

David A. Kopriva

# Implementing Spectral Methods for Partial Differential Equations

Algorithms for Scientists and Engineers

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