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Implementing the Modified Golden Rule? Optimal Ramsey Taxation with Incomplete Markets Revisited

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Abstract

What is the prescription of Ramsey capital taxation in the long run? Aiyagari (1995) addressed the question in a heterogeneous-agent incomplete-markets (HAIM) economy, showing that a positive capital tax should be imposed to implement the so-called modified golden rule (MGR). In deriving the MGR result, Aiyagari (1995) implicitly assumed that the multiplier on the resource constraint of the Ramsey problem converges to a finite positive value in the limit. We first show that this implicit assumption has a strong implication for the shadow price of Ramsey taxation in the limit: it must go to zero. We next show that if the shadow price of Ramsey taxation remains positive rather than goes to zero in the limit, the results differ sharply, including (i) the multiplier on the resource constraint of the Ramsey problem must explode in the limit if a Ramsey steady state exists, (ii) Ramsey steady states may fail to exist, (iii) the MGR does not hold and the corresponding capital tax is non-positive even if a Ramsey steady state exists. The key to our results is embedded in the hallmark of the HAIM economy: the risk-free gross interest rate is lower than the inverse of the preference discount factor in steady state. We briefly explore which feature, convergent or divergent multiplier, is more plausible.

JEL Classification: C61; E22; E62; H21; H30

Key Words: Capital Taxation; Modified Golden Rule; Ramsey Problem; Incomplete Markets; Heterogeneous-Agent

*Corresponding author, YiLi Chien. Email: yilichien@gmail.com. This paper is a complete rewrite of our previous work under the title “Aiyagari Meets Ramsey: Optimal Capital Taxation with Incomplete Markets.” While we find the non-existence of a Ramsey steady state in the current version, our previous work builds on the incorrect premise that a Ramsey steady state always exists. We thank Andrew Atkeson, Dirk Krueger, Tomoyuki Nakajima, Yena Park, Yi Wen and participants at various seminars and conferences for useful comments. The views expressed are those of the individual authors and do not necessarily reflect the official positions of the Federal Reserve Bank of St. Louis, the Federal Reserve System, or the Board of Governors.

1 Introduction

The heterogeneous-agent incomplete-markets (HAIM hereafter) model considers an environment in which households are subject to uninsurable idiosyncratic shocks and borrowing restrictions; in response, households buffer their consumption against adverse shocks via precautionary savings. During the past two decades, the HAIM model has become a standard workhorse for policy evaluations in the current state-of-the-art macroeconomics that jointly addresses aggregate and inequality issues.¹

Given the importance and popularity of the HAIM model, it is natural to ask: what is the prescription of Ramsey capital taxation in the long run for the HAIM economy? The first attempt to answer this question is the work of Aiyagari (1995). He showed that the so-called “modified golden rule” (MGR hereafter) has to hold at the Ramsey steady state.² On the other hand, in steady state, the after-tax gross return on capital, which is equated to the risk-free gross interest rate, R , is always less than the inverse of the discount factor, $1/\beta$, in the HAIM economy. Aiyagari (1995) thus reached the conclusion that a positive capital tax should be imposed to implement the steady-state allocation that satisfies the MGR. The finding by Aiyagari (1995) is important in the optimal taxation literature and, in particular, it represents a distinct departure from the classical result of no permanent capital tax prescribed by Chamley (1986) and Judd (1985).

In his analysis, Aiyagari (1995) made two important assumptions. First, he assumed the existence of a Ramsey steady state explicitly in Assumption 2 (p. 1170), in which policy and all other variables are assumed to converge to a steady state. Second, he assumed implicitly that the multiplier on the resource constraint of the Ramsey problem converges to a finite positive value in the limit in footnote 15 (p. 1171). While imposing Assumption 2 (the existence assumption), Aiyagari (1995, footnote 14) did express his concern: “It seems quite difficult to guarantee that a solution to the optimal tax problem converges to a steady state.”³ In a recent paper, Straub and Werning (2020) revisited the classical Chamley-Judd result, showing that the common assumption that endogenous multipliers associated with the Ramsey problem converge in the limit is not necessarily true at the optimum.

Following the leads provided by Aiyagari (1995) with regard to the possible non-existence of a Ramsey steady state and Straub and Werning (2020) with regard to the possible non-convergence of multipliers with the Ramsey problem, this paper revisits the long-standing issue with respect to the existence of

¹It is also known as the Bewley-Huggett-Aiyagari model. For surveys of the literature, see Heathcote, Storesletten, and Violante (2009), Guvenen (2011), Ljungqvist and Sargent (2012, chapter 18), Quadrini and Ríos-Rull (2015) and Krueger, Mitman, and Perri (2016).

²The Ramsey steady state is defined as a situation where the optimal Ramsey allocation features the steady-state property in the long run. See Definitions 3 and 4 for the details.

³Aiyagari (1995, footnote 14) noted that the existence assumption was also made by Chamley (1986) and Lucas (1990).

Ramsey steady states and the implementation of the MGR in the HAIM economy.⁴

The conclusion reached by Aiyagari (1995) applies to general utility functions.⁵ Working with a specific utility function, i.e., the commonly-used separable isoelastic utility function, this paper revisits the issue. We first show that if the multiplier on the resource constraint of the Ramsey problem converges to a finite positive value in the limit as assumed by Aiyagari (1995), then the shadow price of raising government revenues through distorting taxes must go to zero in the limit. However, if the shadow price remains positive rather than goes to zero in the limit, the results differ sharply, including (i) the multiplier on the resource constraint of the Ramsey problem must explode in the limit if a Ramsey steady state exists, (ii) there is no Ramsey steady state if the elasticity of intertemporal substitution (EIS hereafter) is weakly less than 1, and (iii) a Ramsey steady state is possible if the EIS is larger than 1, but the MGR fails to hold and the corresponding capital tax is non-positive. Our first main result shows that the implicit assumption made by Aiyagari (1995) has a strong implication for the shadow price of Ramsey taxation in the limit. As to our second main result, it overturns the assumptions imposed and the conclusion reached by Aiyagari (1995).

The two main results basically remain robust if replacing the separable isoelastic utility function with the GHH utility function *à la* Greenwood, Hercowitz, and Huffman (1988); see the Online Appendix. Given that our derived two main results are mutually exclusive in some sense, we briefly explore which one is more plausible after their derivation.

Aiyagari (1995) obtained his results mainly in the setting of endogenous rather than exogenous government spending. We demonstrate that our second main result remains robust, regardless of whether government spending is endogenously determined or exogenously given.

It is well known that the steady-state outcome in a competitive equilibrium, $R < 1/\beta$, represents the signature feature of the HAIM model.⁶ Unlike individual households in the face of earnings risk, the Ramsey planner in the HAIM economy (without aggregate shocks) faces no uncertainty in allocating aggregate resources. Given that the planner discounts the future by β , the strict inequality of $R < 1/\beta$ then dictates that the market discounts resources at a lower rate than the planner discounts utility, implying the existence of the planner's desire to improve welfare by front-loading aggregate consumption through policy tools. As will be seen, the feature of $R < 1/\beta$ in steady state plays a key role in driving our results.

⁴It should be noted that the HAIM economy addressed by Aiyagari (1995) and our paper differs qualitatively from the economic environments studied by Straub and Werning (2020).

⁵For more details, see footnote 21 later.

⁶Ljungqvist and Sargent (2012, p. 9) explained that the outcome of $R < 1/\beta$ in steady state can be thought of as follows: it lowers the rate of return on savings enough to offset agents' precautionary savings motive so as to make their asset holdings converge rather than diverge in the limit.

1.1 Methodology

In order to explicitly account for the social benefit or cost of having one extra unit of aggregate consumption and labor supply, the primal approach to the Ramsey problem *à la* Lucas and Stokey (1983) is adopted. Lucas and Stokey (1983) considered a representative-agent setting. Our method follows the work of Werning (2007) and Park (2014) to extend the Lucas-Stokey formulation to the setting of heterogeneous households.

Our methodology first formulates the household problem as a time-zero trading problem of the Arrow-Debreu complete-market economy; however, we impose two additional constraints—one for incomplete markets and the other for borrowing constraints—to take into consideration the key features of the HAIM economy.⁷ Due to the fact that the Ramsey planner also encounters the same incomplete-markets frictions faced by households, the typical implementability condition is not sufficient and hence additional constraints are needed for the characterization of the Ramsey problem. This causes our HAIM Ramsey problem to become a generalization of the RA (representative-agent) Ramsey problem.

The methodology adopted by this paper results in several contributions to the literature on the Ramsey problem. First, our approach is capable of analytically deriving all FOCs of the primal Ramsey problem in the typical HAIM economy, which to our knowledge is unprecedented. Accounting for all the necessary optimal Ramsey conditions is critical to our analysis and findings. Second, our approach offers an advantage in that the Ramsey problem of our HAIM economy would reduce to that of a RA economy if markets were complete rather than incomplete. Given that the meaning and intuition of the Ramsey problem in the RA economy are well-understood, this advantage makes the model mechanism that drives our main results transparent and intuitive. Finally, our methodology allows us to investigate the existence of a Ramsey steady state instead of assuming its existence as in the extant literature as well as to characterize the properties of a Ramsey steady state if it does exist.

1.2 Related Literature

The literature on optimal capital taxation is vast. Here we focus only on a very limited subset of the studies framed in a heterogeneous-agent environment with incomplete markets or market frictions.

Our work is closely related to the recent study by Chien and Wen (2019), who utilized an analytically tractable heterogeneous-agent model with idiosyncratic preference shocks to address the same issue.

⁷This approach of modeling incomplete markets is pioneered by Aiyagari, Marcet, Sargent, and Seppala (2002), who named the additional constraints for incomplete markets as measurability conditions. The later work by Chien, Cole, and Lustig (2011) extends this approach to heterogeneous-agent models in the context of asset pricing.

They demonstrated that the Ramsey planner intends to increase the supply of government bonds until full self-insurance is achieved or an exogenous debt limit binds. However, in order to have an analytical solution, their model makes a few special assumptions and deviates from the standard HAIM model. Hence, their study cannot directly investigate the issue regarding the existence of the Ramsey steady state assumption made by Aiyagari (1995).

Conesa, Kitao, and Krueger (2009) considered optimal capital taxation in a HAIM-type economy but in a life-cycle framework. The quantitative part of their study largely focuses on the steady-state welfare. In an overlapping generations model with two-period-lived households, Krueger and Ludwig (2018) characterized the optimal capital tax of the Ramsey problem. In their analysis, the planner lacks government bonds as a policy tool. In contrast, government bonds play an essential role in our results. Hence, their results do not contradict ours since a Ramsey steady state with a binding government debt limit could exist.

Gottardi, Kajii, and Nakajima (2015) considered an environment deviating from the standard HAIM economy, in that there is risky human capital in addition to physical capital. They derived qualitative and quantitative properties for the solution to the Ramsey problem, showing that the interaction between market incompleteness and risky human capital accumulation provides a justification for taxing physical capital. In this paper, we stick to the standard HAIM economy with idiosyncratic earnings risk and show that a Ramsey steady state can fail to exist.

Dávila, Hong, Krusell, and Ríos-Rull (2012) characterized constrained efficiency for the HAIM economy. To decentralize the constrained efficient allocation, the planner is required to know each agent's realized shocks in order to impose individual-specific capital taxes. We consider flat tax rates applied uniformly to all agents as in the typical Ramsey problem and, as such, the constrained efficient allocation is infeasible to the Ramsey planner.

Recent papers, including Le Grand and Ragot (2017), Açıkgöz, Hagedorn, Holter, and Wang (2018) and Dyrda and Pedroni (2018), numerically solve optimal Ramsey fiscal policy for both the transition path and the steady state of the HAIM economy. While Açıkgöz, Hagedorn, Holter, and Wang (2018) considered separable isoelastic preferences in their numerical analysis, both Le Grand and Ragot (2017) and Dyrda and Pedroni (2018) adopted GHH preferences in their numerical analysis. The results of these papers are basically consistent with the finding in Aiyagari (1995); in particular, they all find that the MGR holds at the Ramsey steady state. Thus, the results of these numerical work are consistent with our first main result rather than our second main result. For their robustness, it seems important in the light of our results to check if the multiplier on the resource constraint of their Ramsey problem converges to a

finite positive value in the limit. Indeed, this is a key assumption made by Aiyagari (1995) to uphold the MGR at the Ramsey steady state.

The rest of the paper is organized as follows. Section 2 and Section 3 introduce our model economy and characterize its competitive equilibrium, respectively. Section 4 formulates the Ramsey problem. Our main findings are demonstrated in Section 5. Section 6 checks and shows the robustness of our results to the endogenous government spending setting, and Section 7 offers a discussion of our findings.

2 Model Economy

The model economy mainly builds on Aiyagari (1994). There is a unit measure of infinitely-lived households who are subject to idiosyncratic labor productivity shocks. There are no aggregate shocks. Markets are incomplete in that there are no state-contingent securities for idiosyncratic shocks. In addition, all households are subject to exogenous borrowing constraints at all times.

Time is discrete and the horizon is infinity, indexed by $t = 0, 1, 2, \dots$. Time 0 is a planning period and actions begin in time 1. All households are ex ante identical and endowed with the same asset holdings. Ex post heterogeneity arises because households experience different histories of the idiosyncratic shock realization. Let θ_t (which takes a positive value in a finite set Θ) denote the incidence of the idiosyncratic labor productivity shock at time t , and let θ^t denote the history of events for the idiosyncratic shock of a household up through and until time t . The shock θ_t is independently and identically distributed across households, and the sequence $\{\theta_t\}$ follows a first-order Markov process over time. We let $\pi_t(\theta^t)$ denote the unconditional probability of θ^t and $\pi(\theta_t|\theta_{t-1})$ denote the conditional probability. We have $\pi_t(\theta^t) = \pi(\theta_t|\theta_{t-1})\pi_{t-1}(\theta^{t-1})$. Because of the independence of productivity shocks across households at any time, a law of large numbers applies so that the probability $\pi_t(\theta^t)$ also represents the fraction of the population that experiences θ^t at time t . We let $\pi_1(\theta^1 = \theta_1) = 1$ for the initial value of θ_1 (the initial realization θ_1 is given). We call a household that has the history θ^t simply “household θ^t .” We also introduce additional notation: $\theta^{t+1} \succ \theta^t$ means that the left-hand-side node is a successor node to the right-hand-side node; and for $s > t$, $\theta^s \succeq \theta^t$ ($\theta^s \succ \theta^t$) represents the set of successor shocks after θ^t up to θ^s including (excluding) θ_t .

Households maximize their lifetime utility

$$U = \sum_{t=1}^{\infty} \beta^t \sum_{\theta^t} \left[u(c_t(\theta^t)) - v\left(\frac{l_t(\theta^t)}{\theta_t}\right) \right] \pi_t(\theta^t),$$

where $\beta \in (0, 1)$ is the discount factor; $c_t(\theta^t)$ and $l_t(\theta^t)$ denote the consumption and the labor supply for household θ^t at time t ; and $l_t(\theta^t)/\theta_t$ is the corresponding “raw” labor supply (hours worked). The assumptions on the functions $u(\cdot)$ and $v(\cdot)$ are standard; in particular, we impose $u'(0) = \infty$, $v'(0) = 0$ and $v'(\infty) = \infty$.

There is a standard neoclassical constant returns-to-scale production technology, denoted by $F(K, L)$, that is operated by a representative firm, where K and L are aggregate capital and labor, respectively. As in Aiyagari (1995), $F(K, L)$ satisfies standard properties such as Inada conditions plus $F(K, L) = 0$ if either $K = 0$ or $L = 0$. The firm produces output by hiring labor and renting capital from households. The firm’s optimal conditions for profit maximization at time t satisfy

$$\begin{aligned} w_t &= F_L(K_t, L_t), \\ r_t &= F_K(K_t, L_t), \end{aligned}$$

where w_t and r_t are the wage rate and the capital rental rate, and F_L and F_K denote the marginal product of labor and capital, respectively. All markets are competitive.

The government is required to finance an exogenous stream of government spending $\{G_t\}$ and it can issue one-period government bonds and levy flat-rate, time-varying labor and capital taxes at rates $\tau_{l,t}$ and $\tau_{k,t}$, respectively. The flow government budget constraint at time t is expressed as

$$\tau_{l,t}w_tL_t + \tau_{k,t}(r_t - \delta)K_t + B_{t+1} = G_t + R_tB_t, \quad (1)$$

where R_t is the risk-free gross interest rate between time $t - 1$ and t , $\delta \in (0, 1)$ is the depreciation rate of capital, and B_t is the amount of government bonds issued at time $t - 1$. The government is assumed to fully commit to a sequence of taxes imposed and debts issued, given the initial amount of government bonds B_1 at time 0. This setup for the government is standard for the Ramsey problem. Section 6 considers an alternative setup where G_t becomes endogenously determined rather than exogenously given. This alternative setup is adopted by Aiyagari (1995).

3 Characterization of Competitive Equilibrium

This section characterizes the competitive equilibrium of the model economy, paving the way for the formulation of the Ramsey problem in the next section. We first describe the household problem.

3.1 Household Problem

We express the household problem as a time-zero trading problem as in an Arrow-Debreu economy but with the imposition of additional constraints to account for the key features of the HAIM economy. As noted in the Introduction, this method facilitates the formulation of the primal Ramsey problem for the HAIM economy.

Denote P_t as the time-zero price of one unit of consumption delivered at time t . We set $P_0 = 1$ as a normalization. Given that K and B are perfect substitutes in the mind of households, the after-tax return on capital has to equal the risk-free rate:

$$\frac{P_t}{P_{t+1}} = R_{t+1} = 1 + (1 - \tau_{k,t+1})(r_{t+1} - \delta), \quad (2)$$

which constitutes a no-arbitrage condition for trades in capital and government bonds.

Let $p_t(\theta^t) = P_t \pi_t(\theta^t)$ be the state-contingent price of one unit of consumption delivered in the event of θ^t at time t . The household's time-zero budget constraint in an Arrow-Debreu economy is expressed as

$$\hat{a}_1 = \sum_{t \geq 1} \sum_{\theta^t} p_t(\theta^t) [c_t(\theta^t) - \hat{w}_t l_t(\theta^t)], \quad (3)$$

where $\hat{w}_t = (1 - \tau_{l,t})w_t$ is the after-tax wage rate at time t and $\hat{a}_1 = K_1 + B_1$, where K_1 and B_1 are the economy's initial capital and government bonds, respectively. All households by assumption have the same initial asset holdings $\hat{a}_1 > 0$.

3.1.1 Measurability Conditions and Borrowing Constraints

Two key features of the HAIM economy are (i) incomplete markets—no state-contingent claims on idiosyncratic shocks (in fact, households can only self-insure through a risk-free asset), and (ii) ad hoc borrowing constraints—a lower bound on households' asset holdings at all times. Both features impose restrictions on the choice of asset holdings across idiosyncratic states over time. We show how to embed these asset-holding restrictions into a time-zero trading problem for the household.

Given the history of shocks θ^t at time t , the asset holdings with complete markets can be written as

$$p_t(\theta^t) a_t(\theta^t) = \sum_{s \geq t} \sum_{\theta^s \succeq \theta^t} p_s(\theta^s) [c_s(\theta^s) - \hat{w}_s l_s(\theta^s)], \quad (4)$$

where $a_t(\theta^t)$ is the amount of the state-contingent claim held by household θ^t at the beginning of time t .

However, markets are incomplete rather than complete and households do not have access to state-contingent markets in the HAIM economy. This implies that the asset holdings at time $t+1$ are measurable only up to the events prior to the realization of shock θ_{t+1} . Formally, households face the following measurability conditions: for $\forall t \geq 0$ and θ^t ,

$$a_{t+1}(\theta^t, \theta_{t+1}) = a_{t+1}(\theta^t, \tilde{\theta}_{t+1}) \text{ for all } \tilde{\theta}_{t+1}, \theta_{t+1} \in \Theta,$$

which practically impose constraints on a household's asset holdings.

For ease of exposition, we rewrite the measurability conditions as follows: for $\forall t \geq 0$ and θ^t ,

$$\frac{a_{t+1}(\theta^t, \theta_{t+1})}{R_{t+1}} = \frac{a_{t+1}(\theta^t, \tilde{\theta}_{t+1})}{R_{t+1}} \equiv \hat{a}_{t+1}(\theta^t) \text{ for all } \tilde{\theta}_{t+1}, \theta_{t+1} \in \Theta, \quad (5)$$

where R_{t+1} is the risk-free gross interest rate between time t and $t+1$. That is, $\hat{a}_{t+1}(\theta^t)$ is defined so that $R_{t+1}\hat{a}_{t+1}(\theta^t) = a_{t+1}(\theta^t, \theta_{t+1}) = a_{t+1}(\theta^t, \tilde{\theta}_{t+1})$ for all $\tilde{\theta}_{t+1}, \theta_{t+1} \in \Theta$. This makes sense because households can hold only a one-period risk-free asset; and their asset holdings at the beginning of time $t+1$ deflated by their asset return, the risk-free gross interest rate, must be equal to the end of time t asset holdings, which are denoted by $\hat{a}_{t+1}(\theta^t)$.

Households also face the following ad hoc borrowing restrictions for $\forall t \geq 0$:

$$\hat{a}_{t+1}(\theta^t) \geq 0, \text{ for all } \theta^t,$$

which can be equivalently expressed as $a_{t+1}(\theta^{t+1}) \geq 0$ for all t and θ^{t+1} , according to (5).

3.1.2 Formulating and Solving the Household Problem

The asset-holding restrictions, such as the measurability conditions and borrowing constraints, are equivalent to the restrictions imposed on the whole sequence of consumption and labor choices.

Using (4), we can restate the measurability conditions as

$$P_{t-1}\hat{a}_t(\theta^{t-1})\pi_t(\theta^t) = \sum_{s \geq t} \sum_{\theta^s \succeq \theta^t} p_s(\theta^s) [c_s(\theta^s) - \hat{w}_s l_s(\theta^s)], \quad \forall t \geq 1, \theta^t, \quad (6)$$

where we have replaced $a_t(\theta^t)$ with $R_t\hat{a}_t(\theta^{t-1})$ as defined in (5) and used $p_t(\theta^t) = P_t\pi_t(\theta^t)$ and the result of $P_{t-1} = P_t R_t$ in (2). Note that, given $P_0 = 1$ and $\pi_1(\theta^1) = 1$, the measurability conditions (6) reduce to the household's time-zero budget constraint (3) as $t = 1$. As to the borrowing constraints, they can be

expressed as

$$\sum_{s \geq t} \sum_{\theta^s \succeq \theta^t} p_s(\theta^s) [c_s(\theta^s) - \widehat{w}_s l_s(\theta^s)] \geq 0, \quad \forall t \geq 2, \theta^t. \quad (7)$$

Given that $\pi_1(\theta^1) = 1$ so that the initial realization $\theta^1 = \theta_1$ is given, it is implicitly assumed that the borrowing constraints do not bite at $t = 1$.

If markets were complete, then households would only face a single constraint (3). The presence of the additional constraints represented by (6) and (7) is due to the incomplete markets and borrowing constraints, respectively.⁸

Given prices $\{\widehat{w}_t, p_t(\theta^t)\}$, the household chooses a sequence of consumption $\{c_t(\theta^t)\}$, labor $\{l_t(\theta^t)\}$, and asset holdings $\{\widehat{a}_{t+1}(\theta^t)\}$ to maximize the lifetime utility as of time zero, subject to the time-zero budget constraint (3), the measurability conditions (6), and the borrowing constraints (7). Let χ be the multiplier on the time-zero budget constraint, $\nu_t(\theta^t)$ the multiplier on the measurability condition in the event of θ^t at time t , and $\varphi_t(\theta^t)$ the multiplier on the borrowing constraint in the event of θ^t at time t . Incorporating all the constraints through these multipliers gives the household's Lagrangian:⁹

$$\begin{aligned} \tilde{L} = & \min_{\{\chi, \nu, \varphi\}} \max_{\{c, l, \widehat{a}\}} \sum_{t=1}^{\infty} \beta^t \sum_{\theta^t} \left[u(c_t(\theta^t)) - v\left(\frac{l_t(\theta^t)}{\theta_t}\right) \right] \pi_t(\theta^t) \\ & + \chi \left\{ \widehat{a}_1 - \sum_{t=1}^{\infty} \sum_{\theta^t} p_t(\theta^t) [c_t(\theta^t) - \widehat{w}_t l_t(\theta^t)] \right\} \\ & + \sum_{t=2}^{\infty} \sum_{\theta^t} \nu_t(\theta^t) \left\{ \sum_{s \geq t} \sum_{\theta^s \succeq \theta^t} p_s(\theta^s) [c_s(\theta^s) - \widehat{w}_s l_s(\theta^s)] - P_{t-1} \widehat{a}_t(\theta^{t-1}) \pi_t(\theta^t) \right\} \\ & + \sum_{t=2}^{\infty} \sum_{\theta^t} \varphi_t(\theta^t) \left\{ \sum_{s \geq t} \sum_{\theta^s \succeq \theta^t} p_s(\theta^s) [c_s(\theta^s) - \widehat{w}_s l_s(\theta^s)] \right\}. \end{aligned}$$

Note that the constraints associated with the multipliers $\{\nu_t(\theta^t)\}$ and $\{\varphi_t(\theta^t)\}$ start from $t = 2$ rather than $t = 1$. This is due to the two features of our model. First, the measurability conditions (6) reduce to the household's time-zero budget constraint (3) as $t = 1$. Second, given that $\pi_1(\theta^1) = 1$, it is implicitly assumed that the borrowing constraints are not binding at $t = 1$.

⁸We show in the Appendix that our formulation of the household constraints is equivalent to the more standard recursive formulation.

⁹There is a technical issue regarding whether the Lagrangian \tilde{L} can be written as an infinite sum so as to allow the application of the standard Lagrange multiplier method. We justify it in the Online Appendix.

Using Abel's summation formula, the Lagrangian \tilde{L} can be rewritten as¹⁰

$$\begin{aligned}\hat{L} &= \min_{\{\chi, \nu, \varphi\}} \max_{\{c, l, \hat{a}\}} \sum_{t=1}^{\infty} \beta^t \sum_{\theta^t} \left[u(c_t(\theta^t)) - v\left(\frac{l_t(\theta^t)}{\theta_t}\right) \right] \pi_t(\theta^t) \\ &\quad - \sum_{t=1}^{\infty} \sum_{\theta^t} \zeta_t(\theta^t) p_t(\theta^t) [c_t(\theta^t) - \hat{w}_t l_t(\theta^t)] + \chi \hat{a}_1 \\ &\quad - \sum_{t=2}^{\infty} \sum_{\theta^t} \nu_t(\theta^t) P_{t-1} \hat{a}_t(\theta^{t-1}) \pi_t(\theta^t),\end{aligned}$$

where $\zeta_t(\theta^t)$ is called the ‘‘cumulative multiplier,’’ and its law of motion is given by

$$\zeta_{t+1}(\theta^{t+1}) = \zeta_t(\theta^t) - \nu_{t+1}(\theta^{t+1}) - \varphi_{t+1}(\theta^{t+1}) \text{ with } \zeta_1 = \chi > 0. \quad (8)$$

Obviously, $\zeta_t(\theta^t)$ is a cumulative sum of all Lagrangian multipliers in the past history from the measurability conditions and the borrowing constraints; it encodes the frequency and severity of both types of constraints over time.¹¹

From the Lagrangian \hat{L} , the FOCs with respect to consumption $c_t(\theta^t)$ and labor supply $l_t(\theta^t)$ are given by

$$\beta^t u'(c_t(\theta^t)) = \zeta_t(\theta^t) P_t, \quad (9)$$

$$\beta^t v' \left(\frac{l_t(\theta^t)}{\theta_t} \right) \frac{1}{\theta_t} = \zeta_t(\theta^t) \hat{w}_t P_t, \quad (10)$$

while the FOC with respect to asset holdings $\hat{a}_{t+1}(\theta^t)$ is given by

$$\sum_{\theta^{t+1} \succ \theta^t} \nu_{t+1}(\theta^{t+1}) \pi(\theta_{t+1} | \theta_t) = 0. \quad (11)$$

From the FOCs (9) and (10), we see that the value of $\zeta_t(\theta^t)$ cannot be negative.

The last FOC requires that the mean of the multipliers on the measurability condition across idiosyncratic states θ^{t+1} be equal to zero, given θ^t . If markets were complete instead, households could have a short position on consumption claims at time t contingent on shock θ_{t+1} being high at time $t + 1$ (‘‘save less for a high state,’’ which is associated with $\nu_{t+1}(\theta^{t+1}) > 0$ in the Lagrangian \tilde{L}), and could have a long position on consumption claims at time t contingent on shock θ_{t+1} being low at time $t + 1$ (‘‘save

¹⁰See Ljungqvist and Sargent (2012, p. 821) for the formula.

¹¹This approach of defining recursive multipliers as in (8) was proposed and developed by Marcet and Marimon (1999, 2019) for solving dynamic problems with forward-looking constraints. Both Aiyagari, Marcet, Sargent, and Seppala (2002) and Chien, Cole, and Lustig (2011) adopted this approach.

more for a low state,” which is associated with $\nu_{t+1}(\theta^{t+1}) < 0$ in the Lagrangian \tilde{L}). However, markets are incomplete and households cannot save at time t , depending on whether shock θ_{t+1} at time $t + 1$ is high or low. As such, the best choice for $\hat{a}_{t+1}(\theta^t)$ at time t is to satisfy an average—that is, the condition (11). Putting together (9), (11) and (2), the motion (8) actually enforces the household’s Euler equation

$$u'(c_t(\theta^t)) \geq \beta R_{t+1} \sum_{\theta^{t+1} \succ \theta^t} u'(c_{t+1}(\theta^{t+1})) \pi(\theta_{t+1} | \theta_t), \quad (12)$$

where the equality holds if $\hat{a}_{t+1}(\theta^t) > 0$.

Note that the household’s Euler equation given by (12) can also be expressed equivalently as

$$\sum_{\theta^{t+1} \succ \theta^t} \zeta_{t+1}(\theta^{t+1}) \pi(\theta_{t+1} | \theta_t) \leq \zeta_t(\theta^t), \quad (13)$$

where the equality holds if the borrowing constraint of the state-contingent asset, $a_{t+1}(\theta^{t+1}) \geq 0$, does not bind for all possible subsequent θ_{t+1} states. To see this, using (11), the summation of the motion (8) over θ^{t+1} gives

$$\sum_{\theta^{t+1} \succ \theta^t} \zeta_{t+1}(\theta^{t+1}) \pi(\theta_{t+1} | \theta_t) = \zeta_t(\theta^t) - \sum_{\theta^{t+1} \succ \theta^t} \varphi_{t+1}(\theta^{t+1}) \pi(\theta_{t+1} | \theta_t), \quad (14)$$

and we know that $\varphi_{t+1}(\theta^{t+1}) \geq 0$ for all θ^{t+1} . Thus, to uphold the equality part of (13), it is required that $\varphi_{t+1}(\theta^{t+1}) = 0$ for all θ^{t+1} in (14). This feature is caused by the measurability condition (5), which effectively ensures that $\varphi_{t+1}(\theta^{t+1}) = 0$ for all θ^{t+1} , provided that $\hat{a}_{t+1}(\theta^t) > 0$.

3.2 Competitive Equilibrium

A competitive equilibrium of the model economy is defined in the standard way.

Definition 1. *Given the initial capital K_1 and initial government bonds B_1 , a competitive equilibrium is defined as sequences of tax rates, government spending and government bonds $\{\tau_{l,t}, \tau_{k,t}, G_t, B_{t+1}\}_{t=1}^\infty$, and sequences of prices $\{w_t, r_t, P_t\}_{t=1}^\infty$, aggregate allocations $\{C_t, L_t, K_{t+1}\}_{t=1}^\infty$ and individual allocation plans $\{c_t(\theta^t), l_t(\theta^t), \hat{a}_{t+1}(\theta^t)\}_{t=1}^\infty$, such that*

1. $\{c_t(\theta^t), l_t(\theta^t), \hat{a}_{t+1}(\theta^t)\}$ solve the household problem.
2. $\{L_t, K_t\}$ solve the representative firm’s problem.

3. The no-arbitrage condition holds: $\frac{P_t}{P_{t+1}} = 1 + (1 - \tau_{k,t+1})(r_{t+1} - \delta)$.

4. The time-zero government budget constraint holds:¹²

$$B_1 = \sum_{t=1}^{\infty} P_t [\tau_{l,t} w_t L_t + \tau_{k,t} (r_t - \delta) K_t - G_t]. \quad (15)$$

5. All markets clear for all t :

$$\begin{aligned} B_{t+1} + K_{t+1} &= \sum_{\theta^t} \hat{a}_{t+1}(\theta^t) \pi_t(\theta^t), \\ L_t &= \sum_{\theta^t} l_t(\theta^t) \pi_t(\theta^t), \\ C_t &= \sum_{\theta^t} c_t(\theta^t) \pi_t(\theta^t), \\ F(K_t, L_t) &= C_t + G_t + K_{t+1} - (1 - \delta) K_t. \end{aligned}$$

3.3 Characterizing the Competitive Equilibrium

This subsection characterizes the competitive equilibrium in terms of the aggregate allocations and the cumulative multipliers of the household problem. This step is critical for the primal Ramsey approach in the HAIM economy. To facilitate the characterization, we work with commonly-used separable isoelastic preferences:¹³

Assumption 1.

$$u(c) = \frac{c^{1-\alpha} - 1}{1-\alpha}, \alpha > 0; \quad v\left(\frac{l}{\theta}\right) = \frac{1}{\gamma} \left(\frac{l}{\theta}\right)^{\gamma}, \gamma > 1.$$

It is known that $1/\alpha$ represents the EIS. As will be seen, the value of the EIS plays an important role in our results.

Proposition 1. *Under Assumption 1, the consumption and labor sharing rules are given, respectively, by*

$$c_t(\theta^t) = \frac{\zeta_t(\theta^t)^{-\frac{1}{\alpha}}}{H_t} C_t, \quad (16)$$

$$l_t(\theta^t) = \frac{\theta_t^{\frac{\gamma}{\gamma-1}} \zeta_t(\theta^t)^{\frac{1}{\gamma-1}}}{J_t} L_t, \quad (17)$$

¹²From the flow government budget constraint (1) to the time-zero one, the transversality condition, $\lim_{t \rightarrow \infty} P_{t-1} B_t = 0$, is imposed.

¹³We consider GHH preferences in the Online Appendix.

where H_t and J_t are defined as

$$\begin{aligned} H_t &\equiv \sum_{\theta^t} \zeta_t(\theta^t)^{\frac{-1}{\alpha}} \pi_t(\theta^t), \\ J_t &\equiv \sum_{\theta^t} \theta_t^{\frac{\gamma}{\gamma-1}} \zeta_t(\theta^t)^{\frac{1}{\gamma-1}} \pi_t(\theta^t). \end{aligned}$$

H_t and J_t , are referred to, respectively, as the consumption and labor aggregate multipliers, which are specific moments of the distribution of the individual cumulative multiplier $\zeta_t(\theta^t)$.¹⁴ In addition, P_t and \widehat{w}_t can be expressed respectively as

$$P_t = \beta^t C_t^{-\alpha} H_t^\alpha \quad (18)$$

and

$$\widehat{w}_t = \frac{L_t^{\gamma-1} J_t^{1-\gamma}}{C_t^{-\alpha} H_t^\alpha}. \quad (19)$$

Finally, with (18), the risk-free rate is given by

$$\frac{1}{R_{t+1}} = \frac{P_{t+1}}{P_t} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\alpha} \left(\frac{H_{t+1}}{H_t} \right)^\alpha. \quad (20)$$

The proofs of our results, including Proposition 1, are all relegated to the Appendix. Equations (16) through (19) show that one can express the individual allocations $\{c_t(\theta^t), l_t(\theta^t)\}$ and the market prices $\{P_t, \widehat{w}_t\}$ of the competitive equilibrium in terms of the aggregate allocations $\{C_t, L_t\}$ and the individual cumulative multipliers $\{\zeta_t(\theta^t)\}$, and the aggregate multipliers $\{H_t, J_t\}$. The following proposition demonstrates that the Ramsey planner can pick a competitive equilibrium by choosing aggregate allocations plus asset holdings and cumulative multipliers that satisfy a set of conditions.¹⁵

For ease of exposition, we define

$$\begin{aligned} \kappa_t(\theta^t) &\equiv \beta^t \left[C_t^{1-\alpha} H_t^{\alpha-1} \zeta_t(\theta^t)^{\frac{-1}{\alpha}} - L_t^\gamma J_t^{-\gamma} \theta_t^{\frac{\gamma}{\gamma-1}} \zeta_t(\theta^t)^{\frac{1}{\gamma-1}} \right] \\ &= P_t (c_t(\theta^t) - \widehat{w}_t l_t(\theta^t)), \end{aligned} \quad (21)$$

which represents the present value of the time- t net savings made by household θ^t . The second equality holds by utilizing equations (16) through (19).

Proposition 2. *Impose Assumption 1. Given the initial capital K_1 , government bonds B_1 , capital tax rate*

¹⁴Similar expressions for consumption can be seen in Nakajima (2005), Werning (2007) and Park (2014).

¹⁵Results similar to Proposition 2 but in different contexts can be seen in Aiyagari, Marcet, Sargent, and Seppala (2002, Proposition 1), Werning (2007, Proposition 1), and Park (2014, Proposition 1).

$\tau_{k,1}$, and a stream of government spending $\{G_t\}$, sequences of aggregate allocations $\{C_t, L_t, K_{t+1}\}$, asset holdings $\{\widehat{a}_{t+1}(\theta^t)\}$, and cumulative multipliers $\{\zeta_t(\theta^t)\}$ (with the associated aggregate multipliers, H_t and J_t) can be supported as a competitive equilibrium if and only if they satisfy the following conditions:¹⁶

1. *Resource constraints:* $F(K_t, L_t) + (1 - \delta)K_t - K_{t+1} \geq C_t + G_t, \forall t \geq 1$.

2. *The implementability condition:*

$$\sum_{t=1}^{\infty} \sum_{\theta^t} \kappa_t(\theta^t) \pi_t(\theta^t) \geq \widehat{a}_1.$$

3. *Measurability conditions:*

$$\sum_{s \geq t} \sum_{\theta^s \succeq \theta^t} \kappa_s(\theta^s) \pi_s(\theta^s) = \beta^{t-1} C_{t-1}^{-\alpha} H_{t-1}^{\alpha} \widehat{a}_t(\theta^{t-1}) \pi_t(\theta^t), \forall t \geq 2, \theta^t.$$

4. *Borrowing constraints:*

$$\sum_{s \geq t} \sum_{\theta^s \succeq \theta^t} \kappa_s(\theta^s) \pi_s(\theta^s) \geq 0, \forall t \geq 2, \theta^t.$$

5. *The evolution of $\zeta_t(\theta^t)$ satisfies $\sum_{\theta^{t+1} \succ \theta^t} \zeta_{t+1}(\theta^{t+1}) \pi(\theta_{t+1} | \theta_t) \leq \zeta_t(\theta^t), \forall t \geq 1, \theta^t$.*

6. *If the borrowing constraint does not bind for $\widehat{a}_{t+1}(\theta^t)$, then*

$$\sum_{\theta^{t+1} \succ \theta^t} \zeta_{t+1}(\theta^{t+1}) \pi(\theta_{t+1} | \theta_t) = \zeta_t(\theta^t),$$

and this property holds for all θ^t and all $t \geq 1$.

Condition 2 of Proposition 2 corresponds to the time-zero household budget constraint (by Walras' law, equivalently, the government time-zero budget constraint), which is conventionally called the “implementability condition” in the formulation of the primal Ramsey problem. When the market is complete without frictions, our model reduces to the RA economy and imposing Conditions 3-6 becomes unnecessary. In particular, since $\zeta_t(\theta^t)$ in (8) equals χ at all times, Conditions 5 and 6 become redundant since they are automatically satisfied.

¹⁶The initial capital tax rate, $\tau_{k,1}$, should be a choice variable for the Ramsey planner. However, given that the initial capital is pre-installed and that households are homogeneous at time zero, taxing the initial capital is essentially the same as allowing a lump-sum tax. As is standard in the literature, we restrict the planner's ability to choose $\tau_{k,1}$ in the Ramsey problem.

4 Ramsey Problem

Different government policies result in different competitive equilibria. We define the Ramsey problem formally:

Definition 2. *The Ramsey problem is to choose a competitive equilibrium that attains the maximization of the household's lifetime utility U .*

On the basis of Proposition 2, the Ramsey problem can be represented as maximizing

$$\sum_{t \geq 1} \beta^t \sum_{\theta^t} \left[\frac{1}{1-\alpha} \left(\left(\frac{\zeta_t(\theta^t)^{\frac{-1}{\alpha}}}{H_t} C_t \right)^{1-\alpha} - 1 \right) - \frac{1}{\gamma} \left(\frac{\theta_t^{\frac{1}{\gamma-1}} \zeta_t(\theta^t)^{\frac{1}{\gamma-1}}}{J_t} L_t \right)^\gamma \right] \pi_t(\theta^t)$$

by choosing $C_t, L_t, K_{t+1}, \{\widehat{a}_{t+1}(\theta^t)\}$, and $\{\zeta_t(\theta^t)\}$ subject to Conditions 1 to 6 stated in Proposition 2 and to H_t and J_t defined earlier, given $K_1, B_1, \tau_{k,1}$ and $\{G_t\}$. The objective of the Ramsey problem is derived by substituting the consumption sharing rule (16) and the labor sharing rule (17) into U .

From (6), the strict inequality of the borrowing constraints (7) can be equivalently expressed as $P_{t-1} \widehat{a}_t(\theta^{t-1}) \pi_t(\theta^t) > 0, \forall t \geq 2, \theta^t$. Using (18), Condition 6 of Proposition 2 can be captured by the complementary slackness condition:

$$\beta^{t-1} C_{t-1}^{-\alpha} H_{t-1}^\alpha \widehat{a}_t(\theta^{t-1}) \pi_t(\theta^t) \left[\sum_{\theta^t > \theta^{t-1}} \zeta_t(\theta^t) \pi(\theta_t | \theta_{t-1}) - \zeta_{t-1}(\theta^{t-1}) \right] = 0, \forall t \geq 2, \theta^t. \quad (22)$$

That is, if $\widehat{a}_t(\theta^{t-1}) > 0$, then the square brackets shown above must equal zero. Thus the Ramsey problem is given by

$$\max_{\substack{\{C_t, L_t, K_{t+1}, \\ \{\widehat{a}_{t+1}(\theta^t)\}, \{\zeta_t(\theta^t)\}\}} \sum_{t \geq 1} \beta^t \sum_{\theta^t} \left[\frac{1}{1-\alpha} \left(\left(\frac{\zeta_t(\theta^t)^{\frac{-1}{\alpha}}}{H_t} C_t \right)^{1-\alpha} - 1 \right) - \frac{1}{\gamma} \left(\frac{\theta_t^{\frac{1}{\gamma-1}} \zeta_t(\theta^t)^{\frac{1}{\gamma-1}}}{J_t} L_t \right)^\gamma \right] \pi_t(\theta^t),$$

subject to

$$\{\beta^t \mu_t\} : F(K_t, L_t) + (1-\delta)K_t \geq C_t + G_t + K_{t+1}, \forall t \geq 1,$$

$$\chi^P : \sum_{t \geq 1} \sum_{\theta^t} \kappa_t(\theta^t) \pi_t(\theta^t) \geq \widehat{a}_1,$$

$$\{\nu_t^P(\theta^t)\} : \sum_{s \geq t} \sum_{\theta^s \succeq \theta^t} \kappa_s(\theta^s) \pi_s(\theta^s) = \beta^{t-1} C_{t-1}^{-\alpha} H_{t-1}^\alpha \widehat{a}_t(\theta^{t-1}) \pi_t(\theta^t), \forall t \geq 2, \theta^t,$$

$$\{\varphi_t^P(\theta^t)\} : \sum_{s \geq t} \sum_{\theta^s \succeq \theta^t} \kappa_s(\theta^s) \pi_s(\theta^s) \geq 0, \forall t \geq 2, \theta^t,$$

$$\{\xi_t(\theta^t)\} : \sum_{\theta^{t+1} \succ \theta^t} \zeta_{t+1}(\theta^{t+1}) \pi(\theta_{t+1} | \theta_t) \leq \zeta_t(\theta^t), \forall t \geq 1, \theta^t,$$

$$\{\phi_t(\theta^t)\} : \beta^{t-1} C_{t-1}^{-\alpha} H_{t-1}^\alpha \widehat{a}_t(\theta^{t-1}) \pi_t(\theta^t) \left[\sum_{\theta^t \succ \theta^{t-1}} \zeta_t(\theta^t) \pi(\theta_t | \theta_{t-1}) - \zeta_{t-1}(\theta^{t-1}) \right] = 0, \forall t \geq 2, \theta^t,$$

where $\{\beta^t \mu_t\}$, χ^P , $\{\nu_t^P(\theta^t)\}$, $\{\varphi_t^P(\theta^t)\}$, $\{\xi_t(\theta^t)\}$, and $\{\phi_t(\theta^t)\}$ are the corresponding multipliers.

Using Abel's summation formula and $\pi_t(\theta^t) = \pi(\theta_t | \theta_{t-1}) \pi_{t-1}(\theta^{t-1})$, the Lagrangian for the Ramsey problem gives

$$\begin{aligned} \mathcal{L} = & \max_{\{C_t, L_t, K_{t+1}, \{\widehat{a}_{t+1}(\theta^t)\}, \{\zeta_t(\theta^t)\}\}} \sum_{t \geq 1} \beta^t W(t) + \sum_{t \geq 1} \beta^t \mu_t [F(K_t, L_t) + (1 - \delta)K_t - K_{t+1} - C_t - G_t] \\ & + \sum_{t \geq 1} \sum_{\theta^t} \xi_t(\theta^t) \left[\zeta_t(\theta^t) - \sum_{\theta^{t+1} \succ \theta^t} \zeta_{t+1}(\theta^{t+1}) \pi(\theta_{t+1} | \theta_t) \right] - \chi^P \widehat{a}_1 \\ & - \sum_{t \geq 2} \beta^{t-1} C_{t-1}^{-\alpha} H_{t-1}^\alpha \sum_{\theta^{t-1}} \widehat{a}_t(\theta^{t-1}) \left(\sum_{\theta^t \succ \theta^{t-1}} \nu_t^P(\theta^t) \pi(\theta_t | \theta_{t-1}) \right) \pi_{t-1}(\theta^{t-1}) \\ & - \sum_{t \geq 2} \beta^{t-1} C_{t-1}^{-\alpha} H_{t-1}^\alpha \sum_{\theta^{t-1}} \widehat{a}_t(\theta^{t-1}) \left(\begin{array}{c} \sum_{\theta^t \succ \theta^{t-1}} \phi_t(\theta^t) \pi(\theta_t | \theta_{t-1}) \times \\ [\sum_{\theta^t \succ \theta^{t-1}} \zeta_t(\theta^t) \pi(\theta_t | \theta_{t-1}) - \zeta_{t-1}(\theta^{t-1})] \end{array} \right) \pi_{t-1}(\theta^{t-1}), \end{aligned}$$

with

$$W(t) \equiv \sum_{\theta^t} \pi_t(\theta^t) \left[\underbrace{\frac{1}{1-\alpha} \left(\left(\frac{\zeta_t(\theta^t)^{\frac{-1}{\alpha}}}{H_t} C_t \right)^{1-\alpha} - 1 \right)}_{\text{Part 1}} - \frac{1}{\gamma} \left(\frac{\theta_t^{\frac{1}{\gamma-1}} \zeta_t(\theta^t)^{\frac{1}{\gamma-1}}}{J_t} L_t \right)^\gamma \right] + \underbrace{\beta^{-t} \eta_t(\theta^t) \kappa_t(\theta^t)}_{\text{Part 2}}, \quad (23)$$

where

$$\eta_{t+1}(\theta^{t+1}) = \eta_t(\theta^t) + \nu_{t+1}^P(\theta^{t+1}) + \varphi_{t+1}^P(\theta^{t+1}), \quad \eta_1 = \chi^P > 0, \quad (24)$$

which is the motion of the Ramsey planner's cumulative multiplier. The Ramsey planner cannot complete the market as typically assumed and is thereby subject to the same market structure of the HAIM economy—that is, the same measurability conditions and borrowing constraints as those facing the household. These market frictions are summarized by the multipliers $\nu_{t+1}(\theta^{t+1})$ and $\varphi_{t+1}(\theta^{t+1})$ in the household problem and by $\nu_{t+1}^P(\theta^{t+1})$ and $\varphi_{t+1}^P(\theta^{t+1})$ in the planner problem. However, note that while we have

the term $\chi \widehat{a}_1$ in the household Lagrangian \widehat{L} , we have the term $-\chi^P \widehat{a}_1$ in the planner Lagrangian \mathcal{L} . The opposite sign is due to the fact that the implementability condition in the Ramsey problem represents the government budget constraint rather than the household budget constraint. As such, while increasing \widehat{a}_1 relaxes the household budget constraint, it tightens the government budget constraint.

4.1 Comparison with the Representative-Agent Model

When the market is complete without frictions as in the RA model, $\zeta_t(\theta^t)$ in (8) equals χ for all t and θ^t . As such, H_t equals $\chi^{-1/\alpha}$, P_t reduces to $\beta^t C_t^{-\alpha} \chi^{-1}$ and J_t becomes $\chi^{\frac{1}{\gamma-1}} \sum_{\theta^t} \theta_t^{\frac{\gamma}{\gamma-1}} \pi_t(\theta^t)$. In addition, from (24), we know that $\eta_t(\theta^t)$ reduces to χ^P all the time. Hence, $W(t)$ defined in (23) reduces to

$$W^{RA}(t) = \underbrace{\frac{C_t^{1-\alpha} - 1}{1-\alpha} - \frac{L_t^\gamma}{\gamma}}_{\text{Part 1}} + \underbrace{\chi^P \chi^{-1} (C_t^{-\alpha} C_t - L_t^{\gamma-1} L_t)}_{\text{Part 2}},$$

which is the corresponding pseudo-utility function in the RA model under Assumption 1.¹⁷ Part 1 of $W^{RA}(t)$ represents the current-period utility. Its Part 2, in terms of $\beta^t W^{RA}(t)$, is given by

$$\chi^P \chi^{-1} \beta^t (C_t^{-\alpha} C_t - L_t^{\gamma-1} L_t) = \chi^P P_t^{RA} (C_t - \widehat{w}_t^{RA} L_t), \quad (25)$$

where $P_t^{RA} = \beta^t C_t^{-\alpha} \chi^{-1}$ is the time-zero price of one unit of consumption at time t , and $\widehat{w}_t^{RA} = L_t^{\gamma-1} C_t^\alpha$ is the after-tax wage rate at time t . Thus, the term $P_t^{RA} (C_t - \widehat{w}_t^{RA} L_t)$ shown in (25) represents the time- t net savings evaluated at the time-zero price in the RA model. The time- t net savings of households also represents the amount of net revenues collected by the government in period t because of Walras' law; hence, the implementability condition multiplier, χ^P , “measures the utility costs of raising government revenues through distorting taxes” (Ljungqvist and Sargent (2012, p. 629)) in the RA framework.

Part 1 of $W(t)$ in our HAIM model also represents the current-period utility. Its Part 2, in terms of $\beta^t W(t)$, is given by

$$\eta_t(\theta^t) \kappa_t(\theta^t) = \eta_t(\theta^t) P_t (c_t(\theta^t) - \widehat{w}_t l_t(\theta^t)), \quad (26)$$

where the equality holds according to (21). Thus, the term $P_t (c_t(\theta^t) - \widehat{w}_t l_t(\theta^t))$ represents the time- t net savings of household θ^t evaluated at time zero in the HAIM economy. The taxes imposed by the Ramsey planner alter household θ^t 's consumption and labor supply and, consequently, distort his/her net savings. The shadow price of this distortion on household θ^t 's net savings is given by the multiplier $\eta_t(\theta^t)$. Note

¹⁷Under the complete-market assumption, our Ramsey planner problem is identical to the one in the RA model, which can be seen in subsection 16.6.1 in Ljungqvist and Sargent (2012, p. 626).

that $\eta_t(\theta^t)$ is no longer a time-invariant constant χ^P , as in the RA model. From the evolution of $\eta_t(\theta^t)$ governed by (24), we see that $\eta_t(\theta^t)$ starts from χ^P ($\eta_1 = \chi^P$), but in a sequence it varies not only across households but also over time, meaning that the utility cost of collecting government revenues is not only household specific but also time varying.

Now consider the steady-state version of equation (20):

$$1 = \beta R \left(\frac{H_{t+1}}{H_t} \right)^\alpha. \quad (27)$$

Given that $H_t = \chi^{-1/\alpha}$ all the time in the RA economy, we see that $H_{t+1}/H_t = 1$ and $\beta R = 1$ are the two sides of the same coin in steady state under the RA economy. In contrast, given that $\beta R < 1$ in steady state in the HAIM economy, we see that $H_{t+1}/H_t > 1$ and $\beta R < 1$ are the two sides of the same coin in steady state under the HAIM economy. Equation (27) tells us that H_t is increasing over time and must diverge to infinity in the limit in the HAIM economy, since $\beta R < 1$ holds in steady state. Put simply, the feature of an increasing and divergent H_t exactly underlies the hallmark of the competitive equilibrium in the HAIM model—the risk-free gross interest rate is lower than the inverse of the discount factor in steady state.

The divergent tendency of H_t , all else equal, makes Part 2 of $W(t)$ converge more slowly than Part 1. As will be seen, this asymmetric convergence between Part 1 and Part 2 of $W(t)$ is the key to our result of showing the non-existence of a Ramsey steady state.

4.2 Optimal Conditions of the Ramsey Problem

From the Lagrangian \mathcal{L} , the necessary FOCs with respect to $\hat{a}_{t+1}(\theta^t)$, C_t , L_t , and K_{t+1} for $t \geq 1$ yield, respectively,

$$\sum_{\theta^{t+1} \succ \theta^t} \left[\nu_{t+1}^P(\theta^{t+1}) + \phi_{t+1}(\theta^{t+1}) \left(\sum_{\theta^{t+1} \succ \theta^t} \zeta_{t+1}(\theta^{t+1}) \pi(\theta_{t+1} | \theta_t) - \zeta_t(\theta^t) \right) \right] \pi(\theta_{t+1} | \theta_t) = 0, \quad (28)$$

$$W_C(t) = \mu_t, \quad (29)$$

$$-W_L(t) = \mu_t F_L(K_t, L_t), \quad (30)$$

$$\mu_t = \beta \mu_{t+1} [F_K(K_{t+1}, L_{t+1}) - \delta + 1], \quad (31)$$

where the derivation of (29) has made use of (28), and $W_C(t)$ and $W_L(t)$ denote the derivatives of $W(t)$ with respect to C_t and L_t , respectively.¹⁸

The explicit expressions of $W_C(t)$ and $W_L(t)$ in the FOCs of the Ramsey problem are crucial to our analysis later. One can derive them from the pseudo-utility $W(t)$ defined in (23). However, to facilitate the proof and discussion hereafter, it is convenient to express W_C and W_L in the following way. First, using the consumption sharing rule (16), $W_C(t)$ in (29) is expressed as

$$W_C(t) = C_t^{-\alpha} \left[\underbrace{\sum_{\theta^t} \left(\frac{c_t(\theta^t)}{C_t} \right) \left(\frac{c_t(\theta^t)}{C_t} \right)^{-\alpha} \pi_t(\theta^t)}_{\text{Part 1}} + \underbrace{(1 - \alpha) H_t^\alpha M_t}_{\text{Part 2}} \right], \quad (32)$$

and using (16)-(19), $W_L(t)$ in (30) is expressed as

$$-W_L(t) = \hat{w}_t C_t^{-\alpha} \left[\underbrace{\sum_{\theta^t} \left(\frac{l_t(\theta^t)}{L_t} \right) \left(\frac{c_t(\theta^t)}{C_t} \right)^{-\alpha} \pi_t(\theta^t)}_{\text{Part 1}} + \underbrace{\gamma H_t^\alpha N_t}_{\text{Part 2}} \right], \quad (33)$$

where $M_t \equiv \sum_{\theta^t} \left(\frac{c_t(\theta^t)}{C_t} \right) \eta_t(\theta^t) \pi_t(\theta^t)$ and $N_t \equiv \sum_{\theta^t} \left(\frac{l_t(\theta^t)}{L_t} \right) \eta_t(\theta^t) \pi_t(\theta^t)$.

Part 1 of $W_C(t)$ and Part 1 of $W_L(t)$ denote the sum of households' "normalized" marginal utility of consumption, $\left(\frac{c_t(\theta^t)}{C_t} \right)^{-\alpha}$, weighted by their consumption shares and labor shares, respectively. They represent the planner's social evaluation of increasing C_t and L_t , respectively. We next explain the meaning of the weighted sum of $\eta_t(\theta^t)$ shown in Part 2 of $W_C(t)$ and of $W_L(t)$. Summing up (26) across all households at time t gives

$$P_t (M_t C_t - N_t \hat{w}_t L_t).$$

Contrasting the above with the corresponding one in the RA model, namely, $P_t^{RA} \chi^P (C_t - \hat{w}_t^{RA} L_t)$, we see that the role of χ^P (i.e., the utility costs of raising government revenues through distorting taxes) has been replaced either by M_t or by N_t , depending on whether distorting the time- t aggregate net savings is through the margin of changing C_t or changing L_t . Since the issue is about collecting government revenues across all households and different households contribute differently to aggregate consumption and labor supply, it is intuitive that these utility costs or shadow prices are weighted (by the consumption or labor share depending on the changed margin) rather than unweighted as given by $\sum_{\theta^t} \eta_t(\theta^t) \pi_t(\theta^t)$.

¹⁸The FOC with respect to $\zeta_t(\theta^t)$ will not be needed for the derivation of our main results.

5 Ramsey Steady State

Before presenting our main results, we define the steady state of the HAIM economy.¹⁹

Definition 3. *The steady state of the HAIM economy meets two conditions:*

1. *Each aggregate variable stays at a finite positive value.*
2. *The cross-sectional distributions of the consumption share $c_t(\theta^t)/C_t$ and of the labor share $l_t(\theta^t)/L_t$ are time invariant.*

As to the Ramsey steady state, it is defined as follows:

Definition 4. *The long-run optimal solution to the Ramsey problem is defined as a Ramsey steady state if it features the steady state of the HAIM economy.*

Given the normalization of $P_0 = 1$, we obtain from (2) that $P_t = \prod_{s=1}^t \frac{1}{R_s}$. Using (32)-(33) and $P_t = \beta^t C_t^{-\sigma} H_t^\sigma$ according to (18), one can then rewrite the FOCs (29) and (30) as

$$\underbrace{\left(\beta^t \prod_{s=1}^t R_s \right) C_t^{-\alpha} \sum_{\theta^t} \left(\frac{c_t(\theta^t)}{C_t} \right)^{1-\alpha} \pi_t(\theta^t)}_{\text{Part 1}} + \underbrace{(1-\alpha)M_t}_{\text{Part 2}} = \underbrace{\left(\beta^t \prod_{s=1}^t R_s \right) \mu_t}_{\text{Part 3}}, \quad (34)$$

$$\widehat{w}_t \left(\underbrace{\left(\beta^t \prod_{s=1}^t R_s \right) C_t^{-\alpha} \sum_{\theta^t} \left(\frac{l_t(\theta^t)}{L_t} \right) \left(\frac{c_t(\theta^t)}{C_t} \right)^{-\alpha} \pi_t(\theta^t)}_{\text{Part 1}} + \underbrace{\gamma N_t}_{\text{Part 2}} \right) = \underbrace{\left(\beta^t \prod_{s=1}^t R_s \right) \mu_t F_L(K_t, L_t)}_{\text{Part 3}}. \quad (35)$$

With (34) and (35) at hand, we now proceed to derive our two main results, which differ sharply depending on whether μ_t converges or $\lim_{t \rightarrow \infty} N_t > 0$ holds. As will be seen, these two results are mutually exclusive, in that a convergent μ_t implies $\lim_{t \rightarrow \infty} N_t = 0$, whereas $\lim_{t \rightarrow \infty} N_t > 0$ implies a divergent

¹⁹As shown by many quantitative studies, the existence of a steady state is not a problem for the HAIM economy. For the existence of steady states in the HAIM economy, see Açıkgöz (2018) and Zhu (2020).

μ_t .²⁰ Recall that μ_t denotes the shadow price of resources in the Ramsey problem and that N_t denotes the shadow price of raising government revenues by distorting the time- t aggregate net savings via the margin of changing L_t . After the derivation of the two main results, we briefly explore which result is more plausible.

5.1 μ_t Converges

Aiyagari (1995, footnote 15) implicitly assumed that the multiplier on the resource constraint, μ_t , converges to a finite positive value in the limit. When this is true so that $\mu_t = \mu_{t+1} > 0$ in the limit, we see from the planner's Euler equation (equation (20) in Aiyagari (1995) or the FOC (31) of our paper) that the MGR, $\beta [F_K(K, L) - \delta + 1] = 1$, holds in steady state. Given that $R < 1/\beta$ in steady state in the HAIM economy, the planner will then levy a positive capital tax in the long run.

The conclusion reached by Aiyagari (1995) above applies to general utility functions.²¹ Working with a specific, but commonly-used, utility function, i.e., the separable isoelastic utility function under Assumption 1, we show below that the implicit assumption made by Aiyagari (1995, footnote 15) has a strong implication for the shadow price of Ramsey taxation in the limit.

Suppose there is a Ramsey steady state with $\beta R < 1$. When μ_t converges to a finite positive value in the limit as implicitly assumed by Aiyagari (1995, footnote 15), both Part 1 and Part 3 of (35) vanish in the Ramsey steady state. This then implies $\lim_{t \rightarrow \infty} N_t = 0$. Analogously, with a convergent μ_t , both Part 1 and Part 3 of (34) vanish in the Ramsey steady state. This then implies $\lim_{t \rightarrow \infty} M_t = 0$. We are ready to state our first main result.

Proposition 3. *Impose Assumption 1. Suppose that there is a Ramsey steady state with $\beta R < 1$ and that the shadow price of resources, μ_t , converges to a finite positive value in the limit. Then $\lim_{t \rightarrow \infty} M_t = \lim_{t \rightarrow \infty} N_t = 0$ at the Ramsey steady state.*

Note that M_t and N_t represent the shadow prices of raising government revenues by distorting the time- t aggregate net savings via the margin of changing C_t and changing L_t , respectively, and that both M_t and N_t are a generalization of the multiplier χ^P on the implementability condition; see the paragraph after (32)-(33) for the details. Thus, Proposition 3 states that if the multiplier on the resource constraint of the Ramsey problem converges to a finite positive value in the limit as assumed by Aiyagari (1995), then

²⁰If $\lim_{t \rightarrow \infty} N_t = 0$, we see from (35) that it does not necessarily imply a convergent μ_t .

²¹Aiyagari (1995) worked with GHH preferences; see his footnote 6 for the details. However, with the two assumptions imposed (namely, the existence of a Ramsey steady state and the convergence of μ_t in the limit), it is clear from the planner's Euler equation (equation (20) in Aiyagari (1995)) that his conclusion is applicable to general utility functions.

the shadow price of Ramsey taxation must go to zero in the limit. This limiting property holds, regardless of whether the tax distortion is through the margin of changing C_t or changing L_t . We show in the Online Appendix that replacing Assumption 1 with the assumption of GHH preferences still reaches the same conclusion; see Proposition 5 in the Online Appendix. We discuss the plausibility of this limiting property later.

We next study the situation where $\lim_{t \rightarrow \infty} N_t > 0$ at the Ramsey optimum.

5.2 $\lim_{t \rightarrow \infty} N_t > 0$

Suppose there is a Ramsey steady state with $\beta R < 1$. The term $C_t^{-\alpha} \sum_{\theta^t} \left(\frac{l_t(\theta^t)}{L_t} \right) \left(\frac{c_t(\theta^t)}{C_t} \right)^{-\alpha} \pi_t(\theta^t)$ in Part 1 of (35) converges in steady state. Given that $\beta R < 1$ in steady state, Part 1 of (35) then vanishes in the limit. Since $\lim_{t \rightarrow \infty} N_t > 0$ by assumption, we then see from (35) that $\lim_{t \rightarrow \infty} (\beta^t \prod_{s=1}^t R_s) \mu_t > 0$ must hold, implying that μ_t must explode in the limit if a Ramsey steady state with $\beta R < 1$ exists.

Although μ_t itself explodes, it is possible for the ratio $\frac{\mu_{t+1}}{\mu_t}$ to converge so as to support a Ramsey steady state. Let us consider $\alpha > 1$, $\alpha = 1$, and $\alpha < 1$, separately.

(i) $\alpha > 1$. The term $C_t^{-\alpha} \sum_{\theta^t} \left(\frac{c_t(\theta^t)}{C_t} \right)^{1-\alpha} \pi_t(\theta^t)$ in Part 1 of (34) converges in steady state. Given that $\beta R < 1$ in steady state, Part 1 of (34) then vanishes in the limit. Since $\lim_{t \rightarrow \infty} (\beta^t \prod_{s=1}^t R_s) \mu_t > 0$ must hold at a Ramsey steady state, no matter whether $\lim_{t \rightarrow \infty} M_t > 0$ or $\lim_{t \rightarrow \infty} M_t = 0$, there is no possibility to satisfy the FOC (34) in steady state with $\alpha > 1$. In other words, there is no Ramsey steady state in this case. It is worth noting that, as observed by Açıkgöz, Hagedorn, Holter, and Wang (2018), $\alpha = 2$ is the one most commonly used in quantitative studies under Assumption 1. Straub and Werning (2020) deemed that $\alpha > 1$ is widely considered more plausible empirically.

(ii) $\alpha = 1$. The FOC (34) reduces to

$$C_t^{-1} = \mu_t.$$

The divergence of μ_t drives C_t to zero in the limit. However, $C_t \rightarrow 0$ is incompatible with the steady state defined by Definition 3, which requires that C_t converge to a positive value in steady state. We conclude that there is no Ramsey steady state in this case.

(iii) $\alpha < 1$. There are two subcases, depending on whether $\lim_{t \rightarrow \infty} M_t = 0$ or $\lim_{t \rightarrow \infty} M_t > 0$ (recall that M_t represents the shadow price of raising government revenues by distorting the time- t aggregate net savings via the margin of changing C_t). If $\lim_{t \rightarrow \infty} M_t = 0$, due to (a) Part 1 of (34) vanishes in the limit and (b) $\lim_{t \rightarrow \infty} (\beta^t \prod_{s=1}^t R_s) \mu_t > 0$ must hold at a Ramsey steady state, there is no possibility for this subcase to satisfy the FOC (34) in steady state. On the other hand, if $\lim_{t \rightarrow \infty} M_t > 0$, it is possible

for this subcase to satisfy the FOC (34) in steady state. However, we show in the Appendix that even if a Ramsey steady state exists, the MGR does not hold at the Ramsey steady state and the corresponding capital tax is non-positive.

To sum up, we state the second main result of the paper.

Proposition 4. *Impose Assumption 1 and suppose $\lim_{t \rightarrow \infty} N_t > 0$.*

1. *The shadow price of resources, μ_t , must explode in the limit if a Ramsey steady state with $\beta R < 1$ exists.*
2. *If $\alpha \geq 1$, there is no Ramsey steady state with $\beta R < 1$.*
3. *If $\alpha < 1$, (i) there is no Ramsey steady state with $\beta R < 1$ if $\lim_{t \rightarrow \infty} M_t = 0$; (ii) a Ramsey steady state with $\beta R < 1$ is possible if $\lim_{t \rightarrow \infty} M_t > 0$, but the MGR fails to hold at the Ramsey steady state and the corresponding capital tax is non-positive.*

We show in the Online Appendix that replacing Assumption 1 with the assumption of GHH preferences reaches similar results; see Proposition 6 in the Online Appendix.

From (27), we see that the result that $R < 1/\beta$ and the result that $H_{t+1}/H_t > 1$ are two sides of the same coin: $H_{t+1}/H_t > 1$ holds in steady state if and only if $R < 1/\beta$ holds in steady state. As shown in the proof of Proposition 4, the increasing and divergent behavior of the H_t term (equivalently, the feature of $R < 1/\beta$ in steady state) is the exact force that undermines the existence of Ramsey steady states when $\alpha \geq 1$.

Aiyagari (1995) adopted the dual approach to formulate the Ramsey problem, letting the planner choose $\{\hat{w}_t, R_t, K_{t+1}\}$. Perhaps because of complications, Aiyagari (1995) did not report the derivation of the FOCs with respect to \hat{w}_t and R_t . The Euler equation for the planner, equation (20) in Aiyagari (1995) (which corresponds to equation (31) in our setting), is the only FOC derived for the Ramsey problem in Aiyagari (1995). It is important to recognize that if we were to confine the analysis *only* to the FOC (31) and assume the convergence of μ_t at the Ramsey steady state as Aiyagari (1995) did, we would have the exact conclusion reached by Aiyagari (1995); namely, the MGR holds at the optimum and capital income should be taxed in the long run. Put differently, we would not be able to derive Proposition 4. This recognition highlights the importance of taking into account the necessary Ramsey FOCs other than (31). To our knowledge, the analytical form of the expression for the term $W_C(t)$ or $W_L(t)$ that appears in the Ramsey FOCs (29)-(30) has never been derived before.

The intuition underlying our second main result can be understood as follows. Unlike households in the face of idiosyncratic income shocks, the Ramsey planner faces no uncertainty in allocating aggregate

resources. The strict inequality $R < 1/\beta$ at the steady state of the HAIM economy then dictates an asymmetric discounting; that is, the market discounting rate is always lower than the preference discounting rate. This feature of asymmetric discounting impels a desire for the planner to front-load aggregate consumption. Such a desire persists permanently since the strict inequality $R < 1/\beta$ holds at the steady state of the HAIM economy.

Proposition 4 indicates that the existence of a Ramsey steady state depends on the value of α , which controls the EIS. The following intends to provide additional explanations and intuition for such a dependence. As discussed in Subsection 4.1, the utility costs of implementing a policy hinge on its effects over the net savings of households (equivalently, by Walras' law, the amount of net tax revenues collected by the government). Let us consider the impact of changing aggregate consumption on the net savings (government revenues) through consumption spending. There is only a term involving C_t in Part 2 of $W(t)$ given by (23). Expressed in $\beta^t W(t)$ and by omitting $\eta_t(\theta^t)$, this term equals

$$\beta^t C_t^{1-\alpha} H_t^{\alpha-1} \zeta_t(\theta^t)^{\frac{-1}{\alpha}} = P_t C_t \frac{\zeta_t(\theta^t)^{\frac{-1}{\alpha}}}{H_t},$$

which represents household θ^t 's consumption spending at time t according to the consumption sharing rule (16). From (18), we have $P_t C_t = \beta^t C_t^{1-\alpha} H_t^\alpha$ and so $\partial(P_t C_t)/\partial C_t = (1 - \alpha)\beta^t C_t^{-\alpha} H_t^\alpha$. Thus a drop in aggregate consumption C_t , all else equal, will raise, lower, or not change individual consumption spending via altering $P_t C_t$ if α is larger than, less than, or equal to 1, respectively. This implies that a reduction in aggregate consumption over time (front-loading consumption) will render the government constraint associated with $\eta_t(\theta^t)$ in (23) looser, tighter, or unchanged, depending on whether α is larger than, less than, or equal to 1, respectively. Since front-loading aggregate consumption relaxes the government constraint by increasing its revenues if $\alpha > 1$, it actually enforces the planner's desire to front-load aggregate consumption in the presence of $R < 1/\beta$ in steady state. In contrast, since front-loading aggregate consumption tightens the government constraint by reducing its revenues if $\alpha < 1$, it counterbalances the planner's desire to front-load aggregate consumption in the presence of $R < 1/\beta$ in steady state.

When $\alpha = 1$, neither enforcement (associated with $\alpha > 1$) nor counterbalance (associated with $\alpha < 1$) occurs. We then see a clean case of front-loading aggregate consumption in the presence of $R < 1/\beta$ in steady state. From the proof of Proposition 4, we know that μ_t is increasing and divergent because H_t is increasing and divergent. If $\alpha = 1$, we have $W_C(t) = C_t^{-1}$ from (32). Thus, given that μ_t increases over time, it is apparent that the optimal C_t determined by the FOC (29), namely, $C_t^{-1} = \mu_t$, will decrease over time.

5.3 Convergent or Divergent μ_t ?

Propositions 3 and 4 are mutually exclusive, in that a convergent μ_t implies $\lim_{t \rightarrow \infty} M_t = \lim_{t \rightarrow \infty} N_t = 0$, whereas $\lim_{t \rightarrow \infty} N_t > 0$ implies a divergent μ_t . Which one is more plausible? We briefly explore the issue.²² Our exploration focuses on the implausibility of $\lim_{t \rightarrow \infty} M_t = \lim_{t \rightarrow \infty} N_t = 0$.

Proposition 2 characterizes the competitive equilibrium of our HAIM economy by its stated Conditions 1-6. Let us imagine a different economy in which its competitive equilibrium can be characterized simply by Conditions 5-6 plus Condition 1 of Proposition 2. Then the Ramsey problem for this different economy is given by

$$\max_{\substack{\{C_t, L_t, K_{t+1}, \\ \{\hat{a}_{t+1}(\theta^t)\}, \{\zeta_t(\theta^t)\}\}} \sum_{t \geq 1} \beta^t \sum_{\theta^t} \left[\frac{1}{1-\alpha} \left(\left(\frac{\zeta_t(\theta^t)^{\frac{-1}{\alpha}}}{H_t} C_t \right)^{1-\alpha} - 1 \right) - \frac{1}{\gamma} \left(\frac{\theta_t^{\frac{1}{\gamma-1}} \zeta_t(\theta^t)^{\frac{1}{\gamma-1}}}{J_t} L_t \right)^\gamma \right] \pi_t(\theta^t),$$

subject to

$$\{\beta^t \hat{\mu}_t\} : F(K_t, L_t) + (1-\delta)K_t \geq C_t + G_t + K_{t+1}, \forall t \geq 1,$$

$$\{\hat{\xi}_t(\theta^t)\} : \sum_{\theta^{t+1} \succ \theta^t} \zeta_{t+1}(\theta^{t+1}) \pi(\theta_{t+1} | \theta_t) \leq \zeta_t(\theta^t), \forall t \geq 1, \theta^t,$$

$$\{\hat{\phi}_t(\theta^t)\} : \beta^{t-1} C_{t-1}^{-\alpha} H_{t-1}^\alpha \hat{a}_t(\theta^{t-1}) \pi_t(\theta^t) \left[\sum_{\theta^t \succ \theta^{t-1}} \zeta_t(\theta^t) \pi(\theta_t | \theta_{t-1}) - \zeta_{t-1}(\theta^{t-1}) \right] = 0, \forall t \geq 2, \theta^t,$$

where $\{\beta^t \hat{\mu}_t\}$, $\{\hat{\xi}_t(\theta^t)\}$, and $\{\hat{\phi}_t(\theta^t)\}$ are the corresponding multipliers. This Ramsey problem differs from our Ramsey problem by leaving out the implementability condition, measurability conditions, and borrowing constraints as stated in Proposition 2.

The necessary FOCs of this Ramsey problem with respect to $\hat{a}_{t+1}(\theta^t)$, C_t , L_t , and K_{t+1} for $t \geq 1$ yield, respectively,²³

$$\sum_{\theta^{t+1} \succ \theta^t} \hat{\phi}_{t+1}(\theta^{t+1}) \left(\sum_{\theta^{t+1} \succ \theta^t} \zeta_{t+1}(\theta^{t+1}) \pi(\theta_{t+1} | \theta_t) - \zeta_t(\theta^t) \right) \pi(\theta_{t+1} | \theta_t) = 0, \quad (36)$$

$$\widehat{W}_C(t) = \hat{\mu}_t, \quad (37)$$

$$-\widehat{W}_L(t) = \hat{\mu}_t F_L(K_t, L_t), \quad (38)$$

$$\hat{\mu}_t = \beta \hat{\mu}_{t+1} [F_K(K_{t+1}, L_{t+1}) - \delta + 1], \quad (39)$$

²²From (35), we see that $\lim_{t \rightarrow \infty} N_t < 0$ cannot be true.

²³The derivation of (37) has made use of (36).

where

$$\widehat{W}_C(t) = C_t^{-\alpha} \sum_{\theta^t} \left(\frac{c_t(\theta^t)}{C_t} \right) \left(\frac{c_t(\theta^t)}{C_t} \right)^{-\alpha} \pi_t(\theta^t), \quad (40)$$

$$-\widehat{W}_L(t) = \widehat{w}_t C_t^{-\alpha} \sum_{\theta^t} \left(\frac{l_t(\theta^t)}{L_t} \right) \left(\frac{c_t(\theta^t)}{C_t} \right)^{-\alpha} \pi_t(\theta^t). \quad (41)$$

The above FOCs with respect to the aggregate allocation $\{C_t, L_t, K_{t+1}\}$ are not different from the FOCs of our Ramsey problem in essence, except that $W_C(t)$ and $W_L(t)$ of (32)-(33) are replaced with $\widehat{W}_C(t)$ and $\widehat{W}_L(t)$ of (40)-(41). Note that if we let $M_t = N_t = 0$, then $W_C(t)$ and $W_L(t)$ will reduce to $\widehat{W}_C(t)$ and $\widehat{W}_L(t)$, respectively.

We have shown that, as far as the aggregate allocation $\{C_t, L_t, K_{t+1}\}$ is concerned, letting $M_t = N_t = 0$ is equivalent to leaving out the implementability condition, measurability conditions, and borrowing constraints altogether as stated in Proposition 2 in the formulation of the Ramsey problem. This indicates the implausibility of $M_t = N_t = 0$.

The above is about $M_t = N_t = 0$. Nevertheless, replacing $M_t = N_t = 0$ with $\lim_{t \rightarrow \infty} M_t = \lim_{t \rightarrow \infty} N_t = 0$, the result equally applies to the limiting case.

As we have explained earlier, the shadow price χ^P in the RA economy has been replaced either by M_t or by N_t in our HAIM economy, depending on whether distorting the time- t aggregate net savings is through the margin of changing C_t or changing L_t . The feature of $\lim_{t \rightarrow \infty} M_t = \lim_{t \rightarrow \infty} N_t = 0$ implies that, in the long run, there will be no distorting cost either through the margin of changing C_t or through the margin of changing L_t . Ljungqvist and Sargent (2012, p. 659) noted that the shadow price of raising government revenues “remains strictly positive so long as the government must resort to distortionary taxation either in the current period or for some realization of the state in a future period.” It would seem no particular reason why the government here can free from the imposition of distortionary taxes in the long run.

6 Endogenous Government Spending

This section checks the robustness of our Proposition 4 findings by altering the model setup from exogenous to endogenous government spending, which is the main setting considered by Aiyagari (1995). We show here that even with endogenous government spending, our results concerning the existence of a Ramsey steady state are robust and remain unchanged.

Following Aiyagari (1995), the household lifetime utility U is modified to

$$U^G = \sum_{t=1}^{\infty} \beta^t \sum_{\theta^t} \left[u(c_t(\theta^t)) - v\left(\frac{l_t(\theta^t)}{\theta_t}\right) + V(G_t) \right] \pi_t(\theta^t),$$

where $V(\cdot)$ is the utility function of public consumption G_t , which is assumed to be common for all households. The usual assumptions are applied to $V(\cdot)$. This modification of the setup does not change the household problem, since the determination of G_t is exogenous to households. However, the Ramsey problem is only changed slightly because G_t is now a choice variable to the Ramsey planner. As long as G_t is non-negative (which could be ensured by assuming $V'(0) = \infty$), G_t can be chosen to satisfy the time- t resource constraint so that Proposition 2 still applies. The Lagrangian for the Ramsey problem is identical to the previous Lagrangian \mathcal{L} , except for the replacement of $W(t)$ by $W(t) + V(G_t)$. As a result, the FOCs with respect to aggregate consumption, labor, and capital remain the same as before. The additional FOC with respect to G_t is given by

$$V'(G_t) = \mu_t, \tag{42}$$

which together with FOC (31) does imply the MGR if a Ramsey steady state is assumed. This is essentially the procedure for obtaining the MGR in Aiyagari (1995); see equation (20) of his Proposition 1 on page 1170.

However, the introduction of endogenous G_t does not alter the fundamental force that drives the results of Proposition 4. The marginal social benefit of having one extra unit of aggregate consumption, namely, $W_C(t)$, could still diverge in the long-run given that the Ramsey outcome of $R\beta = 1$ is infeasible in steady state. With the additional government tool—endogenous government spending—the extra output can be expended either on government spending or on private consumption, and hence the marginal benefits to the social welfare by exercising these two options have to be equalized at the optimum. Indeed, putting (29) and (42) together gives rise to $V'(G_t) = W_C(t)$ and hence the optimal choice of G_t has to respect and be consistent with the divergent behavior of μ_t . This equality casts doubt on the convergence assumption of G_t to a finite positive value made by Aiyagari (1995). In brief, it is the erroneously assumed convergence of μ_t , not the endogenous government spending assumption, which is the root of the problem.

7 Discussion

We show that a Ramsey steady state may fail to exist in the HAIM economy, but we fall short of explaining in terms of policy tools why it fails to exist. This section provides a brief discussion.

In the proof of Proposition 2, an implicit but standard assumption is that, given the sequence of capital stocks $\{K_{t+1}\}_{t=1}^{\infty}$ and households' asset holdings $\{\hat{a}_{t+1}(\theta^t)\}_{t=1}^{\infty}$, it is always feasible to pick a sequence of government bonds, $\{B_{t+1}\}_{t=1}^{\infty}$, so as to clear the asset market in each time period. Aiyagari (1995) made the same assumption in his analysis; see equation (19) of his paper and the discussion about it. Albeit standard in the literature, this feasibility assumption is not innocuous here since the planner may implement front-loading aggregate consumption induced by $R < 1/\beta$ via issuing an ever increasing amount of government bonds. In the same context as our paper, Chien and Wen (2019) utilized an analytically tractable HAIM model to demonstrate that the Ramsey planner intends to increase the supply of government bonds until full self-insurance is achieved or an exogenous debt limit binds. Their work suggests that the feasibility assumption with no quantity restriction on the planner's issuing government bonds could be the culprit for the non-existence of a Ramsey steady state in our economy. If this is indeed true, then imposing an upper bound on the issuance of government bonds should provide a mechanism to restore the existence of Ramsey steady states. Following Aiyagari, Marcet, Sargent, and Seppala (2002), let us impose an exogenous upper bound $\bar{B} < \infty$ on the issuance of government bonds:

$$B_{t+1} \leq \bar{B}, \forall t \geq 1.$$

It can be shown that incorporating the above additional constraints into the Ramsey problem does provide an offset to the increasing and divergent force of the H_t term and offers an opportunity for the existence of Ramsey steady states.

The constraints above are known as the ad hoc debt limit. By analogy with the household savings problem in Aiyagari (1994), Aiyagari, Marcet, Sargent, and Seppala (2002) also considered the so-called natural debt limit, which is defined as “the maximum debt that could be repaid almost surely under an optimal tax policy” (p. 1225). What will happen if the government's natural debt limit is incorporated into the Ramsey problem? This is an interesting question. However, answering the question is a formidable task, in that the government's natural debt limit in the Ramsey problem is endogenously determined and evolves dynamically according to the policy choice in a non-trivial way.²⁴ At any rate, explaining this and other related issues in detail is beyond the scope of the present study and we leave it to future works.

²⁴In the absence of capital, Aiyagari, Marcet, Sargent, and Seppala (2002) were able to derive the natural debt limit explicitly; see p. 1232 of their paper. However, in the presence of capital, the derivation becomes much more difficult.

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A Appendix

A.1 Deriving the Household Flow Budget Constraints

The measurability conditions (6) at time t and $t + 1$ equal, respectively

$$P_{t-1}\widehat{a}_t(\theta^{t-1})\pi_t(\theta^t) = \sum_{s \geq t} \sum_{\theta^s \succeq \theta^t} p_s(\theta^s) [c_s(\theta^s) - \widehat{w}_s l_s(\theta^s)], \quad (43)$$

$$P_t\widehat{a}_{t+1}(\theta^t)\pi_{t+1}(\theta^{t+1}) = P_t\widehat{a}_{t+1}(\theta^t)\pi(\theta_{t+1}|\theta_t)\pi_t(\theta^t) = \sum_{s \geq t+1} \sum_{\theta^s \succeq \theta^{t+1}} p_s(\theta^s) [c_s(\theta^s) - \widehat{w}_s l_s(\theta^s)]. \quad (44)$$

Summing over θ_{t+1} on both sides of (44) gives

$$\begin{aligned} P_t\widehat{a}_{t+1}(\theta^t)\pi_t(\theta^t) &= \sum_{s \geq t+1} \left(\sum_{\theta_{t+1}} \sum_{\theta^s \succeq (\theta^t, \theta_{t+1})} p_s(\theta^s) [c_s(\theta^s) - \widehat{w}_s l_s(\theta^s)] \right) \\ &= \sum_{s \geq t+1} \sum_{\theta^s \succeq \theta^t} p_s(\theta^s) [c_s(\theta^s) - \widehat{w}_s l_s(\theta^s)]. \end{aligned} \quad (45)$$

Subtracting (45) from (43) leads to

$$P_{t-1}\widehat{a}_t(\theta^{t-1})\pi_t(\theta^t) - P_t\widehat{a}_{t+1}(\theta^t)\pi_t(\theta^t) = p_t(\theta^t) [c_t(\theta^t) - \widehat{w}_t l_t(\theta^t)] = P_t [c_t(\theta^t) - \widehat{w}_t l_t(\theta^t)] \pi_t(\theta^t).$$

Using (2) with $P_{t-1}/P_t = R_t$, we obtain from the above equation

$$c_t(\theta^t) + \widehat{a}_{t+1}(\theta^t) = \widehat{w}_t l_t(\theta^t) + R_t \widehat{a}_t(\theta^{t-1}),$$

which represents the household flow budget constraints.

Putting (6) and (7) together yields $\widehat{a}_{t+1}(\theta^t) \geq 0, \forall \theta^t, t$.

A.2 Proof of Proposition 1

With the imposition of Assumption 1, the FOC for consumption (9) yields

$$c_t(\theta^t) = \left(\frac{\zeta_t(\theta^t) P_t}{\beta^t} \right)^{-\frac{1}{\alpha}}.$$

Summing $c_t(\theta^t)$ over θ^t gives the aggregate consumption at time t :

$$\begin{aligned} C_t &= \sum_{\theta^t} c_t(\theta^t) \pi_t(\theta^t) = \sum_{\theta^t} \left(\frac{\zeta_t(\theta^t) P_t}{\beta^t} \right)^{-\frac{1}{\alpha}} \pi_t(\theta^t) \\ &= \left(\frac{P_t}{\beta^t} \right)^{-\frac{1}{\alpha}} \sum_{\theta^t} \zeta_t(\theta^t)^{-\frac{1}{\alpha}} \pi_t(\theta^t) = \left(\frac{P_t}{\beta^t} \right)^{-\frac{1}{\alpha}} H_t, \end{aligned}$$

which gives (18). Plugging (18) back into (9) gives (16).

From (10), we have

$$l_t(\theta^t) = \left(\frac{\theta_t^\gamma \zeta_t(\theta^t) \widehat{w}_t P_t}{\beta^t} \right)^{\frac{1}{\gamma-1}}.$$

Summing $l_t(\theta^t)$ over θ^t gives the aggregate labor supply at time t :

$$\begin{aligned} L_t &= \sum_{\theta^t} l_t(\theta^t) \pi_t(\theta^t) = \sum_{\theta^t} \left(\frac{\theta_t^\gamma \zeta_t(\theta^t) \widehat{w}_t P_t}{\beta^t} \right)^{\frac{1}{\gamma-1}} \pi_t(\theta^t) \\ &= \left(\frac{\widehat{w}_t P_t}{\beta^t} \right)^{\frac{1}{\gamma-1}} \sum_{\theta^t} \theta_t^{\frac{\gamma}{\gamma-1}} \zeta_t(\theta^t)^{\frac{1}{\gamma-1}} \pi_t(\theta^t) = \left(\frac{\widehat{w}_t P_t}{\beta^t} \right)^{\frac{1}{\gamma-1}} J_t, \end{aligned}$$

which together with (18) gives (19). Plugging (19) back into (10) gives (17).

A.3 Proof of Proposition 2

“Only if” part: Condition 1 of Proposition 2—the resource constraints—is implied by a competitive equilibrium since it is part of the definition. Note also that Conditions 5 and 6 of Proposition 2 are implied by (8) and (11) from the household problem in a competitive equilibrium.

The remaining proof is to show that the time-zero budget constraint (3), the measurability conditions (6), and the borrowing constraints (7) in the household problem can be re-expressed as Conditions 2-4 of Proposition 2. Substituting (2), (16)-(17) and (18)-(20), all of which build on the household’s optimal behavior, into (3)-(7), we obtain Conditions 2-4.

“If” part: Suppose the sequence of asset holdings $\{\widehat{a}_{t+1}(\theta^t)\}_{t=1}^\infty$, aggregate allocations $\{C_t, L_t, G_t, K_{t+1}\}_{t=1}^\infty$, and cumulative multipliers $\{\zeta_t(\theta^t)\}_{t=1}^\infty$ with the associated aggregate multipliers $\{H_t, J_t\}_{t=1}^\infty$ satisfy Conditions 1-6 stated in Proposition 2. We show that a competitive equilibrium of the HAIM economy can be constructed in the following way.

First, we pick the prices and taxes as defined below:

$$r_t = F_K(K_t, L_t), \quad (46)$$

$$w_t = F_L(K_t, L_t), \quad (47)$$

$$P_t = \beta^t C_t^{-\alpha} H_t^\alpha, \quad (48)$$

$$1 - \tau_{k,t+1} = \frac{\frac{P_t}{P_{t+1}} - 1}{F_K(K_{t+1}, L_{t+1}) - \delta} = \frac{\frac{1}{\beta} \left(\frac{C_t}{C_{t+1}} \right)^{-\alpha} \left(\frac{H_t}{H_{t+1}} \right)^\alpha - 1}{F_K(K_{t+1}, L_{t+1}) - \delta}, \quad (49)$$

$$1 - \tau_{l,t} = \frac{L_t^{\gamma-1} J_t^{1-\gamma}}{F_L(K_t, L_t) C_t^{-\alpha} H_t^\alpha}. \quad (50)$$

Note that (46)-(47) correspond to the profit-maximization conditions of the representative firm and that (49) ensures that the no-arbitrage condition (2) holds.

Second, we show that the household problem can be solved. Let the individual consumption and labor allocations be given by (16) and (17). Then, individual consumption and labor allocations together with prices defined in (46)-(50) satisfy the first-order conditions, (9) and (10), of the household problem. To derive the household's Euler equation, we combine the individual consumption allocations, prices defined in (46)-(50), and Conditions 5-6. The time-zero budget constraint (3), the measurability conditions (6), and the borrowing constraints (7) in the household problem can be obtained by using (46)-(50) plus Conditions 2-4.

Third, we need to make sure that all markets clear. Plugging in individual consumption allocations (16) into Condition 1 implies that the market clearing condition of the goods market is satisfied. The labor market clearing condition is achieved by aggregating (17) across all households. For the asset market, we pick $\{B_{t+1}\}_{t=1}^\infty$ such that

$$B_{t+1} = \sum_{\theta^t} \hat{a}_{t+1}(\theta^t) - K_{t+1},$$

which ensures that the asset market clears in each time period.

The last condition to be met in competitive equilibrium is the government budget constraint. From

(3), we have

$$\begin{aligned}
B_1 + K_1 &= \widehat{a}_1 = \sum_{t \geq 1} P_t \sum_{\theta^t} [c_t(\theta^t) \pi_t(\theta^t) - \widehat{w}_t l_t(\theta^t) \pi_t(\theta^t)] \\
&= \sum_{t \geq 1} P_t [C_t - w_t L_t + \tau_{l,t} w_t L_t] \\
&= \sum_{t \geq 1} P_t [C_t + r_t K_t - F(K_t, L_t) + \tau_{l,t} w_t L_t],
\end{aligned}$$

where the derivation has made use of $\widehat{w}_t = w_t(1 - \tau_{l,t})$ and $F(K_t, L_t) = w_t L_t + r_t K_t$. Utilizing the resource constraint and the no-arbitrage condition (2) then gives

$$\begin{aligned}
B_1 + K_1 &= \sum_{t \geq 1} P_t [r_t K_t - K_{t+1} + (1 - \delta) K_t + \tau_{l,t} w_t L_t - G_t] \\
&= \sum_{t \geq 1} P_t [(1 + (1 - \tau_{k,t})(r_t - \delta)) K_t - K_{t+1} + \tau_{k,t}(r_t - \delta) K_t + \tau_{l,t} w_t L_t - G_t] \\
&= \sum_{t \geq 1} P_t \left[\frac{P_{t-1}}{P_t} K_t - K_{t+1} + \tau_{k,t}(r_t - \delta) K_t + \tau_{l,t} w_t L_t - G_t \right] \\
&= P_0 K_1 + \sum_{t \geq 1} P_t [\tau_{k,t}(r_t - \delta) K_t + \tau_{l,t} w_t L_t - G_t],
\end{aligned}$$

which leads to the time-zero government budget constraint since we normalize $P_0 = 1$.

A.4 Proof for the Case of $\alpha < 1$ with $\lim_{t \rightarrow \infty} M_t > 0$

Part 1 of both (34) and (35) vanish in steady state because of $\beta R < 1$. Part 2 of (35) is positive in steady state because of $\lim_{t \rightarrow \infty} N_t > 0$. Given $\alpha < 1$, Part 2 of (34) is also positive because of $\lim_{t \rightarrow \infty} M_t > 0$ by presumption. Thus, given $\beta R < 1$, the divergent μ_t implied by $\lim_{t \rightarrow \infty} (\beta^t \prod_{s=1}^t R_s) \mu_t > 0$ contradicts neither (34) nor (35) in steady state. We conclude that both the divergent μ_t and a convergent μ_{t+1}/μ_t can coexist and be consistent with the FOCs (29)-(31) in steady state.

Using (34), we have

$$\begin{aligned}
\frac{\mu_{t+1}}{\mu_t} &= \frac{C_{t+1}^{-\alpha} \sum_{\theta^{t+1}} \left(\frac{c_{t+1}(\theta^{t+1})}{C_{t+1}} \right)^{1-\alpha} \pi_{t+1}(\theta^{t+1}) + (1 - \alpha) M_{t+1} / \beta^{t+1} \prod_{s=1}^{t+1} R_s}{C_t^{-\alpha} \sum_{\theta^t} \left(\frac{c_t(\theta^t)}{C_t} \right)^{1-\alpha} \pi_t(\theta^t) + (1 - \alpha) M_t / \beta^t \prod_{s=1}^t R_s} \\
&= \frac{(\beta^t \prod_{s=1}^t R_s) C_{t+1}^{-\alpha} \sum_{\theta^{t+1}} \left(\frac{c_{t+1}(\theta^{t+1})}{C_{t+1}} \right)^{1-\alpha} \pi_{t+1}(\theta^{t+1}) + (1 - \alpha) M_{t+1} / \beta R_{t+1}}{(\beta^t \prod_{s=1}^t R_s) C_t^{-\alpha} \sum_{\theta^t} \left(\frac{c_t(\theta^t)}{C_t} \right)^{1-\alpha} \pi_t(\theta^t) + (1 - \alpha) M_t}.
\end{aligned}$$

Since both $(\beta^t \prod_{s=1}^t R_s) C_{t+1}^{-\alpha} \sum_{\theta^{t+1}} \left(\frac{c_{t+1}(\theta^{t+1})}{C_{t+1}}\right)^{1-\alpha} \pi_{t+1}(\theta^{t+1})$ and $(\beta^t \prod_{s=1}^t R_s) C_t^{-\alpha} \sum_{\theta^t} \left(\frac{c_t(\theta^t)}{C_t}\right)^{1-\alpha} \pi_t(\theta^t)$ equal zero in the limit with $\beta R < 1$ in steady state, we obtain

$$\lim_{t \rightarrow \infty} \frac{\mu_{t+1}}{\mu_t} = \frac{1}{\beta R} \lim_{t \rightarrow \infty} \frac{M_{t+1}}{M_t},$$

where $\lim_{t \rightarrow \infty} \frac{M_{t+1}}{M_t}$ is a constant in steady state. The FOC (31) in steady state then yields

$$1 = (1/R) \lim_{t \rightarrow \infty} \frac{M_{t+1}}{M_t} [F_K(K, L) - \delta + 1], \quad (51)$$

which fails to satisfy the MGR, unless $\lim_{t \rightarrow \infty} \frac{M_{t+1}}{M_t} = \beta R < 1$. However, the result of $\lim_{t \rightarrow \infty} \frac{M_{t+1}}{M_t} < 1$ implies that M_t itself goes to zero in the limit, which contradicts our presumption of $\lim_{t \rightarrow \infty} M_t > 0$. Given that $\lim_{t \rightarrow \infty} \frac{M_{t+1}}{M_t} \geq 1$ must hold, we obtain from (2) and (51)

$$R = [1 + (1 - \tau_k)(r - \delta)] \geq [F_K(K, L) - \delta + 1],$$

which shows that $\tau_k \leq 0$ at the Ramsey steady state.

Online Appendix to
Implementing the Modified Golden Rule? Optimal
Ramsey Taxation with Incomplete Markets Revisited

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1 Justify our infinite-dimensional Lagrangian

One possible way of justification is to follow the work of Alvarez and Jermann (2000). In terms of our notation, they mainly imposed two conditions:

(i) The implied interest rates for the allocation $\{c_t(\theta^t)\}$ are high, i.e., $\sum_t \sum_{\theta^t} p_t c_t(\theta^t) < \infty$ (see Definition 3.4 of their paper).

(ii) For each θ^t , there is a constant $b(\theta^t)$ such that $|u(c_t(\theta^t))| \leq b(\theta^t) u'(c_t(\theta^t)) c_t(\theta^t)$ for all t (see equation (4.6) of their paper).

With the imposition of these two conditions, Alvarez and Jermann (2000) showed that the resulting Lagrangian of their problem is finite. Condition (ii) is satisfied under our Assumption 1 if $\alpha \neq 1$. However, the log utility (i.e., $\alpha = 1$) fails to satisfy the condition (see Remark 1 after equation (4.6) in their paper). In view of the fact that our Assumption 1 allows for the log utility case, we turn to the work of Le Van and Saglam (2004), which provides a set of sufficient conditions to justify expressing the Lagrangian \tilde{L} as an infinite sum.¹

Let ℓ^∞ denote the space of bounded sequences and ℓ^1 the space of summable sequences. Le Van and Saglam (2004) considered the following optimization problem (P):

$$\min f(x) \text{ s.t. } g(x) \leq 0 \text{ with } x \in \ell^\infty,$$

where $f : \ell^\infty \rightarrow \mathbb{R} \cup \{+\infty\}$, $g(x) = \{g_t(x)\}_{t=0}^\infty$ with each $g_t : \ell^\infty \rightarrow \mathbb{R} \cup \{+\infty\}$, and f and g_t are convex functions. Define $F = \{x \in \ell^\infty \mid f(x) < +\infty\}$ and $\Gamma = \{x \in \ell^\infty \mid g_t(x) < +\infty, \forall t\}$. Since the domain of the problem P belongs to ℓ^∞ , in applying the method of Lagrange multipliers to solve the problem, there are questions with regard to whether the Lagrange multipliers exist and whether they can be represented by a summable sequence in ℓ^1 .

It is important to recognize that f and g_t are functions from ℓ^∞ to $\mathbb{R} \cup \{+\infty\}$. Dechert (1982) considered the situation where f and g_t are functions from ℓ^∞ to \mathbb{R} . Le Van and Saglam (2004) extended it to the situation where f and g_t are functions from ℓ^∞ to $\mathbb{R} \cup \{+\infty\}$. This extension is important, in that it allows for the case where $-\log c$ goes to infinity as $c \rightarrow 0$ and the case where $g_t(x)$ may go to infinity.

To present multipliers as a summable sequence of real numbers in the infinite dimensional space, Le Van and Saglam (2004) put restrictions on the asymptotic behavior of the objective functional $f(x)$ and the constraint functions $g(x)$. For $x, y \in \ell^\infty$ and $T \in \mathbb{N}$, define $x^T(x, y) = x_t$ if $t \leq T$ and $x^T(x, y) = y_t$ if $t > T$. Le Van and Saglam (2004) proposed the following:

¹Other relevant works include Dechert (1982) and Rustichini (1998). Golosov, Tsyvinski, and Werquin (2016) provided a survey on the issue.

(1) **Assumption f.** If $x \in F$, $y \in \ell^\infty$ and $x^T(x, y) \in F$ for all T large enough, then $\lim_{T \rightarrow \infty} f(x^T(x, y)) = f(x)$.

(2) **Assumption g.** If $x, y \in \Gamma$ and $x^T(x, y) \in \Gamma$ for all T large enough, then

(a) $\lim_{T \rightarrow \infty} g_t(x^T(x, y)) = g_t(x), \forall t$.

(b) $\lim_{t \rightarrow \infty} [g_t(x^T(x, y)) - g_t(y)] = 0$ for all T large enough.

(c) $\exists M$ such that $\|g(x^T(x, y))\| \leq M$ for all T large enough.

Since the sequence $x^T(x, y)$ differs from x only for $t > T$, Assumption g-(a) basically requires that $g_t(x)$ at t be little affected by changes in the distant future $T \gg t$. Since the sequence $x^T(x, y)$ differs from y only for a finite number of times, Assumption g-(b) basically requires that $g_t(x^T(x, y))$ not be very different from $g_t(y)$ when t is large. Assumption g-(c) is satisfied when g is continuous on Γ .

(3) **Slater condition.** $\exists x^0 \in \ell^\infty$ such that $\sup_t g_t(x^0) < 0$.

This condition requires that the feasible set of the problem P have an interior point.

We have the following result.

Theorem (Le Van and Saglam (2004)). Suppose (i) $g(x) \in \ell^\infty, \forall x \in \Gamma$, (ii) Assumptions f and g are satisfied, and (iii) $x^0 \in \ell^\infty$ satisfies the Slater condition. If x^* is a solution of the problem P and $x^T(x^*, x^0) \in \Gamma \cap F$ for all T large enough, then there exists $\Lambda \in \ell_+^1$ such that:

$$\forall x \in \ell^\infty, f(x) + \Lambda g(x) \geq f(x^*) + \Lambda g(x^*) \text{ with } \Lambda g(x^*) = 0.$$

Let us incorporate an additional constraint $g'(x) = 0$ into the problem P. If $g'(x)$ is linear and $g'(x) = 0$ holds as an equality all the time, then the Slater condition becomes: $\exists x^0 \in \ell^\infty$ such that $\sup_t g_t(x^0) < 0$ and $g'(x^0) = 0$; see Boyd and Vandenberghe (2004).

We proceed to show that the Le Van-Saglam theorem applies here.

Let $x = \{c_t(\theta^t), l_t(\theta^t), \hat{a}_t(\theta^{t-1})\}_{t=1}^\infty$, $f(x) = -U$, and $g(x) = \{g_1(x), g_{2t}(x), g'_t(x)\}$ with

$$g_1(x) = \sum_{t \geq 1} \sum_{\theta^t} p_t(\theta^t) [c_t(\theta^t) - \hat{w}_t l_t(\theta^t)] - \hat{a}_1,$$

$$g_{2t}(x) = - \sum_{s \geq t} \sum_{\theta^s \geq \theta^t} p_s(\theta^s) [c_s(\theta^s) - \hat{w}_s l_s(\theta^s)], \forall t \geq 2, \theta^t,$$

$$g'_t(x) = \sum_{s \geq t} \sum_{\theta^s \geq \theta^t} p_s(\theta^s) [c_s(\theta^s) - \widehat{w}_s l_s(\theta^s)] - P_{t-1} \widehat{a}_t(\theta^{t-1}) \pi_t(\theta^t), \forall t \geq 2, \theta^t,$$

where $g_1(x)$ is the household's time-zero budget constraint, $\{g_{2t}(x)\}$ corresponds to the borrowing constraints, and $\{g'_t(x)\}$ to the measurability conditions.

(i) By the definition of Γ , it is obvious that $g(x) \in \ell^\infty, \forall x \in \Gamma$.

(ii) Assumptions f and g.

Assumption f.

Let $\hat{x} = \{c_t(\theta^t), l_t(\theta^t)\}_{t=1}^\infty \in F$ and $\hat{y} = \{c'_t(\theta^t), l'_t(\theta^t)\}_{t=1}^\infty \in \ell^\infty$ with $x^T(\hat{x}, \hat{y}) \in F$ for all T large enough. We have

$$f(x^T(\hat{x}, \hat{y})) = -U(x^T(\hat{x}, \hat{y})) = \underbrace{-\sum_{t=1}^T \beta^t \sum_{\theta^t} \left[u(c_t(\theta^t)) - v\left(\frac{l_t(\theta^t)}{\theta_t}\right) \right] \pi_t(\theta^t)}_{\text{Part 1}} - \underbrace{\sum_{t=T+1}^\infty \beta^t \sum_{\theta^t} \left[u(c'_t(\theta^t)) - v\left(\frac{l'_t(\theta^t)}{\theta_t}\right) \right] \pi_t(\theta^t)}_{\text{Part 2}}.$$

Because $\hat{y} = \{c'_t(\theta^t), l'_t(\theta^t)\}_{t=1}^\infty \in \ell^\infty$ (bounded sequences), $x^T(\hat{x}, \hat{y}) \in F = \{x \in \ell^\infty \mid f(x) < +\infty\}$, and $\beta^T \rightarrow 0$ as $T \rightarrow \infty$, Part 2 of the above equation will vanish as $T \rightarrow \infty$ and, therefore, we have $\lim_{T \rightarrow \infty} f(x^T(\hat{x}, \hat{y})) = f(\hat{x})$.

Assumption g.

Let $x = \{c_t(\theta^t), l_t(\theta^t), \widehat{a}_t(\theta^{t-1})\}_{t=1}^\infty$ and $y = \{c'_t(\theta^t), l'_t(\theta^t), \widehat{a}'_t(\theta^{t-1})\}_{t=1}^\infty$, where $x, y \in \Gamma$ and $x^T(x, y) \in \Gamma$ for all T large enough.

(a) Holding t fixed, we have $g_t(x^T(x, y)) \rightarrow g_t(x)$ for T sufficiently large. This is true because y belongs to ℓ^∞ (bounded sequences), $x^T(x, y) \in \Gamma = \{x \in \ell^\infty \mid g_t(x) < +\infty, \forall t\}$, and $P_T = \prod_{s=1}^T \frac{1}{R_s}$ with $P_T \rightarrow 0$ as $T \rightarrow \infty$.

(b) For a fixed T , it is clear that $g_t(x^T(x, y)) = g_t(y)$ as $t > T$.

(c) $g_t(x)$ is differential on Γ and hence is continuous on Γ .

(iii) Slater condition.

Let $x^0 = \{c_t^0(\theta^t), l_t^0(\theta^t), \widehat{a}_t^0(\theta^{t-1})\}_{t=1}^\infty$ with $\{c_t^0(\theta^t)\}_{t=1}^\infty = (c_1, \bar{c}, \bar{c}, \dots), \forall \theta^t, \{l_t^0(\theta^t)\}_{t=1}^\infty = (l_1, \bar{l}, \bar{l}, \dots), \forall \theta^t$, and $\{\widehat{a}_t^0(\theta^{t-1})\}_{t=1}^\infty$ specified below. Given $\{P_t, \widehat{w}_t\}_{t=1}^\infty$, there exist $0 < \bar{c} < \infty$ and $0 < \bar{l} < \infty$ such that

$$0 < \sum_{s \geq t} P_s (\bar{c} - \widehat{w}_s \bar{l}) < \infty, \forall t \geq 2, \quad (1)$$

where $P_s = \prod_{j=1}^s \frac{1}{R_j}$. Given $\widehat{a}_1 > 0$ by our setup, let us pick $\{\widehat{a}_t^0(\theta^{t-1})\}_{t \geq 2} = \{\widehat{a}_t^0\}_{t \geq 2}, \forall \theta^{t-1}$, to satisfy

$$P_{t-1}\widehat{a}_t^0 = \sum_{s \geq t} P_s (\bar{c} - \widehat{w}_s \bar{l}), \forall t \geq 2. \quad (2)$$

By (1) and (2), we know that $0 < \widehat{a}_2^0 < \infty$. Thus, there exist $0 < c_1 < \infty$ and $0 < l_1 < \infty$ such that

$$\widehat{a}_1 > P_1 (c_1 - \widehat{w}_1 l_1 + \widehat{a}_2^0). \quad (3)$$

Given x^0 that satisfies (1)-(3), we verify the Slater condition. First, we have

$$\begin{aligned} g_{2t}(x^0) &= - \sum_{s \geq t} \sum_{\theta^s \succeq \theta^t} p_s(\theta^s) (\bar{c} - \widehat{w}_s \bar{l}) \\ &= - \sum_{s \geq t} \sum_{\theta^s \succeq \theta^t} P_s \pi_s(\theta^s) (\bar{c} - \widehat{w}_s \bar{l}) < 0, \forall t \geq 2, \theta^t, \end{aligned}$$

where the last strict inequality holds because of (1).

Turning to $g'_t(x^0)$, we have

$$\begin{aligned} g'_t(x^0) &= \sum_{s \geq t} \sum_{\theta^s \succeq \theta^t} p_s(\theta^s) (\bar{c} - \widehat{w}_s \bar{l}) - P_{t-1} \widehat{a}_t^0 \pi_t(\theta^t) \\ &= \sum_{s \geq t} \sum_{\theta^s \succeq \theta^t} P_s \pi_{s-t}(\theta^s | \theta^t) \pi_t(\theta^t) (\bar{c} - \widehat{w}_s \bar{l}) - P_{t-1} \widehat{a}_t^0 \pi_t(\theta^t) \\ &= \pi_t(\theta^t) \left[\sum_{s \geq t} P_s (\bar{c} - \widehat{w}_s \bar{l}) - P_{t-1} \widehat{a}_t^0 \right], \forall t \geq 2, \theta^t, \end{aligned}$$

where the last equality utilizes $\sum_{\theta^s \succeq \theta^t} \pi_{s-t}(\theta^s | \theta^t) = 1$. Because of (2), the last equality implies $g'_t(x^0) = 0, \forall t \geq 2, \theta^t$.

As for $g_1(x^0)$, we have

$$\begin{aligned} g_1(x^0) &= \sum_{\theta^1} \pi_1(\theta^1) \left[P_1 (c_1 - \widehat{w}_1 l_1) + \sum_{s \geq 2} \sum_{\theta^s \succeq \theta^1} P_s \pi_{s-1}(\theta^s | \theta^1) (\bar{c} - \widehat{w}_s \bar{l}) \right] - \widehat{a}_1 \\ &= [P_1 (c_1 - \widehat{w}_1 l_1) + \widehat{a}_2^0] - \widehat{a}_1, \end{aligned}$$

where the last equality invokes $\sum_{s \geq 2} \sum_{\theta^s \succeq \theta^1} P_s \pi_{s-1}(\theta^s | \theta^1) (\bar{c} - \widehat{w}_s \bar{l}) = \sum_{s \geq 2} P_s (\bar{c} - \widehat{w}_s \bar{l}) = P_1 \widehat{a}_2^0$.

By (3), we obtain $g_1(x^0) < 0$.

Finally, since $\{c_t^0(\theta^t), l_t^0(\theta^t)\}_{t=1}^\infty \in F$, it is clear that $x^T(x^*, x^0) \in \Gamma \cap F$ for all T large enough.

2 GHH Preferences

We replace Assumption 1 (separable isoelastic preferences) with Assumption 2 (GHH preferences) and report the results derived.

Households maximize their lifetime utility

$$U = \sum_{t=1}^{\infty} \beta^t \sum_{\theta^t} \left[u(c_t(\theta^t), \frac{l_t(\theta^t)}{\theta_t}) \right] \pi_t(\theta^t),$$

where $u(\cdot)$ takes the following form:

Assumption 2. $u(c, \frac{l}{\theta}) = \frac{1}{1-\alpha} \left(c - \frac{1}{\gamma} \left(\frac{l}{\theta} \right)^\gamma \right)^{1-\alpha}$, $\alpha > 0, \gamma > 1$.

2.1 Household Problem

The household Lagrangian is given by

$$\begin{aligned} \hat{L} = & \min_{\{\chi, \nu, \varphi\}} \max_{\{c, l, \hat{a}\}} \sum_{t=1}^{\infty} \beta^t \sum_{\theta^t} \left[u(c_t(\theta^t), \frac{l_t(\theta^t)}{\theta_t}) \right] \pi_t(\theta^t) \\ & - \sum_{t=1}^{\infty} \sum_{\theta^t} \zeta_t(\theta^t) p_t(\theta^t) [c_t(\theta^t) - \hat{w}_t l_t(\theta^t)] + \chi \hat{a}_1 \\ & - \sum_{t=2}^{\infty} \sum_{\theta^t} \nu_t(\theta^t) P_{t-1} \hat{a}_t(\theta^{t-1}) \pi_t(\theta^t), \end{aligned}$$

where

$$\zeta_{t+1}(\theta^{t+1}) = \zeta_t(\theta^t) - \nu_{t+1}(\theta^{t+1}) - \varphi_{t+1}(\theta^{t+1}) \text{ with } \zeta_1 = \chi > 0.$$

The FOCs (9)-(11) become

$$\beta^t \left(c_t(\theta^t) - \frac{1}{\gamma} \left(\frac{l_t(\theta^t)}{\theta_t} \right)^\gamma \right)^{-\alpha} = \zeta_t(\theta^t) P_t,$$

$$\beta^t \left(c_t(\theta^t) - \frac{1}{\gamma} \left(\frac{l_t(\theta^t)}{\theta_t} \right)^\gamma \right)^{-\alpha} l_t(\theta^t)^{\gamma-1} \theta_t^{-\gamma} = \hat{w}_t \zeta_t(\theta^t) P_t,$$

$$\sum_{\theta^{t+1} > \theta^t} \nu_{t+1}(\theta^{t+1}) \pi(\theta_{t+1} | \theta_t) = 0.$$

The consumption and labor sharing rules are given, respectively, by

$$cs(\theta^t) \equiv \frac{c_t(\theta^t) - \frac{1}{\gamma} \left(\frac{l_t(\theta^t)}{\theta_t} \right)^\gamma}{C_t - \frac{1}{\gamma} \frac{L_t^\gamma}{x_t^{\gamma-1}}} = \frac{\zeta_t(\theta^t)^{\frac{-1}{\alpha}}}{H_t},$$

$$ls(\theta^t) \equiv \frac{l_t(\theta^t)}{L_t} = \frac{\theta_t^{\frac{\gamma}{\gamma-1}}}{x_t},$$

where $H_t \equiv \sum_{\theta^t} \zeta_t(\theta^t)^{\frac{-1}{\alpha}} \pi_t(\theta^t)$ and $x_t \equiv \sum_{\theta^t} \theta_t^{\frac{\gamma}{\gamma-1}} \pi_t(\theta^t)$.

The prices (18)-(20) become

$$P_t = \beta^t H_t^\alpha \left(C_t - \frac{1}{\gamma} \frac{L_t^\gamma}{x_t^{\gamma-1}} \right)^{-\alpha},$$

$$\widehat{w}_t = \left(\frac{L_t}{x_t} \right)^{\gamma-1},$$

$$\frac{1}{R_{t+1}} = \frac{P_{t+1}}{P_t} = \beta \left(\frac{C_{t+1} - \frac{1}{\gamma} \frac{L_{t+1}^\gamma}{x_{t+1}^{\gamma-1}}}{C_t - \frac{1}{\gamma} \frac{L_t^\gamma}{x_t^{\gamma-1}}} \right)^{-\alpha} \left(\frac{H_{t+1}}{H_t} \right)^\alpha.$$

Equation (21) becomes

$$\begin{aligned} \kappa_t(\theta^t) &\equiv P_t (c_t(\theta^t) - \widehat{w}_t l_t(\theta^t)) \\ &= \beta^t H_t^\alpha \left(C_t - \frac{1}{\gamma} \frac{L_t^\gamma}{x_t^{\gamma-1}} \right)^{-\alpha} \left[\left(C_t - \frac{1}{\gamma} \frac{L_t^\gamma}{x_t^{\gamma-1}} \right) \frac{\zeta_t(\theta^t)^{\frac{-1}{\alpha}}}{H_t} - \frac{\gamma-1}{\gamma} \left(\frac{L_t}{x_t} \right)^\gamma \theta_t^{\frac{\gamma}{\gamma-1}} \right]. \end{aligned}$$

2.2 Ramsey Problem

The planner Lagrangian is given by

$$\begin{aligned}
\mathcal{L} = & \max_{\{C_t, L_t, K_{t+1}, \{\hat{a}_{t+1}(\theta^t)\}, \{\zeta_t(\theta^t)\}\}} \sum_{t \geq 1} \beta^t W(t) + \sum_{t \geq 1} \beta^t \mu_t [F(K_t, L_t) + (1 - \delta)K_t - K_{t+1} - C_t - G_t] \\
& + \sum_{t \geq 1} \sum_{\theta^t} \xi_t(\theta^t) \left[\zeta_t(\theta^t) - \sum_{\theta^{t+1} > \theta^t} \zeta_{t+1}(\theta^{t+1}) \pi(\theta_{t+1} | \theta_t) \right] - \chi^P \hat{a}_1 \\
& - \sum_{t \geq 2} \beta^{t-1} H_{t-1}^\alpha \left(C_{t-1} - \frac{1}{\gamma} \frac{L_{t-1}^\gamma}{x_{t-1}^{\gamma-1}} \right)^{-\alpha} \sum_{\theta^{t-1}} \hat{a}_t(\theta^{t-1}) \left(\sum_{\theta^t > \theta^{t-1}} \nu_t^P(\theta^t) \pi(\theta_t | \theta_{t-1}) \right) \pi_{t-1}(\theta^{t-1}) \\
& - \sum_{t \geq 2} \beta^{t-1} H_{t-1}^\alpha \left(C_{t-1} - \frac{1}{\gamma} \frac{L_{t-1}^\gamma}{x_{t-1}^{\gamma-1}} \right)^{-\alpha} \sum_{\theta^{t-1}} \hat{a}_t(\theta^{t-1}) \left(\frac{\sum_{\theta^t > \theta^{t-1}} \phi_t(\theta^t) \pi(\theta_t | \theta_{t-1}) \times}{[\sum_{\theta^t > \theta^{t-1}} \zeta_t(\theta^t) \pi(\theta_t | \theta_{t-1}) - \zeta_{t-1}(\theta^{t-1})]} \right) \pi_{t-1}(\theta^{t-1})
\end{aligned}$$

with

$$\tilde{W}(t) \equiv \sum_{\theta^t} \pi_t(\theta^t) \left[\underbrace{\frac{1}{1-\alpha} \left(\frac{\zeta_t(\theta^t)^{-\frac{1}{\alpha}}}{H_t} \left[C_t - \frac{1}{\gamma} \frac{L_t^\gamma}{x_t^{\gamma-1}} \right] \right)^{1-\alpha}}_{\text{Part 1}} + \underbrace{\beta^{-t} \eta_t(\theta^t) \kappa_t(\theta^t)}_{\text{Part 2}} \right],$$

where

$$\eta_{t+1}(\theta^{t+1}) = \eta_t(\theta^t) + \nu_{t+1}^P(\theta^{t+1}) + \varphi_{t+1}^P(\theta^{t+1}), \quad \eta_1 = \chi^P > 0.$$

The FOCs (28)-(31) remain unchanged, but $W_C(t)$ of (32) and $-W_L(t)$ of (33) are modified to be

$$\begin{aligned}
\tilde{W}_C(t) = & \left(C_t - \frac{1}{\gamma} \frac{L_t^\gamma}{x_t^{\gamma-1}} \right)^{-\alpha} \sum_{\theta^t} cs(\theta^t)^{1-\alpha} \pi_t(\theta^t) \\
& + (1 - \alpha) H_t^\alpha \left(C_t - \frac{1}{\gamma} \frac{L_t^\gamma}{x_t^{\gamma-1}} \right)^{-\alpha} \tilde{M}_t \\
& + \frac{\alpha(\gamma-1)}{\gamma} H_t^\alpha \left(C_t - \frac{1}{\gamma} \frac{L_t^\gamma}{x_t^{\gamma-1}} \right)^{-\alpha-1} \left(\frac{L_t}{x_t} \right)^\gamma x_t \tilde{N}_t,
\end{aligned}$$

$$\begin{aligned}
-\tilde{W}_L(t) = & \left(\frac{L_t}{x_t} \right)^{\gamma-1} \left(C_t - \frac{1}{\gamma} \frac{L_t^\gamma}{x_t^{\gamma-1}} \right)^{-\alpha} \sum_{\theta^t} cs(\theta^t)^{1-\alpha} \pi_t(\theta^t) \\
& + (1 - \alpha) H_t^\alpha \left(C_t - \frac{1}{\gamma} \frac{L_t^\gamma}{x_t^{\gamma-1}} \right)^{-\alpha} \left(\frac{L_t}{x_t} \right)^{\gamma-1} \tilde{M}_t \\
& + (\gamma - 1) H_t^\alpha \left(C_t - \frac{1}{\gamma} \frac{L_t^\gamma}{x_t^{\gamma-1}} \right)^{-\alpha} \left[\left(\frac{L_t}{x_t} \right)^{\gamma-1} + \frac{\alpha}{\gamma} \left(C_t - \frac{1}{\gamma} \frac{L_t^\gamma}{x_t^{\gamma-1}} \right)^{-1} \left(\frac{L_t}{x_t} \right)^{2\gamma-1} x_t \right] \tilde{N}_t,
\end{aligned}$$

where $\tilde{M}_t \equiv \sum_{\theta^t} c_s(\theta^t)\eta_t(\theta^t)\pi_t(\theta^t)$ and $\tilde{N}_t \equiv \sum_{\theta^t} l_s(\theta^t)\eta_t(\theta^t)\pi_t(\theta^t)$; \tilde{M}_t and \tilde{N}_t correspond to M_t and N_t under Assumption 1.

Using $\tilde{W}_C(t)$ and $-\tilde{W}_L(t)$ plus $P_t = \beta^t H_t^\alpha \left(C_t - \frac{1}{\gamma} \frac{L_t^\gamma}{x_t^{\gamma-1}}\right)^{-\alpha}$ and $P_t = \prod_{s=1}^t \frac{1}{R_s}$, one can rewrite the FOCs (29) and (30) as

$$\left(\begin{array}{l} H_t^{-\alpha} \sum_{\theta^t} c_s(\theta^t)^{1-\alpha} \pi_t(\theta^t) + (1-\alpha) \tilde{M}_t \\ + \frac{\alpha(\gamma-1)}{\gamma} \left(C_t - \frac{1}{\gamma} \frac{L_t^\gamma}{x_t^{\gamma-1}}\right)^{-1} \left(\frac{L_t}{x_t}\right)^\gamma x_t \tilde{N}_t \end{array} \right) = \left(\beta^t \prod_{s=1}^t R_s\right) \mu_t, \quad (4)$$

$$\hat{w}_t \left(\begin{array}{l} H_t^{-\alpha} \sum_{\theta^t} c_s(\theta^t)^{1-\alpha} \pi_t(\theta^t) + (1-\alpha) \tilde{M}_t \\ + \frac{\alpha(\gamma-1)}{\gamma} \left(C_t - \frac{1}{\gamma} \frac{L_t^\gamma}{x_t^{\gamma-1}}\right)^{-1} \left(\frac{L_t}{x_t}\right)^\gamma x_t \tilde{N}_t + (\gamma-1) \tilde{N}_t \end{array} \right) = \left(\beta^t \prod_{s=1}^t R_s\right) \mu_t F_L(K_t, L_t), \quad (5)$$

where we have used $\hat{w}_t = \left(\frac{L_t}{x_t}\right)^{\gamma-1}$ in expressing the second equation. Putting (4) and (5) together leads to

$$\left(\beta^t \prod_{s=1}^t R_s\right) \mu_t = \frac{\hat{w}_t (\gamma-1) \tilde{N}_t}{F_L(K_t, L_t) - \hat{w}_t}. \quad (6)$$

A finite positive μ_t in the limit leads to $\lim_{t \rightarrow \infty} \tilde{N}_t = 0$ according to (6). Given $\lim_{t \rightarrow \infty} H_t = \infty$, we then obtain from (4) that $\lim_{t \rightarrow \infty} \tilde{M}_t = 0$ at the Ramsey steady state. We have the following result.

Proposition 5 *Impose Assumption 2. Suppose that there is a Ramsey steady state with $\beta R < 1$ and that the shadow price of resources, μ_t , converges to a finite positive value in the limit. Then $\lim_{t \rightarrow \infty} \tilde{M}_t = \lim_{t \rightarrow \infty} \tilde{N}_t = 0$ at the Ramsey steady state.*

By contrast, if $\lim_{t \rightarrow \infty} \tilde{N}_t > 0$. This then implies from (6) that $\lim_{t \rightarrow \infty} (\beta^t \prod_{s=1}^t R_s) \mu_t > 0$, which in turn implies that μ_t must explode in the limit, given $\beta R < 1$ in steady state.

To be consistent with $\lim_{t \rightarrow \infty} (\beta^t \prod_{s=1}^t R_s) \mu_t > 0$, it requires that the left-hand side of both (4) and (5) be strictly positive in the limit. This is possible since $\lim_{t \rightarrow \infty} \tilde{N}_t > 0$. Thus, unlike under Assumption 1, a Ramsey steady state with $\beta R < 1$ is possible for the case of $\alpha \geq 1$ as well as for the case of $\alpha < 1$ under Assumption 2.

Finally, we examine whether the MGR holds at the Ramsey steady state. Using $F_L(K_t, L_t) - \hat{w}_t = w_t \tau_{l,t}$, (6) gives

$$\lim_{t \rightarrow \infty} \frac{\mu_{t+1}}{\mu_t} = \lim_{t \rightarrow \infty} \frac{1}{\beta R_{t+1}} \left[\frac{\frac{\hat{w}_{t+1}(\gamma-1)\tilde{N}_{t+1}}{w_{t+1}\tau_{l,t+1}}}{\frac{\hat{w}_t(\gamma-1)\tilde{N}_t}{w_t\tau_{l,t}}} \right] = \frac{1}{\beta R} \lim_{t \rightarrow \infty} \frac{\tilde{N}_{t+1}}{\tilde{N}_t},$$

where we have utilized $\hat{w}_t = \hat{w}_{t+1} > 0$ and $w_t \tau_{l,t} = w_{t+1} \tau_{l,t+1} > 0$ in steady state.² The FOC (31) in

²Equation (6) implies that $\tau_{l,t} \geq 0$. We can further rule out $\tau_{l,t} = \tau_{l,t+1} = 0$ in steady state. Using $\hat{w}_t = w_t(1 - \tau_{l,t})$,

steady state then yields

$$1 = (1/R) \lim_{t \rightarrow \infty} \frac{\tilde{N}_{t+1}}{\tilde{N}_t} [F_K(K, L) - \delta + 1], \quad (7)$$

which fails to satisfy the MGR, unless $\lim_{t \rightarrow \infty} \frac{\tilde{N}_{t+1}}{\tilde{N}_t} = \beta R < 1$. However, the result of $\lim_{t \rightarrow \infty} \frac{\tilde{N}_{t+1}}{\tilde{N}_t} < 1$ implies that \tilde{N}_t itself goes to zero in the limit, which contradicts the result of $\lim_{t \rightarrow \infty} \tilde{N}_t > 0$. Given that $\lim_{t \rightarrow \infty} \frac{\tilde{N}_{t+1}}{\tilde{N}_t} \geq 1$ must hold, we obtain from (2) and (7)

$$R = [1 + (1 - \tau_k)(r - \delta)] \geq [F_K(K, L) - \delta + 1],$$

which shows that $\tau_k \leq 0$ at the Ramsey steady state.

To sum up, we state:

Proposition 6 *Impose Assumption 2.*

1. *The shadow price of resources, μ_t , must explode in the limit if a Ramsey steady state with $\beta R < 1$ exists.*
2. *A Ramsey steady state with $\beta R < 1$ is possible, regardless of whether $\alpha > 1$, $\alpha = 1$, or $\alpha < 1$; however, the MGR fails to hold at the Ramsey steady state and the corresponding capital tax is non-positive.*

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equation (5) yields

$$\left(\begin{array}{l} H_t^{-\alpha} \sum_{\theta^t} cs(\theta^t)^{1-\alpha} \pi_t(\theta^t) + (1 - \alpha) \tilde{M}_t \\ + \frac{\alpha(\gamma-1)}{\gamma} \left(C_t - \frac{1}{\gamma} \frac{L_t^\gamma}{x_t^{\gamma-1}} \right)^{-1} \left(\frac{L_t}{x_t} \right)^\gamma x_t \tilde{N}_t \\ + (\gamma - 1) \tilde{N}_t \end{array} \right) = \frac{(\beta^t \prod_{s=1}^t R_s) \mu_t}{1 - \tau_{l,t}}.$$

Given that $\lim_{t \rightarrow \infty} \tilde{N}_t > 0$, the result of $\tau_{l,t} = \tau_{l,t+1} = 0$ in steady state will lead to a contradiction between the above equality and equation (4).

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