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Implicit Functions and Solution Mappings

A View from Variational Analysis

With 12 Illustrations

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Preface

Setting up equations and solving them has long been so important that, in popular imagination, it has virtually come to describe what mathematical analysis and its applications are all about. A central issue in the subject is whether the solution to an equation involving parameters may be viewed as a function of those parameters, and if so, what properties that function might have. This is addressed by the classical theory of implicit functions, which began with single real variables and progressed through multiple variables to equations in infinite dimensions, such as equations associated with integral and differential operators.

A major aim of the book is to lay out that celebrated theory in a broader way than usual, bringing to light many of its lesser known variants, for instance where standard assumptions of differentiability are relaxed. However, another major aim is to explain how the same constellation of ideas, when articulated in a suitably expanded framework, can deal successfully with many other problems than just solving equations.

These days, forms of modeling have evolved beyond equations, in terms, for example, of problems of minimizing or maximizing functions subject to constraints which may include systems of inequalities. The question comes up of whether the solution to such a problem may be expressed as a function of the problem's parameters, but differentiability no longer reigns. A function implicitly obtainable this manner may only have one-sided derivatives of some sort, or merely exhibit Lipschitz continuity or something weaker. Mathematical models resting on equations are replaced by "variational inequality" models, which are further subsumed by "generalized equation" models.

The key concept for working at this level of generality, but with advantages even in the context of equations, is that of the set-valued *solution mapping* which assigns to each instance of the parameter element in the model *all* the corresponding solutions, if any. The central question is whether a solution mapping can be localized graphically in order to achieve single-valuedness and in that sense produce a function, the desired *implicit function*.

In modern variational analysis, set-valued mappings are an accepted workhorse in problem formulation and analysis, and many tools have been developed for handling them. There are helpful extensions of continuity, differentiability, and regularity of several types, together with powerful results about how they can be applied. A corresponding further aim of this book is to bring such ideas to wider attention by demonstrating their aptness for the fundamental topic at hand.

In line with classical themes, we concentrate primarily on local properties of solution mappings that can be captured metrically, rather than on results derived from topological considerations or involving exotic spaces. In particular, we only briefly discuss the Nash–Moser inverse function theorem. We keep to finite dimensions in Chapters 1 to 4, but in Chapters 5 and 6 provide bridges to infinite dimensions. Global implicit function theorems, including the classical Hadamard theorem, are not discussed in the book.

In Chapter 1 we consider the implicit function paradigm in the classical case of the solution mapping associated with a parameterized equation. We give two proofs of the classical inverse function theorem and then derive two equivalent forms of it: the implicit function theorem and the correction function theorem. Then we gradually relax the differentiability assumption in various ways and even completely exit from it, relying instead on the Lipschitz continuity. We also discuss situations in which an implicit function fails to exist as a graphical localization of the solution mapping, but there nevertheless exists a function with desirable properties serving locally as a selection of the set-valued solution mapping. This chapter does not demand of the reader more than calculus and some linear algebra, and it could therefore be used by both teachers and students in analysis courses.

Motivated by optimization problems and models of competitive equilibrium, Chapter 2 moves into wider territory. The questions are essentially the same as in the first chapter, namely, when a solution mapping can be localized to a function with some continuity properties. But it is no longer an equation that is being solved. Instead it is a condition called a generalized equation which captures a more complicated dependence and covers, as a special case, variational inequality conditions formulated in terms of the set-valued normal cone mapping associated with a convex set. Although our prime focus here is variational models, the presentation is self-contained and again could be handled by students and others without special background. It provides an introduction to a subject of great applicability which is hardly known to the mathematical community familiar with classical implicit functions, perhaps because of inadequate accessibility.

In Chapter 3 we depart from insisting on localizations that yield implicit functions and approach solution mappings from the angle of a "varying set." We identify continuity properties which support the paradigm of the implicit function theorem in a set-valued sense. This chapter may be read independently from the first two. Chapter 4 continues to view solution mappings from this angle but investigates substitutes for classical differentiability. By utilizing concepts of generalized derivatives, we are able to get implicit mapping theorems that reach far beyond the classical scope.

Chapter 5 takes a different direction. It presents extensions of the Banach open mapping theorem which are shown to fit infinite-dimensionally into the paradigm of the theory developed finite-dimensionally in Chapter 3. Some background in basic functional analysis is required. Chapter 6 goes further down that road and illustrates how some of the implicit function/mapping theorems from earlier in the book can be used in the study of problems in numerical analysis.

This book is targeted at a broad audience of researchers, teachers and graduate students, along with practitioners in mathematical sciences, engineering, economics and beyond. In summary, it concerns one of the chief topics in all of analysis, historically and now, an aid not only in theoretical developments but also in methods for solving specific problems. It crosses through several disciplines such as real and functional analysis, variational analysis, optimization, and numerical analysis, and can be used in part as a graduate text as well as a reference. It starts with elementary results and with each chapter, step by step, opens wider horizons by increasing the complexity of the problems and concepts that generate implicit function phenomena.

Many exercises are included, most of them supplied with detailed guides. These exercises complement and enrich the main results. The facts they encompass are sometimes invoked in the subsequent sections.

Each chapter ends with a short commentary which indicates sources in the literature for the results presented (but is not a survey of all the related literature). The commentaries to some of the chapters additionally provide historical overviews of past developments.

Whidbey Island, Washington — *Asen L. Dontchev*
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The authors

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