# Implicit Partial Differential Equations

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# Preface

Nonlinear partial differential equations has become one of the main tools of modern mathematical analysis; in spite of seemingly contradictory terminology, the subject of nonlinear differential equations finds its origins in the theory of linear differential equations, and a large part of functional analysis derived its inspiration from the study of linear pdes.

In recent years, several mathematicians have investigated nonlinear equations, particularly those of the second order, both linear and nonlinear and either in divergence or nondivergence form. Quasilinear and fully nonlinear differential equations are relevant classes of such equations and have been widely examined in the mathematical literature.

In this work we present a new family of differential equations called "*implicit partial differential equations*", described in detail in the introduction (c.f. Chapter 1). It is a class of nonlinear equations that does not include the family of fully nonlinear elliptic pdes. We present a new functional analytic method based on the Baire category theorem for handling the existence of almost everywhere solutions of these implicit equations. The results have been obtained for the most part in recent years and have important applications to the calculus of variations, nonlinear elasticity, problems of phase transitions and optimal design; some results have not been published elsewhere.

The book is essentially self-contained, and includes some background material on viscosity solutions, different notions of convexity involved in the vectorial calculus of variations, singular values, Vitali type covering theorems, and the approximation of Sobolev functions by piecewise affine functions. Also, a comparison is made with other methods — notably the method of viscosity solutions and briefly that of convex integration. Many mathematical examples stemming from applications to the material sciences are thoroughly discussed.

The book is divided into four parts. In Part 1 we consider the *scalar* case for first (Chapter 2) and second (Chapter 3) order equations. We also compare (Chapter 4) our approach for obtaining existence results with the celebrated viscosity method. While most of our existence results obtained in this part of the book are consequences of *vectorial* results considered in the second part, we have avoided (except for very briefly in Section 3.3) vectorial machinery in order to make the material more readable.

In Part 2 we first (Chapter 5) recall basic results on generalized notions of convexity, such as quasiconvexity, as well as on some important lower semicontinuity theorems of the calculus of variations. Central existence results of Part 2 are in Chapter 6, where $N$th order vectorial problems are discussed.

In Part 3 we study in great detail applications of vectorial existence results to important problems originating, for example, from geometry or from the material sciences. These applications concern singular values, potential wells and the complex eikonal equation.

Finally, in Part 4 we gather some nonclassical Vitali type covering theorems, as well as several fine results on the approximation of Sobolev functions by piecewise affine or polynomial functions. These last results may be relevant in other contexts, such as numerical analysis.

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We wish to recall here that recently we dedicated an article, on the same subject of this book, respectively to the memory of Ennio De Giorgi and to Stefan Hildebrandt on his 60th birthday.

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# 1
# Introduction

## 1.1 The first order case

### *1.1.1 Statement of the problem*

One of the main purposes of this book is to study the Dirichlet problem

$$\begin{cases} F_i(x, u(x), Du(x)) = 0, & \text{a.e. } x \in \Omega, \quad i = 1, \ldots, I \\ u(x) = \varphi(x), & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is an open set, $u : \Omega \to \mathbb{R}^m$ and therefore $Du \in \mathbb{R}^{m \times n}$ (if $m = 1$ we say that the problem is *scalar* and otherwise we say that it is *vectorial*), $F_i : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times n} \to \mathbb{R}$, $F_i = F_i(x, s, \xi)$, $i = 1, \ldots, I$, are given. The boundary condition $\varphi$ is prescribed (depending of the context it will be either continuously differentiable or only Lipschitz-continuous).

As is well known, it is not reasonable to expect the solutions to be $C^1(\Omega; \mathbb{R}^m)$ (even when $m = n = 1$). We will however investigate throughout this book the existence of $W^{1,\infty}(\Omega; \mathbb{R}^m)$ solutions of (1.1). The nature of the question excludes automatically from our investigation quasilinear problems (i.e., equations where the derivatives appear linearly) since as well known solutions of such problems cannot satisfy the Dirichlet boundary condition. The equations that we will consider in this monograph will therefore be called of *implicit type*, i.e., they exclude the quasilinear case. The approach we will discuss here is a functional analytic method based on the Baire category theorem and on weak lower semicontinuity of convex and quasiconvex integrals.

### *1.1.2 The scalar case*

We now discuss the case $m = 1$, i.e., there is only one unknown scalar function $u$. This case is much simpler than the vectorial one and has received much more attention. The prototype of first order implicit equations is the *eikonal equation* which is of importance in geometrical optics.

**Example 1.1 (Eikonal equation)** The problem is to find $u \in W^{1,\infty}(\Omega)$ satisfying

$$\begin{cases} |Du(x)| = a\left(x, u(x)\right), & \text{a.e. in } \Omega \\ u = \varphi, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega$ is an open set (bounded or unbounded) of $\mathbb{R}^n$, $a : \Omega \times \mathbb{R} \to \mathbb{R}$ is a bounded continuous function and the boundary datum $\varphi \in W^{1,\infty}(\Omega)$ satisfies the compatibility condition

$$|D\varphi(x)| \leq a\left(x, \varphi(x)\right), \quad \text{a.e. in } \Omega.$$

A natural generalization of this example leads to the following result (c.f. Theorem **??**, for a more general version).

**Theorem 1.2** *Let $\Omega$ be an open set of $\mathbb{R}^n$. Let $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function, convex with respect to the last variable and coercive (i.e., $\lim F(x, u, \xi) = +\infty$, if $|\xi| \to \infty$ uniformly with respect to $(x, u)$). Let $\varphi \in W^{1,\infty}(\Omega)$ be a function satisfying*

$$F\left(x, \varphi(x), D\varphi(x)\right) \leq 0, \quad \text{a.e. in } \Omega. \tag{1.2}$$

*For every $\varepsilon > 0$ there exists $u \in W^{1,\infty}(\Omega)$ such that $\|u - \varphi\|_{L^\infty} \leq \varepsilon$ and*

$$\begin{cases} F(x, u(x), Du(x)) = 0, & \text{a.e. in } \Omega \\ u = \varphi, & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Apart from the conclusion *on the density* and from the fact that no other hypothesis than continuity on the behavior of the function $F$ with respect to the variable $u$ is made, this theorem is well known (and also much more precise, because some explicit formulas for a solution are known in special cases) since the pioneering work of Hopf [188], Lax [211], Kruzkov [208] (see also Benton [39], Crandall-Lions [96] and for a thorough treatment Lions [218]). We will come back to it below when we will briefly speak of *viscosity solutions* of (1.3). Theorem 1.2 is a consequence of the general results obtained in this book (c.f. also [111] and De Blasi-Pianigiani [127]).

We will also be able to treat a generalization of the eikonal equation, which we call the *eikonal system* of the following type.

**Example 1.3 (Eikonal system)** We look for solutions $u \in W^{1,\infty}(\Omega)$ of the following problem

$$\begin{cases} \left|\frac{\partial u}{\partial x_i}\right| = a_i(x, u(x)), & i = 1, \ldots, n, \quad \text{a.e. in } \Omega \\ u = \varphi, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega$ is an open set of $\mathbb{R}^n$, $a_i : \Omega \times \mathbb{R} \to \mathbb{R}$, with $a_i \geq a_0 > 0$, $i = 1, \ldots, n$, are bounded continuous functions and the boundary datum $\varphi \in W^{1,\infty}(\Omega)$ satisfies the compatibility conditions

$$\left|\frac{\partial \varphi}{\partial x_i}\right| < a_i(x, \varphi(x)), \quad i = 1, \ldots, n, \quad \text{a.e. in } \Omega.$$

The example can be considered either as a nonconvex version of (1.3) by setting, for instance,

$$F(x, s, \xi) = -\sum_{i=1}^{n} ||\xi_i| - a_i(x, s)|$$

or as a system of convex functions (in the gradient variable) of the implicit type (1.1), with $F_i(x, s, \xi) = |\xi_i| - a_i(x, s)$.

In the nonconvex case our approach will lead to the following theorem (c.f. Theorem **??**), which is in optimal form when the Hamiltonian $F$ is independent of the lower order terms $(x, u)$. Setting

$$E = \{\xi \in \mathbb{R}^n : F(\xi) = 0\} \tag{1.4}$$

the problem is then transformed into a differential inclusion.

**Theorem 1.4** *Let $\Omega \subset \mathbb{R}^n$ be open and $E \subset \mathbb{R}^n$. Let $\varphi \in W^{1,\infty}(\Omega)$ satisfy*

$$D\varphi(x) \in E \cup \operatorname{int}\operatorname{co} E, \quad a.e.\ x \in \Omega; \tag{1.5}$$

*then there exists (a dense set of) $u \in W^{1,\infty}(\Omega)$ such that*

$$\begin{cases} Du(x) \in E, & a.e.\ x \in \Omega \\ u(x) = \varphi(x), & x \in \partial\Omega. \end{cases} \tag{1.6}$$

**Remark 1.5** (i) The interior of the convex hull of $E$ is denoted by $\operatorname{int}\operatorname{co} E$. Observe also that (when compared with Section 1.1.4 for the vectorial case)

$$\operatorname{int}\operatorname{co} E = \operatorname{int}\overline{\operatorname{co} E}.$$

(ii) The density, in the $L^\infty$-norm, is to be understood in the sense of the Baire category theorem.

(iii) Note that when $F$ is convex and coercive and $E$ is given by (1.4) then

$$E \cup \operatorname{int} \operatorname{co} E = \{\xi \in \mathbb{R}^n : \ F(\xi) \leq 0\}.$$

Similarly, in Example 1.3,

$$\operatorname{int} \operatorname{co} E = \{\xi \in \mathbb{R}^n : |\xi_i| < a_i, \quad i = 1, \ldots, n\}.$$

Observe also that if $F$ is linear, then $\operatorname{int} \operatorname{co} E = \emptyset$ and thus, as already mentioned, our analysis excludes linear (and quasilinear) equations.

This theorem has been established by many authors depending on further assumptions on the boundary datum. When $\varphi$ is linear an explicit construction can be made, which by analogy with the case $n = 2$ we call a *pyramid,* c.f. Cellina [78], Friesecke [162]. When $\varphi$ is nonlinear we refer to [108], [110] and to Bressan-Flores [55] and De Blasi-Pianigiani [127]. Theorem 1.4 extends to the case with explicit dependence on $(x, u)$, c.f. Theorem **??**.

Returning to Theorem 1.4 one should observe that the compatibility condition (1.5) is also necessary in the sense described below (c.f. Section 2.4). First note that if $\varphi$ is linear, i.e.,

$$\varphi(x) = \langle \xi_0; x \rangle + q$$

for some $\xi_0 \in \mathbb{R}^n$ and $q \in \mathbb{R}$, then necessarily any solution of $Du(x) \in E \subset \operatorname{co} E$ verifies, by the Jensen inequality,

$$\xi_0 = \frac{1}{\operatorname{meas} \Omega} \int_\Omega Du(x)\, dx \in \overline{\operatorname{co} E}.$$

Thus a necessary condition for the solvability of problem (1.6) is

$$D\varphi \in \overline{\operatorname{co} E}. \tag{1.7}$$

Moreover, in Section 2.4, we show in an example that in general the condition

$$\xi_0 = D\varphi(x) \in E \cup \operatorname{int} \operatorname{co} E, \quad \text{a.e. } x \in \Omega$$

cannot be replaced by (1.7). The necessary condition (1.7) can also be considered when $\varphi$ is nonlinear and we refer to the discussion of Section **??** for more details.

### *1.1.3 Some examples in the vectorial case*

When we turn to the *vectorial case* the problem becomes more delicate because the classical notion of convexity is too strong and has to be replaced by weaker notions such as *quasiconvexity* and *rank one convexity.* Before entering into some details about the extension of Theorem 1.4 to the vectorial case, we point out some examples that will be treated in this book.

The first example (c.f. Chapter 7) that we will consider is the problem of *singular values*. It is of importance in *nonlinear elasticity* and in *optimal design* (c.f. [107]).

We recall that, given a matrix $\xi \in \mathbb{R}^{n\times n}$, we denote by $0 \le \lambda_1(\xi) \le \lambda_2(\xi) \le \cdots \le \lambda_n(\xi)$ its singular values; these are the eigenvalues of the symmetric matrix $\left(\xi^t \xi\right)^{1/2}$. They satisfy

$$|\xi|^2 = \sum_{i,j=1}^{n} \xi_{ij}^2 = \sum_{i=1}^{n} (\lambda_i(\xi))^2 ,$$

$$|adj_s \xi|^2 = \sum_{i_1 < \cdots < i_s} (\lambda_{i_1}(\xi))^2 \cdot \ldots \cdot (\lambda_{i_s}(\xi))^2 ,$$

$$|\det \xi| = \prod_{i=1}^{n} \lambda_i(\xi) ,$$

where $adj_s \xi \in \mathbb{R}^{\binom{n}{s}\times\binom{n}{s}}$ denotes the matrix obtained by forming all the $s \times s$ minors, $2 \le s \le n-1$, of the matrix $\xi$ (if $n = 3$, $adj_2 \xi \in \mathbb{R}^{3\times 3}$ is the usual adjugate matrix). In particular, if $n = 2$, then

$$|\xi|^2 = \sum_{i,j=1}^{2} \xi_{ij}^2 = (\lambda_1(\xi))^2 + (\lambda_2(\xi))^2 ,$$

$$|\det \xi| = \lambda_1(\xi)\, \lambda_2(\xi) .$$

We will then get the following existence theorem (c.f. Theorem **??**).

**Theorem 1.6** *Let $\Omega \subset \mathbb{R}^n$ be an open set, $a_i : \Omega\times\mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, n$, be bounded continuous functions satisfying $0 < c \le a_1(x,s) \le \ldots \le a_n(x,s)$ for some constant c and for every $(x,s) \in \Omega \times \mathbb{R}^n$. Let $\varphi \in C^1\left(\overline{\Omega};\mathbb{R}^n\right)$ (or piecewise $C^1$) satisfy*

$$\prod_{i=\nu}^{n} \lambda_i(D\varphi(x)) < \prod_{i=\nu}^{n} a_i(x,\varphi(x)) , \quad x \in \Omega, \quad \nu = 1, \ldots, n$$

*(in particular $\varphi \equiv 0$ satisfies the above condition); then there exists (a dense set of) $u \in W^{1,\infty}(\Omega;\mathbb{R}^n)$ such that*

$$\begin{cases} \lambda_i(Du(x)) = a_i(x,u(x)) , & a.e.\ x \in \Omega, \quad i = 1, \ldots, n \\ u(x) = \varphi(x) , & x \in \partial\Omega. \end{cases}$$

**Remark 1.7** The above theorem has been established in [108], [109], [110], [111] when $n = 2$ and, with the same proof, in [117] for the general case. When $n = 3$, $a_i \equiv 1$ and $\varphi \equiv 0$, this theorem can be found in Cellina-Perrotta [80]; see also Celada-Perrotta [74].

It is interesting to see some implications of the theorem when $n = 2$. The problem is then equivalent to

$$\begin{cases} |Du(x)|^2 = a_1^2 + a_2^2, & \text{a.e. } x \in \Omega \\ |\det Du(x)| = a_1 a_2, & \text{a.e. } x \in \Omega \\ u(x) = \varphi(x), & x \in \partial\Omega. \end{cases} \tag{1.8}$$

Therefore the system can be seen as a combination of the *vectorial eikonal equation* and of the *equation of prescribed absolute value of the Jacobian determinant*. The first equation is, as we just saw, at the origin of the study of nonlinear first order pdes and an important motivation for the introduction of the notion of *viscosity solutions*. The second equation says that the absolute value of the Jacobian determinant is given. This last equation, without the absolute value, has also been studied (c.f. Moser [247] and in many other articles since then and in particular Dacorogna-Moser [115]); it has important applications for example in *dynamical systems* and in *nonlinear elasticity*.

If we now consider in (1.8) the case $a_1 = a_2 = 1$ and if we set

$$u = u(x, y) = \left(u^1, u^2\right), \qquad Du = \begin{pmatrix} u_x^1 & u_y^1 \\ u_x^2 & u_y^2 \end{pmatrix},$$

we find that (1.8) implies that

$$\begin{cases} \left[\left(u_x^1 - u_y^2\right)^2 + \left(u_y^1 + u_x^2\right)^2\right]\left[\left(u_x^1 + u_y^2\right)^2 + \left(u_y^1 - u_x^2\right)^2\right] = 0, & \text{a.e. in } \Omega \\ \left(u^1, u^2\right) = \left(\varphi^1, \varphi^2\right), & \text{on } \partial\Omega. \end{cases}$$

The theorem then means that we can find, under appropriate compatibility conditions, a Lipschitz map $u$ that is either *conformal* or *anticonformal* (i.e., it satisfies either the *Cauchy-Riemann equation* or the *anti Cauchy-Riemann equation*) and on the boundary of the domain has both real and imaginary parts given. Of course if we have classical complex analysis in mind this result is quite surprising.

The second example that will be treated in detail in this book (c.f. Chapter 8) is the problem of two *potential wells* in two dimensions.

First let us introduce some notation. We let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $SO(n)$ (the set of *special orthogonal matrices*) denote the set of matrices $U \in \mathbb{R}^{n \times n}$ such that $U^t U = U U^t = I$ and $\det U = 1$.

Let us be given $N$ matrices $A_i \in \mathbb{R}^{n \times n}$. The problem of potential wells consists in finding $u \in \varphi + W_0^{1,\infty}(\Omega; \mathbb{R}^n)$ such that

$$\begin{cases} Du(x) \in E = \bigcup_{i=1}^{N} SO(n) A_i \\ u(x) = \varphi(x), \quad x \in \partial\Omega. \end{cases} \tag{1.9}$$

The $N$ *wells* are $SO(n) A_i$, $1 \leq i \leq N$.

Before going further we should note that in the case of singular values considered in the preceding example, if we take $a_i = 1$ for every $i = 1, \ldots, n$, then the problem is also of potential wells type, i.e., $N = 2$ and

$$E = SO(n)I \cup SO(n)I_{-}, \quad \text{where} \quad I_{-} = diag\,(-1, 1, 1, \ldots, 1)\,,$$

or, in other words $E = O(n)$ (the set of orthogonal matrices).

The general problem of potential wells has been intensively studied by many authors in conjunction with *crystallographic models* involving fine micro-structures. The reference papers on the subject are Ball-James [31], [32]; see also Bhattacharya-Firoozye-James-Kohn [42], De Simone-Dolzmann [131], Dolzmann-Müller [135], Ericksen [145], [146], Firoozye-Kohn [153], Fonseca-Tartar [158], Kinderlehrer-Pedregal [200], Kohn [204], Luskin [219], Müller-Sverak [249], Pipkin [263], Sverak [289].

The mathematical problem (1.9) is very difficult and the difficulty increases drastically with the dimension and/or the number of wells. One of the main difficulties is to characterize the *quasiconvex* (or the *rank one convex*) *hull* of the set $E$; c.f. below for the definition of these hulls. The case that is best understood is when $n = N = 2$, i.e., the case of two potential wells in two dimensions. For (1.9) we will prove an existence result of Lipschitz solutions under the appropriate compatibility condition on the boundary datum (c.f. Theorem 1.16 and Theorem **??**). The same result has also been obtained by Müller-Sverak in [249] using *convex integration* (c.f. Section 1.3.2 below).

The third example (c.f. Chapter 9) that we want to mention is the *complex eikonal* equation. The problem has recently been introduced by Magnanini-Talenti [221], motivated by the study of harmonic functions in 3 dimensions and by problems of *geometrical optics* with *diffraction*. The question under consideration is to find a complex function

$$w\,(x, y) = u\,(x, y) + iv\,(x, y)$$

such that

$$\begin{cases} w_x^2 + w_y^2 + f^2 = 0, & \text{a.e in } \Omega \\ w = \varphi, & \text{on } \partial\Omega, \end{cases}$$

where $f : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ $(f = f\,(x, y, u, v))$ is continuous and $\Omega \subset \mathbb{R}^2$ is an open set. This is therefore equivalent to the system

$$\begin{cases} |Dv|^2 = |Du|^2 + f^2, & \text{a.e in } \Omega \\ \langle Du; Dv \rangle = 0, & \text{a.e in } \Omega \\ (u, v) = (\varphi_1, \varphi_2)\,, & \text{on } \partial\Omega. \end{cases} \tag{1.10}$$

We will prove the following existence result (c.f. Theorem **??**)

**Theorem 1.8** *For every $\varphi \in W^{1,\infty}\left(\Omega; \mathbb{R}^2\right)$ there exists a (dense set of) function $w = (u, v) \in W^{1,\infty}\left(\Omega; \mathbb{R}^2\right)$ satisfying (1.10).*

### *1.1.4 Convexity conditions in the vectorial case*

In the context of vectorial problems we need to replace the notion of convexity by the concepts of quasiconvexity, rank one convexity or polyconvexity. We now introduce the first two notions and we refer to Chapter 5 for more details.

We say that a Borel measurable function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is *quasiconvex* if

$$f(A) \leq \frac{1}{\operatorname{meas}\Omega}\int_{\Omega} f(A + D\varphi(x))\,dx\,,$$

for every bounded domain $\Omega \subset \mathbb{R}^n$, every $A \in \mathbb{R}^{m\times n}$, and every $\varphi \in W_0^{1,\infty}(\Omega;\mathbb{R}^m)$. Moreover, a function $f : \mathbb{R}^{m\times n} \to \overline{\mathbb{R}} = \mathbb{R}\cup\{+\infty\}$ is said to be *rank one convex* if

$$f(tA + (1-t)B) \leq tf(A) + (1-t)f(B)\ ,$$

for every $t \in [0,1]$ and every $A, B \in \mathbb{R}^{m\times n}$ with $\operatorname{rank}\{A - B\} = 1$.

It is well known that

$$f \text{ convex} \Longrightarrow f \text{ quasiconvex} \Longrightarrow f \text{ rank one convex.}$$

Note also that when $m = 1$ (i.e., in the scalar case) the three notions are equivalent. The classical example of a quasiconvex (and also rank one convex) function that is not convex is (when $m = n$)

$$f(A) = \det A,\ A \in \mathbb{R}^{n\times n}.$$

Given a set $E \subset \mathbb{R}^{m\times n}$, the convex hull of $E$, denoted $\operatorname{co} E$, is classically defined as the smallest convex set that contains $E$. By analogy we define the *rank one convex hull* of $E$, denoted $\operatorname{Rco} E$, to be the smallest rank one convex set that contains $E$; more precisely

$$\operatorname{Rco} E = \left\{ \begin{array}{c} \xi \in \mathbb{R}^{m\times n} : f(\xi) \leq 0, \quad \forall f : \mathbb{R}^{m\times n} \to \overline{\mathbb{R}} = \mathbb{R}\cup\{+\infty\}\,, \\ f|_E = 0\ , \quad f \text{ rank one convex} \end{array} \right\}.$$

Similarly we define the (closure of the) *quasiconvex hull* of $E$ as

$$\overline{\operatorname{Qco} E} = \left\{ \begin{array}{c} \xi \in \mathbb{R}^{m\times n} : f(\xi) \leq 0, \quad \forall\, f : \mathbb{R}^{m\times n} \to \mathbb{R}, \\ f|_E = 0\ , \quad f \text{ quasiconvex} \end{array} \right\}.$$

In the first example (c.f. Theorem 1.6) considered above we have

$$\overline{\operatorname{Qco} E} = \operatorname{Rco} E = \left\{ \xi \in \mathbb{R}^{n\times n} : \ \prod_{i=\nu}^{n} \lambda_i(\xi) \leq \prod_{i=\nu}^{n} a_i\,, \quad \nu = 1,\ldots,n \right\}.$$

These two concepts allow us to discuss extensions of Theorem 1.4 to the vectorial case. The natural generalization is: given $E \subset \mathbb{R}^{m\times n}$ and $\varphi \in W^{1,\infty}(\Omega;\mathbb{R}^m)$ satisfying

$$D\varphi(x) \in E \cup \operatorname{int}\overline{\operatorname{Qco} E}, \quad \text{a.e. } x \in \Omega,$$

there exists $u \in W^{1,\infty}(\Omega;\mathbb{R}^m)$ such that

$$\begin{cases} Du(x) \in E, & \text{a.e. } x \in \Omega \\ u(x) = \varphi(x), & x \in \partial\Omega. \end{cases}$$

However there are several obstacles to obtaining such a theorem in this full generality. The first problem concerns the regularity assumption on the boundary datum $\varphi$. In most of our theorems, with some exceptions such as Theorem 1.8, we will be obliged to assume that $\varphi \in C^1\left(\overline{\Omega};\mathbb{R}^m\right)$ (or *piecewise* $C^1$, denoted by $C^1_{piec}$) since we lack in the vectorial case good approximation theorems by piecewise affine functions (c.f. Chapter 10 for more details). The main problem is however that quasiconvex hulls are poorly understood, contrary to the convex ones. We will therefore need in our theorems to require some further structure on the quasiconvex hulls. With such restrictions we will be able to obtain the claimed generalization.

### *1.1.5 Some typical existence theorems in the vectorial case*

We have selected two results that are relatively simple to express and that apply to the first and third examples quoted above (c.f. Theorem 1.6 and Theorem 1.8). The proofs of these results can be found in Section 6.5 of Chapter 6. The first one is (c.f. Theorem **??**).

**Theorem 1.9** *Let $\Omega \subset \mathbb{R}^n$ be open. Let $F_i : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$, $F_i = F_i(x,s,\xi)$, $i = 1,\ldots,I$, be continuous with respect to $(x,s) \in \Omega \times \mathbb{R}^m$ and quasiconvex and positively homogeneous of degree $\alpha_i > 0$ with respect to the last variable $\xi \in \mathbb{R}^{m\times n}$.*

*Let $a_i : \Omega \times \mathbb{R}^m \to \mathbb{R}$, $i = 1,\ldots,I$, be bounded continuous functions satisfying for a certain $a_0 > 0$*

$$a_i(x,s) \geq a_0 > 0, \quad i = 1,\ldots,I, \quad \forall\, (x,s) \in \Omega \times \mathbb{R}^m.$$

*Assume that, for every $(x,s) \in \Omega \times \mathbb{R}^m$,*

$$\begin{aligned} &\operatorname{Rco}\left\{\xi \in \mathbb{R}^{m\times n} : \ F_i(x,s,\xi) = a_i(x,s), \quad i = 1,\ldots,I\right\} \\ &\quad = \left\{\xi \in \mathbb{R}^{m\times n} : \ F_i(x,s,\xi) \leq a_i(x,s), \quad i = 1,\ldots,I\right\} \end{aligned}$$

*and is bounded in $\mathbb{R}^{m\times n}$ uniformly with respect to $x \in \Omega$ and to $s$ in a bounded set of $\mathbb{R}^m$. If $\varphi \in C^1_{piec}(\overline{\Omega};\mathbb{R}^m)$ satisfies*

$$F_i(x,\varphi(x),D\varphi(x)) < a_i(x,\varphi(x)), \quad a.e.\ x \in \Omega, \quad i = 1,\ldots,I,$$

*then there exists (a dense set of) $u \in W^{1,\infty}(\Omega;\mathbb{R}^m)$ such that*

$$\begin{cases} F_i(x,u(x),Du(x)) = a_i(x,u(x)), \quad a.e.\ x \in \Omega, \quad i = 1,\ldots,I \\ u(x) = \varphi(x), \quad x \in \partial\Omega. \end{cases}$$

The second result is (c.f. Theorem **??** and Remark **??**).

**Theorem 1.10** *Let* $\Omega \subset \mathbb{R}^n$ *be open. Let* $F_i : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$, $F_i = F_i(x, s, \xi)$, $i = 1, \ldots, I$, *be continuous with respect to* $(x, s) \in \Omega \times \mathbb{R}^m$ *and convex with respect to the last variable* $\xi \in \mathbb{R}^{m\times n}$.

*Assume that, for every* $(x, s) \in \Omega \times \mathbb{R}^m$,

$$\begin{aligned} &\operatorname{Rco}\left\{\xi \in \mathbb{R}^{m\times n} : \ F_i(x, s, \xi) = 0, \quad i = 1, \ldots, I\right\} \\ &\quad = \left\{\xi \in \mathbb{R}^{m\times n} : \ F_i(x, s, \xi) \leq 0, \quad i = 1, \ldots, I\right\} \end{aligned}$$

*and is bounded in* $\mathbb{R}^{m\times n}$ *uniformly with respect to* $x \in \Omega$ *and* $s$ *in a bounded set of* $\mathbb{R}^m$. *Let* $\varphi \in C^1_{piec}(\overline{\Omega}; \mathbb{R}^m)$ *satisfy*

$$F_i(x, \varphi(x), D\varphi(x)) < 0, \quad a.e.\ x \in \Omega, \quad i = 1, \ldots, I,$$

*or* $\varphi \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ *be such that*

$$F_i(x, \varphi(x), D\varphi(x)) \leq -\theta, \quad a.e.\ x \in \Omega, \quad i = 1, \ldots, I,$$

*for a certain* $\theta > 0$.

*Then there exists (a dense set of)* $u \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ *such that*

$$\begin{cases} F_i(x, u(x), Du(x)) = 0, & a.e.\ x \in \Omega, \quad i = 1, \ldots, I \\ u(x) = \varphi(x), & x \in \partial\Omega. \end{cases}$$

## 1.2 Second and higher order cases

In fact, second order equations can be reduced to a system of first order equations and therefore the problems considered in this section are vectorial even though they might appear as if they were scalar.

Vectorial calculus of variations gives an interesting motivation to study second order *implicit* pdes. An example is proposed in Section 1.4.4, in the application to optimal design of an existence theorem for some second order implicit differential problem.

### *1.2.1 Dirichlet-Neumann boundary value problem*

We consider second order equations (in Chapters **??** and **??** we will also deal with second order systems) of the form

$$F(x, u(x), Du(x), D^2u(x)) = 0, \quad x \in \Omega, \tag{1.11}$$

where $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n}_s \to \mathbb{R}$ is a continuous function. Since the matrix $D^2u(x)$ of the second derivatives is *symmetric*, then for every fixed $x \in \Omega$ this matrix is an element of the subset

$$\mathbb{R}^{n\times n}_s = \left\{\xi \in \mathbb{R}^{n\times n} : \ \xi = \xi^t\right\}$$

of the $n \times n$ matrices $\mathbb{R}^{n\times n}$.

We say that (1.11) is a second order partial differential equation of *implicit type*, since our hypotheses exclude that it is a quasilinear equation, i.e., it is not possible to write it as an equivalent equation which is linear with respect to the matrix of the second derivatives $D^2u(x)$.

We can consider, for example, the equation

$$|\Delta u| = 1, \quad \text{a.e. in } \Omega, \tag{1.12}$$

together with a boundary datum $u = \varphi$ on $\partial\Omega$. Instead, we could simply solve the Dirichlet problem with the same boundary datum for the linear equation $\Delta u = 1$. But, the interesting fact is that, if we remain with the original nonlinear equation, then we can solve even a Dirichlet-Neumann problem of the type

$$\begin{cases} |\Delta u| = 1, \quad \text{a.e. in } \Omega \\ u = \varphi, \quad \text{on } \partial\Omega \\ \partial u/\partial\nu = \psi, \quad \text{on } \partial\Omega. \end{cases}$$

Independently of the differential equation, if a smooth function $u$ is given on a smooth boundary $\partial\Omega$, then its tangential derivative is automatically determined. Therefore to prescribe Dirichlet and Neumann conditions at the same time is equivalent to give $u$ and $Du$ together.

This means that the Dirichlet-Neumann problem that we consider will be written, in the specific context of (1.12), under the form

$$\begin{cases} |\Delta u| = 1, \quad \text{a.e. in } \Omega \\ u = \varphi, \quad \text{on } \partial\Omega \\ Du = D\varphi, \quad \text{on } \partial\Omega \end{cases} \tag{1.13}$$

(note the compatibility condition that we have imposed on the boundary gradient to be equal to the gradient $D\varphi$ of the boundary datum $\varphi$; of course we assume that $\varphi$ is defined all over $\overline{\Omega}$). In terms of Sobolev spaces the boundary condition is to be understood as $u - \varphi \in W_0^{2,\infty}(\Omega)$.

Returning to the equation (1.11), we will consider Dirichlet-Neumann problems in Chapter **??** of the form (1.14)

$$\begin{cases} F(x, u(x), Du(x), D^2u(x)) = 0, \quad \text{a.e. in } \Omega \\ u = \varphi, \quad Du = D\varphi, \quad \text{on } \partial\Omega. \end{cases} \tag{1.14}$$

We look for solutions $u$ in the class $W^{2,\infty}(\Omega)$ and in general we cannot expect that $u \in C^2(\Omega)$.

Before stating an existence theorem, we need to introduce the notion of coercivity in a rank one direction for the function $F$. We say that $F(x, s, p, \xi)$ is *coercive* with respect to the last variable $\xi$ *in the rank one direction* $\lambda$, if $\lambda \in \mathbb{R}_s^{n\times n}$ with rank $\{\lambda\} = 1$, and for every $x \in \Omega$ and every bounded set of $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}_s^{n\times n}$, there exist constants $m, q > 0$, such that

$$F(x, s, p, \xi + t\,\lambda) \geq m\,|t| - q \tag{1.15}$$

for every $t \in \mathbb{R}$, $x \in \Omega$ and for every $(s, p, \xi)$ that vary on the bounded set of $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}_s^{n \times n}$.

The function involved in (1.12), namely $F(\xi) = |\text{trace}(\xi)| - 1$, is indeed coercive in the rank one direction $e_1 \otimes e_1$ where

$$e_1 \otimes e_1 = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

The theorem that we will obtain, following [112], is (c.f. Theorem **??**)

**Theorem 1.11** *Let $\Omega \subset \mathbb{R}^n$ be open. Let $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}_s^{n \times n} \to \mathbb{R}$ be a continuous function, convex with respect to the last variable and coercive in a rank one direction $\lambda$. Let $\varphi \in C_{piec}^2(\overline{\Omega})$ be such that*

$$F\left(x, \varphi(x), D\varphi(x), D^2\varphi(x)\right) \leq 0, \quad a.e.\ x \in \Omega \tag{1.16}$$

*or $\varphi \in W^{2,\infty}(\Omega)$ satisfy, for a certain $\theta > 0$*

$$F\left(x, \varphi(x), D\varphi(x), D^2\varphi(x)\right) \leq -\theta, \quad a.e.\ x \in \Omega.$$

*Then there exists (a dense set of) $u \in W^{2,\infty}(\Omega)$ such that*

$$\begin{cases} F(x, u(x), Du(x), D^2u(x)) = 0, & a.e.\ in\ \Omega \\ u = \varphi,\ Du = D\varphi, & on\ \partial\Omega. \end{cases}$$

We need the compatibility condition (1.16) first to be sure that the function $F$ is equal to zero somewhere (consequence of the compatibility condition and the coercivity assumption). More relevant, however is the implication by the convexity assumption through the Jensen inequality: in fact, for example, if we assume that the problem (1.14) without the lower order terms and with special boundary datum $\varphi$ equal to a polynomial of degree two (i.e., $D^2\varphi(x) = \xi_0$ for some $\xi_0 \in \mathbb{R}_s^{n \times n}$ and for every $x \in \Omega$) has a solution $u \in W^{2,\infty}(\Omega)$, then, since $F\left(D^2u\right) = 0$ a.e. in $\Omega$ and $Du = D\varphi$ on $\partial\Omega$, we obtain the necessary compatibility condition

$$\begin{aligned} 0 &= \frac{1}{|\Omega|} \int_\Omega F\left(D^2u(x)\right) dx \geq F\left(\frac{1}{|\Omega|} \int_\Omega D^2u(x)\, dx\right) \\ &= F\left(\frac{1}{|\Omega|} \int_\Omega D^2\varphi(x)\, dx\right) = F(\xi_0) = F(D^2\varphi)\,. \end{aligned}$$

### *1.2.2 Fully nonlinear partial differential equations*

Let us make a remark related to the important case of second order elliptic *fully nonlinear* partial differential equations. The coercivity condition that we will assume (c.f. (1.15)) prohibits the equations we consider here to be *elliptic* in the sense of Caffarelli-Nirenberg-Spruck [65], Crandall-Ishii-Lions [95], Evans [147], Trudinger [301]. To prove this claim we first recall that *ellipticity* of $F = F(\xi)$ where $\xi \in \mathbb{R}_s^{n\times n}$, means

$$F(\xi) < F(\xi + \eta), \quad \forall\, \eta \geq 0,\ \eta \neq 0, \tag{1.17}$$

where the notation $\eta \geq 0$ means that $\eta \in \mathbb{R}_s^{n\times n}$ is a positive semidefinite matrix (note that some authors use the same definition with $F$ replaced by $-F$). If $F$ is differentiable, it turns out (see for example Trudinger [301]) that (1.17) is equivalent to the *positivity* of the $n \times n$ matrix $DF$, that is

$$\sum_{ij} \frac{\partial F}{\partial \xi_{ij}} \lambda_i \lambda_j > 0, \quad \forall\, (\lambda_i) \in \mathbb{R}^n - \{0\}. \tag{1.18}$$

We now show that (1.18) *excludes* the coercivity of $F$ in any rank one direction. In fact we have that

$$\sum_{ij} \frac{\partial F(\xi)}{\partial \xi_{ij}} \lambda_i \lambda_j = \frac{d}{dt} F(\xi + t\lambda)\Big|_{t=0}$$

where $\xi = (\xi_{ij})$ is a generic $n \times n$ matrix while $\lambda = (\lambda_i \lambda_j)$ is a generic $n \times n$ matrix of *rank one*. Therefore the condition (1.18) means that $F$ is monotone in *all* directions $\lambda$ of rank one; while coercivity in a rank one direction $\lambda$ implies that $F$ is *not* monotone in this direction.

### *1.2.3 Singular values*

The preceding result can be extended to systems and we give here only one example (c.f. Theorem **??**). We recall that, for $\xi \in \mathbb{R}_s^{n\times n}$, we denote by $0 \leq \lambda_1(\xi) \leq \ldots \leq \lambda_n(\xi)$ its singular values, which are now, because of the symmetry of the matrix, the absolute value of the eigenvalues.

**Theorem 1.12** *Let $\Omega \subset \mathbb{R}^n$ be an open set, $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, n$ be continuous bounded functions satisfying*

$$0 < c \leq a_1(x,s,p) \leq \ldots \leq a_n(x,s,p)$$

*for some constant $c$ and for every $(x,s,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$. Let $\varphi \in C^2_{piec}(\overline{\Omega})$ be such that*

$$\lambda_i\left(D^2\varphi(x)\right) < a_i(x, \varphi(x), D\varphi(x)), \quad a.e.\ x \in \Omega, \quad i = 1, \ldots, n \tag{1.19}$$

*(in particular $\varphi \equiv 0$). Then there exists (a dense set of) $u \in W^{2,\infty}(\Omega)$ such that*

$$\begin{cases} \lambda_i\left(D^2u(x)\right) = a_i(x, u(x), Du(x)), & a.e.\ x \in \Omega, \quad i = 1, \ldots, n \\ u(x) = \varphi(x),\ Du(x) = D\varphi(x), & x \in \partial\Omega. \end{cases} \tag{1.20}$$

As a consequence we find that the following Dirichlet-Neumann problem (1.21) admits a solution.

**Corollary 1.13** *Let $\Omega \subset \mathbb{R}^n$ be open. Let $f : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that*

$$f(x, s, p) \geq f_0 > 0,$$

*for some constant $f_0$ and for every $(x, s, p) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$. Let $\varphi \in C^2(\overline{\Omega})$ (or $C^2_{piec}(\overline{\Omega})$) satisfy*

$$\left|\det D^2\varphi(x)\right| < f(x, \varphi(x), D\varphi(x)), \quad x \in \overline{\Omega}.$$

*Then there exists (a dense set of) $u \in W^{2,\infty}(\Omega)$ such that*

$$\begin{cases} \left|\det D^2u(x)\right| = f(x, u(x), Du(x)), & a.e.\ x \in \Omega, \\ u = \varphi,\ Du = D\varphi, & on\ \partial\Omega. \end{cases} \tag{1.21}$$

Observe that because of the Dirichlet-Neumann boundary data, the above problem cannot be handled as a corollary of the results on the Monge-Ampère equation.

### *1.2.4 Some extensions*

The results on second order equations carry to higher order equations (c.f. Chapter 6). We give here only one example which concerns the *Nth order eikonal equation*. Let us first introduce the following notation for $u : \mathbb{R}^n \to \mathbb{R}$; we let

$$D^N u = \left(\frac{\partial^N u}{\partial x_{j_1} \ldots \partial x_{j_N}}\right)_{1 \leq j_1, \ldots, j_N \leq n}$$

and

$$D^{[N-1]}u = \left(u, Du, \ldots, D^{N-1}u\right).$$

Finally $\mathbb{R}^M_s$ denotes the space where $D^{[N-1]}u$ lies (see Chapter 5 for more details).

**Theorem 1.14** *Let $\Omega \subset \mathbb{R}^n$ be open. Let $a : \Omega \times \mathbb{R}^M_s \to \mathbb{R}_+$ be bounded and continuous and $\varphi \in C^N_{piec}(\overline{\Omega})$ satisfy*

$$\left|D^N\varphi(x)\right| \leq a\left(x, D^{[N-1]}\varphi(x)\right), \quad a.e.\ x \in \Omega;$$

*then there exists (a dense set of) $u \in W^{N,\infty}(\Omega)$ satisfying*

$$\begin{cases} \left|D^N u(x)\right| = a\left(x, D^{[N-1]}u(x)\right), & a.e.\ x \in \Omega \\ D^\alpha u(x) = D^\alpha \varphi(x), \quad x \in \partial\Omega, & \alpha = 0, \ldots, N-1. \end{cases}$$

## 1.3 Different methods

There are, roughly speaking, three general methods to deal with the problems that we consider in this book and we will describe them briefly now. The third one will be the one used throughout this monograph. Of course for some particular examples there are some ad hoc methods; we think, for instance, of the *pyramidal construction* mentioned above (c.f. also Chapter 2) or of the *confocal ellipses construction* of Murat and Tartar [295] (c.f. Chapter 3 and for applications of this construction [107] and Section 1.4).

### *1.3.1 Viscosity solutions*

The first method is the oldest and the one that has received the most attention. It deals essentially with scalar problems, although there are some results on some particular vectorial equations. We here discuss only viscosity solutions of first order equations since it is mainly in this case that the two methods, which will be discussed below, are comparable. The advantage over those two other methods is that it gives much more information than existence of solutions; for instance, uniqueness, stability, maximality and, last but not least, explicit formulas (such as the Hopf-Lax formula, which in the case of the eikonal equation will be given below). However, because of the many extra properties that it carries with it, the viscosity approach applies to many fewer equations than the two other methods that we will present below. The justification of this last statement is the purpose of Chapter 4 and will be briefly discussed now.

We recall that the problem under consideration is

$$\begin{cases} F\left(x, u\left(x\right), Du\left(x\right)\right) = 0, & \text{a.e. } x \in \Omega \\ u\left(x\right) = \varphi\left(x\right), & x \in \partial\Omega, \end{cases} \tag{1.22}$$

where $\Omega \subset \mathbb{R}^n$ is an open set, $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and $\varphi$ is a given function.

We should immediately point out that in this monograph we will be concerned only with viscosity solutions that are locally Lipschitz (the definition has been extended to functions that are even discontinuous) and that satisfy the boundary condition everywhere.

The notion of viscosity solution arose in the pde context by attempting to find solutions as limits of solutions of

$$\begin{cases} F\left(x, u^{\varepsilon}\left(x\right), Du^{\varepsilon}\left(x\right)\right) = \varepsilon \Delta u^{\varepsilon}\left(x\right), & \text{a.e. } x \in \Omega \\ u^{\varepsilon}\left(x\right) = \varphi\left(x\right), & x \in \partial\Omega, \end{cases}$$

when $\varepsilon \to 0$; hence the name of *viscosity solutions*. The concept of viscosity solution is now, following Crandall-Lions [96] and Crandall-Ishii-Lions [95], more general, and we will give the precise definition in Chapter 4. It turns out that in optimal control the value function of certain problems is a viscosity solution of (1.22). For example if we consider the *eikonal equation*

$$\begin{cases} |Du(x)| = 1, & \text{a.e. in } \Omega \\ u = \varphi, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega$ is a bounded, open and convex set of $\mathbb{R}^n$ and the boundary datum $\varphi \in W^{1,\infty}(\Omega)$ satisfies the compatibility condition

$$|D\varphi(x)| \leq 1, \quad \text{a.e. in } \Omega,$$

we find that the viscosity solution is then given by

$$u(x) = \inf_{y \in \partial\Omega} \{\varphi(y) + |x - y|\},$$

which is, when $\varphi = 0$, nothing but the distance to the boundary, namely

$$u(x) = \operatorname{dist}(x; \partial\Omega).$$

In Chapter 4 we will recall the definition of viscosity solutions, give some examples, properties, and discuss the Hopf-Lax formula. We do not intend to give any detailed presentation of this method; there are several excellent articles and books on this subject and we mention only a few of them: Bardi-Capuzzo Dolcetta [34], Barles [35], Benton [39], Capuzzo Dolcetta-Evans [67], Capuzzo Dolcetta-Lions [68], Crandall-Evans-Lions [94], Crandall-Ishii-Lions [95], Crandall-Lions [96], Douglis [137], Fleming-Soner [154], Frankowska [160], Hopf [188], Ishii [193], Kruzkov [208], Lax [211], Lions [218] and Subbotin [286].

We now come back, following Cardaliaguet-Dacorogna-Gangbo-Georgy [71], to the fact that, if we are only interested in existence of locally Lipschitz functions of (1.22), then the viscosity approach is too restrictive. To be more precise, we will discuss the case where $F$ does not depend explicitly on $x$ and $u$, namely

$$\begin{cases} F(Du(x)) = 0, & \text{a.e. } x \in \Omega \\ u(x) = \varphi(x), & x \in \partial\Omega. \end{cases} \tag{1.23}$$

We have seen in Theorem 1.4 that if

$$E = \{\xi \in \mathbb{R}^n : F(\xi) = 0\}$$

and if $\varphi \in C^1(\overline{\Omega})$ is such that

$$D\varphi(x) \in E \cup \operatorname{int} \operatorname{co} E, \quad \forall x \in \Omega, \tag{1.24}$$

then (1.23) has a (dense set of) $W^{1,\infty}$ solutions (we recall that $\operatorname{int} \operatorname{co} E$ denotes the interior of the convex hull of $E$). This condition is close to necessary, therefore a natural question is to know whether, under this condition, a $W^{1,\infty}$ viscosity solution exists. We will show in Chapter 4 that the answer is in general negative unless strong geometric restrictions on $\Omega$ and $\varphi$ are assumed.

For instance, if we consider the example (c.f. Example 1.3)

$$\begin{cases} -\left[\left(\frac{\partial u}{\partial x_1}\right)^2 - 1\right]^2 - \left[\left(\frac{\partial u}{\partial x_2}\right)^2 - 1\right]^2 = 0, & \text{a.e. in } \Omega \\ u = 0, \quad \text{on } \partial\Omega \end{cases} \tag{1.25}$$

then, since $0 \in \operatorname{int} \operatorname{co} E$, we have by Theorem 1.4 that there are $W^{1,\infty}$ solutions of (1.25); but we will show (c.f. Theorem **??**) that, if $\Omega$ is convex, there is no $W^{1,\infty}$ *viscosity* solutions unless $\Omega$ is a rectangle whose normals are elements of $E = \{(\pm 1, \pm 1)\}$ ; in this case the viscosity solution will be

$$u(x_1, x_2) = \inf_{(y_1, y_2) \in \partial\Omega} \{|x_1 - y_1| + |x_2 - y_2|\} .$$

In particular, for any smooth domain (such as the unit disk) the Dirichlet problem (1.25) has no viscosity solution. This example shows also that the existence of viscosity solutions does not depend on the smoothness of the data (in the case where $\Omega$ is the unit disk, then all the data are analytic).

### *1.3.2 Convex integration*

This method is due to Gromov [181] (see also the notion of *$P$-convexity* in Section 2.4.11 in the book by Gromov [182], where *partial differential relations* are considered). It was introduced for solving some problems of geometry and topology, in particular the Nash-Kuiper $C^1$ *isometric immersion theorem*. Gromov's method was developed essentially to get smooth solutions, although Lipschitz solutions are also considered in the context of isometric immersions. We refer to the book of Spring [283] for an other presentation of the method (see Chapter 9 of [283] for the treatment of *systems of partial differential equations*, where in particular *underdetermined systems*, *triangular systems* and *$C^1$-isometric immersions* are studied). We will discuss here only the first order case, but the method applies also to higher orders.

Müller-Sverak [249] (see also Celada-Perrotta [74], De Simone-Dolzmann [131]) have applied this method for solving the problem of two *potential wells* in two dimensions that we presented in (1.9). We now sketch their approach, which is more analytical in its presentation than the one of Gromov.

We first introduce the following notion. We say that a set $K \subset \mathbb{R}^{m \times n}$ admits an *in-approximation* by open sets $U_i$ if the three following properties hold:

(i) $U_i \subset \operatorname{Rco} U_{i+1}$ ($\operatorname{Rco} U$ stands for the rank one convex hull of $U$ defined above);

(ii) the $U_i$ are uniformly bounded;

(iii) if a sequence $\xi_i \in U_i$ converges to $\xi$ as $i \to \infty$, then $\xi \in K$.

A typical theorem (c.f. Müller-Sverak [249], [250]) that can be established is then the following.

**Theorem 1.15** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $K \subset \mathbb{R}^{m\times n}$ admit an in-approximation by open sets $U_i$. Let $\varphi \in C^1\left(\overline{\Omega};\mathbb{R}^m\right)$ such that*

$$D\varphi\left(x\right) \in U_1.$$

*Then there exists $u \in W^{1,\infty}\left(\Omega;\mathbb{R}^m\right)$ such that*

$$\begin{cases} Du\left(x\right) \in K, & a.e.\ x \in \Omega \\ u\left(x\right) = \varphi\left(x\right), & x \in \partial\Omega. \end{cases}$$

The difficulty rests on the fact that the sets $K$ and $U_i$, $i \in \mathbb{N}$, are not convex, not even rank one convex. Thus, if $u_i$ is a generic sequence of approximate solutions such that $Du_i(x) \in U_i$ a.e. $x \in \Omega$, $i \in \mathbb{N}$, since by (ii) $Du_i$ are uniformly bounded in $L^\infty\left(\Omega;\mathbb{R}^{m\times n}\right)$ then, up to a subsequence, $u_i$ weakly$^*$ converges to a function $u$. However the weak$^*$ convergence is not enough to guarantee that $Du\left(x\right) \in K$, a.e. $x \in \Omega$, because, as already said, $K$ is not a quasiconvex set.

The proof of the theorem is obtained instead by constructing an appropriate sequence $u_i$ such that $Du_i(x) \in U_i$ a.e. $x \in \Omega$ and show *strong convergence* in $W^{1,1}\left(\Omega;\mathbb{R}^m\right)$ of this sequence to a solution $u$.

Of course a main difficulty is to find an *in-approximation*. The papers quoted above ([249], [74], [131]) deal with such a construction in some particular examples. We now present a typical result that can be obtained by this method. It concerns the problem of *two potential wells* in dimension two described in (1.9) (c.f. Theorem **??**).

**Theorem 1.16** *Let $\Omega \subset \mathbb{R}^2$ be open. Let $A, B \in \mathbb{R}^{2\times 2}$ be two matrices such that* $\operatorname{rank}\{A-B\} = 1$ *and* $\det B > \det A > 0$. *Let $\varphi \in C^1_{piec}(\overline{\Omega};\mathbb{R}^2)$ satisfy*

$$D\varphi(x) \in \operatorname{int}\left\{ \begin{array}{c} \xi \in \mathbb{R}^{2\times 2} :\ \xi = \alpha R_a A + \beta R_b B,\ R_a,\ R_b \in SO(2), \\ 0 \le \alpha \le \frac{\det B - \det \xi}{\det B - \det A},\quad 0 \le \beta \le \frac{\det \xi - \det A}{\det B - \det A} \end{array} \right\},$$

*for almost every $x \in \Omega$. Then there exists $u \in W^{1,\infty}(\Omega;\mathbb{R}^2)$ such that*

$$\begin{cases} Du(x) \in SO(2)A \cup SO(2)B & a.e.\ in\ \Omega \\ u(x) = \varphi(x) & on\ \partial\Omega. \end{cases}$$

The representation formula for the rank one convex hull is due to Sverak [289], while the theorem has been proved by Müller-Sverak [249], using convex integration, and by the authors (in [109], [111]), using the method presented in this book (c.f. Chapter 8).

### 1.3.3 The Baire category method

The approach that we will present can be characterized as a functional analytic method, in contrast with the more geometrical one of Gromov, although some constructions are very similar. It is based on the Baire category theorem. It was introduced by Cellina [76] to prove density (in the sense of the Baire category theorem) of solutions for the differential inclusion

$$\begin{cases} x'(t) \in \{-1, 1\}, & \text{a.e. } t > 0 \\ x(0) = x_0. \end{cases}$$

The method, still for differential inclusions, was further developed by De Blasi-Pianigiani [125], [126] and by Bressan-Flores [55]. The authors of the present book, in a series of papers [108], [109], [110], [111] and [112], extended the method to the present framework.

We will now very roughly present the idea of the proof in the simplest case which is the one of Theorem 1.2. We recall that $\varphi \in W^{1,\infty}(\Omega)$ satisfies

$$F(x, \varphi(x), D\varphi(x)) \leq 0, \quad \text{a.e. in } \Omega \tag{1.26}$$

and that we wish to show the existence of (a dense set of) $u \in W^{1,\infty}(\Omega)$ such that

$$\begin{cases} F(x, u(x), Du(x)) = 0, & \text{a.e. in } \Omega \\ u = \varphi, & \text{on } \partial\Omega. \end{cases} \tag{1.27}$$

We start by introducing the functional space

$$V = \left\{ u \in \varphi + W_0^{1,\infty}(\Omega) : \ F(x, u(x), Du(x)) \leq 0, \quad \text{a.e. } x \in \Omega \right\}$$

which in this particular case is the set of subsolutions of (1.27). Note also that $V$ is nonempty since (1.26) holds. We next endow $V$ with the $C^0$ metric. We claim that $V$ is then a complete metric space. This follows from the coercivity and the convexity of $F$. Indeed the coercivity condition ensures that any Cauchy sequence in $V$ has uniformly bounded gradient and therefore has a subsequence that converges weak* in $W^{1,\infty}$ to a limit. Since the convexity of $F$ implies lower semicontinuity, we get that the limit is indeed in $V$.

We next introduce, for every integer $k$, the subset $V^k$ of $V$

$$V^k = \left\{ u \in V : \ \int_\Omega F(x, u(x), Du(x))\, dx > -\frac{1}{k} \right\}.$$

The same argument as above implies that $V^k$ is *open* in $V$. The difficult step is then to show that $V^k$ is *dense* in $V$; the proof of this property is in the spirit of the necessary conditions for weak lower semicontinuity and of relaxation theorems in the calculus of variations (c.f. below for some historical comments).

Once these results have been established, we can conclude from the *Baire category theorem* (see for example Brezis [57] or Yosida [306]) that

$$\begin{aligned}\bigcap_k V^k &= \left\{u \in V : \int_\Omega F\left(x, u(x), Du\left(x\right)\right) dx \geq 0\right\} \\ &= \left\{u \in \varphi + W_0^{1,\infty}\left(\Omega\right) : \ F\left(x, u(x), Du\left(x\right)\right) = 0, \quad \text{a.e. } x \in \Omega\right\}\end{aligned}$$

is dense, and hence nonempty, in $V$.

This is the outline of the proof of Theorem 1.2 and of the method used throughout this book.

The proof of the density resembles the *in-approximation* of the convex integration method outlined above, but for those familiar with the calculus of variations it looks, as mentioned above, much more like a *relaxation* type result, or the study of *necessary conditions* for weak lower semicontinuity (convexity in the scalar case and quasiconvexity in the vectorial case) which are well known since the pioneering work of Leonida Tonelli in 1921.

More precisely, the *convexity* of $F$, with respect to the gradient variable, as a necessary condition for weak lower semicontinuity in the scalar case $m = 1$, was first discovered by Tonelli ([299], Section 1 of Chapter X) for $n = 1$ and then obtained by Caccioppoli-Scorza Dragoni [64] for $n = 2$ and by McShane [220] for general $n \geq 1$ in the smooth case (see also the book by L.C. Young [307]); while Carathéodory functions $F$ have been treated by Ekeland-Témam [142] and Marcellini-Sbordone [232]. Moreover Morrey [245] (see also Theorems 4.4.2 and 4.4.3 in the book by Morrey [246], the papers by Acerbi-Fusco [3], Marcellini [227] and the books by Dacorogna [101] and Giusti [178]) introduced the concept of *quasiconvexity* of $F$, with respect to the gradient variable, to prove that it is a necessary condition for weak lower semicontinuity in the vector-valued case $m > 1$. Finally, *relaxation* results of the integral of $F$, as appearing in (1.28), concern

$$\begin{aligned}\text{either} \quad \int_\Omega F^{**}\left(x, u(x), Du\left(x\right)\right) dx \quad \text{when } m &= 1, \\ \text{or} \quad \int_\Omega QF\left(x, u(x), Du\left(x\right)\right) dx \quad \text{if } m &> 1,\end{aligned}$$

where $F^{**}$ and $QF$ are respectively the *convex* and the *quasiconvex envelope* of $F$ (c.f. Chapter 5). In this context when $m = 1$, we refer to Ekeland-Témam ([142], Chapter X), Marcellini-Sbordone [231], [232]; while if $m > 1$ we quote Dacorogna [100] [101] and Acerbi-Fusco [3] (see also some related results by Buttazzo-Dal Maso [62], Goffman-Serrin [179], Rockafellar [272], Serrin [281]).

To conclude, we should stress that the main reason for getting density of $V^k$ in $V$ is that the equations under consideration possess, locally, more than one solution. This is why linear and uniformly elliptic equations are excluded from our analysis.

## 1.4 Applications to the calculus of variations

Our first motivation for studying first order implicit equations, besides their intrinsic interest, comes from the calculus of variations. In this context, first order pdes have been intensively used, c.f. for example the monographs of Carathéodory [69], Giaquinta-Hildebrandt [173] and Rund [275].

We start with a heuristic consideration, explaining the link between the existence of minimizers of integrals of the calculus of variations and first order implicit differential equations. Let $\overline{u} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$, $n, m \geq 1$, be a minimizer in a Sobolev class of functions of an integral of the calculus of variations of the form

$$\int_\Omega f\left(x, u(x), Du(x)\right)\, dx\ . \tag{1.28}$$

Then, if $f$ is not quasiconvex with respect to the gradient variable, direct methods do not apply. In this case we denote by $Qf$ the *quasiconvex envelope* of $f$ (c.f. Chapter 5), i.e.,

$$Qf\left(x, s, \xi\right) = \sup\left\{g\left(x, s, \xi\right) :\ g \leq f,\ \ g\left(x, s, \xi\right)\ \text{quasiconvex in}\ \xi\right\}.$$

In the scalar case, when $m = 1$, then $Qf = f^{**}$ is the classical convex envelope of $f$ (see for example Ekeland-Témam [142] and Rockafellar [273]).

A general relaxation theorem (due to Dacorogna [100] [101] and to Acerbi-Fusco [3], who extended to the vector-valued case a result proved in the scalar case by Ekeland-Témam [142] and Marcellini-Sbordone [232]) states that, in the given class of functions,

$$\inf\left\{\int_\Omega f\left(x, u(x), Du(x)\right)\, dx\right\} = \inf\left\{\int_\Omega Qf\left(x, u(x), Du(x)\right)\, dx\right\}\ .$$

Therefore any minimizer $\overline{u}$ of the integral in (1.28) satisfies

$$\int_\Omega f\left(x, \overline{u}(x), D\overline{u}(x)\right)\, dx\ =\ \int_\Omega Qf\left(x, \overline{u}(x), D\overline{u}(x)\right)\, dx\ ,$$

which implies, since $f \geq Qf$, that

$$f\left(x, \overline{u}(x), D\overline{u}(x)\right) = Qf\left(x, \overline{u}(x), D\overline{u}(x)\right),\quad \text{a.e. } x \in \Omega. \tag{1.29}$$

This is a first order equation for $\overline{u}$ which holds almost everywhere in $\Omega$.

We will show below that (1.29) can be fitted into our general theory of first order implicit differential equations and systems. We will also show that, in the vector-valued case $m > 1$, we are led in some cases to study implicit partial differential equations of order $N$ greater than 1.

These heuristic considerations can be made precise, in the form of theorems, in some special cases; see in particular Theorems 1.17 and 1.18 below.

### *1.4.1 Some bibliographical notes*

As already mentioned above, we will briefly describe some problems in the *calculus of variations* which may or may not have a solution, depending on the context and on the assumptions. The main characteristic of the variational problems that we consider in this section is the *lack of convexity* (even the lack of *quasiconvexity* in the vector-valued case $m > 1$) of the integrand with respect to the gradient variable. We will study some model problems of this type in the next subsections.

We follow (in particular for the vector-valued case) the authors' approach in [107], [114], although we recall that the mathematical literature on this subject is broad, a large part of it being dedicated to the one dimensional scalar case $n = m = 1$, the vectorial case $n, m > 1$ being at the moment understood only in special situations. We quote for example: Allaire-Francfort [9], Aubert-Tahraoui [21], [22], [23], [24], Ball-James [31], [32], Bauman-Phillips [36], Buttazzo-Ferone-Kawohl [63], Celada-Perrotta [75], Cellina [77], [78], Cellina-Colombo [79], Cellina-Zagatti [82], [81], Cesari [84], [85], Chipot-Kinderlehrer [86], Cutrì[98], Dacorogna [99], [101], Dacorogna-Marcellini [107], Ekeland-Témam [142], Firoozye-Kohn [153], Fonseca-Tartar [158], Fusco-Marcellini-Ornelas [165], Friesecke [162], Giachetti-Schianchi [171], Kinderlehrer-Pedregal [200], Kohn [204], Kohn-Strang [205], Marcellini [224], [225], [226], [230], Mascolo [235], Mascolo-Schianchi [238], [239], [240], Monteiro Marques-Ornelas [244], Müller [248], Müller-Sverak [249], Olech [256], Ornelas [259], Raymond [265], [266], [267], [268], Sverak [289], Sychev [290], Tahraoui [292], [293], Treu [300], Zagatti [309].

### *1.4.2 The variational problem*

Similar to the first part of this section, we could study integrals of $f(x, u(x), Du(x))$, related to a function $f$ depending on $x$ and $u(x)$ too. However we have chosen to consider here (and below in this section) only dependence on the gradient variable $Du(x)$ as in (1.30), with the aim of proposing the variational problem in the simplest context. It would be of interest to generalize these results to a wider class of integrals with $f = f(x, u(x), Du(x))$, and in fact some partial results have been already obtained in the literature on this subject quoted in the previous subsection.

Let $\Omega$ be a bounded open set of $\mathbb{R}^n$ $(n \geq 1)$. In general we will consider a variational problem related to vector-valued unknown functions $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$, $m \geq 1$, and to an integrand $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ that we assume to be lower semicontinuous in $\mathbb{R}^{m\times n}$, not necessarily convex, and satisfying the condition $f(\xi) \geq c_1 |\xi|^p - c_2$ for some constants $c_1 > 0$, $c_2 \in \mathbb{R}$ and $p > 1$. The variational problem that we study is: to minimize the functional integral

$$\int_{\Omega} f\left(Du(x)\right)\, dx \tag{1.30}$$

in the class of vector-valued functions

$$u \in u_0 + W_0^{1,p}\left(\Omega;\mathbb{R}^m\right),$$

where $u_0 \in W^{1,p}\left(\Omega;\mathbb{R}^m\right)$ is a given boundary datum.

Because of the lack of quasiconvexity of $f$, the integral functional in (1.30) is not lower semicontinuous in the weak topology of $W^{1,p}\left(\Omega;\mathbb{R}^m\right)$. Thus it is not possible to apply the *direct methods* (based on lower semicontinuity and on the relative compactness of minimizing sequences in the weak topology of $W^{1,p}\left(\Omega;\mathbb{R}^m\right)$) in order to obtain the existence of the minimum. Nevertheless the integral functional in (1.30) may have a minimum in spite of the lack of (quasi)convexity.

In the next subsection we first consider the (nonconvex) scalar case $m = 1$, and we give some sufficient conditions (which are also necessary in some cases) to obtain the existence of the minimum; under some assumptions we will find solutions in the class $u \in W^{1,\infty}(\Omega)$, i.e., with $p = +\infty$.

In the last subsection we study an application to *optimal design* in the vector-valued case. We note explicitly that nonconvex (and even not quasiconvex) variational problems in the vector-valued case are far from being solved in a general context.

### *1.4.3 The scalar case*

In general we can lack solutions for a *nonconvex* variational problem. Well known is the classical example of Bolza (see Section 2.5) in the one dimensional scalar case $n = m = 1$ for integrals of $f(u, u')$ (note that, when $n = 1$, then the dependence of the integrand $f$ on $u$, other than $u'$, is necessary to exhibit examples of lack of attainment of minima of coercive integrals). Other examples for $n > 1$ are proposed in Section 2.5.

Here we consider a bounded open set $\Omega \subset \mathbb{R}^n$ for some $n \geq 2$. Let us also assume that $\Omega$ is a *uniformly convex set*, in the sense that there exists a positive constant $c$ and, for every $x_0 \in \partial\Omega$, a hyperplane $\pi_{x_0}$ containing $x_0$ such that

$$\operatorname{dist}\left(x;\pi_{x_0}\right) \geq c \cdot \left|x - x_0\right|^2, \quad \forall\, x \in \partial\Omega\,.$$

Note that, for every $x_0 \in \partial\Omega$, $\pi_{x_0}$ is a *supporting hyperplane*, i.e., it is a hyperplane passing through $x_0$ and leaving the set $\Omega$ on one of the two half spaces delimited by $\pi_{x_0}$. A ball is a uniformly convex set.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a lower semicontinuous function, not necessarily convex, bounded from below. Let us denote by $f^{**}$ the largest convex function which is less than or equal to $f$ on $\mathbb{R}^n$. We assume that $f^{**}$ is *affine* on the (open) set $A$, where $f \neq f^{**}$, i.e., there exist $\eta \in \mathbb{R}^n$ and $q \in \mathbb{R}$ such that

$$\begin{cases} f^{**}(\xi) = \langle \eta; \xi \rangle + q, & \forall\, \xi \in A = \{\xi \in \mathbb{R}^n :\ f(\xi) > f^{**}(\xi)\}\,, \\ f^{**}(\xi) = \ f(\xi), & \forall\, \xi \in \mathbb{R}^n \ - A\,. \end{cases}$$

We also assume that $A$ is bounded (for more general assumptions see Theorem **??**). Then, in Chapter 2, we will prove the following existence result.

**Theorem 1.17** *Under the stated assumptions, for every boundary datum* $u_0 \in C^2\left(\overline{\Omega}\right)$*, the integral*

$$\int_\Omega f\left(Du(x)\right)\, dx \tag{1.31}$$

*has a minimizer in the class of functions* $u \in u_0 + W_0^{1,\infty}(\Omega)$.

The proof starts with the minimization of the associated *relaxed* variational problem related to the integral over $\Omega$ of $f^{**}\left(Du(x)\right)$. If we denote by $u^{**}$ a minimizer of the relaxed problem, then we are led to solve the differential problem

$$\begin{cases} Du\,(x) \in \partial A, & \text{a.e. } x \in \Omega' \\ u\,(x) = u^{**}\,(x)\,, & x \in \partial\Omega'\,, \end{cases} \tag{1.32}$$

where $\Omega'$ is a suitable *open* subset of $\Omega$. Moreover, the boundary datum $u^{**}$ in (1.32) satisfies the compatibility condition

$$Du^{**}(x) \in A \subset \operatorname{int}\operatorname{co}\partial A, \quad \text{a.e. } x \in \Omega'\,.$$

We can apply Theorem 1.6 (c.f. Theorem **??**) with $E = \partial A$ and obtain the existence of a function $u \in W^{1,\infty}(\Omega')$ which solves (1.32). This function $u$, extended equal to $u^{**}$ out of $\Omega'$, is a minimizer of the integral in (1.31) in the class $u_0 + W_0^{1,\infty}(\Omega)$. Further details of the proof can be found in Section 2.5.

Theorem 1.17 is specific for the scalar case $n \geq 2$ and it generalizes similar results obtained by Marcellini [225], Mascolo-Schianchi [238], [239], [240], Mascolo [235], Cellina [77] and Friesecke [162]. Theorem 1.17 has been recently proved by Sychev [290] in the form presented here (see also Zagatti [309]). In particular Mascolo-Schianchi pointed out the condition of affinity of the function $f^{**}$on the set where $f \neq f^{**}$, while Cellina and Friesecke proved the necessity of this condition of affinity for linear boundary data $u_0$.

### *1.4.4 Application to optimal design in the vector-valued case*

Following Kohn-Strang [205], we consider the two dimensional case $n = 2$ and $m = 2$ (here for simplicity we limit ourselves to $m = 2$; see [107] and

Kohn-Strang [205] for a discussion of the case $m > 2$; see also Allaire-Francfort [9] for the case $n, m \geq 2$). More explicitly we consider a variational problem in *optimal design*, related to the lower semicontinuous (nonconvex) function $f : \mathbb{R}^{2\times 2} \to \mathbb{R}$

$$f(\xi) = \begin{cases} 1 + |\xi|^2 & \text{if } \xi \neq 0 \\ 0 & \text{if } \xi = 0. \end{cases} \tag{1.33}$$

Kohn-Strang computed in [205] the quasiconvex envelope $Qf : \mathbb{R}^{2\times 2} \to \mathbb{R}$ of $f$ ($Qf$ is the largest quasiconvex function on $\mathbb{R}^{2\times 2}$ less than or equal to $f$; c.f. Chapter 5). It turns out that $Qf$ is given by

$$Qf(\xi) = \begin{cases} 1 + |\xi|^2 & \text{if } |\xi|^2 + 2|\det \xi| \geq 1 \\ 2\sqrt{|\xi|^2 + 2|\det \xi|} - 2|\det \xi| & \text{if } |\xi|^2 + 2|\det \xi| < 1. \end{cases} \tag{1.34}$$

We consider a bounded open set $\Omega$ of $\mathbb{R}^2$ and a boundary datum $u_0$ linear in $\Omega$, with $\det Du_0 \neq 0$ and, just to consider one case, we assume that $\det Du_0 > 0$. To avoid the trivial situation $Qf(Du_0) = f(Du_0)$, we also assume that $u_0$ satisfies the condition

$$|Du_0|^2 + 2|\det Du_0| < 1. \tag{1.35}$$

Finally, we assume that $Du_0$ is a *symmetric* $2\times 2$ matrix. This implies that there exists $\varphi$, polynomial of degree 2, such that

$$u_0 = \begin{pmatrix} \varphi_x \\ \varphi_y \end{pmatrix}, \quad \text{with} \quad \det D^2\varphi(x) = \det Du_0 > 0.$$

By considering explicitly the components of $u \in W^{1,\infty}(\Omega; \mathbb{R}^2)$, a generic function with $\det Du \geq 0$, we have

$$Du = \begin{pmatrix} u_x^1 & u_y^1 \\ u_x^2 & u_y^2 \end{pmatrix}, \quad |Du|^2 + 2|\det Du| = \left(u_x^1 + u_y^2\right)^2 + \left(u_y^1 - u_x^2\right)^2.$$

A crucial step in the resolution of the variational problem that we consider here, related to the integrand $f$ in (1.33), is obtained by restricting ourselves to vector-valued functions $u$ which are gradients of functions $v \in W^{2,\infty}(\Omega)$; i.e., $u = \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$; thus we obtain

$$Du = \begin{pmatrix} v_{xx} & v_{xy} \\ v_{xy} & v_{yy} \end{pmatrix}, \quad |Du|^2 + 2|\det Du| = (v_{xx} + v_{yy})^2 = (\Delta v)^2. \tag{1.36}$$

The compatibility condition (1.35) on the boundary datum $\varphi$ becomes

$$\varphi \in C^2\left(\overline{\Omega}\right) \quad \text{and} \quad 0 < \Delta\varphi(x) < 1, \;\; \det D^2\varphi(x) > 0. \tag{1.37}$$

By applying Theorem **??** of Chapter 3 with $a = 0$ and $b = 1$, we can find $w \in \varphi + W_0^{2,\infty}(\Omega)$ such that

$$\begin{cases} \Delta w(x) \in \{0,1\}, & \text{a.e. } x \in \Omega, \\ \det D^2 w(x) \geq 0, & \text{a.e. } x \in \Omega. \end{cases} \tag{1.38}$$

Since either $\Delta w = 0$ or $\Delta w = 1$, a.e. in $\Omega$, by (1.34), (1.36) we obtain

$$Qf\left(D^2 w(x)\right) = f\left(D^2 w(x)\right), \quad \text{a.e. } x \in \Omega.$$

Then, as stated in Theorem 1.18, we can easily prove (see Section 3.3.3 for more details) that the function $\overline{u} = \begin{pmatrix} w_x \\ w_y \end{pmatrix}$ is a minimizer of the integral $\int_\Omega f(Du(x))\,dx$ in the class of functions $u \in W^{1,\infty}\left(\Omega;\mathbb{R}^2\right)$ such that $u = u_0 = D\varphi$ on $\partial\Omega$.

**Theorem 1.18** *Let $\Omega$ be a bounded open set of $\mathbb{R}^2$. Let $u_0 : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear boundary datum, such that $Du_0$ is a constant symmetric $2\times 2$ matrix satisfying the conditions*

$$0 < \operatorname{trace} Du_0 < 1, \quad \det Du_0 > 0.$$

*Let $f$ be defined in (1.33). Then the nonconvex variational problem*

$$\min\left\{\int_\Omega f(Du(x))\,dx \; : \; u \in W^{1,\infty}\left(\Omega;\mathbb{R}^2\right), \quad u = u_0 \text{ on } \partial\Omega\right\}$$

*has a solution $\overline{u} \in u_0 + W_0^{1,\infty}\left(\Omega;\mathbb{R}^2\right)$. Moreover there exists $w \in W^{2,\infty}(\Omega)$ satisfying (1.38) such that $\overline{u} = Dw$.*

## 1.5 Some unsolved problems

In this section we propose some open problems that are related to the material of this book.

### *1.5.1 Selection criterion*

The Baire category approach, as well as the convex integration method, are purely “existential” contrary to the viscosity method, which in the convex scalar case gives, among other properties, uniqueness.

A natural question, particularly in the vectorial context, is the choice, among the many solutions, of a special one.

In some scalar cases the viscosity solution is the pointwise maximal (or minimal) solution among all Lipschitz ones. Another characterization of viscosity solutions is by passing to the limit, using the maximum principle, in some elliptic regularized problems; indeed this is the historical approach. The maximum principle and the notion of maximality are not clearly defined for vectors.

The question is whether one can find a simple criterion of selection in the vectorial case or, incidentally, in the scalar case when there is no viscosity solution. The selection of one special solution is, of course, of importance also for numerical purposes.

### *1.5.2 Measurable Hamiltonians*

Consider the problem

$$\begin{cases} F(x, u(x), Du(x)) = 0, & \text{a.e. in } \Omega \\ u = \varphi, & \text{on } \partial\Omega, \end{cases} \tag{1.39}$$

where $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function $F = F(x, s, \xi)$, i.e., $F$ is measurable in $x$ and continuous in $(s, \xi)$.

The question is: does there exist $W^{1,\infty}$ solutions of (1.39)?

In this book we consider *continuous* functions $F$. Almost the same proofs could handle *semicontinuity* with respect to $x$ but not general measurability. This problem also arises in the viscosity context, even for the eikonal equation

$$\begin{cases} |Du(x)| = a(x), & \text{a.e. in } \Omega \\ u = \varphi, & \text{on } \partial\Omega \end{cases} \tag{1.40}$$

(see for example Newcomb-Su [255] for bounded *lower semicontinuous* functions $a$).

The same problem can be posed either in the vectorial context, or for systems, or for higher order equations.

### *1.5.3 Lipschitz boundary data*

Most of our *vectorial* existence theorems require the boundary datum to be $C^1$ or piecewise $C^1$ ($C^N$ in the $N$th order case). Only those involving convex sets (c.f. Theorem 1.8, 1.10 and 1.11) allow for $W^{1,\infty}$ data ($W^{N,\infty}$ in the $N$th order case), with in addition a compactness inclusion.

The question is: can we treat $W^{1,\infty}$ compatibility conditions? In the scalar case this can be achieved, c.f. Theorem 1.4.

### *1.5.4 Approximation of Lipschitz functions by smooth functions*

Related to the previous question is the following one concerning approximation of $W^{1,\infty}$ functions by either smooth functions or piecewise affine ones, under some constraints. Before formulating precisely the problem, we start with the scalar case. In Corollary **??** we prove that given open sets $\Omega, A \subset \mathbb{R}^n$, $\varepsilon > 0$, and a function $u \in W^{1,\infty}(\Omega)$ with

$$Du(x) \in A, \text{ a.e. } x \in \Omega,$$

there exists a function $v \in W^{1,\infty}(\Omega)$ such that

$$\begin{cases} v \text{ is piecewise affine on } \Omega; \\ v = u \text{ on } \partial\Omega; \\ \|v - u\|_{L^\infty(\Omega)} < \varepsilon; \\ Dv(x) \in A, \text{ a.e. } x \in \Omega. \end{cases} \tag{1.41}$$

Can this be done in the vectorial case? A similar question can be asked for approximation by smooth functions instead of piecewise affine ones. We achieve this, in the vectorial context, (c.f. Corollary **??**) only when $A$ is convex and $Du$ is compactly contained in the interior of $A$.

### *1.5.5 Extension of Lipschitz functions and compatibility conditions*

When solving, for example, a problem of the form

$$\begin{cases} Du(x) \in E, & \text{a.e. } x \in \Omega \\ u(x) = \varphi(x), & x \in \partial\Omega, \end{cases}$$

we require that the boundary datum $\varphi \in W^{1,\infty}(\Omega)$ satisfies

$$D\varphi(x) \in E \cup \operatorname{int} \overline{\operatorname{co} E}, \text{ a.e. } x \in \Omega,$$

or, in the vectorial case (with some extra hypotheses),

$$D\varphi(x) \in E \cup \operatorname{int} \overline{\operatorname{Qco} E}, \text{ a.e. } x \in \Omega.$$

Of course it is, a priori, not completely natural to ask that the boundary datum $\varphi$ be defined on the whole of $\overline{\Omega}$; one should give necessary and/or sufficient conditions only in terms of values of $\varphi$ given on the boundary $\partial\Omega$. This can be achieved (c.f. Section 2.4) when $\varphi$ is scalar; for example, for the eikonal equation (when the domain $\Omega$ is convex)

$$\begin{cases} |Du(x)| = 1, & \text{a.e. } x \in \Omega \\ u(x) = \varphi(x), & x \in \partial\Omega; \end{cases}$$

the condition is the Lipschitz continuity of $\varphi$ with constant 1, i.e.,

$$|\varphi(x) - \varphi(y)| \leq |x - y|, \quad \forall\, x, y \in \partial\Omega.$$

However in the vectorial case, it is an open problem to give necessary and/or sufficient conditions only in terms of values of $\varphi$ on the boundary $\partial\Omega$, except in some special cases; c.f. Kirszbraun theorem (Theorem 2.10.43 in Federer [151]).

### 1.5.6 Existence under quasiconvexity assumption

We have already pointed out that the natural condition to solve

$$\begin{cases} Du(x) \in E, & \text{a.e. } x \in \Omega \\ u(x) = \varphi(x), & x \in \partial\Omega \end{cases}$$

could be

$$D\varphi(x) \in E \cup \operatorname{int} \overline{\operatorname{Qco} E}, \text{ a.e. } x \in \Omega. \tag{1.42}$$

In the present book we are able to do this only under further assumptions on the quasiconvex hull of $E$; in particular we require the so-called relaxation property which is, in general, difficult to verify. The question is therefore to know if (1.42) is sufficient for existence.

### 1.5.7 Problems with constraints

We start by mentioning one case which might be relevant to nonlinear elasticity, although the question of constraints is more general.

Given $\varphi \in W^{1,\infty}(\Omega; \mathbb{R}^n)$ satisfying

$$\begin{cases} F(x, \varphi(x), D\varphi(x)) \leq 0, & \text{a.e. } x \in \Omega, \\ \det D\varphi(x) > 0, & \text{a.e. } x \in \Omega, \end{cases} \tag{1.43}$$

with some appropriate hypotheses on $F$, we ask if we can find a function $u \in W^{1,\infty}(\Omega; \mathbb{R}^n)$ such that

$$\begin{cases} F(x, u(x), Du(x)) = 0, & \text{a.e. } x \in \Omega, \\ \det Du(x) > 0, & \text{a.e. } x \in \Omega, \\ u(x) = \varphi(x), & x \in \partial\Omega. \end{cases} \tag{1.44}$$

We achieve this result (c.f. Theorem **??**; see also (1.38)) in a particular case of second order equations.

A similar question arises if we assume that

$$\begin{cases} F(x, \varphi(x), D\varphi(x)) \leq 0, & \text{a.e. } x \in \Omega, \\ \det D\varphi(x) = 1, & \text{a.e. } x \in \Omega; \end{cases} \tag{1.45}$$

in this case we look for a function $u \in W^{1,\infty}(\Omega; \mathbb{R}^n)$ such that

$$\begin{cases} F(x, u(x), Du(x)) = 0, & \text{a.e. } x \in \Omega, \\ \det Du(x) = 1, & \text{a.e. } x \in \Omega, \\ u(x) = \varphi(x), & x \in \partial\Omega. \end{cases} \tag{1.46}$$

In a more general context, under appropriate compatibility conditions on the boundary datum $\varphi$, the question is to find a map $u \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ satisfying

$$\begin{cases} F(x, u(x), Du(x)) = 0, & \text{a.e. } x \in \Omega, \\ G(x, u(x), Du(x)) < 0, & \text{a.e. } x \in \Omega, \\ u(x) = \varphi(x), & x \in \partial\Omega. \end{cases}$$

A similar question arises if we replace the constraint with strict inequality by either $G(x,u,Du) \leq 0$ or by $G(x,u,Du) = 0$.

The problem (1.46) can be considered as a case where

$$\operatorname{int} \overline{\operatorname{Qco} E} = \emptyset.$$

This phenomenon also happens in the linear (or quasilinear) case. For example, second order problems can be considered as first order systems with the linear constraints

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial u_j}{\partial x_i}, \quad i,j = 1,2,\ldots,n;$$

consequently second order equations, when seen as first order systems, have $\operatorname{int} \overline{\operatorname{Qco} E} = \emptyset$. The last one is a case already solved in this book.

### *1.5.8 Potential wells*

The problem of *potential wells* is described in Chapter 8 (see also Section 1.1.3). Under the notation of Chapter 8, the problem of potential wells consists in finding a function $u \in \varphi + W_0^{1,\infty}(\Omega;\mathbb{R}^n)$, $\Omega \subset \mathbb{R}^n$, satisfying the differential problem (the $N$ *wells* are $SO(n)\,A_i$, $1 \leq i \leq N$)

$$\begin{cases} Du(x) \in E = \bigcup_{i=1}^{N} SO(n)\,A_i \\ u(x) = \varphi(x), \quad x \in \partial\Omega. \end{cases} \tag{1.47}$$

The problem has been solved when $N = 2$ (i.e., *two potential wells*) and $n = 2$ (i.e., *dimension two*). The question is: can problem (1.47) be solved when $N \geq 3$ and/or $n \geq 3$?

The problem is already at the algebraic level of computing the rank one convex hull.

### *1.5.9 Calculus of variations*

A question in the *scalar case* is: can Theorem 1.17 be generalized to integrands $f$ which also depend on $(x,u)$, searching more generally for $W^{1,p}$ solutions?

In the *vectorial case*, can we give a sufficiently general class of nonquasiconvex functions for which there is attainment of the minimum?

For example, when $n = m \geq 2$, integrals of the calculus of variations related to functions of the form

$$f(\xi) = g(\det \xi),$$

even with $g$ not convex, are relatively well understood (c.f. [107]). However, for $n, m \geq 2$, functions of the type

$$f(\xi) = g(|\xi|),$$

with $g$ not convex, are treated only in some particular cases, such as the one of Theorem 1.18. See also [107] for some necessary conditions.

Relevant functions for applications, which combine the two previous cases, when $n = m \geq 2$ are of the form

$$f(\xi) = g(|\xi|, \det \xi). \tag{1.48}$$

If $g$ is not convex, the question is to find sufficient conditions on $g$ to obtain minimizers of the related integral.

In particular, the phenomenon of *cavitation* in nonlinear elasticity (introduced by Ball [28]) enter in this context. Realistic mathematical assumptions for the problem of cavitation, related to a nonconvex function $g$ in (1.48), have been introduced and studied by Marcellini [229], [230] (see also Section 2.6.3, Volume 2, of the recent book by Giaquinta, Modica and Soucek [176]). The existence of minimizers under realistic assumptions is still an open problem.

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