}.$$

We say that a portfolio process π is *admissible* if $\pi \in \mathcal{A}$.

The state price density process is $H(t) := \exp\left(-\left(r + \frac{1}{2}\theta^2\right)t - \theta W(t)\right)$, for each $t \in [0, T]$. A portfolio π must satisfy the *budget constraint* that

$$\mathbb{E}\left(H(T)X^\pi(T)\right) \leq x_0 + g(0). \tag{3.4}$$

The utility function of the investor is the power utility function

$$U(x) := \frac{1}{\gamma}x^\gamma, \qquad x > 0,$$

for a fixed constant $\gamma \in (-\infty, 1) \setminus \{0\}$. The investor seeks to maximise the expected utility of their terminal wealth, subject to constraints on the range of values of the terminal wealth.

Define the constant

$$A := \frac{\theta}{\sigma(1-\gamma)}.$$

A **is a measure of the market price of risk per unit of deviation of an investment asset corrected by risk aversion.** We also define the process

$$Z(t) = \exp\left(\left(r + \theta\sigma A - \frac{1}{2}\sigma^2 A^2\right) t + \sigma A W(t)\right), \quad \forall t \in [0, T]. \tag{3.5}$$

$Z(t)$ represents the wealth of an investor at time t who has no constraints on their investment strategy or wealth process, and invests 1 at time 0 with no savings plan.

3.3 Problem with a lower and an upper bound

Donnelly et al. (2015) introduce the constrained problem with an upper bound only, in which the investor seeks to maximize the expected utility of their terminal wealth, subject to the wealth being bounded above by the upper bound $K_U > 0$.

Here we extend the problem to include a lower bound K_L, below which the terminal wealth must not fall. Combined with the upper bound K_U, this means that the investor's terminal wealth lies in the range $[K_L, K_U]$. Additionally, one can have an equivalence between the upper and lower bounds of terminal wealth and those of the annuities or pensions.

The addition of a lower bound has already been well studied in the literature (for example, see Basak (1995)).

In order to avoid both the uninteresting case that the investor can immediately be assured of maximizing the terminal utility and to avoid the need to breach the non-arbitrage condition, we assume that

Assumption 3.1. $K_L < (x_0 + g(0))\, e^{rT} < K_U$.

Problem 3.2. Find $\pi \in \mathcal{A}$ such that

$$\mathbb{E}\left(U(X^\pi(T))\right) = \sup_{\pi \in \mathcal{A}} \{\mathbb{E}\left(U(X^\pi(T))\right)\},$$

and $X^\pi(T) \in [K_L, K_U]$, a.s.

The next proposition gives an expression for the optimal terminal wealth for Problem 3.2, when there is both a lower and upper bound constraint on the terminal wealth.

Proposition 3.3. *A solution to the constrained problem at the terminal time T is*

$$\begin{aligned} X^\star(T) =&(y_0+g(0))Z(T) - \max\{0, (y_0+g(0))Z(T) - K_U\} \\ &+ \max\{0, K_L - (y_0+g(0))Z(T)\}, \end{aligned} \tag{3.6}$$

with the shadow wealth $y_0 > 0$ *chosen so that the budget constraint (3.4) is satisfied with equality by $X^\star(T)$, given the investor's initial wealth $X^\star(0) = x_0$, a.s. and savings plan g.*

Proof. The proof is found in Appendix A. □

The value at time t of the maturity value $\max\{0, (y_0+g(0))Z(T) - K_U\}$ was shown in Donnelly et al. (2015) to be $c(t, Y(t); K_U)$ with the process

$$Y(t) := (y_0+g(0))Z(t) - g(t),$$

and, using Φ to denote the cumulative distribution function of the standard normal, the real-valued function

$$c(t, y; K_U) := y\Phi(d_+(t, y; K_U)) - K_U e^{-r(T-t)}\Phi(d_-(t, y; K_U)),$$

in which

$$d_\pm(t, y; K) := \frac{1}{\sigma A\sqrt{T-t}}\left(\ln\left(\frac{y}{K}\right) + \left(r \pm \frac{1}{2}\sigma^2 A^2\right)(T-t)\right), \tag{3.7}$$

for all $y > 0$ and for all $t \in [0, T]$, for each $K > 0$. Next we derive the value and replicating portfolio of the put option with maturity value $\max\{0, K_L - (y_0+g(0))Z(T)\}$.

Lemma 3.4. *The price at time $t \in [0, T]$ of a European put option with maturity value $\max\{0, K_L - (y_0+g(0))Z(T)\}$ is given by $p(t, Y(t); K_L)$ with*

$$p(t, y; K_L) := K_L e^{-r(T-t)}\Phi(-d_-(t, y; K_L)) - y\Phi(-d_+(t, y; K_L)).$$

The replicating portfolio for the put option is to hold in the risky asset at time t the amount $\pi_p(t, Y(t); K_L)$, with

$$\pi_p(t, y; K_L) := -Ay\,\Phi(-d_+(t, y; K_L)), \quad \forall t \in [0, T], \quad y > 0 \tag{3.8}$$

and the remaining amount $p(t, Y(t); K_L) - \pi_p(t, Y(t); K_L)$ in the risk-free bond.

Proof. The proof is found in Appendix A. □

The optimal strategy for Problem 3.2 is given in the next proposition.

Proposition 3.5. *An optimal investment strategy for Problem 3.2 is to invest the amount*

$$\pi(t) := A\left[1 - \Phi(d_+\left(t, Y(t); K_U\right) - \Phi(-d_+\left(t, Y(t); K_L\right))\right] Y(t) \tag{3.9}$$

in the risky stock and the amount $X^\pi(t) - \pi(t)$ in the risk-free bond.

The wealth process corresponding to this optimal investment strategy is

$$X^\pi(t) = Y(t) - g(t) - c(t, Y(t); K_U) + p(t, Y(t); K_L). \tag{3.10}$$

In particular, the relationship between the investor's initial wealth $X^\pi(0) = x_0$ and the shadow initial wealth y_0 is

$$x_0 = y_0 - c(0, y_0 + g(0); K_U) + p(0, y_0 + g(0); K_L). \tag{3.11}$$

Proof. The proof follows trivially from the previous lemmas. □

Note that the more risk averse the investor, the smaller the value of the constant A and the less the investor puts in the risky stock. The relative value of the shadow initial wealth y_0 over the investor's actual initial wealth x_0 has a concrete interpretation when there is only an upper bound, i.e. $K_U > 0$, and no lower bound, i.e. $K_L = 0$. For the p-quantiles of the constrained terminal wealth that fall below the target wealth K_U, it gives their uplift over those for the unconstrained terminal wealth.

To see this, we calculate the p-quantiles under the constrained strategy. For the constrained strategy, there is a probability mass at the target wealth K_U and at the lower bound K_L. For this reason we use the following generalised definition of the p-quantile.

Definition 3.6. The p-quantile for a random variable X is

$$q_p(X) = \inf\left\{y \in \mathbb{R} : \mathbb{P}\left[X \leq y\right] \geq p\right\},$$

with the convention that $\inf\{\emptyset\} = \infty$.

A result on the quantile can also be derived in this framework.

Proposition 3.7 (p-quantiles)**.** *Suppose an investor has initial wealth $x_0 > 0$ and follows the savings plan g. Define*

$$\beta_p := \sigma A\sqrt{T}\,\Phi^{-1}(p) + \left(r + \theta\sigma A - \frac{1}{2}\sigma^2 A^2\right) T. \tag{3.12}$$

If the investor follows the optimal constrained strategy, i.e. the terminal wealth is constrained to lie in the range $[K_L, K_U]$, then the p-quantile of the investor's terminal wealth $X(T)$ is

$$\mathcal{Q}_p(X(T); (K_L, K_U)) = \max\left\{K_L, \min\left\{K_U, (y_0 + g(0))e^{\beta_p}\right\}\right\}. \tag{3.13}$$

Proof. The proof is found in Appendix A. □

4 Practical implementation

In order to implement an algorithm that can utilize the theoretical results from the previous section, we have assumed that information on the parameters is available, where r is the risk-free rate, μ is the average return of risky assets, σ is volatility, A has to do with risk aversion, T is the time horizon, g is the payment stream or savings plan, x_0 is the initial wealth and K_L and K_U are lower and upper bounds which can either be expressed in terms of the terminal wealth $X(T)$ or as constraints on the yearly pensions. The uplift exists, but it is not a simple constant proportion, when both lower and upper bounds apply, i.e. when $0 < K_L < K_U$.

The algorithm introduces a new constraint on the investment strategy. The investor is not permitted to short-sell the stock, or borrow from the bond in order to invest in the stock. These are realistic restrictions for pension savers.

In the first step, we compute y_0 from equation 3.11, i.e. $x_0 = y_0 - c(0, y_0 + g(0); K_U) + p(0, y_0 + g(0); K_L)$. As $Y(0) = y_0$, this allows us to calculate the amount to be held in the risky stock at time 0, namely

$$\begin{aligned}\pi(0) = \max\Bigg\{ \min\Bigg\{ & A\left[1 - \Phi(d_+(0, Y(0); K_U) - \Phi(-d_+(0, Y(0); K_L))\right] Y(0), \\ & x_0 \Bigg\}, 0 \Bigg\}.\end{aligned} \tag{4.1}$$

The maximum and minimum bounds ensure that the investor does not short-sell the stock, nor borrow from the bond in order to invest in the stock.

Define $RealWealth(t)$ to be the sum of what the investor has in the stock and the bond at time t, with $RealWealth(0) = x_0$. The investor puts $\pi(0)$ in the risky stock and the remaining wealth, $RealWealth(0) - \pi(0)$, in the bond at time 0. Note that, due to the investment restrictions on the amount in the stock, the value of $RealWealth(t)$ may deviate from the theoretical wealth value obtained by not having the investment restrictions.

Next, for each $t = 1, \ldots, T$ we roll forward the values of the investment in the stock and bond, which were made at time $t-1$. The amount in the stock will increase by $S_1(t)/S_1(t-1)$ and the amount in the bond by $S_0(t)/S_0(t-1)$.

To update the strategy at time t, we calculate the amount to be invested in the risky stock as

$$\begin{aligned}\pi(t) = \max\Bigg\{ \min\Bigg\{ & A\left[1 - \Phi(d_+(t, Y(t); K_U) - \Phi(-d_+(t, Y(t); K_L))\right] Y(t), \\ & RealWealth(t)\Bigg\}, 0\Bigg\},\end{aligned} \tag{4.2}$$

where $Y(t) = (y_0 + g(0))Z(t) - g(t)$. The value invested in bonds is whatever the current real wealth less the amount in stocks, namely $RealWealth(t) - \pi(t)$.

When implementing this algorithm in practice, the adjustments can be done yearly, monthly or at higher frequencies. In the following illustration we have also considered weekly and daily time periods[2].

5 Numerical illustration

5.1 Presentation of the base case

In this section, we show how the practical implementation approximates the theoretical results on the quantiles of the stochastic distribution of terminal wealth or annuities. This allows us to examine the effect of the investment strategy constraints on the investment strategy. We perform $2,000$ simulations in order to compare the quantiles from the practical implementation with the theoretical quantiles of the distribution of terminal wealth $X(T)$.

The parameter values are $r = 0$, $\mu = 0.0343$, $\sigma = 0.1544$, $A = 1$, $T = 30$, $g = 0$ and $x_0 = 100$. The choice of the parameters implies that the investor's risk aversion constant

[2]The authors implemented the algorithm in R (R Core Team, 2014).

is $\gamma = -0.44$.

The values of μ and σ (equal to 0.0343 and 0.1544 respectively) are the same as those used in Guillen et al. (2013), where the authors calculate the parameter σ based on historical volatility. Namely, to estimate parameter σ the authors refer to the data found in Dimson et al. (2002) that provide an overview of the long-term performance of individual market for each of 16 countries, and estimate total returns on equities, bonds, bills, currencies, inflation and risk premia for all 101 years from 1900 to 2000.

The values of K_L and K_U are set to 83.3 and 138.3, respectively. The reason for choosing these values is that they **approximately correspond to the 15% and 35% quantiles** of an unrestricted investment over 30 years (**unrestricted in the sense that there are no upper and lower bounds**). This has a meaning. In implementing a savings mechanism with bounds the investor would avoid extremes. Other possible values would depend on the preferences and needs of the consumers. We have analyzed other cases for illustration, but the final conclusions do not change. For these values of K_L and K_U, we calculate the shadow initial wealth y_0 satisfying equation (3.11), which results in $y_0 = 100.7542$. We implement the practical algorithm outlined in Section 4, that calculates the real wealth accumulated by the investor at the end of the investment period subject to realistic constraints on the investment strategy. $2,000$ simulations are done.

When the algorithm is implemented, we calculate the quantiles of the real wealth at $T = 30$. We also consider $T = 30$ for $K_L = 83.3$ and $K_U = 138.3$, for an initial investment of $x_0 = 100$. We calculate the quantiles of the simulated final wealth at $T = 30$ for different periodicity of updates: yearly, monthly, weekly and daily. The quantiles, obtained by simulating the final wealth are compared to the theoretical ones, i.e. those resulting from expression (3.13). The results are shown in Figure 1. Note how the values of the quantiles converge to the theoretical values as the frequency of the update increases. When we use daily adjustments, the resulting quantiles of the approximated distribution are almost identical to the theoretical distribution.

The results are also presented in Table 1. The first column shows the values of p for the p-quantiles. The second column shows the theoretical

quantiles obtained by evaluating expression (3.13) for the given parameters. The next four columns are the practical approximation, which does not allow the investor to either short-sell the stock or the bond.

Figure 1: Graph showing the p-quantiles of the final wealth for the theoretical distribution and simulated values for the yearly, monthly, weekly and daily updates, also showed in T

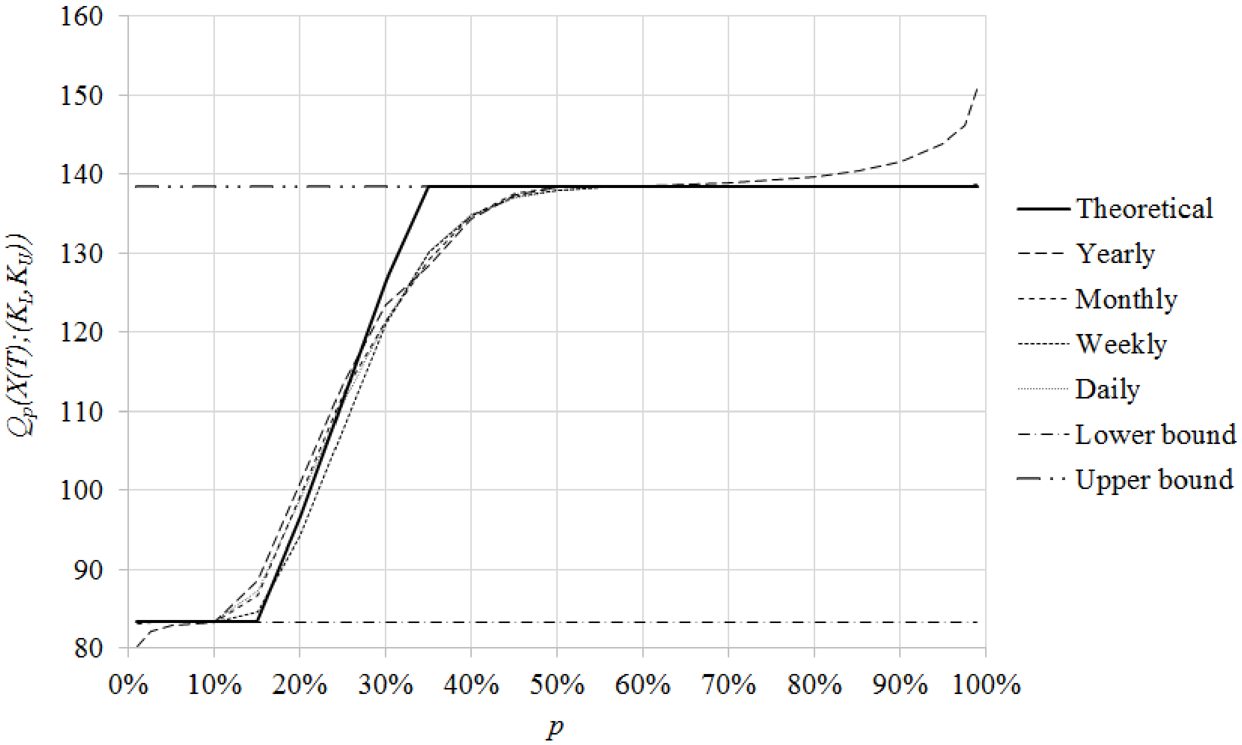

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Table 1: Table showing the p-quantiles of the final wealth assuming that there is a limit to the amount invested in stocks (it must be between 0% and 100 % of your wealth). The table shows the theoretical distribution and the simulated practical implementation of the algorithm to calculate the p-quantiles of real wealth at $T = 30$ for an initial investment equal to 100 with a lower bound $K_L = 83.3$ and an upper bound $K_U = 138.3$. Adjustment for updates every year, month, week and day are showed. $A = 1$, $\sigma = 0.1544$, $\gamma = -0.438797$ and 2,000 runs.

Update:	Theoretical	Yearly	Monthly	Weekly	Daily
p	$\mathcal{Q}_p(X(T))$				
1%	83.33	80.17	83.17	83.33	83.33
2.5%	83.33	82.10	83.30	83.33	83.33
5%	83.33	82.86	83.33	83.33	83.33
10%	83.33	83.32	83.33	83.34	83.43
15%	83.33	88.70	86.70	84.66	87.27
20%	96.77	100.71	99.09	94.16	98.65
25%	111.46	113.54	112.31	107.84	111.18
30%	126.55	123.52	121.36	120.96	121.45
35%	138.33	128.50	129.23	130.14	130.06
40%	138.33	134.37	134.72	134.66	134.96
45%	138.33	137.39	137.61	137.14	136.95
50%	138.33	138.34	138.28	137.98	137.93
55%	138.33	138.40	138.33	138.30	138.28
60%	138.33	138.50	138.33	138.33	138.33
65%	138.33	138.65	138.33	138.33	138.33
70%	138.33	138.89	138.33	138.33	138.33
75%	138.33	139.21	138.33	138.33	138.33
80%	138.33	139.70	138.34	138.33	138.33
85%	138.33	140.43	138.34	138.33	138.33
90%	138.33	141.56	138.35	138.33	138.33
95%	138.33	143.95	138.38	138.33	138.33
97.5%	138.33	146.14	138.46	138.33	138.33
99%	138.33	150.79	138.70	138.33	138.33
Prob. hit K_U	66.24%				
Prob. hit K_L	15.42%				
Mean		124.48	124.21	123.52	124.01
Variance		464.46	438.40	464.16	441.60
Skewness		-0.96	-1.13	-1.06	-1.10
Kurtosis		2.34	2.56	2.35	2.49

The results shown in Table 1 indicate that daily adjustments of the investment strategy algorithm presented in Section 4 lead to almost identical results to the distribution of the theoretical quantiles of the annuities. This fact implies that pension savers can use this method to achieve a pension that lies between an upper and a lower level, as presented in the introduction.

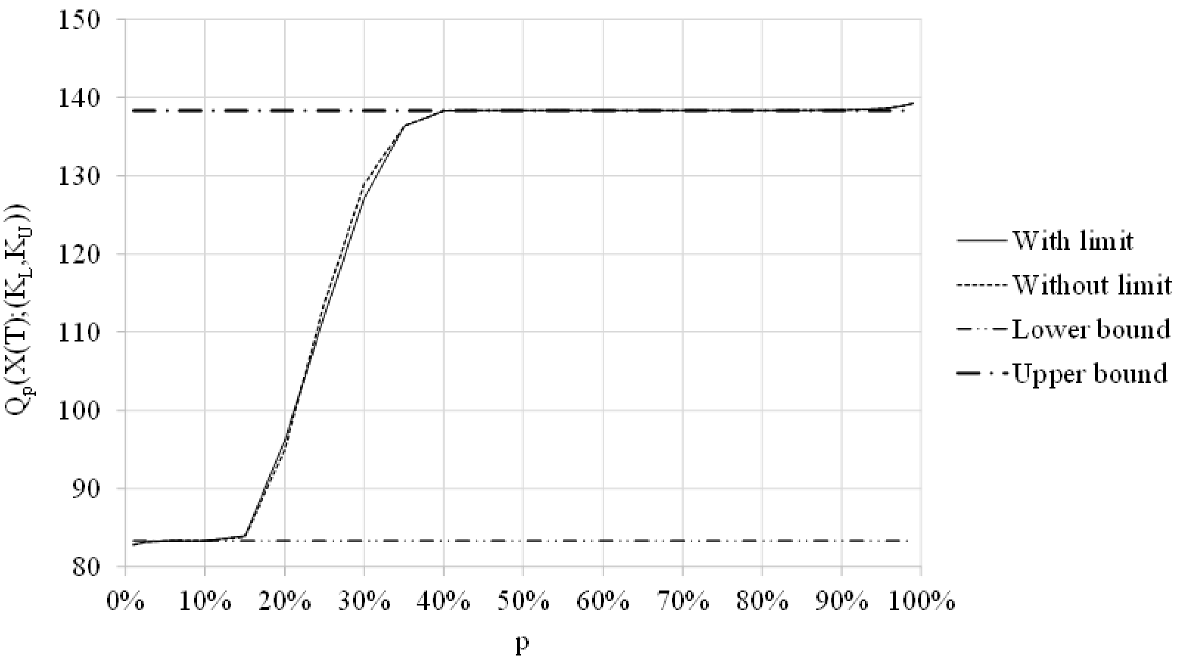
The table corresponds to an initial investment of $x_0 = 100$, lower bound $K_L = 83.3$ and upper bound $K_U = 138.3$. If we multiply all three of these parameters by the same constant, then the values shown in the table would also be scaled by that constant. In fact, this is a general result. So, this table covers a whole range of possibilities.

However, in the next subsections we extend our numerical example in order to compare the results that we get in different scenarios.

5.2 Limited and unlimited investment in stocks

In this section we compare the case where there is a limit to the amount invested in stocks (between 0% and 100% of your wealth) and the case without this limit. In order to do that, we have calculated the $p-$quantiles of the final wealth for the case where $A = 1.5$, and compare the results with and without this limit. The values of the rest of parameters are the same as those used to produce Table 1. **The results are shown in Figure 2 where we represent the p-quantiles of the final wealth for the monthly updates. We observe that the $p-$quantiles are very similar in both cases, only small differences are observed for quantiles between 20% and 45%.**

Figure 2: Graph comparing the p-quantiles of the final wealth for the simulated values for the monthly updates for the case where there is a limit to the amount invested in stocks (between 0% and 100% of your wealth) and without this limit. We consider $T = 30$, an initial investment $X_0 = 100$, a lower bound $K_L = 83.3$ and an upper bound $K_U = 138.3$. $A = 1.5$, $\sigma = 0.1544$, $\gamma = 0.040802$ and 2000 runs. These results are also provided in Tables A1 and A3 in the Appendix.

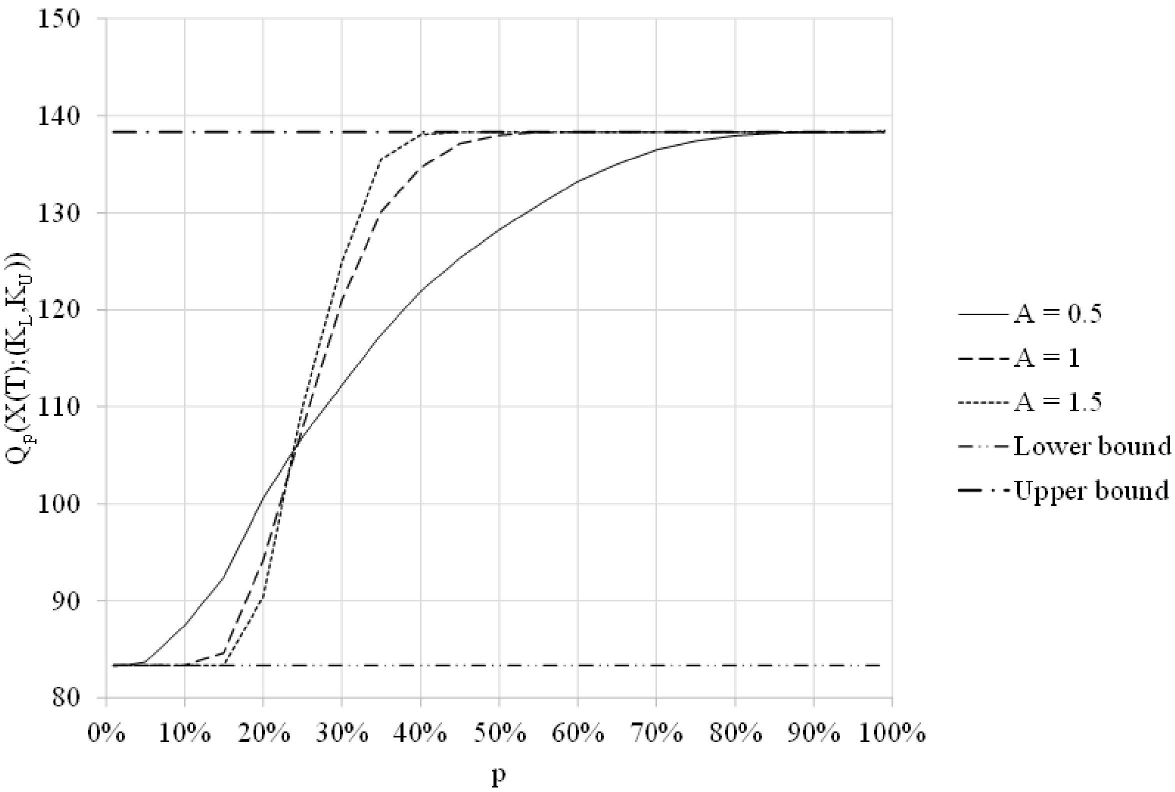


The results are also shown in Tables A1 and A3 in the Appendix. In the case that $A = 0.5$ or $A = 1$, the values of the $p-$quantiles are the same with and without this limit.

5.3 Comparing results with different values of A (therefore, different γ)

Now we compare the $p-$quantiles obtained with different values of A, namely $A = 0.5$, $A = 1$ and $A = 1.5$. We consider again that there is a limit to the amount invested in stocks (between 0% and 100% of the investor's wealth). **The results for $A = 0.5$, $A = 0.5$ and $A = 1.5$ using weekly updates are represented in Figure 3.**

Figure 3: Graph comparing the p-quantiles of the final wealth for the simulated values for the weekly updates for different values of A. We consider $T = 30$, an initial investment equal to 100 with a lower bound $K_L = 83.3$ and an upper bound $K_U = 138.3$. $\sigma = 0.1544$ and 2000 runs. These results are also provided in Tables 1, A1 and A2 in the Appendix.



We observe that the results are quite similar for $A = 1$ and $A = 1.5$, but very different to the case where $A = 0.5$. For $A = 0.5$ the upper bound is reached at higher quantiles and the lower bound is reached at lower quantiles, unlike the cases where $A = 1.5$ or $A = 1$. As the value of A decreases, risk aversion increases, it becomes less likely that the investor reaches the upper wealth constraints (around 50% for $A = 0.5$ versus 66% for the other two values of A). However, correspondingly, the investor is less likely to reach the lower wealth constraint (9% for $A = 0.5$ versus 15% for the other two values of A). **The results are also shown in Table A2 in the Appendix (case $A = 0.5$), Table 1 (case $A = 1$) and Table A1 in the Appendix (case $A = 1.5$).**

5.4 Comparing results with different values of the upper and lower bounds

In Table 2 we show the p-quantiles of the final wealth for the simulated practical implementation of the algorithm with $A = 1$ and different values of K_L and K_U. We consider again that there is a limit to the amount invested in stocks (between 0% and 100% of the investor's wealth). Adjustment for weekly updates are showed.

Namely, on the one hand we compare the case where $K_L = 83.3$ and $K_U = 183.3$ with the case where $K_L = 83.3$ and $K_U = \infty$. As expected, when the upper wealth is unbounded (i.e. $K_U = \infty$), the terminal wealth is also unbounded. Thus the values of the p-quantiles sharply increase as p increases, rather than being capped. In the simulation, they reach the value of 884.62 for $p = 99\%$. Imposing an upper bound on the terminal wealth (i.e. $K_U = 183.3$), the p-quantiles reach the target wealth of 183.3 for $p >= 60\%$. However, for nearly all p-quantiles below $p = 60\%$, the upper bounded wealth has higher quantiles than the unbounded wealth. This illustrates the advantage of imposing an upper bound, as the investor increases the certainty of higher wealth values below the bound.

Table 2: Table showing the p-quantiles of the final wealth assuming that there is a limit to the amount invested in stocks (between 0% and 100% of your wealth). The table shows the theoretical distribution and the simulated practical implementation of the algorithm to calculate the p-quantiles of real wealth at $T = 30$ for an initial investment equal to 100 with extreme values of K_L and K_U. Adjustment for weekly updates are showed. $A = 1$, $\sigma = 0.1544$, $\gamma = -0.438797$, 2,000 runs.

$K_L =$	83.3	83.3	0
$K_U =$	138.3	∞	138.3
p	$\mathcal{Q}_p(X(T))$		
1%	83.33	81.86	45.79
2.5%	83.33	83.16	61.54
5%	83.33	83.30	76.41
10%	83.34	83.33	104.68
15%	84.66	83.33	122.06
20%	94.16	83.33	133.51
25%	107.84	83.33	137.34
30%	120.96	83.33	138.18
35%	130.14	89.41	138.32
40%	134.66	100.62	138.33
45%	137.14	111.53	138.33
50%	137.98	124.24	138.33
55%	138.30	139.54	138.33
60%	138.33	158.24	138.33
65%	138.33	177.66	138.33
70%	138.33	198.59	138.33
75%	138.33	221.71	138.33
80%	138.33	259.39	138.33
85%	138.33	306.68	138.33
90%	138.33	362.52	138.33
95%	138.33	494.03	138.33
97.5%	138.33	631.33	138.33
99%	138.33	884.62	138.34
Mean	123.52	189.11	130.17
Variance	464.16	32008.75	408.03
Skewness	-1.06	4.65	-2.88
Kurtosis	2.35	42.78	10.98

We also consider the case where there is no lower bound on the terminal wealth ($K_L = 0$) but there is an upper bound (i.e. $K_U = 183.3$). In that case, as expected, we observe lower values for the lower quantiles, namely the 1%-quantile equals 45.79. The target wealth $K_U = 183.3$ is more likely to be reached than in the other cases, namely at $p = 40\%$ instead of at $p > 50\%$. Moreover, without the lower bound on the terminal wealth, the values of the lower p-quantiles, i.e. for $p < 40\%$, are higher in the two cases that do have a lower bound. This shows the cost to the investor of imposing a lower bound: while the investor gains certainty about the minimum value of their wealth at retirement, this comes at the cost of a potential loss in investment performance.

5.5 Comparing quantiles with different volatilities

In this section we compare the p-quantiles of the simulated final wealth for different volatilities. We consider the example shown in **Figure 1 and Table 1** for the theoretical quantiles, where $\sigma = 0.1544$ and $A = 1$ (which results in $\gamma = -0.4388$). In Table 3 we have considered two alternative values for the volatility σ, namely 0.0772 and 0.3088 (so half and twice the corresponding original volatility). The values of A are now equal to 4 and 0.25 respectively. We consider again that there is a limit to the amount invested in stocks (between 0% and 100% of the investor's wealth). We observe that the higher the volatility, the higher the probability to hit K_L for the theoretical distributions.

Table 3: Table showing the theoretical distribution of real wealth at $T = 30$ for an initial investment equal to 100 with a lower bound $K_L = 83.3$ and an upper bound $K_U = 138.3$. $\gamma = -0.438797$, and 2,000 runs. The theoretical distribution is shown for different volatilities (and different A).

$p - quantiles$:	Theoretical	
	$A = 4$	$A = 0.25$
p	$\sigma = 0.0722$	$\sigma = 0.3088$
1%	83.33	83.33
2.5%	101.81	83.33
5%	138.33	83.33
10%	138.33	83.33
15%	138.33	83.33
20%	138.33	83.33
25%	138.33	84.69
30%	138.33	90.24
35%	138.33	95.71
40%	138.33	101.20
45%	138.33	106.81
50%	138.33	112.64
55%	138.33	118.79
60%	138.33	125.38
65%	138.33	132.57
70%	138.33	138.33
75%	138.33	138.33
80%	138.33	138.33
85%	138.33	138.33
90%	138.33	138.33
95%	138.33	138.33
97.5%	138.33	138.33
99%	138.33	138.33
Prob. hit K_U	96.23%	31.35%
Prob. hit K_L	1.88%	23.8%

5.6 Simulated quantiles with real returns

In that section we have used real S&P500 yearly returns (for the period $1982 - 2012$) to simulate by bootstrap the p-quantiles. For comparative purposes, we have subtracted 0.0599 from the real S&P500 returns, so that they have the same average as considered in our previous examples ($\mu = 0.0343$). The results are showed in Table 4 for yearly updates. As noted by one of the reviewers this method may not capture serial correlation that occurs in times of high volatility[3].

Table 4: Table showing the p-quantiles of the final wealth assuming that there is a limit to the amount invested in stocks (between 0% and 100 % of your wealth). The table shows the p-quantiles of real wealth at $T = 30$ for an initial investment equal to 100 with a lower bound $K_L = 83.3$ and an upper bound $K_U = 138.3$, which have been obtained by bootstrap (real S&P500 yearly returns are used). Adjustment for updates every year are showed. $\gamma = -0.438797$ and 2,000 runs.

Volatility:	0.164585
A**:**	0.88
p	$\mathcal{Q}_p(X(T))$
1%	75.47
2.5%	79.51
5%	81.87
10%	82.99
15%	83.32
20%	92.58
25%	108.09
30%	117.47
35%	124.87
40%	130.95
45%	135.02
50%	137.49
55%	138.34
60%	138.35
65%	138.39
70%	138.43
75%	138.50
80%	138.59
85%	138.73
90%	138.95
95%	139.54
97.5%	139.88
99%	140.63
Mean	122.19
Variance	491.46
Skewness	-0.93
Kurtosis	2.18

In that sense, 0.164585 is the volatility of the real returns. Results for different volatilities are available upon request. We consider $A = 1$, $K_L = 83.3$ and $K_U = 138.3$

[3] A backtest considering different 30 year periods (e.g. 1960-1990, 1965-1995, 1970-2000, 1975-2005, 1980-2010, 1985-2015) would shed more light on the issue.

and we assume that there is a limit to the amount invested in stocks (between 0% and 100% of the investor's wealth). The results are again very similar to the theoretical expressions.

6 Conclusion

Investing for retirement involves many sources of uncertainty that go beyond the traditional maximization principles which characterize short-term investment operations. For short-term investments the main issue is to maximize risk-adjusted expected returns. In contrast, pension savers need to cope with long-term market evolution and their own possibilities of providing a sustained contribution to the savings fund. The fear of losing their accumulated wealth just before retirement may explain why investors are generally very conservative. Indeed, the prevailing attitude is to reduce investment risk when approaching the age of retirement in favor of positions that invest the majority of savings in risk-free assets or bonds. Fees for managing the saving funds also reduce the potential gains. So, we advocate moving towards saving strategies that guarantee a smoothed pension while concentrating on the choice of profitable investment opportunities, rather than reducing investment risk to near zero. This is automatically done by the algorithm shown in this paper.

We have investigated the practical implementation of an investment strategy that has the advantage of constraining the pension annuities, or equivalently final wealth accumulated after the investment period, which should be located between a lower and an upper bound. The practical implementation is illustrated in an extensive numerical example where different scenarios are considered and where we have also used real returns. **Future research may include a beefed-up numerical analysis on the implications of g different from 0, the presentation and discussion of the development of the stock weight over time, and an analysis of how results depend on the investment horizon.**

The main advantage of the proposed strategy is that the pension saver is protected against extreme values, by providing an smoothing savings mechanism which includes an embedded guarantee on the retirement pension. Additionally, the portfolio is easily rebalanced in practice so that the accumulated wealth at any moment is constrained by the lower and upper bounds. Then, the accumulated wealth can be easily translated into a life-long annuity that the investor will receive (as the annuity values are proportional to the retirement values), which is easy to understand and communicate, increasing the transparency of the investment mechanism. Results showing values expressed as life-long annuities are available upon request. Other retirement smoothing mechanisms for the payout phase can be found in the literature, like the contribution by Maurer et al. (2016) and other previous research (see, Guillen et al. (2006)).

Everything has been presented in nominal values, rather than real values, because we do not want to introduce the uncertainty of inflation. The pension saver could choose to invest everything inflation-hedged. However, over the long run most pension savers are better off taking on some more risk to get a higher expected return.

We follow Merton and develop a spread defining the possible outcomes of the future annuity of the customer. There is an upper bound or a target giving the highest possible pension income the pensioner wishes to achieve. And there is a lower bottom describing the lowest possible outcome. The suggested mechanism of this paper is to sell any upside above the highest possible pension income and to buy the downside below the lowest possible income.

The spread indicates the risk the pension saver wishes to take. If the spread is high, then the pension saver is risk-seeking and vice versa if the spread is low. The lowest possible spread is of course, when the pension savers entire income is inflation-hedged and, in that case, the upper bound equals the lower bound.

Notice that we distinguish between the risk in the control of inflation and the risk of more risky investments. Our point of view being that the risk of the inflation control is best monitored by investment experts, while the risk involved in the more risky assets should be defined from a pension saver's individual risk preferences. Longevity or future mortality play a similar role: there is a risk involved, but it is most likely too expensive to control over the long run rather than allow the saver to bear. The individual pension saver is probably better off with some conservative, deterministic mortality forecasts. This paper suggests a solution to Merton's pension vision when the inflation hedge and future mortality are not guaranteed, but cautiously estimated for the long run by the investment experts and the actuaries.

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A Appendix

Proof of Proposition 3.3

Proof. The proof is an adaption of the proof of (Donnelly et al., 2015, Proposition 4.3).

Assume that we have chosen $y_0 > 0$ so that the budget constraint (3.4) is satisfied with equality by $X^\star(T)$.

For the investor's utility function, the first derivative $U'(x) = x^{\gamma-1}$, which is a strictly decreasing function, has a strictly decreasing inverse $I(y) := y^{\frac{1}{\gamma-1}}$, for $y > 0$. We can show that for the constant

$$y := (y_0 + g(0))^{\gamma-1} e^{\left(\gamma r + \frac{1}{2}\frac{\gamma}{1-\gamma}\theta^2\right)T},$$

we have $(y_0 + g(0))Z(T) = I(yH(T))$.

We work with $I(y(y_0)H(T))$ in the proof, rather than with $(y_0 + g(0))Z(T)$ due to the properties of $I(x)$ and $U'(x)$: they are both strictly decreasing functions of x.

Let $X(T) \in [K_L, K_U]$, a.s. be any attainable final wealth so that $\mathbb{E}\left(H(T)X(T)\right) \leq x_0$. We show that

$$\mathbb{E}\left(U(X(T))\right) \leq \mathbb{E}\left(U(X^\star(T))\right),$$

in which

$$X^\star(T) = \begin{cases} K_L & I(yH(T)) \leq K_L \\ I(yH(T)) & \text{if } I(yH(T)) \in (K_L, K_U) \\ K_U & I(yH(T)) \geq K_U. \end{cases}$$

As I and U' are strictly decreasing functions we can write:

$$X^\star(T) = \begin{cases} K_L & yH(T) \geq U'(K_L) \\ I(yH(T)) & \text{if } yH(T) \in (U'(K_L), U'(K_U)) \\ K_U & \text{if } yH(T) \leq U'(K_U) \end{cases}$$

As U is a concave function then for any $a, b \in \mathbb{R}$, $U(a) - U(b) \leq U'(b) \cdot (a - b)$. In particular,

$$U(X(T)) - U(X^\star(T)) \leq U'(X^\star(T)) \cdot (X(T) - X^\star(T)), \quad \text{a.s.}$$

Take expectations:

$$
\begin{aligned}
&\mathbb{E}\left(U(X(T)) - U(X^\star(T))\right)\\
\leq &\mathbb{E}\left(U'(X^\star(T)) \cdot (X(T) - X^\star(T))\right)\\
\leq &\mathbb{E}\left(U'(X^\star(T)) \cdot (X(T) - X^\star(T)) \,|\, yH(T) \geq U'(K_L)\right) \cdot \mathbb{P}\left[yH(T) \geq U'(K_L)\right]\\
&+ \mathbb{E}\left(U'(X^\star(T)) \cdot (X(T) - X^\star(T)) \,|\, yH(T) \in (U'(K_L), U'(K_U))\right)\\
&\qquad\qquad \cdot \mathbb{P}\left[yH(T) \in (U'(K_L), U'(K_U))\right]\\
&+ \mathbb{E}\left(U'(X^\star(T)) \cdot (X(T) - X^\star(T)) \,|\, yH(T) \leq U'(K_U)\right) \cdot \mathbb{P}\left[yH(T) \leq U'(K_U)\right].
\end{aligned}
$$

Observe that on the event $[yH(T) \in (U'(K_L), U'(K_U))]$,

$$
U'(X^\star(T)) = U'(I(yH(T))) = yH(T)
$$

so that

$$
\begin{aligned}
&\mathbb{E}\left(U'(X^\star(T)) \cdot (X(T) - X^\star(T)) \,|\, yH(T) > U'(K_U)\right)\\
=&\mathbb{E}\left(yH(T) \cdot (X(T) - X^\star(T)) \,|\, yH(T) > U'(K_U)\right).
\end{aligned}
$$

Next observe that on the event $[yH(T) \leq U'(K_U)]$, as $X(T) \in [K_L, K_U]$ a.s, then

$$
X(T) - X^\star(T) = X(T) - K_U \leq 0
$$

and

$$
U'(X^\star(T)) = U'(K_U) \geq yH(T).
$$

The negative sign of $X(T) - X^\star(T)$ reverses the inequality $U'(X^\star(T)) \geq yH(T)$, giving that on the event $[yH(T) \leq U'(K_U)]$,

$$
U'(X^\star(T)) \cdot (X(T) - X^\star(T)) \leq yH(T) \cdot (X(T) - X^\star(T)).
$$

On the event $[yH(T) \geq U'(K_L)]$, as $X(T) \in [K_L, K_U]$ a.s, then

$$
X(T) - X^\star(T) = X(T) - K_L \geq 0
$$

and

$$
U'(X^\star(T)) = U'(K_L) \leq yH(T).
$$

Due to the positive sign of $X(T) - X^\star(T)$, the inequality $U'(X^\star(T)) \leq yH(T)$ is maintained, giving

$$
U'(X^\star(T)) \cdot (X(T) - X^\star(T)) \leq yH(T) \cdot (X(T) - X^\star(T)).
$$

In summary, we find that

$$\mathbb{E}\left(U(X(T)) - U(X^\star(T))\right) \le \mathbb{E}\left(yH(T)\cdot(X(T) - X^\star(T))\right).$$

As both solutions satisfy the budget constraint (3.4), the last line in the above inequality can be evaluated as

$$\mathbb{E}\left(yH(T)\cdot(X(T) - X^\star(T))\right) \le y\cdot\left((x_0+g(0)) - (x_0+g(0))\right) = 0,$$

which means

$$\mathbb{E}\left(U(X(T)) - U(X^\star(T))\right) \le 0.$$

Hence

$$\mathbb{E}\left(U(X^\pi(T))\right) = \sup_{\pi\in\mathcal{A}} \mathbb{E}\left(U(X^\pi(T))\right) \le \mathbb{E}\left(U(X^\star(T))\right) \le \mathbb{E}\left(U(X^\pi(T))\right),$$

i.e. $X^\pi(T) = X^\star(T)$, a.s.

□

Proof of Lemma 3.4

Proof. From (Donnelly et al., 2015, Lemma 4.4), a European call option with maturity value $\max\{0, (y_0+g(0))Z(T) - K_L\}$ is given by $c(t, Y(t); K_L)$ with

$$Y(t) := (y_0 + g(0))\, Z(t), \tag{A.1}$$

and

$$c(t, y; K_L) := y\Phi(d_+(t,y;K_L)) - K_L e^{-r(T-t)}\Phi(d_-(t,y;K_L)),$$

in which the functions $d_\pm(t,y;K_L)$ are defined by equation (3.7) and Φ denotes the cumulative distribution function of the standard normal.

Thus by put-call parity, the value of the put option with the same strike price K_L satisfies

$$p(t,y;K_L) = c(t,y;K_L) + K_L e^{-r(T-t)} - y.$$

To find the replicating portfolio, we differentiate the put pricing function p to get

$$p_t(t,y;K_L) = -\frac{y\phi(d_+(t,y;K_L))\sigma A}{2\sqrt{T-t}} + rK_L e^{-r(T-t)}\Phi(-d_-(t,y;K_L))$$

$$p_y(t,y;K_L) = \Phi(d_+(t,y;K_L)) - 1, \qquad p_{yy}(t,y) = \frac{\phi(d_+(t,y;K_L))}{y\sigma A\sqrt{T-t}},$$

where ϕ denotes the density function of the standard normal. By Ito's formula,

$$dp(t,Y(t)) = p_t(t,Y(t))dt + p_y(t,Y(t))dY(t) + \frac{1}{2}p_{yy}(t,Y(t))d\left[Y\right](t),$$

in which

$$dY(t) = (r+\theta\sigma A)\,Y(t)dt + \sigma AY(t)dW(t).$$

Substituting for: the derivatives of the pricing function p, the dynamics of Y and the candidate replicating portfolio $\pi_p\,(t,Y(t)) := -AY(t)\,\Phi(-d_+(t,Y(t)))$, we find that the dynamics of the pricing function c satisfy the wealth equation (3.2). Hence $\pi_p\,(t,Y(t))$ is the amount to be invested in the risky stock at time t in order to replicate the payoff of the European put option.

□

Proof of Proposition 3.7

Proof. Fix $p \in (0,1)$. From (Donnelly et al., 2015, Lemma 4.9), with no lower bound on the terminal wealth,

$$\mathcal{Q}_p(X(T);(0,K_U)) = \min\left\{K_U,(y_0+g(0))e^{\beta_p}\right\}.$$

It is useful to consider another investor who has the same savings plan g and the same upper bound K_U as the first investor. However, this second investor has no lower bound on the terminal wealth, i.e. $K_L = 0$, and starts with an initial wealth $\tilde{x}_0$ that satisfies

$$\tilde{x}_0 = y_0 - c(0,y_0+g(0);K_U).$$

This second investor follows the optimal constrained strategy. Then, as $g(T) = 0$, the wealth at time T of the second investor is

$$\tilde{X}(T) = (y_0+g(0))Z(T) - g(T) - c(T,Y(T);K_U) = \min\left\{K_U,Y(T)\right\}.$$

Thus the terminal wealth of the constrained investor, who has a lower bound K_L on their terminal wealth, is related to that of the second unconstrained investor by

$$X(T) = \begin{cases} \tilde{X}(T) & \text{if } \tilde{X}(T) \geq K_L \\ K_L & \text{if } \tilde{X}(T) < K_L. \end{cases}$$

The desired expression (3.13) follows by consideration of the last expression.

□

Tables showing the p-quantiles of the final wealth

Table A1: Table showing the p-quantiles of the final wealth assuming that there is a limit to the amount invested in stocks (between 0% and 100 % of your wealth). The table shows the theoretical distribution and the simulated practical implementation of the algorithm to calculate the p-quantiles of real wealth at $T = 30$ for an initial investment equal to 100 with a lower bound $K_L = 83.3$ and an upper bound $K_U = 138.3$. Adjustment for updates every year, month, week and day are showed. $A = 1.5$, $\sigma = 0.1544$, $\gamma = 0.04080199$ and 2,000 runs.

Update:	Theoretical	Yearly	Monthly	Weekly	Daily
p	$\mathcal{Q}_p(X(T))$				
1%	83.33	79.26	82.85	83.29	83.33
2.5%	83.33	81.34	83.18	83.33	83.33
5%	83.33	82.60	83.30	83.33	83.33
10%	83.33	83.27	83.33	83.33	83.33
15%	83.33	87.31	83.92	83.33	83.63
20%	91.38	100.66	96.16	90.40	95.81
25%	112.96	113.15	112.12	110.02	112.22
30%	136.65	125.09	127.13	124.87	126.77
35%	138.33	133.76	136.31	135.55	136.07
40%	138.33	138.34	138.31	138.03	137.97
45%	138.33	138.40	138.33	138.33	138.32
50%	138.33	138.49	138.33	138.33	138.33
55%	138.33	138.63	138.33	138.33	138.33
60%	138.33	138.83	138.33	138.33	138.33
65%	138.33	139.08	138.34	138.33	138.33
70%	138.33	139.44	138.34	138.33	138.33
75%	138.33	139.86	138.34	138.33	138.33
80%	138.33	140.53	138.35	138.33	138.33
85%	138.33	141.31	138.37	138.33	138.33
90%	138.33	142.61	138.41	138.33	138.33
95%	138.33	145.79	138.56	138.34	138.33
97.5%	138.33	148.78	138.87	138.37	138.34
99%	138.33	154.74	139.25	138.52	138.47
Prob. hit K_U	69.66%				
Prob. hit K_L	18.02%				
Mean		125.10	124.67	123.98	124.56
Variance		477.09	471.50	494.04	472.93
Skewness		-1.00	-1.16	-1.09	-1.16
Kurtosis		2.44	2.53	2.36	2.53

Table A2: Table showing the p-quantiles of the final wealth assuming that there is a limit to the amount invested in stocks (between 0% and 100% of your wealth). The table shows the theoretical distribution and the simulated practical implementation of the algorithm to calculate the p-quantiles of real wealth at $T = 30$ for an initial investment equal to 100 with a lower bound $K_L = 83.3$ and an upper bound $K_U = 138.3$. Adjustment for updates every year, month, week and day are showed. $A = 0.5$, $\sigma = 0.1544$, $\gamma = -1.877594$ and 2,000 runs.

Update:	Theoretical	Yearly	Monthly	Weekly	Daily
p	$\mathcal{Q}_p(X(T))$				
1%	83.33	82.82	83.33	83.33	83.33
2.5%	83.33	83.17	83.35	83.34	83.35
5%	83.33	83.47	83.90	83.68	84.16
10%	84.74	88.14	88.43	87.47	88.35
15%	93.99	95.23	95.53	92.45	94.81
20%	102.06	102.31	102.36	100.59	101.64
25%	109.54	107.62	108.77	106.90	107.20
30%	116.71	113.05	113.73	112.23	112.32
35%	123.78	118.48	118.90	117.46	117.37
40%	130.89	123.20	122.97	121.92	122.23
45%	138.15	125.86	126.55	125.33	125.62
50%	138.33	128.38	129.57	128.26	129.28
55%	138.33	133.03	132.13	130.81	131.94
60%	138.33	135.10	134.10	133.25	134.20
65%	138.33	136.96	135.70	135.03	135.72
70%	138.33	138.02	136.81	136.50	136.78
75%	138.33	138.35	137.58	137.42	137.50
80%	138.33	138.49	138.04	137.94	137.92
85%	138.33	138.75	138.27	138.21	138.20
90%	138.33	139.28	138.33	138.31	138.30
95%	138.33	140.56	138.33	138.33	138.33
97.5%	138.33	141.97	138.33	138.33	138.33
99%	138.33	143.70	138.33	138.33	138.33
Prob. hit K_U	54.87%				
Prob. hit K_L	9.32%				
Mean		121.90	121.52	120.64	121.16
Variance		367.05	340.80	356.32	346.26
Skewness		-0.79	-0.87	-0.80	-0.81
Kurtosis		2.25	2.34	2.19	2.23

Table A3: Table showing the p-quantiles of the final wealth assuming that there is no limit to the amount invested in stocks. The table shows the theoretical distribution and the simulated practical implementation of the algorithm to calculate the p-quantiles of real wealth at $T = 30$ for an initial investment equal to 100 with a lower bound $K_L = 83.3$ and an upper bound $K_U = 138.3$. Adjustment for updates every year, month, week and day are showed. $A = 1.5$, $\sigma = 0.1544$, $\gamma = 0.04080199$ and 2,000 runs.

Update:	Theoretical	Yearly	Monthly	Weekly	Daily
p	$\mathcal{Q}_p(X(T))$				
1%	83.33	79.13	82.82	83.29	83.33
2.5%	83.33	81.35	83.17	83.33	83.33
5%	83.33	82.60	83.30	83.33	83.33
10%	83.33	83.27	83.33	83.33	83.33
15%	83.33	86.99	83.80	83.33	83.62
20%	91.38	100.46	95.01	89.08	92.98
25%	112.96	113.38	113.68	110.08	113.76
30%	136.65	125.32	128.94	127.51	127.42
35%	138.33	134.05	136.33	135.55	136.07
40%	138.33	138.34	138.31	138.02	137.97
45%	138.33	138.40	138.33	138.33	138.32
50%	138.33	138.49	138.33	138.33	138.33
55%	138.33	138.63	138.33	138.33	138.33
60%	138.33	138.83	138.33	138.33	138.33
65%	138.33	139.08	138.33	138.33	138.33
70%	138.33	139.44	138.34	138.33	138.33
75%	138.33	139.86	138.34	138.33	138.33
80%	138.33	140.54	138.35	138.33	138.33
85%	138.33	141.31	138.37	138.33	138.33
90%	138.33	142.63	138.41	138.33	138.33
95%	138.33	145.79	138.56	138.34	138.33
97.5%	138.33	149.19	138.87	138.37	138.34
99%	138.33	154.90	139.24	138.52	138.47
Prob. hit K_U	69.66%				
Prob. hit K_L	18.02%				
Mean		125.10	124.71	124.07	124.56
Variance		477.09	474.97	497.43	477.00
Skewness		-1.00	-1.17	-1.11	-1.16
Kurtosis		2.44	2.55	2.38	2.53