IMPLICIT-EXPLICIT MULTISTEP METHODS FOR QUASILINEAR PARABOLIC EQUATIONS

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Abstract. Efficient combinations of implicit and explicit multistep methods for nonlinear parabolic equations were recently studied in [1]. In this note we present a refined analysis to allow more general nonlinearities. The abstract theory is applied to a quasilinear parabolic equation.

Dedicated to Professor Vidar Thomée on the occasion of his 65^{th} *birthday, August 20, 1998*

1. Introduction

In this paper we extend our study of implicit-explicit multistep finite element schemes for parabolic problems to quasilinear equations. In particular, we establish abstract convergence results for these methods under weaker stability and consistency conditions. Thus the abstract theory can be applied to various nonlinear parabolic problems yielding convergence under mild meshconditions.

We consider problems of the form: Given $T > 0$ and $u^0 \in H$, find $u : [0, T] \to D(A)$ such that

$$\begin{aligned} &u'(t) + Au(t) = B(t, u(t)), \quad 0 < t < T,\\ &u(0) = u^0, \end{aligned} \tag{1.1}$$

with A a positive definite, selfadjoint, linear operator on a Hilbert space $(H, (\cdot, \cdot))$ with domain $D(A)$ dense in H, and $B(t, \cdot) : D(A) \to H$, $t \in [0, T]$, a (possibly) nonlinear operator. To motivate the construction of the fully discrete schemes, we first consider the semidiscrete problem approximating (1.1): For a given finite dimensional subspace V_h of V, $V = D(A^{1/2})$, we seek a function u_h, $u_h(t) \in V_h$, defined by

$$\begin{aligned} &u_h'(t) + A_h u_h(t) = B_h(t, u_h(t)), \quad 0 < t < T,\\ &u_h(0) = u_h^0; \end{aligned} \tag{1.2}$$

here $u_h^0 \in V_h$ is a given approximation to u^0, and A_h, B_h are appropriate operators on V_h with A_h a positive definite, selfadjoint, linear operator.

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Following [1] and [5], we let (α,β) be a strongly $A(0)-$stable $q-$step scheme and (α,γ) be an explicit $q-$step scheme, characterized by three polynomials α,β and γ,

$$\alpha(\zeta)=\sum_{i=0}^{q}\alpha_i\zeta^i\,,\quad \beta(\zeta)=\sum_{i=0}^{q}\beta_i\zeta^i\,,\quad \gamma(\zeta)=\sum_{i=0}^{q-1}\gamma_i\zeta^i\,.$$

Letting $N\in\mathbb{N}$, $k=\frac{T}{N}$ be the time step, and $t^n=nk, n=0,\dots,N$, we combine the (α,β) and (α,γ) schemes to obtain an (α,β,γ) scheme for discretizing (1.2) in time, and define a sequence of approximations U^n, $U^n\in V_h$, to $u^n:=u(t^n)$, by

$$\sum_{i=0}^{q}\alpha_i U^{n+i}+k\sum_{i=0}^{q}\beta_i A_h U^{n+i}=k\sum_{i=0}^{q-1}\gamma_i B_h(t^{n+i},U^{n+i}). \tag{1.3}$$

Given $U^0,\dots,U^{q-1}$ in $V_h, U^q,\dots,U^N$ are well defined by the (α,β,γ) scheme, see [1]. The scheme (1.3) is efficient, its implementation to advance in time requires solving a linear system with the same matrix for all time levels.

Stability and consistency assumptions. Let $|\cdot|$ denote the norm of H, and introduce in V the norm $\|\cdot\|$ by $\|v\|:=|A^{1/2}v|$. We identify H with its dual, and denote by V' the dual of V, again by $(\cdot,\cdot)$ the duality pairing on V' and V, and by $\|\cdot\|_\star$ the dual norm on V'. Let T_u be a tube around the solution u, $T_u:=\{v\in V:\min_t\|u(t)-v\|\le 1\}$, say. For stability purposes, we assume that $B(t,\cdot)$ can be extended to an operator from V into V', [1] and an estimate of the form

$$\|B(t,v)-B(t,w)\|_\star\le\lambda\|v-w\|+\mu|v-w|\qquad\forall v,w\in T_u \tag{1.4}$$

holds, uniformly in t, with two constants λ and μ. It is essential for our analysis that

$$\lambda<1/\sup_{x>0}\max_{|\zeta|=1}\Big|\frac{x\gamma(\zeta)}{(\alpha+x\beta)(\zeta)}\Big|, \tag{1.5}$$

while the tube T_u is defined in terms of the norm of V for concreteness. Under these conditions we will show convergence, provided that a mild meshcondition is satisfied, see Theorem 2.1. The proof can be easily modified to yield convergence under conditions analogous to (1.4) for v and w belonging to tubes defined in terms of other norms, not necessarily the same for both arguments; milder or stronger meshconditions, respectively, are required if the tubes are defined in terms of weaker or stronger norms, cf. Remark 2.2 and Section 3.

We will assume in the sequel that (1.1) possesses a solution which is sufficiently regular for our results to hold. Local uniqueness of smooth solutions follows easily in view of (1.4).

For the space discretization we use a family V_h, $0<h<1$, of finite dimensional subspaces of V. In the sequel the following discrete operators will play an essential role:

[1]this is actually the condition needed, but for simplicity we have also assumed that $B(\cdot,t):D(A)\to H$

Define $P_o : V' \to V_h$, $A_h : V \to V_h$ and $B_h(t,\cdot) : V \to V_h$ by

$$\begin{aligned}(P_o v, \chi) &= (v, \chi) \quad \forall \chi \in V_h\\ (A_h\varphi, \chi) &= (A\varphi, \chi) \quad \forall \chi \in V_h\\ (B_h(t,\varphi), \chi) &= (B(t,\varphi), \chi) \quad \forall \chi \in V_h.\end{aligned}$$

Let $B(t,\cdot) : V \to V'$ be differentiable, and assume that the linear operator $M(t)$, $M(t) := A - B'(t, u(t)) + \sigma I$, is uniformly positive definite, for an appropriate constant σ. We introduce the 'elliptic' projection $R_h(t) : V \to V_h, t \in [0,T]$, by

$$P_o M(t) R_h(t) v = P_o M(t) v. \tag{1.6}$$

We will show consistency of the (α, β, γ) scheme for $R_h(t)u(t)$; to this end we shall use approximation properties of the elliptic projection operator $R_h(t)$. We assume that $R_h(t)$ satisfies the estimates

$$|u(t) - R_h(t)u(t)| + h^{d/2}\|u(t) - R_h(t)u(t)\| \le Ch^r, \tag{1.7}$$

$$|\frac{d}{dt}[u(t) - R_h(t)u(t)]| \le Ch^r, \tag{1.8}$$

with two integers r and d, $2 \le d \le r$. We further assume that

$$\|\frac{d^j}{dt^j}[R_h(t)u(t)]\| \le C, \quad j = 1, \dots, p+1, \tag{1.9}$$

p being the order of both multistep schemes.

For consistency purposes, we assume for the nonlinear part the estimate

$$\|B(t, u(t)) - B(t, R_h(t)u(t)) - B'(t, u(t))(u(t) - R_h(t)u(t))\|_\star \le Ch^r. \tag{1.10}$$

Then, under some mild meshconditions and for appropriate starting values $U^0, \dots,$ U^{q-1}, we shall derive optimal order error estimates in $|\cdot|$.

Implicit-explicit multistep methods for linear parabolic equations with time dependent coefficients were first introduced and analyzed in [5]. Recently, [1], we analyzed implicit-explicit multistep finite element methods for nonlinear parabolic problems, under stronger conditions on the nonlinearity. More precisely, we took B independent of t, and assumed for stability purposes the global condition

$$|(B'(v)w, \omega)| \le \lambda\|w\|\,\|\omega\| + \mu(v)|w|\,|\omega| \qquad \forall v, w, \omega \in V \tag{1.4$'$}$$

with a sufficiently small constant λ and a functional $\mu(v)$ bounded for v bounded in V, and for consistency purposes that

$$\|B(u(t)) - B(R_h u(t))\|_\star \le Ch^r \tag{1.10$'$}$$

with elliptic projection operator R_h defined, in terms of the linear operator A only, by $(AR_h v, \chi) = (Av, \chi)\ \forall \chi \in V_h$.

It is easily seen that (1.4) follows from (1.4′). Besides the fact that (1.4) is local, in contrast to the global condition (1.4′), the major difference between the two conditions consists in the norm of ω used in their last term: in (1.4′) the $H-$norm while in (1.4), implicitly, the $V-$norm is used.

Condition (1.10′) restricts essentially the order of the derivatives contained in B to $d/2$, if A is a differential operator of order d. It was already mentioned in [1] that, for some concrete differential equations, one can get by with a less stringent condition by taking into account in the definition of the elliptic projection operator the terms of B of order higher than $d/2$; an attempt in this direction is the definition of the elliptic projection considered in this note. Condition (1.10) may be satisfied even if A and B are differential operators of the same order.

To emphasize that the new stability and consistency conditions do indeed allow more general nonlinearities than the corresponding conditions used in [1], we mention two simple examples of initial and boundary value problems in one space variable in a bounded interval. It is easily seen that condition (1.4′) is satisfied for the equation

$$u_t - u_{xx} = (f(u))_x,$$

provided that f' is uniformly bounded by a small constant; condition (1.4) on the other hand is satisfied with $\lambda = 0$ for any smooth function f. Next we consider the equation

$$u_t - u_{xx} = (a(x,t,u)u_x)_x.$$

It is easily seen in this case that condition (1.10′) is not satisfied whereas condition (1.10) is satisfied, cf. Section 3. These two examples are particular cases of the quasi-linear equation

$$u_t = \operatorname{div}(c(x,t,u)\nabla u + g(x,t,u)) + f(x,t,u)$$

which will be considered in Section 3.

An outline of the paper is as follows: Section 2 is devoted to the abstract analysis of the implicit-explicit multistep schemes. Explicit bounds for λ are derived for some implicit-explicit schemes of order up to 6. In the last section, we apply our abstract results to a quasilinear parabolic partial differential equation.

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2. Multistep schemes

In this section we shall analyze implicit-explicit multistep schemes for the abstract parabolic initial value problem (1.1).

Let (α,β) be an implicit strongly $A(0)-$stable $q-$step scheme, and (α,γ) be an explicit $q-$step scheme. We assume that both methods (α,β) and (α,γ) are of order p, i.e.,

$$\sum_{i=0}^{q} i^\ell \alpha_i \;=\; \ell \sum_{i=0}^{q} i^{\ell-1}\beta_i \;=\; \ell \sum_{i=0}^{q-1} i^{\ell-1}\gamma_i, \qquad \ell = 0,1,\dots,p.$$

For examples of (α,β,γ) schemes satisfying these stability and consistency properties we refer to [1] and the references therein; see also Remark 2.4.

Our main concern in this section is to analyze the approximation properties of the sequence $\{U^n\}$. As an intermediate step, we shall show consistency of the scheme (1.3) for the elliptic projection W of the solution u of (1.1), $W(t) = R_h(t)u(t)$.

Consistency. The consistency error E^n of the scheme (1.3) for W is given by

$$kE^n = \sum_{i=0}^{q}(\alpha_i I + k\beta_i A_h)W^{n+i} - k\sum_{i=0}^{q-1}\gamma_i B_h(t^{n+i}, W^{n+i})\,, \tag{2.1}$$

$n = 0, \dots, N-q$. Using (1.6), the definition of A_h and B_h, and (1.1), and letting $\gamma_q := 0$, we split E^n as $E^n = E_1^n + E_2^n + E_3^n + E_4^n$, with

$$kE_1^n = \sum_{i=0}^{q}\alpha_i[R_h(t^{n+i}) - P_o]u^{n+i}\,, \tag{2.2i}$$

$$kE_2^n = P_o\sum_{i=0}^{q}[\alpha_i u^{n+i} - k\gamma_i u'(t^{n+i})]\,, \tag{2.2ii}$$

$$E_3^n := \sum_{i=0}^{q}(\beta_i - \gamma_i)A_h W^{n+i}\,, \tag{2.2iii}$$

and

$$E_4^n := \sum_{i=0}^{q}\gamma_i\{A_h W^{n+i} - P_o A u^{n+i} + P_o B(t^{n+i}, u^{n+i}) - B_h(t^{n+i}, W^{n+i})\}\,. \tag{2.2iv}$$

First, we will estimate E_1^n. Using (1.8) and the fact that $\alpha_0 + \cdots + \alpha_q = 0$, it is easily seen that

$$\max_{0\le n\le N-q}|E_1^n| \le Ch^r\,. \tag{2.3i}$$

Further, in view of the consistency properties of (α, γ),

$$\Big|\sum_{i=0}^{q}[\alpha_i u^{n+i} - k\gamma_i u'(t^{n+i})]\Big| \le Ck^{p+1}\,,$$

i.e.,

$$\max_{0\le n\le N-q}|E_2^n| \le Ck^p\,. \tag{2.3ii}$$

Now, using (1.9) and the consistency properties of (α, β) and (α, γ), we have

$$\max_{0\le n\le N-q}\|E_3^n\|_\star \le Ck^p\,. \tag{2.3iii}$$

Finally, we will estimate E_4^n. First, from (1.6) we deduce that

$$[A_h - B_h'(t, u(t)) + \sigma I]R_h(t)u(t) = P_o[A - B_h'(t, u(t)) + \sigma I]u(t)$$

and rewrite (2.2iv) as

$$\begin{aligned}E_4^n = P_o \sum_{i=0}^{q} \gamma_i \{B(t^{n+i}, u^{n+i}) - B(t^{n+i}, W^{n+i}) - B'(t^{n+i}, u^{n+i})(u^{n+i} - W^{n+i})\}\\ + \sigma P_o \sum_{i=0}^{q} \gamma_i (u^{n+i} - W^{n+i}) .\end{aligned}$$

Then, in view of (1.10) and (1.7), we obtain

$$\max_{0\le n\le N-q} \|E_4^n\|_\star \le C h^r . \tag{2.3iv}$$

Thus, we have the following estimate for the consistency error E^n,

$$\max_{0\le n\le N-q} \|E^n\|_\star \le C(k^p + h^r) . \tag{2.4}$$

Convergence. In the sequel assume that we are given initial approximations U^0, U^1, $\dots$, $U^{q-1} \in V_h$ to $u^0, \dots, u^{q-1}$ such that

$$\sum_{j=0}^{q-1} \big(|W^j - U^j| + k^{1/2}\|W^j - U^j\|\big) \le C(k^p + h^r). \tag{2.5}$$

Let $U^n \in V_h$, $n = q, \dots, N$, be recursively defined by the (α, β, γ) scheme (1.3). Let $\vartheta^n = W^n - U^n, n = 0, \dots, N$. Then (2.1) and (1.3) yield the error equation for ϑ^n

$$\begin{aligned}\sum_{i=0}^{q}(\alpha_i I + k\beta_i A_h)\vartheta^{n+i} =& k \sum_{i=0}^{q-1} \gamma_i \{B_h(t^{n+i}, W^{n+i}) - B_h(t^{n+i}, U^{n+i})\}\\ &+ kE^n, \quad n = 0, \dots, N - q.\end{aligned} \tag{2.6}$$

The rational functions $e(\ell, \cdot)$ and $f(\ell, \cdot)$ defined from the expansions

$$\begin{aligned}\big(\alpha(\zeta) + x\beta(\zeta)\big)^{-1} &= \sum_{\ell\in\mathbb{Z}} e(\ell, x)\ \zeta^{-\ell},\\ \big(\alpha(\zeta) + x\beta(\zeta)\big)^{-1}\gamma(\zeta) &= \sum_{\ell\in\mathbb{Z}} f(\ell, x)\ \zeta^{-\ell},\end{aligned} \tag{2.7}$$

will play an important role in the stability analysis. Due to the strong $A(0)-$stability, for all $x \in (0, \infty]$, the modulus of all roots of $\alpha(\cdot) + x\beta(\cdot)$ is less than one. Therefore, the expansions are valid for all $|\zeta| \ge 1$ and we have $e(\ell, \cdot) = 0$ for $\ell \le q - 1$ and $f(\ell, \cdot) = 0$ for $\ell \le 0$. We also note that the only pole of these rational functions is $-\alpha_q/\beta_q < 0$ and that they vanish at ∞. Thus, we can define $e(\ell, kA_h)$ and $f(\ell, kA_h)$. We let $b^\ell := B_h(t^\ell, W^\ell) - B_h(t^\ell, U^\ell)$, and set

$$\vartheta_1^0 = 0, \quad \vartheta_1^n = k \sum_{\ell=0}^{n-1} f(n - \ell, kA_h) b^\ell,$$

$$\vartheta_2^n = k \sum_{\ell=0}^{n-q} e(n - \ell, kA_h) E^\ell.$$

Then, in view of (2.7), we have

$$\sum_{i=0}^{q}(\alpha_i I + k\beta_i A_h)(\vartheta_1^{n+i} + \vartheta_2^{n+i}) = k\sum_{i=0}^{q-1}\gamma_i\ b^{n+i} + kE^n, \quad n = 0,\dots,N-q,$$

cf., e.g., [9, pp. 242–244]. Therefore, the sequence ϑ_3^n, $\vartheta_3^n = \vartheta^n - \vartheta_1^n - \vartheta_2^n$, satisfies the relation

$$\sum_{i=0}^{q}\ (\alpha_i I + k\beta_i A_h)\vartheta_3^{n+i} = 0, \qquad n \geq 0,$$

and, consequently, with $g_j(n,x) = \sum_{\ell=j+1}^{q} e(n+\ell-j,x)(\alpha_\ell + x\beta_\ell)$,

$$\vartheta_3^n = \sum_{j=0}^{q-1} g_j(n, kA_h)\vartheta_3^j, \quad n \geq 0.$$

It is easily seen that $\vartheta_2^j = 0$, for $j \leq q-1$; therefore $\vartheta_3^0,\dots,\vartheta_3^{q-1}$, and thus all ϑ_3^n, depend only on the initial entries $W^0,\dots,W^{q-1},U^0,\dots,U^{q-1}$.

Using a spectral expansion in terms of the eigenvectors of A_h and Parseval's identity we prove the following result. Similar techniques are used in [10] and [11].

Lemma 2.1. *There exist positive constants K_1, K_2, M_1, M_2, N_1 and N_2, depending only on α, β and γ, such that for any n, $0 \leq n \leq N$, the following estimates are valid*

$$k\sum_{\ell=0}^{n}\|\vartheta_1^\ell\|^2 \leq K_1^2\ k\sum_{\ell=0}^{n-1}\|b^\ell\|_\star^2, \tag{2.8i}$$

$$|\vartheta_1^n|^2 \leq K_2\ k\sum_{\ell=0}^{n-1}\|b^\ell\|_\star^2, \tag{2.8ii}$$

$$k\sum_{\ell=0}^{n}\|\vartheta_2^\ell\|^2 \leq M_1^2\ k\sum_{\ell=0}^{n-q}\|E^\ell\|_\star^2, \tag{2.9i}$$

$$|\vartheta_2^n|^2 \leq M_2\ k\sum_{\ell=0}^{n-q}\|E^\ell\|_\star^2, \tag{2.9ii}$$

and

$$k\sum_{\ell=0}^{n}\|\vartheta_3^\ell\|^2 \leq qN_1\ \sum_{j=0}^{q-1}(|\vartheta_3^j|^2 + k\|\vartheta_3^j\|^2), \tag{2.10i}$$

$$|\vartheta_3^n| \leq N_2\ \sum_{j=0}^{q-1}|\vartheta_3^j|. \tag{2.10ii}$$

In particular, with $m_1(x,\zeta) = \frac{x}{(\alpha+x\beta)(\zeta)}$ and $k_1(x,\zeta) = m_1(x,\zeta)\gamma(\zeta)$,

$$K_1 = \sup_{x>0}\max_{|\zeta|=1}\ |k_1(x,\zeta)|, \qquad K_2 = \sup_{x>0}\int_0^1\ |\frac{1}{\sqrt{x}}k_1(x,e^{-2i\pi t})|^2 dt,$$

$$M_1 = \sup_{x>0} \max_{|\zeta|=1} |m_1(x,\zeta)|, \qquad M_2 = \sup_{x>0} \int_0^1 |\frac{1}{\sqrt{x}} m_1(x, e^{-2i\pi t})|^2 dt,$$

$$N_1 = \max_{0\le j\le q-1} \sup_{x>0} \int_0^1 \frac{x\ |\delta_j(e^{-2i\pi t}, x)|^2}{1+x} dt,$$

$$N_2 = \max_{0\le j\le q-1} \sup_{n\ge q} \sup_{x>0} |g_j(n,x)|,$$

where

$$\delta_j(\zeta, x) = \frac{\sum_{\ell=j+1}^q (\alpha_\ell + x\beta_\ell)\zeta^\ell}{\sum_{\ell=0}^q (\alpha_\ell + x\beta_\ell)\zeta^\ell}.$$

Proof. It suffices to show the estimates for $b^\ell = 0$ for $\ell \ge n$, $E^\ell = 0$ for $\ell \ge n-q+1$, and n replaced by ∞ on the right-hand sides. The proof now consists of two parts: First we derive the bounds as stated and then show that $K_1, \dots, N_2$ are indeed finite.

We introduce

$$\widehat{E}(t) = \sum_{\ell=0}^\infty E^\ell\ e^{2i\pi\ell t} \quad \text{and} \quad \widehat{\vartheta}_j(t) = \sum_{\ell=0}^\infty \vartheta_j^\ell\ e^{2i\pi\ell t},\ j=2,3;$$

from the definition of ϑ_2 and (2.7), we deduce

$$\widehat{\vartheta}_2(t) = k\big(\alpha(e^{-2i\pi t})I + \beta(e^{-2i\pi t})kA_h\big)^{-1}\widehat{E}(t).$$

Therefore, we have $\|\widehat{\vartheta}_2(t)\| \le M_1\|\widehat{E}(t)\|_\star$, and, using Parseval's identity,

$$\sum_{\ell=0}^\infty \|\vartheta_2^\ell\|^2 = \int_0^1 \|\widehat{\vartheta}_2(t)\|^2\ dt \le M_1^2 \int_0^1 \|\widehat{E}(t)\|_\star^2\ dt = M_1^2 \sum_{\ell=0}^\infty \|E^\ell\|_\star^2,$$

i.e. (2.9i) holds. Using similar arguments we prove (2.8i). In order to prove (2.10i), we first note that, in view of (2.7),

$$\begin{aligned}
\widehat{\vartheta}_3(t) &= \sum_{j=0}^{q-1}\sum_{\ell=0}^{\infty}\sum_{s=j+1}^{q} e(\ell+s-j, kA_h)e^{2i\pi(\ell+s-j)t}\ (\alpha_s + \beta_s kA_h)e^{-2i\pi st}\vartheta_3^j e^{2i\pi jt} \\
&= \sum_{j=0}^{q-1}\sum_{\ell\in\mathbb{Z}} e(\ell, kA_h)e^{2i\pi\ell t} \sum_{s=j+1}^{q}\ (\alpha_s + \beta_s kA_h)e^{-2i\pi st}\vartheta_3^j e^{2i\pi jt} \\
&= \sum_{j=0}^{q-1} \delta_j(e^{-2i\pi t}, kA_h)\vartheta_3^j e^{2i\pi jt}.
\end{aligned}$$

Further

$$k\int_0^1 \|\delta_j(e^{-2i\pi t}, kA_h)\vartheta_3^j e^{2i\pi jt}\|^2\ dt \le N_1\ (|\vartheta_3^j|^2 + k\|\vartheta_3^j\|^2),$$

and, therefore,

$$k\int_0^1 \|\widehat{\vartheta}_3(t)\|^2\ dt \le qN_1 \sum_{j=0}^{q-1}(|\vartheta_3^j|^2 + k\|\vartheta_3^j\|^2),$$

which immediately yields (2.10i). For the estimate (2.9ii), let $\{w_m\}$ be an $H-$orthonormal basis of V_h consisting of eigenfunctions of A_h, $A_h w_m = \lambda_m w_m$. Then $\widehat{E}(t)$ can be expressed as

$$\widehat{E}(t) = \sum_m \hat{e}_m(t) w_m;$$

with $x_m = k\lambda_m$, we have

$$\begin{aligned}
\vartheta_2^n &= \int_0^1 \ \widehat{\vartheta}_2(t) e^{-2i\pi n t} \ dt \\
&= \sqrt{k} \sum_\ell \frac{1}{\sqrt{\lambda_\ell}} \int_0^1 \frac{\sqrt{x_\ell}}{(\alpha + x_\ell \beta)(e^{-2i\pi t})} \hat{e}_\ell(t) e^{-2i\pi n t} \ dt \ w_\ell.
\end{aligned}$$

Therefore, we conclude, using the Cauchy–Schwarz inequality,

$$\begin{aligned}
|\vartheta_2^n|^2 &= k \sum_\ell \frac{1}{\lambda_\ell} \Big| \int_0^1 \ \frac{\sqrt{x_\ell}}{(\alpha + x_\ell \beta)(e^{-2i\pi t})} \hat{e}_\ell(t) e^{-2i\pi n t} \ dt \ \Big|^2 \\
&\leq k M_2 \sum_\ell \frac{1}{\lambda_\ell} \int_0^1 \ |\hat{e}_\ell(t)|^2 \ dt = k M_2 \ \int_0^1 \ \|\widehat{E}(t)\|_\star^2 \ dt,
\end{aligned}$$

and (2.9ii) follows. Using similar arguments we prove (2.8ii).

To complete the proof it remains to verify that K_1, K_2, M_1, M_2, N_1 and N_2 are finite. For N_2 we refer to [7]. Let us next consider the map k_1 which is continuous from the compact set $[0, +\infty] \times S_1$ into $\mathbb{C}$, except if $x = 0$ and ζ is a root of α. Therefore, in order to prove boundedness of K_1, it suffices to show that k_1 is bounded in a neighborhood of these points. From the Dahlquist $0-$stability condition, i.e., "$\alpha(0) = 1$ and the roots of modulus 1 of α are simple", we deduce that there exist r analytic functions $\zeta_1, \dots, \zeta_r$ from $[0, \eta]$ into $\mathbb{C}$, such that $\zeta_j(x)$ are roots of $\alpha + x\beta$, and $\zeta_j = \zeta_j(0)$, $j = 1, \dots, r$, are the unimodular roots of α. Then, we can write

$$k_1(x, \zeta) = \sum_{j=1}^r \frac{x a_j(x)}{\zeta - \zeta_j(x)} + b(x, \zeta),$$

where the functions a_j as well as the coefficients of the rational function $b(x, \cdot)$ are analytic on $[0, \eta]$. We observe that, for $\zeta \in S_1$,

$$\frac{|\zeta - \zeta_j(x)|}{x} \geq \frac{1 - |\zeta_j(x)|}{x} \to - \operatorname{Re} \frac{\zeta_j'(0)}{\zeta_j(0)} \qquad (\text{as } x \to 0).$$

The strong $A(0)-$stability means that, for all $x \in (0, \infty]$, the modulus of all roots of $\alpha + x\beta$ is less than one, and the "growth factors" $\operatorname{Re} \frac{\zeta_j'(0)}{\zeta_j(0)}$ of the principal roots ζ_j, $j = 1, \dots, r$, of α satisfy $\operatorname{Re} \frac{\zeta_j'(0)}{\zeta_j(0)} < 0$. Therefore, K_1 is bounded. Similarly, we can show that M_1 is finite. For K_2, we note that, in view of Minkowski's inequality, it suffices to verify that, for $x \in [0, \eta]$ and $j = 1, \dots, r$,

$$A_j = \int_0^1 \ \frac{x |a_j(x)|^2}{|e^{-2i\pi t} - \zeta_j(x)|^2} \ dt = \frac{x |a_j(x)|^2}{1 - |\zeta_j(x)|^2}$$

is bounded; this follows from the proof for K_1. In a similar way, one can see that M_2 and N_1 are finite as well. $\square$

In our main result, Theorem 2.1, we will need to estimate ϑ^n. Part of it, namely $\vartheta_2^n + \vartheta_3^n$, can be estimated in terms of $\vartheta^0, \dots, \vartheta^{q-1}$ and the consistency errors $E^0, \dots, E^{N-q}$.

Lemma 2.2. *There exists a constant C such that, for $n = 0, \dots, N$,*

$$|\vartheta^n - \vartheta_1^n|^2 + k\sum_{\ell=0}^{n} \|\vartheta^\ell - \vartheta_1^\ell\|^2 \le C\Big\{\sum_{j=0}^{q-1}(|\vartheta^j|^2 + k\|\vartheta^j\|^2) + k\sum_{\ell=0}^{n-q}\|E^\ell\|_\star^2\Big\}. \tag{2.11}$$

Proof. Since $\vartheta_2^j = 0$ for $j = 0, \dots, q-1$, we have

$$\vartheta_3^j = \vartheta^j - k\sum_{\ell=0}^{j-1} f(j-\ell, kA_h)b^\ell, \quad j = 0, \dots, q-1.$$

Therefore

$$|\vartheta_3^j| \le |\vartheta^j| + \sqrt{k}\sum_{\ell=0}^{j-1} m_{j-\ell}\|b^\ell\|_\star, \qquad \text{and} \qquad \|\vartheta_3^j\| \le \|\vartheta^j\| + \sum_{\ell=0}^{j-1} n_{j-\ell}\|b^\ell\|_\star,$$

with

$$m_\ell = \sup_{x>0}|\sqrt{x}f(\ell,x)|, \qquad \text{and} \qquad n_\ell = \sup_{x>0}|xf(\ell,x)|.$$

Then (2.11) follows from the relation $\vartheta^n - \vartheta_1^n = \vartheta_2^n + \vartheta_3^n$, and from (2.9) and (2.10). $\square$

The main result in this paper is given in the following theorem:

Theorem 2.1. *Let k and $h^{2r}k^{-1}$ be sufficiently small. Then, we have the local stability estimate*

$$|\vartheta^n|^2 + k\sum_{\ell=0}^{n}\|\vartheta^\ell\|^2 \le Ce^{c\mu^2 t^n}\Big\{\sum_{j=0}^{q-1}\big(|\vartheta^j|^2 + k\|\vartheta^j\|^2\big) + k\sum_{\ell=0}^{n-q}\|E^\ell\|_\star^2\Big\}, \tag{2.12}$$

$n = q-1, \dots, N$, and the error estimate

$$\max_{0\le n\le N}|u(t^n) - U^n| \le C(k^p + h^r). \tag{2.13}$$

Proof. Let $\rho^n = u^n - W^n, n = 0, \dots, N$. Then, according to (1.7),

$$\max_{0\le n\le N}|\rho^n| \le Ch^r \tag{2.14}$$

and, for sufficiently small h,

$$\max_{0\le n\le N}\|\rho^n\| \le 1/2, \tag{2.15}$$

i.e., in particular, $W^n \in T_u, n = 0, \dots, N$. Now, assuming for the time being that (2.12) holds, using (2.5) and (2.4), we obtain

$$\max_{0\le n\le N}|\vartheta^n| \le C(k^p + h^r)\,, \tag{2.16}$$

and (2.13) follows immediately from (2.14) and (2.16). Thus, it remains to prove (2.12). According to (2.5) and (2.4), there exists a constant $C_\star$ such that the right-hand side of (2.12) can be estimated by $C_\star^2(k^p+h^r)^2$,

$$Ce^{c\mu^2T}\Big\{\sum_{j=0}^{q-1}(|\vartheta^j|^2+k\|\vartheta^j\|^2)+k\sum_{\ell=0}^{N-q}\|E^\ell\|_\star^2\Big\}\le C_\star^2(k^p+h^r)^2. \tag{2.17}$$

The estimate (2.12) is obviously valid for $n=q-1$. Assume that it holds for $q-1,\dots,n-1$, $q\le n\le N$. Then, according to (2.17) and the induction hypothesis, we have, for k and $h^{2r}k^{-1}$ small enough,

$$\max_{0\le j\le n-1}\|\vartheta^j\|\le C_\star(k^{p-1/2}+k^{-1/2}h^r)\le 1/2,$$

i.e., using also (2.15),

$$U^j\in T_u,\quad j=0,\dots,n-1. \tag{2.18}$$

Therefore, in view of (1.4) and Minkowski's inequality,

$$\Big(k\sum_{\ell=0}^{n-1}\|b^\ell\|_\star^2\Big)^{1/2}\le\Big(k\sum_{\ell=0}^{n-1}(\lambda\|\vartheta^\ell\|+\mu|\vartheta^\ell|)^2\Big)^{1/2}\le\lambda\ a_{n-1}+\mu\ d_{n-1}+e_{n-1}$$

with

$$a_n=\Big(k\sum_{\ell=0}^{n}\|\vartheta_1^\ell\|^2\Big)^{1/2},\ d_n=\Big(k\sum_{\ell=0}^{n}|\vartheta_1^\ell|^2\Big)^{1/2},$$
$$\text{and }\ e_n=\Big(k\sum_{\ell=0}^{n}(\lambda\|\vartheta^\ell-\vartheta_1^\ell\|+\mu|\vartheta^\ell-\vartheta_1^\ell|)^2\Big)^{1/2}.$$

Thus, (2.8i) and (2.8ii) yield, for $n\ge 1$,

$$a_n\le K_1(\lambda\ a_{n-1}+\mu\ d_{n-1}+e_{n-1})\le K_1(\lambda\ a_n+\mu\ d_{n-1}+e_{n-1}), \tag{2.19}$$

and

$$\frac{d_n^2-d_{n-1}^2}{k}\le K_2(\lambda\ a_n+\mu\ d_{n-1}+e_{n-1})^2;$$

therefore, in view of (1.5), we have $\lambda K_1<1$ and

$$\frac{d_n^2-d_{n-1}^2}{k}\le K_2\big(\frac{\mu d_{n-1}+e_{n-1}}{1-\lambda K_1}\big)^2\le 2c(\mu^2d_{n-1}^2+e_{n-1}^2),$$

with $c=\frac{K_2}{(1-\lambda K_1)^2}$. Hence, we deduce (note that $d_0=0$)

$$d_n^2\le 2ck\sum_{\ell=0}^{n-1}e^{2c\mu^2(t^{n-1}-t^\ell)}e_\ell^2\le 2ck\ \frac{e^{2c\mu^2t^n}-1}{e^{2c\mu^2k}-1}e_{n-1}^2\le\ \frac{e^{2c\mu^2t^n}-1}{\mu^2}e_{n-1}^2.$$

Thus, we have $\mu d_n\le e^{c\mu^2t^n}e_{n-1}$ and

$$a_n\le\frac{K_1}{1-K_1\lambda}(1+e^{c\mu^2t^n})e_{n-1}, \tag{2.20i}$$

and

$$|\vartheta_1^n| \le \sqrt{c}\ (\mu d_{n-1} + e_{n-1}) \le \sqrt{c}\ (1 + e^{c\mu^2 t^n}) e_{n-1}. \tag{2.20ii}$$

Now, (2.20) and (2.11) yield

$$\begin{aligned} &|\vartheta_1^n|^2 + k \sum_{\ell=0}^{n} \|\vartheta_1^\ell\|^2 \le \\ &\qquad C e^{c\mu^2 t^n} \Big\{ \sum_{j=0}^{q-1} \big(|\vartheta^j|^2 + k\|\vartheta^j\|^2\big) + k \sum_{\ell=0}^{n-q} \|E^\ell\|_\star^2 \Big\}. \end{aligned} \tag{2.21}$$

From (2.21) and (2.11) it easily follows that (2.12) holds for n as well, and the proof is complete. □

Remark 2.1. Let $\tau \in \mathbb{R}$ be such that $A + \tau I$ is positive semidefinite. It is then easily seen that the results of Theorem 2.1 hold also for the scheme

$$\sum_{i=0}^{q} \alpha_i U^{n+i} + k \sum_{i=0}^{q} \beta_i (A_h U^{n+i} + \tau U^{n+i}) = k \sum_{i=0}^{q-1} \gamma_i [B_h(t^{n+i}, U^{n+i}) + \tau U^{n+i}].$$

Remark 2.2. The weak meshcondition "$k^{-1}h^{2r}$ small" is used in the proof of Theorem 2.1 only to show that $\|\vartheta^n\| \le 1/2$ which implies (2.18). If the estimate (1.4) holds in tubes around u defined in terms of weaker norms, not necessarily the same for both arguments v and w, one may get by with an even weaker meshcondition. Assume, for instance, that (1.4) holds for $v, w \in T_u^\star := \{\omega \in V : \min_t \|u(t) - \omega\|^\star \le 1\}$ —or for $v \in T_u$, cf. (2.15), and $w \in T_u^\star$— and the norm $\|\cdot\|^\star$ satisfies an inequality of the form

$$\|v\|^\star \le |v| + |v|^{1-a}\|v\|^a, \quad v \in V,$$

for some constant $a, 0 \le a < 1$. Then, a condition of the form "k and $k^{-a}h^{2r}$ sufficiently small" suffices for (2.12) and (2.13) to hold.

Similarly, when the relation (1.4) is satisfied in tubes around u defined in terms of stronger norms, not necessarily the same for both arguments, the convergence result of Theorem 2.1 may still be valid but under *stronger* meshconditions, cf. [1]; this fact will be used in the next section.

Remark 2.3. The condition (1.5) is sharp. Indeed, assume that $\lambda K_1 > 1$. Since $\lim_{|\zeta|\to\infty} x\gamma(\zeta)/[\alpha(\zeta) + x\beta(\zeta)] = 0$, we can find $x > 0$ and $\zeta \in \mathbb{C}$ with $|\zeta| > 1$ satisfying

$$\Big|\frac{\lambda x \gamma(\zeta)}{\alpha(\zeta) + x\beta(\zeta)}\Big| = 1;$$

thus, there exists a $\Theta \in \mathbb{R}$ such that

$$\alpha(\zeta) + x\big(\beta(\zeta) - \lambda e^{i\Theta}\gamma(\zeta)\big) = 0.$$

Choosing then $B(t, u) = \lambda e^{i\Theta} A u$, condition (1.4) is satisfied. According to the von Neumann criterion, a necessary stability condition is that, if ν is an eigenvalue of A, the solutions of

$$\sum_{i=0}^{q}[\alpha_i + k\nu(\beta_i - \lambda e^{i\Theta}\gamma_i)]v^{n+i} = 0,$$

are bounded; for $k\nu = x$ this is not the case, since the root condition is not satisfied; therefore, the scheme is not unconditionally stable.

Remark 2.4. The (α, β, γ) methods given by the polynomials

$$\alpha(\zeta) = \sum_{j=1}^{q} \frac{1}{j}\zeta^{q-j}(\zeta - 1)^j, \quad \beta(\zeta) = \zeta^q, \quad \text{and} \quad \gamma(\zeta) = \zeta^q - (\zeta - 1)^q$$

satisfy our assumptions with the order $p = q$. The corresponding implicit (α, β) schemes are the well-known B.D.F. methods which are strongly $A(0)-$stable for $1 \le q \le 6$. In this case, $K_1 = 2^q - 1$. First, clearly,

$$2^q - 1 = \lim_{x\to\infty} |k_1(x, -1)| \le K_1.$$

Further, with $d(\zeta) := \sum_{j=1}^{q} \frac{1}{j}(1 - \zeta^{-1})^j$,

$$k_1(x, \zeta) = \frac{1 - (1 - \zeta^{-1})^q}{1 + d(\zeta)/x}.$$

Then, for $\zeta \in S_1$ such that $\operatorname{Re} d(\zeta) \ge 0$,

$$|k_1(x, \zeta)| \le |1 - (1 - \zeta^{-1})^q| \le 2^q - 1.$$

Thus, $K_1 \le 2^q - 1$, for $q = 1$ and 2, since $\operatorname{Re} d(\zeta)$ is nonnegative in this case. For $\operatorname{Re} d(\zeta) < 0$,

$$\sup_{x>0} |k_1(x, \zeta)| = \frac{|d(\zeta)|}{|\operatorname{Im} d(\zeta)|}\, |1 - (1 - \zeta^{-1})^q|,$$

and, for $q = 3, 4, 5, 6$, we have computationally checked that the right-hand side is bounded by $2^q - 5$. Thus $K_1 \le 2^q - 1$. Consequently, in this case condition (1.5) reads $\lambda < \frac{1}{2^q-1}$.

Remark 2.5. Assume we discretize problem (1.1) by an implicit $A(\Theta)-$stable (α, β) scheme, which corresponds to taking $\gamma = \beta$ in our framework. Then, it easily follows from our analysis that the resulting scheme is stable and our estimates hold, provided that $\lambda < 1 - \cos\Theta$.

3. Application to a quasilinear equation

In this section we shall apply our results to a class of quasilinear equations: Let $\Omega \subset \mathbb{R}^\nu$, $\nu \le 3$, be a bounded domain with smooth boundary $\partial\Omega$. For $T > 0$ we seek a real-valued function u, defined on $\bar{\Omega} \times [0, T]$, satisfying

$$\begin{aligned} &u_t - \operatorname{div}(a(x)\nabla u) = \operatorname{div}(b(x,t,u)\nabla u + g(x,t,u)) + f(x,t,u) && \text{in } \Omega \times [0,T], \\ &u = 0 && \text{on } \partial\Omega \times [0,T], \\ &u(\cdot, 0) = u^0 && \text{in } \Omega, \end{aligned} \tag{3.1}$$

with $a : \bar{\Omega} \to (0,\infty)$, $b, f : \bar{\Omega}\times[0,T]\times\mathbb{R} \to \mathbb{R}$, $g : \bar{\Omega}\times[0,T]\times\mathbb{R} \to \mathbb{R}^\nu$, and $u^0 : \bar{\Omega} \to \mathbb{R}$ given smooth functions. We are interested in approximating smooth solutions of this problem, and assume therefore that the data are smooth and compatible such that (3.1) gives rise to a sufficiently regular solution. We assume that $-\mathrm{div}([a(x) + b(x,t,u)]\nabla\cdot)$ is an elliptic operator.

Let $H^s = H^s(\Omega)$ be the usual Sobolev space of order s, and $\|\cdot\|_{H^s}$ be the norm of H^s. The inner product in $H := L^2(\Omega)$ is denoted by $(\cdot,\cdot)$, and the induced norm by $|\cdot|$; the norm of $L^s(\Omega)$, $1 \le s \le \infty$, is denoted by $\|\cdot\|_{L^s}$. Let $Av := -\mathrm{div}(a\nabla v)$ and $B(t,v) := \mathrm{div}(b(\cdot,t,v)\nabla v) + \mathrm{div} g(\cdot,t,v) + f(\cdot,t,v)$. Obviously, $V = H_0^1 = H_0^1(\Omega)$ and the norm $\|\cdot\|$ in V, $\|v\| = |\sqrt{a}\nabla v|$, is equivalent to the H^1-norm.

Let

$$\widetilde{T}_u := \{v \in V \cap L^\infty : \min_t \|u(t) - v\|_{L^\infty} \le 1\},$$

$$\widehat{T}_u := \{v \in V \cap W^1_\infty : \min_t \|u(t) - v\|_{W^1_\infty} \le 1\},$$

and

$$\lambda := \sup\{|b(x,t,y)|/a(x) : x \in \Omega, t \in [0,T], y \in \mathcal{U}\}$$

with $\mathcal{U} := [-1 + \min_{x,t} u, 1 + \max_{x,t} u]$.

Now, for $v, w, \varphi \in V$,

$$\begin{aligned}(B(t,v) - B(t,w), \varphi) = &-(b(\cdot,t,w)\nabla(v-w), \nabla\varphi) - ([b(\cdot,t,v) - b(\cdot,t,w)]\nabla v, \nabla\varphi)\\ &-(g(\cdot,t,v) - g(\cdot,t,w), \nabla\varphi) + (f(\cdot,t,v) - f(\cdot,t,w), \varphi)\,,\end{aligned}$$

and we easily see that

$$\|B(t,v) - B(t,w)\|_\star \le \lambda\|v - w\| + \mu|v - w| \qquad v \in \widehat{T}_u, w \in \widetilde{T}_u. \tag{3.2}$$

Thus, a stability condition of the form (1.4) is satisfied for $v \in \widehat{T}_u$ and $w \in \widetilde{T}_u$.

Further,

$$\begin{aligned}B'(t,v)w = &\,\mathrm{div}(b(\cdot,t,v)\nabla w) + \mathrm{div}(\partial_3 b(\cdot,t,v)w\nabla v)\\ &+ \mathrm{div}(\partial_3 g(\cdot,t,v)w) + \partial_3 f(\cdot,t,v)w,\end{aligned}$$

and, therefore, $A - B'(t,u(t)) + \sigma I$ is, for an appropriate constant σ, uniformly positive definite in H_0^1.

Let V_h be the subspace of V defined on a regular finite element partition $\mathcal{T}_h$ of Ω, and consisting of piecewise polynomial functions of degree at most $r-1$, $r \ge 2$. Let h_K denote the diameter of an element $K \in \mathcal{T}_h$, and $h := \max_{K\in\mathcal{T}_h} h_K$. We define the elliptic projection operator $R_h(t)$, $R_h(t) : V \to V_h$, $t \in [0,T]$, by

$$\begin{aligned}&([a(\cdot) + b(\cdot,t,u(\cdot,t))]\nabla(v - R_h(t)v), \nabla\chi)\\ &\quad + ([\partial_3 b(\cdot,t,u(\cdot,t))]\nabla u(\cdot,t) + \partial_3 g(\cdot,t,u(\cdot,t))](v - R_h(t)v), \nabla\chi)\\ &\quad - ([\partial_3 f(\cdot,t,u(\cdot,t)) - \sigma](v - R_h(t)v), \chi) = 0 \quad \forall\chi \in V_h.\end{aligned}$$

It is well known from the error analysis for elliptic problems that

$$|v - R_h(t)v| + h\|v - R_h(t)v\| \le Ch^r\|v\|_{H^r}, \quad v \in H^r \cap H_0^1, \tag{3.3}$$

i.e., the estimate (1.7) is satisfied with $d = 2$. Further,

$$|\frac{d}{dt}[u(\cdot,t) - R_h(t)u(\cdot,t)]| \le Ch^r, \tag{3.4}$$

and

$$|\frac{d^j}{dt^j}R_h(t)v| + h\|\frac{d^j}{dt^j}R_h(t)v\| \le Ch^r\|v\|_{H^r}, \quad v \in H^r \cap H_0^1, \quad j = 1, \dots, p+1, \tag{3.5}$$

cf., e.g., [4]; thus (1.8) and (1.9) are valid. We further assume, cf. [12], [15], that

$$\sup_t \|u(\cdot,t) - R_h(t)u(\cdot,t)\|_{W^1_\infty} \le \frac{1}{2}. \tag{3.6}$$

Next, we will verify (1.10). We have

$$\begin{aligned} &B(t,u(t)) - B(t,R_h(t)u(t)) - B'(t,u(t))(R_h(t)u(t) - u(t)) = \\ &\qquad = -\int_0^1 \tau B''\big(t, R_h(t)u(t) - \tau[R_h(t)u(t) - u(t)]\big)d\tau[R_h(t)u(t) - u(t)]^2 \end{aligned} \tag{3.7i}$$

and

$$\begin{aligned} B''(t,v)w^2 &= \mathrm{div}(\partial_3^2 b(\cdot,t,v)w^2\nabla v) + 2\mathrm{div}(\partial_3 b(\cdot,t,v)w\nabla w) \\ &\quad + \mathrm{div}(\partial_3^2 g(\cdot,t,v)w^2) + \partial_3^2 f(\cdot,t,v)w^2. \end{aligned} \tag{3.7ii}$$

It easily follows from (3.7) and (3.3), in view of (3.6), that

$$\|B(t,u(t)) - B(t,R_h(t)u(t)) - B'(t,u(t))(u(t) - R_h(t)u(t))\|_{H^{-1}} \le Ch^r, \tag{3.8}$$

i.e., (1.10) is satisfied.

Now, let $W(t) := R_h(t)u(t)$, and assume that we are given approximations $U^0, \dots, U^{q-1} \in V_h$ to $u^0, \dots, u^{q-1}$ such that

$$\sum_{j=0}^{q-1} \big(|W^j - U^j| + k^{1/2}\|W^j - U^j\|\big) \le c(k^p + h^r). \tag{3.9}$$

Then, we define $U^n \in V_h$, $n = q, \dots, N$, recursively by the (α,β,γ) scheme

$$\begin{aligned} &\sum_{i=0}^q \alpha_i(U^{n+i},\chi) + k\sum_{i=0}^q \beta_i(a(\cdot)\nabla U^{n+i},\nabla\chi) = \\ &\qquad = k\sum_{i=0}^{q-1}\gamma_i\{-(b(\cdot,t^{n+i},U^{n+i})\nabla U^{n+i} + g(\cdot,t^{n+i},U^{n+i}),\nabla\chi) \\ &\qquad\qquad + (f(\cdot,t^{n+i},U^{n+i}),\chi)\}, \quad \forall\chi \in V_h, \quad n = 0,\dots,N-q, \end{aligned} \tag{3.10}$$

with (α,β) and (α,γ) multistep schemes of order p, and (α,β) strongly $A(0)-$stable. Then, Theorem 2.1 yields, in view of (3.6), for sufficiently small k and provided that the approximate solutions U^n are in $\widetilde{T}_u$, the error estimate

$$\max_n |u^n - U^n| \le c(k^p + h^r). \tag{3.11}$$

To ensure that $U^n \in \widetilde{T}_u, n = 0, \dots, N$, we define $\underline{h} := \min_{K\in\mathcal{T}_h} h_K$ and will distinguish three cases: $\nu = 1$, $\nu = 2$ and $\nu = 3$.

i. $\nu = 1$. First, since the H^1-norm dominates the $L^\infty-$norm in one space dimension, we have

$$\max_{0\le j\le n+q-1} \|\vartheta^j\|_{L^\infty} \le C \max_{0\le j\le n+q-1} \|\vartheta^j\|,$$

and thus, according to (2.16),

$$\max_{0\le j\le n+q-1} \|\vartheta^j\|_{L^\infty} \le C(k^{p-1/2} + k^{-1/2}h^r).$$

Therefore, for k and $k^{-1}h^{2r}$ sufficiently small, in view of (3.6), $U^j \in \widetilde{T}_u, j = 0,\dots,n+q-1$. We easily conclude that the convergence result holds.

ii. $\nu = 2$. First, we note that

$$\|\chi\|_{L^\infty} \le C|\log(\underline{h})|^{1/2}\|\chi\|_{H^1} \qquad \forall\chi \in V_h,$$

cf. [14, p. 68]. It is then easily seen that the convergence result holds, if k and h are chosen such that $|\log(\underline{h})|k^{2p-1}$ and $|\log(\underline{h})|k^{-1}h^{2r}$ are sufficiently small.

iii. $\nu = 3$. In this case,

$$\|\chi\|_{L^\infty} \le C\underline{h}^{-1/2}\|\chi\|_{H^1} \qquad \forall\chi \in V_h,$$

and the result (3.11) holds, provided that $\underline{h}^{-1}k^{2p-1}$ and $k^{-1}\underline{h}^{-1}h^{2r}$ are sufficiently small.

Remark 3.1. Let the quasilinear equation be given in the form

$$u_t = \operatorname{div}(c(x,t,u)\nabla u + g(x,t,u)) + f(x,t,u).$$

It can then be written in the form used in (3.1) by letting, say, $a(x) := c(x,0,u^0)$ and $b(x,t,u) := c(x,t,u) - a(x)$.

Different splittings might be used on a finite number of subintervals of $[0,T]$. Assume, for instance, that an approximation U to $u(\cdot,t_a)$ has been computed. Then, the splitting $a(x) := c(x,t_a,U)$ and $b(x,t,u) := c(x,t,u)-a(x)$ may be used on a time interval $[t_a,t_b]$.

Remark 3.2. As mentioned in the introduction, the stability assumption (1.4) is weaker than (1.4′) which was used in [1]. For smooth B, (1.4) implies

$$|(B'(v)w,\omega)| \le \lambda\|w\|\|\omega\| + \mu(v)|w|\,\|\omega\| \qquad \forall v,w,\omega \in V. \tag{1.4$''$}$$

The use of (1.4″) may lead to improvements in the analysis of the applications in [1, Section 4]. In particular, the convergence results of [1, Section 4.2] for the Cahn–Hilliard equation in one space dimension will now hold without any meshconditions. Also, in [1, Section 4.3] a reaction diffusion equation with power nonlinearities that grow no faster than $|\xi|^\rho, \rho \le 4$, in $\mathbb{R}^3$ was considered. A more refined analysis shows that the stability hypothesis (1.4″) is now satisfied for $\rho < 5$ in $\mathbb{R}^3$.

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