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Article

Improper Integrals Involving Powers of Inverse Trigonometric and Hyperbolic Functions

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Abstract: Three classes of improper integrals involving higher powers of arctanh, arctan, and arcsin are examined using the recursive approach. Numerous explicit formulae are established, which evaluate these integrals in terms of π, $\ln 2$, the Riemann zeta function, and the Dirichlet beta function.

Keywords: integration by parts; trigonometric functions; Fourier series; Riemann zeta function; Dirichlet beta function

MSC: 11M35; 33B10; 33B30

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1. Introduction and Outline

The evaluation of integrals is an important subject in mathematics, physics and applied sciences. In the mathematical literature (see, for example, the monographs by Boros and Moll [1], and Vălean [2]), there are numerous intriguing integrals. We reproduce, for instance, the following elegant integrals involving inverse trigonometric and hyperbolic functions, where G denotes the Catalan's constant:

$$\text{Entry (a)} \int_0^1 \frac{\operatorname{arctanh}^3 x}{x} dx = \frac{\pi^4}{64}, \qquad [3]$$

$$\text{Entry (b)} \int_0^1 \frac{\operatorname{arctanh}^3 x}{x^2} dx = \frac{3}{2}\zeta(3), \qquad [3]$$

$$\text{Entry (c)} \int_0^1 \frac{\arctan x}{x} dx = G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}, \qquad [4–7]$$

$$\text{Entry (d)} \int_0^1 \frac{\arctan^2 x}{x} dx = \frac{\pi G}{2} - \frac{7}{8}\zeta(3), \qquad ([8], (A.289)) \text{ and } [3,5,6]$$

$$\text{Entry (e)} \int_0^1 \frac{\arcsin x}{x} dx = \frac{\pi}{2}\ln 2, \qquad ([9], §4.521: \text{Equation (1)})$$

$$\text{Entry (f)} \int_0^1 \frac{\arcsin^2 x}{x} dx = \frac{\pi^2}{4}\ln 2 - \frac{7}{8}\zeta(3). \qquad ([1], \text{Equation 6.6.25})$$

Some related integrals of log-trigonometric functions are highlighted as follows:

$$\text{Entry (g) } G = \int_0^{\frac{\pi}{4}} \ln(\cot x) dx = 2\int_0^{\frac{\pi}{4}} \ln(2\cos x) dx, \qquad [5,7,10,11]$$

$$\text{Entry (h) } \frac{\pi}{2}\ln 2 = \int_0^{\frac{\pi}{2}} \ln(\sec x) dx = \int_0^{\frac{\pi}{2}} \ln(\csc x) dx. \qquad [4,12]$$

Euler (1772) discovered the identity (h) and the following remarkable value

$$\int_0^{\frac{\pi}{2}} x \ln(\sin x) dx = \frac{7\zeta(3)}{16} - \frac{\pi^2}{8}\ln 2$$

by making use of the Fourier series

$$\ln(\sin x) = -\ln 2 - \sum_{n=1}^{\infty} \frac{\cos(2nx)}{n}.$$

Koyama and Kurokawa [13] evaluated the integral below as well as the related indefinite integrals:

$$\int_0^{\frac{\pi}{2}} x^2 \ln(\sin x) dx = \frac{3\zeta(3)}{16} - \frac{\pi^3}{24} \ln 2.$$

Further integral identities of a similar nature can be found in the papers [14–18].

Motivated by these elegant formulae, we shall primarily investigate the following improper integrals with two integer parameters in this article:

$$\begin{aligned}
\mathrm{H}(m,n) &:= \int_0^1 \frac{\operatorname{arctanh}^m x}{x^n} dx, \\
\mathrm{T}(m,n) &:= \int_0^1 \frac{\arctan^m x}{x^n} dx, \\
\mathrm{S}(m,n) &:= \int_0^1 \frac{\arcsin^m x}{x^n} dx;
\end{aligned}$$

where $m \in \mathbb{N}_0$ and $n \in \mathbb{Z}$, subject to $m \geq n$, so that all these integrals are convergent. By making use of recurrence relations and Fourier series expansions, we shall explicitly evaluate, in the next three sections, these three classes of integrals. Two classes of subsidiary integrals $\Theta(m)$ and $\Lambda(m)$ regarding log-cosine and log-tangent functions will also be examined. Finally, the paper will conclude with a brief discussion of more integral evaluations in Section 5.

Throughout the paper, we shall utilize the following notations. Let $\mathbb{Z}$ and $\mathbb{N}$ stand, respectively, for the sets of integers and natural numbers with $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $n \in \mathbb{N}_0$ and an indeterminate x, the rising and falling factorials are defined by $(x)_0 = \langle x \rangle_0 = 1$ and

$$\left.\begin{aligned}
(x)_n &= x(x+1)\cdots(x+n-1) \\
\langle x \rangle_n &= x(x-1)\cdots(x-n+1)
\end{aligned}\right\} \quad \text{for} \quad n \in \mathbb{N}.$$

Given $i, j \in \mathbb{Z}$ and $m \in \mathbb{N}$, the symbol "$i \equiv_m j$" represents that "i is congruent to j modulo m". The logical function χ will also be employed with $\chi(\text{true}) = 1$ and $\chi(\text{false}) = 0$. In addition, we need the following four functions:

- Riemann zeta function: $\displaystyle \zeta(x) = \sum_{n=0}^{\infty} \frac{1}{(n+1)^x} \quad (\Re(x) > 1);$
- Dirichlet lambda function: $\displaystyle \lambda(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^x} \quad (\Re(x) > 1);$
- Dirichlet eta function: $\displaystyle \eta(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^x} \quad (\Re(x) > 0);$
- Dirichlet beta function: $\displaystyle \beta(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^x} \quad (\Re(x) > 0).$

2. Evaluation of $\mathbf{H}(m,n)$

When $n \neq 1$, the following algebraic equality holds:

$$\frac{(n-1)-(n-3)x^2}{(n-1)x^n} = \frac{1}{x^n} - \frac{n-3}{(n-1)x^{n-2}}.$$

Then, we have the integral equation below

$$\begin{aligned}\mathfrak{H}(m,n) &= \int_0^1 \frac{\operatorname{arctanh}^m x\{(n-1)-(n-3)x^2\}}{(n-1)x^n}dx \\ &= \int_0^1 \frac{\operatorname{arctanh}^m x}{x^n}dx - \frac{n-3}{n-1}\int_0^1 \frac{\operatorname{arctanh}^m x}{x^{n-2}}dx \\ &= \mathrm{H}(m,n) - \frac{n-3}{n-1}\mathrm{H}(m,n-2).\end{aligned}$$

Considering that

$$\frac{d}{dx}\frac{1-x^2}{(1-n)x^{n-1}} = \frac{(n-1)-(n-3)x^2}{(n-1)x^n},$$

we can alternatively express the integral $\mathfrak{H}(m,n)$, by integration by parts, as

$$\mathfrak{H}(m,n) = \frac{m}{n-1}\int_0^1 \frac{\operatorname{arctanh}^{m-1} x}{x^{n-1}}dx = \frac{m}{n-1}\mathrm{H}(m-1,n-1).$$

Combining the two expressions of $\mathfrak{H}(m,n)$ results in the following three-term recurrence relation

$$(n-1)\mathrm{H}(m,n) - m\mathrm{H}(m-1,n-1) - (n-3)\mathrm{H}(m,n-2) = 0. \tag{1}$$

According to this relation, to compute all the values $\mathrm{H}(m,n)$ for $m \in \mathbb{N}_0$ and $n \in \mathbb{Z}$ with $m \geq n$, we have to determine the boundary values $\{\mathrm{H}(0,n), \mathrm{H}(m,0), \mathrm{H}(m,1)\}$.

2.1. $\mathrm{H}(0,n)$

For $n \leq 0$, we have the following obvious values:

$$\mathrm{H}(0,n) = \int_0^1 x^{-n}dx = \frac{1}{1-n}. \tag{2}$$

2.2. $\mathrm{H}(m,0)$

When $m = 0$, it is easy to see that $\mathrm{H}(0,0) = 1$. For $m \geq 1$, by changing the variable $x \to \frac{1-y}{1+y}$, we have

$$\mathrm{H}(m,0) = \int_0^1 \left(\frac{1}{2}\ln\frac{1+x}{1-x}\right)^m dx = \frac{(-1)^m}{2^{m-1}}\int_0^1 \frac{\ln^m y}{(1+y)^2}dy.$$

According to the power series expansion

$$(1+y)^{-2} = \sum_{k=0}^{\infty}(-1)^k(k+1)y^k \qquad (|y|<1);$$

we can express

$$\mathrm{H}(m,0) = \frac{(-1)^m}{2^{m-1}}\sum_{k=0}^{\infty}(-1)^k(k+1)\int_0^1 y^k \ln^m y dy.$$

By repeatedly applying integration by parts, we can evaluate the last integral

$$\int_0^1 y^k \ln^m y dy = \sum_{i=0}^{m}\frac{(-1)^i\langle m\rangle_i}{(k+1)^{i+1}}y^{k+1}\ln^{m-i} y\Big|_0^1 = \frac{(-1)^m m!}{(k+1)^{m+1}}. \tag{3}$$

Hereafter, exchanges in the order of summation and integration are justified by Lebesgue's dominated convergence theorem ([19], §11.32). By substitution, we can obtain the closed formula

$$\mathrm{H}(m,0) = \frac{m!}{2^{m-1}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^m} = \frac{m!}{2^{m-1}} \eta(m). \tag{4}$$

2.3. $\mathrm{H}(m,1)$

We can also evaluate $\mathrm{H}(m,1)$ by carrying out the same procedure as for $\mathrm{H}(m,0)$. In fact, for $m \geq 1$, by making the change in variable $x \to \frac{1-y}{1+y}$, we can express

$$\mathrm{H}(m,1) = \int_0^1 \frac{1}{x}\left(\frac{1}{2}\ln\frac{1+x}{1-x}\right)^m dx = \frac{(-1)^m}{2^{m-1}} \int_0^1 \frac{\ln^m y}{1-y^2} dy.$$

We take the above integral as an example to show how to justify the term-by-term integration by making use of Lebesgue's dominated convergence theorem. For any fixed x with $0 < x < 1$, we have the following power series expansion

$$\frac{1}{1-y^2} = \sum_{k=0}^{\infty} y^{2k}, \quad \text{where} \quad 0 \leq y \leq x.$$

Now, define the following sequence of functions

$$\varphi_n(y) = \ln^m y \sum_{k=0}^{n} y^{2k} \quad \text{such that} \quad \lim_{n\to\infty} \varphi_n(y) = \frac{\ln^m y}{1-y^2} \quad \text{for} \quad y \in [0,x].$$

When $0 \leq y \leq x$, we have

$$\left|\varphi_n(y)\right| < \left|\frac{\ln^m y}{1-y^2}\right| \leq (-1)^m \frac{\ln^m y}{1-x^2},$$

where the rightmost function is dominating and integrable over $[0,x]$. According to Lebesgue's dominated convergence theorem, we can proceed using term-by-term integration

$$\begin{aligned}\int_0^x \frac{\ln^m y}{1-y^2} dy &= \lim_{n\to\infty} \int_0^x \varphi_n(y) dy = \lim_{n\to\infty} \sum_{k=0}^{n} \int_0^x y^{2k} \ln^m y dy \\ &= \sum_{k=0}^{\infty} \sum_{i=0}^{m} \frac{(-1)^i \langle m \rangle_i}{(2k+1)^{i+1}} x^{2k+1} \ln^{m-i} x,\end{aligned}$$

where we have employed integral formula (3). Observing that the last series is uniformly convergent for $x \in [\frac{1}{2}, 1]$, we can evaluate the series through the term-by-term limit, as follows:

$$\begin{aligned}\mathrm{H}(m,1) &= \frac{(-1)^m}{2^{m-1}} \lim_{x\to 1^-} \int_0^x \frac{\ln^m y}{1-y^2} dy \\ &= \frac{(-1)^m}{2^{m-1}} \lim_{x\to 1^-} \sum_{k=0}^{\infty} \sum_{i=0}^{m} \frac{(-1)^i \langle m \rangle_i}{(2k+1)^{i+1}} x^{2k+1} \ln^{m-i} x \\ &= \frac{(-1)^m}{2^{m-1}} \sum_{k=0}^{\infty} \sum_{i=0}^{m} \frac{(-1)^i \langle m \rangle_i}{(2k+1)^{i+1}} \lim_{x\to 1^-} x^{2k+1} \ln^{m-i} x \\ &= \frac{m!}{2^{m-1}} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{m+1}}.\end{aligned}$$

This gives rise to the below formula

$$\mathrm{H}(m,1) = \frac{m!}{2^{m-1}}\lambda(m+1). \tag{5}$$

In conclusion, we have shown the following general theorem. Its special case $\mathrm{H}(3,n)$ was studied by Sofo and Nimbran [3].

Theorem 1 ($m \in \mathbb{N}_0$ and $n \in \mathbb{Z}$ with $m \geq n$).

$$\boxed{n=0} \quad \mathrm{H}(m,0) = \begin{cases} 1, & m=0; \\ \dfrac{m!}{2^{m-1}}\eta(m), & m \geq 1; \end{cases}$$

$$\boxed{n=1} \quad \mathrm{H}(m,1) = \frac{m!}{2^{m-1}}\lambda(m+1),\ m \geq 1;$$

$$\boxed{n\geq 2} \quad \mathrm{H}(m,n) = \frac{m}{n-1}\mathrm{H}(m-1,n-1) + \frac{n-3}{n-1}\mathrm{H}(m,n-2),\ m \geq n;$$

$$\boxed{n<0} \quad \mathrm{H}(m,n) = \begin{cases} \dfrac{1}{1-n}, & m=0; \\ \dfrac{n+1}{n-1}\mathrm{H}(m,n+2) - \dfrac{m}{n-1}\mathrm{H}(m-1,n+1), & m \geq 1. \end{cases}$$

Keeping in mind that $\eta(k)$ and $\lambda(k)$ can be written in terms of the zeta function (except for $\eta(1) = \ln 2$), we assert, according to this theorem, that $\mathrm{H}(m,n)$ is always expressible as a linear combination of $\ln 2$ and zeta values. The integral values for $\mathrm{H}(m,n)$ with $1 \leq m \leq 5$ and $-5 \leq n \leq 5$ are recorded in Table 1.

Table 1. Values for $\mathrm{H}(m,n)$.

$n\backslash m$	1	2	3	4	5
-5	$\frac{23}{90}$	$\frac{19}{180}+\frac{23\ln 2}{45}$	$\frac{1}{3}+\frac{23\pi^2}{360}$	$\frac{1}{30}+\frac{23\zeta(3)}{20}+\frac{4\ln 2}{3}$	$\frac{1}{6}+\frac{5\pi^2}{18}+\frac{161\pi^4}{4320}$
-4	$\frac{3}{20}+\frac{\ln 2}{5}$	$\frac{1}{3}+\frac{\pi^2}{60}$	$\frac{1}{20}+\frac{9\zeta(3)}{40}+\ln 2$	$\frac{1}{5}+\frac{\pi^2}{6}+\frac{7\pi^4}{1200}$	$\frac{15\zeta(3)}{4}+\frac{45\zeta(5)}{32}+\ln 2$
-3	$\frac{1}{3}$	$\frac{1}{12}+\frac{2\ln 2}{3}$	$\frac{1}{4}+\frac{\pi^2}{12}$	$\frac{3\zeta(3)}{2}+\ln 2$	$\frac{5\pi^2}{24}+\frac{7\pi^4}{144}$
-2	$\frac{1}{6}+\frac{\ln 2}{3}$	$\frac{1}{3}+\frac{\pi^2}{36}$	$\frac{3\zeta(3)}{8}+\ln 2$	$\frac{\pi^2}{6}+\frac{7\pi^4}{720}$	$\frac{15\zeta(3)}{4}+\frac{75\zeta(5)}{32}$
-1	$\frac{1}{2}$	$\ln 2$	$\frac{\pi^2}{8}$	$\frac{9\zeta(3)}{4}$	$\frac{7\pi^4}{96}$
0	$\ln 2$	$\frac{\pi^2}{12}$	$\frac{9\zeta(3)}{8}$	$\frac{7\pi^4}{240}$	$\frac{225\zeta(5)}{32}$
1	$\frac{\pi^2}{8}$	$\frac{7\zeta(3)}{8}$	$\frac{\pi^4}{64}$	$\frac{93\zeta(5)}{32}$	$\frac{\pi^6}{128}$
2	★	$\frac{\pi^2}{6}$	$\frac{3\zeta(3)}{2}$	$\frac{\pi^4}{30}$	$\frac{15\zeta(5)}{2}$
3	★	★	$\frac{\pi^2}{4}$	$3\zeta(3)$	$\frac{\pi^4}{12}$
4	★	★	★	$\frac{\pi^2}{3}+\frac{\pi^4}{90}$	$5\zeta(3)+\frac{5\zeta(5)}{2}$
5	★	★	★	★	$\frac{5\pi^2}{12}+\frac{\pi^4}{18}$

(★ indicates that the corresponding integral diverges.)

3. Evaluation of $\mathrm{T}(m,n)$

Supposing $n \neq 1$, by making use of integration by parts, we can obtain

$$\mathrm{T}(m,n) = \frac{1}{1-n}\left(\frac{\pi}{4}\right)^m + \frac{m}{n-1}\int_0^1 \frac{\arctan^{m-1} x}{x^{n-1}(1+x^2)}dx. \quad (6)$$

When $m \neq 0$, the above integral can further be manipulated as

$$\begin{aligned}\int_0^1 \frac{\arctan^{m-1} x}{x^{n-1}(1+x^2)}dx &= \int_0^1 \frac{\arctan^{m-1} x}{x^{n-1}}dx - \int_0^1 \frac{\arctan^{m-1} x}{x^{n-3}(1+x^2)}dx \\ &= \mathrm{T}(m-1,n-1) - \frac{n-3}{m}\mathrm{T}(m,n-2) - \frac{1}{m}\left(\frac{\pi}{4}\right)^m.\end{aligned}$$

Substituting this into (6), and then simplifying the resulting expression, we can derive the following three-term recurrence relation

$$(n-1)\mathrm{T}(m,n) - m\mathrm{T}(m-1,n-1) + (n-3)\mathrm{T}(m,n-2) + 2\left(\frac{\pi}{4}\right)^m = 0. \quad (7)$$

Based on this relation, to calculate all the $\mathrm{T}(m,n)$ for $m \in \mathbb{N}_0$ and $n \in \mathbb{Z}$ with $m \geq n$, it is sufficient to determine the boundary values $\{\mathrm{T}(0,n), \mathrm{T}(m,0), \mathrm{T}(m,1)\}$.

3.1. $\mathrm{T}(0,n)$

Firstly, it is trivial to check for $n \leq 0$ that

$$\mathrm{T}(0,n) = \int_0^1 x^{-n}dx = \frac{1}{1-n}. \quad (8)$$

3.2. $\mathrm{T}(m,0)$

Then for $m = 0,\ 1$, we have no difficulty evaluating

$$\begin{aligned}\mathrm{T}(0,0) &= \int_0^1 dx = 1, \\ \mathrm{T}(1,0) &= \int_0^1 \arctan x dx = \frac{\pi}{4} - \frac{\ln 2}{2}.\end{aligned}$$

When $m \geq 2$, applying integration by parts twice shows that

$$\begin{aligned}\mathrm{T}(m,0) = \int_0^1 \arctan^m x dx &= \left(\frac{\pi}{4}\right)^m - m\int_0^1 \frac{x\arctan^{m-1} x}{1+x^2}dx \\ &= \left(\frac{\pi}{4}\right)^m - \frac{m\ln 2}{2}\left(\frac{\pi}{4}\right)^{m-1} + \frac{m(m-1)}{2}\int_0^1 \frac{\ln(1+x^2)}{1+x^2}\arctan^{m-2} dx.\end{aligned}$$

Then under the change in variable $x \to \tan y$, the last expression becomes

$$\mathrm{T}(m,0) = \left(\frac{\pi}{4}\right)^m - \frac{m\ln 2}{2}\left(\frac{\pi}{4}\right)^{m-1} - m(m-1)\Theta(m-2), \quad (9)$$

where $\Theta(m)$ stands for the parametric integral

$$\Theta(m) = \int_0^{\frac{\pi}{4}} y^m \ln(\cos y)dy \quad \text{with} \quad m \geq 0. \quad (10)$$

3.3. $\Theta(m)$

To evaluate the integral $\Theta(m)$, we recall the following Fourier series (cf. [9], §1.441)

$$\ln(\cos y) = -\ln 2 - \sum_{k=1}^{\infty} \frac{(-1)^k}{k}\cos(2ky), \quad \text{where} \quad |y| < \frac{\pi}{2}.$$

Then, using Lebesgue's dominated convergence theorem ([19], §11.32), we can express

$$\Theta(m)=\int_0^{\frac{\pi}{4}} y^m \ln(\cos y)dy=\frac{-\ln 2}{m+1}\Big(\frac{\pi}{4}\Big)^{m+1}+\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\int_0^{\frac{\pi}{4}} y^m\cos(2ky)dy.$$

Applying integration by parts for m times, we can evaluate the last integral as follows:

$$\begin{aligned}\mathbf{I}(m,k)=\int_0^{\frac{\pi}{4}} y^m\cos(2ky)dy&=\sum_{j=0}^{m}\frac{\langle m\rangle_j}{(2k)^{j+1}}y^{m-j}\sin\Big(2ky+\frac{j\pi}{2}\Big)\Big|_0^{\frac{\pi}{4}}\\&=\frac{m!}{(2k)^{m+1}}\sin\Big(\frac{m+2}{2}\pi\Big)+\sum_{j=0}^{m}\frac{\langle m\rangle_j}{(2k)^{j+1}}\Big(\frac{\pi}{4}\Big)^{m-j}\sin\Big(\frac{k+j}{2}\pi\Big).\end{aligned}$$

Taking into account the trigonometric identity

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta, \tag{11}$$

we can reformulate the infinite series

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\mathbf{I}(m,k)=&\frac{m!}{2^{m+1}}\sin\Big(\frac{m+2}{2}\pi\Big)\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^{m+2}}\\&+\sum_{j=0}^{m}\frac{\langle m\rangle_j}{2^{j+1}}\Big(\frac{\pi}{4}\Big)^{m-j}\cos\Big(\frac{j\pi}{2}\Big)\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^{j+2}}\sin\Big(\frac{k\pi}{2}\Big)\\&+\sum_{j=0}^{m}\frac{\langle m\rangle_j}{2^{j+1}}\Big(\frac{\pi}{4}\Big)^{m-j}\sin\Big(\frac{j\pi}{2}\Big)\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^{j+2}}\cos\Big(\frac{k\pi}{2}\Big).\end{aligned}$$

Observing that

$$\sin\Big(\frac{k\pi}{2}\Big)=(-1)^{\frac{k-1}{2}}\chi(k\equiv_2 1)\quad\text{and}\quad\cos\Big(\frac{k\pi}{2}\Big)=(-1)^{\frac{k}{2}}\chi(k\equiv_2 0), \tag{12}$$

we have

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^{j+2}}\sin\Big(\frac{k\pi}{2}\Big)&=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^{j+2}}=\beta(j+2),\\\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^{j+2}}\cos\Big(\frac{k\pi}{2}\Big)&=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k)^{j+2}}=\frac{\eta(j+2)}{2^{j+2}}.\end{aligned}$$

By substitution, we obtain

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\mathbf{I}(m,k)=&\frac{m!}{2^{m+1}}\eta(m+2)\sin\Big(\frac{m+2}{2}\pi\Big)\\&+\sum_{j=0}^{m}\frac{\langle m\rangle_j}{2^{j+1}}\Big(\frac{\pi}{4}\Big)^{m-j}\beta(j+2)\cos\Big(\frac{j\pi}{2}\Big)\\&+\sum_{j=0}^{m}\frac{\langle m\rangle_j}{2^{2j+3}}\Big(\frac{\pi}{4}\Big)^{m-j}\eta(j+2)\sin\Big(\frac{j\pi}{2}\Big).\end{aligned}$$

Keeping in mind of (12), we have established the following explicit formula.

Proposition 1 ($m \in \mathbb{N}_0$)**.**

$$\Theta(m) = \frac{m!}{2^{m+1}}\eta(m+2)\sin\left(\frac{m+2}{2}\pi\right) + \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor}(-1)^j \frac{\langle m \rangle_{2j}}{2^{2j+1}}\left(\frac{\pi}{4}\right)^{m-2j}\beta(2j+2)$$
$$-\frac{\ln 2}{m+1}\left(\frac{\pi}{4}\right)^{m+1} - \sum_{j=1}^{\lfloor \frac{m+1}{2} \rfloor}(-1)^j \frac{\langle m \rangle_{2j-1}}{2^{4j+1}}\left(\frac{\pi}{4}\right)^{m+1-2j}\eta(2j+1).$$

From this proposition, we claim that $\Theta(m)$ can always be expressed in terms of $\ln 2$ and values of ζ-function and β-function (particularly $G = \beta(2)$). The initial values for small m are recorded below, where we can locate $\Theta(0)$ in Moll ([20], §8.4), and both $\Theta(0)$ and $\Theta(1)$ in Vălean ([2], Equations 3.87 and 3.113).

$$\begin{aligned}
\Theta(0) &= \frac{G}{2} - \frac{\pi \ln 2}{4}, \\
\Theta(1) &= \frac{\pi G}{8} - \frac{21\zeta(3)}{128} - \frac{\pi^2 \ln 2}{32}, \\
\Theta(2) &= \frac{\pi^2 G}{32} - \frac{\beta(4)}{4} + \frac{3\pi\zeta(3)}{256} - \frac{\pi^3 \ln 2}{192}, \\
\Theta(3) &= \frac{\pi^3 G}{128} - \frac{3\pi\beta(4)}{16} + \frac{1395\zeta(5)}{4096} + \frac{9\pi^2\zeta(3)}{2048} - \frac{\pi^4 \ln 2}{1024}, \\
\Theta(4) &= \frac{\pi^4 G}{512} - \frac{3\pi^2\beta(4)}{32} + \frac{3\beta(6)}{4} - \frac{45\pi\zeta(5)}{4096} + \frac{3\pi^3\zeta(3)}{2048} - \frac{\pi^5 \ln 2}{5120}, \\
\Theta(5) &= \tfrac{\pi^5 G}{2048} - \tfrac{5\pi^3\beta(4)}{128} + \tfrac{15\pi\beta(6)}{16} - \tfrac{120015\zeta(7)}{65536} - \tfrac{225\pi^2\zeta(5)}{32768} + \tfrac{15\pi^4\zeta(3)}{32768} - \tfrac{\pi^6 \ln 2}{24576}.
\end{aligned}$$

3.4. $\mathrm{T}(m,1)$

Applying integration by parts, we have

$$\mathrm{T}(m,1) = \int_0^1 \frac{\arctan^m x}{x}dx = -m\int_0^1 \frac{\ln x}{1+x^2}\arctan^{m-1} x dx.$$

Then, making the change of variable $x \to \tan y$, we can express

$$\mathrm{T}(m,1) = -m\int_0^{\frac{\pi}{4}} y^{m-1}\ln(\tan y)dy = -m\Lambda(m-1). \tag{13}$$

Henceforth, $\Lambda(m)$ is defined by the parametric integral

$$\Lambda(m) = \int_0^{\frac{\pi}{4}} y^m \ln(\tan y)dy \quad \text{for} \quad m \in \mathbb{N}_0. \tag{14}$$

3.5. $\Lambda(m)$

Recalling another known Fourier series (cf. ([9], §1.442))

$$\ln(\tan y) = -2\sum_{k=1}^{\infty}\frac{\cos(4k-2)y}{2k-1}, \quad \text{where} \quad 0 < y < \frac{\pi}{2},$$

we can reformulate the integral $\Lambda(m)$ as

$$\Lambda(m) = -2\sum_{k=1}^{\infty}\frac{1}{2k-1}\int_0^{\frac{\pi}{4}} y^m \cos(4k-2)y dy.$$

Denote the last integral by $\mathbf{J}(m,k)$. Applying integration by parts for m times, we can evaluate this as follows:

$$\begin{aligned}\mathbf{J}(m,k) &= \int_0^{\frac{\pi}{4}} y^m \cos(4k-2)y dy = \sum_{j=0}^{m} \frac{\langle m\rangle_j}{(4k-2)^{j+1}} y^{m-j} \sin\left\{(4k-2)y + \frac{j\pi}{2}\right\}\Big|_0^{\frac{\pi}{4}} \\ &= \sum_{j=0}^{m} \frac{\langle m\rangle_j}{(4k-2)^{j+1}} \left(\frac{\pi}{4}\right)^{m-j} \sin\left(\frac{2k+j-1}{2}\pi\right) - \frac{m!}{(4k-2)^{m+1}} \sin\left(\frac{m\pi}{2}\right).\end{aligned}$$

By making use of the trigonometric identity (11), we can proceed

$$\begin{aligned}\Lambda(m) = -2\sum_{k=1}^{\infty} \frac{\mathbf{J}(m,k)}{2k-1} &= \frac{m!}{2^m} \sin\left(\frac{m\pi}{2}\right) \sum_{k=1}^{\infty} \frac{1}{(2k-1)^{m+2}} \\ &- \sum_{j=0}^{m} \frac{\langle m\rangle_j}{2^j} \left(\frac{\pi}{4}\right)^{m-j} \cos\left(\frac{j\pi}{2}\right) \sum_{k=1}^{\infty} \frac{\sin\left(\frac{2k-1}{2}\pi\right)}{(2k-1)^{j+2}} \\ &- \sum_{j=0}^{m} \frac{\langle m\rangle_j}{2^j} \left(\frac{\pi}{4}\right)^{m-j} \sin\left(\frac{j\pi}{2}\right) \sum_{k=1}^{\infty} \frac{\cos\left(\frac{2k-1}{2}\pi\right)}{(2k-1)^{j+2}}.\end{aligned}$$

Keeping in mind of (12), we can reduce the two trigonometric sums

$$\begin{aligned}&\sum_{k=1}^{\infty} \frac{\sin\left(\frac{2k-1}{2}\pi\right)}{(2k-1)^{j+2}} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^{j+2}} = \beta(j+2), \\ &\sum_{k=1}^{\infty} \frac{\cos\left(\frac{2k-1}{2}\pi\right)}{(2k-1)^{j+2}} = 0.\end{aligned}$$

Therefore, we have proved the following simplified expression

$$\Lambda(m) = \frac{m!}{2^m} \sin\left(\frac{m\pi}{2}\right)\lambda(m+2) - \sum_{j=0}^{m} \frac{\langle m\rangle_j}{2^j} \left(\frac{\pi}{4}\right)^{m-j} \beta(j+2) \cos\left(\frac{j\pi}{2}\right),$$

which is equivalent to the formula below.

Proposition 2 ($m \in \mathbb{N}_0$)**.**

$$\Lambda(m) = \frac{m!}{2^m} \lambda(m+2) \sin\left(\frac{m\pi}{2}\right) - \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^j \frac{\langle m\rangle_{2j}}{2^{2j}} \left(\frac{\pi}{4}\right)^{m-2j} \beta(2j+2).$$

It should be pointed out that a similar formula for $\int_0^{\frac{\pi}{2}} x^m \ln(\tan x) dx$ was found by Elaissaoui and Guennoun [21] by integrating the product of $\ln(\tan x)$ and the Euler polynomials.

In accordance with this proposition, we affirm that $\Lambda(m)$ can always be expressed by ζ-function and β-function values. The first few values for small m are displayed as follows, where the value for $\Lambda(1)$ can be found in ([2], Page 130).

$$
\begin{aligned}
\Lambda(0) &= -G, \\
\Lambda(1) &= \frac{7\zeta(3)}{16} - \frac{\pi G}{4}, \\
\Lambda(2) &= \frac{\beta(4)}{2} - \frac{\pi^2 G}{16}, \\
\Lambda(3) &= \frac{3\pi\beta(4)}{8} - \frac{\pi^3 G}{64} - \frac{93\zeta(5)}{128}, \\
\Lambda(4) &= \frac{3\pi^2\beta(4)}{16} - \frac{3\beta(6)}{2} - \frac{\pi^4 G}{256}, \\
\Lambda(5) &= \frac{5\pi^3\beta(4)}{64} - \frac{15\pi\beta(6)}{8} - \frac{\pi^5 G}{1024} + \frac{1905\zeta(7)}{512}.
\end{aligned}
$$

Summing up, we have proved the following general theorem.

Theorem 2 ($m \in \mathbb{N}_0$ and $n \in \mathbb{Z}$ with $m \geq n$)**.** *Let $\Theta(m)$ and $\Lambda(m)$ be as in Propositions 1 and 2, respectively. The integral values for $\mathrm{T}(m,n)$ are determined as follows:*

$$
\boxed{n=0}\ \mathrm{T}(m,0) = \begin{cases} 1, & m=0; \\ \dfrac{\pi}{4} - \dfrac{\ln 2}{2}, & m=1; \\ \left(\dfrac{\pi}{4}\right)^m - \dfrac{m\ln 2}{2}\left(\dfrac{\pi}{4}\right)^{m-1} - m(m-1)\Theta(m-2), & m \geq 2; \end{cases}
$$

$$
\boxed{n=1}\ \mathrm{T}(m,1) = -m\Lambda(m-1), \quad m \geq 1;
$$

$$
\boxed{n \geq 2}\ \mathrm{T}(m,n) = \tfrac{m}{n-1}\mathrm{T}(m-1,n-1) - \tfrac{n-3}{n-1}\mathrm{T}(m,n-2) - \tfrac{2}{n-1}\left(\tfrac{\pi}{4}\right)^m,\ m \geq n;
$$

$$
\boxed{n < 0}\ \mathrm{T}(m,n) = \begin{cases} \tfrac{1}{1-n}, & m=0; \\ \tfrac{m}{n-1}\mathrm{T}(m-1,n+1) - \tfrac{n+1}{n-1}\mathrm{T}(m,n+2) - \tfrac{2}{n-1}\left(\tfrac{\pi}{4}\right)^m, & m \geq 1. \end{cases}
$$

Some particular results of this theorem are commented as follows:

- Both $\mathrm{T}(2,1)$ and $\mathrm{T}(2,2)$ can be found in Boyadzhiev ([8], Equations (A.289) and (5.54)).
- Kobayashi [6] evaluated $\mathrm{T}(1,1)$, $\mathrm{T}(2,1)$ and further

$$
\int_0^1 \frac{\arctan x \operatorname{arccot} x}{x} = \frac{\pi}{2}\mathrm{T}(1,1) - \mathrm{T}(2,1) = \frac{7}{8}\zeta(3).
$$

- Sofo and Nimbran [3] examined cases for $\mathrm{T}(2,n)$ and $\mathrm{T}(3,n)$.
- When $m = 1$ and $n < 0$, the recurrence relation in Theorem 2 reads as

$$
\mathrm{T}(1,n) = \frac{1+n}{1-n}\mathrm{T}(1,n+2) + \frac{\pi}{2-2n} + \frac{1}{n(1-n)}. \tag{15}
$$

Repeating this relation yields the next equation

$$
\mathrm{T}(1,n) = \frac{n+3}{n-1}\mathrm{T}(1,n+4) - \frac{2}{n(n-1)(n+2)},
$$

which is equivalent to a known recursion due to Chen [22].

By iterating (15) further, we can recover the following explicit formula, recorded by Gradshteyn and Ryzhik ([9], §4.532: Equation (1)):

$$\mathrm{T}(1,n)=\frac{\pi+\psi(\frac{2-n}{4})-\psi(\frac{4-n}{4})}{4-4n}. \tag{16}$$

where γ and ψ stand, respectively, for the Euler–Mascheroni constant and the digamma function (cf. Rainville ([23], §9))

$$\psi(x)=-\gamma+\sum_{n=0}^{\infty}\left\{\frac{1}{n+1}-\frac{1}{n+x}\right\}.$$

In view of this theorem, $\mathrm{T}(m,n)$ can always be expressed in terms of π, $\ln 2$ and values of ζ-function and β-function. The values for $\mathrm{T}(m,n)$ with $1\le m\le 3$ and $-3\le n\le 3$ are given in Table 2.

Table 2. Values for $\mathrm{T}(m,n)$.

$n\backslash m$	1	2	3
-3	$\frac{1}{6}$	$\frac{1}{12}+\frac{\pi}{12}-\frac{\ln 2}{3}$	$\frac{\pi}{8}-\frac{1}{4}+\frac{\pi\ln 2}{4}+\frac{\pi^2}{32}-G$
-2	$\frac{\pi}{12}-\frac{1}{6}+\frac{\ln 2}{6}$	$\frac{1}{3}-\frac{\pi}{6}-\frac{\pi\ln 2}{12}+\frac{\pi^2}{48}+\frac{G}{3}$	$\frac{\pi}{4}-\frac{\ln 2}{2}-\frac{\pi^2}{16}-\frac{\pi^2\ln 2}{32}+\frac{\pi^3}{192}+\frac{\pi G}{4}-\frac{21\zeta(3)}{64}$
-1	$\frac{\pi}{4}-\frac{1}{2}$	$\frac{\pi^2}{16}-\frac{\pi}{4}+\frac{\ln 2}{2}$	$\frac{\pi^3}{64}-\frac{3\pi^2}{32}-\frac{3\pi\ln 2}{8}+\frac{3G}{2}$
0	$\frac{\pi}{4}-\frac{\ln 2}{2}$	$\frac{\pi^2}{16}+\frac{\pi\ln 2}{4}-G$	$\frac{\pi^3}{64}+\frac{3\pi^2\ln 2}{32}-\frac{3\pi G}{4}+\frac{63\zeta(3)}{64}$
1	G	$\frac{\pi G}{2}-\frac{7\zeta(3)}{8}$	$\frac{3\pi^2 G}{16}-\frac{3\beta(4)}{2}$
2	★	$\frac{\pi\ln 2}{4}-\frac{\pi^2}{16}+G$	$\frac{3\pi^2\ln 2}{32}-\frac{\pi^3}{64}+\frac{3\pi G}{4}-\frac{105\zeta(3)}{64}$
3	★	★	$\frac{3\pi\ln 2}{8}-\frac{\pi^3}{64}-\frac{3\pi^2}{32}+\frac{3G}{2}$

(★ indicates that the corresponding integral diverges.)

4. Evaluation of $\mathbf{S}(m,n)$

When $n\neq 1$, we have by integration by parts

$$\mathrm{S}(m,n)=\frac{\arcsin^m x}{(1-n)x^{n-1}}\Big|_0^1+\frac{m}{n-1}\int_0^1\frac{\arcsin^{m-1}x}{x^{n-1}\sqrt{1-x^2}}dx,$$

which can alternatively be expressed as

$$\mathfrak{S}(m,n):=\int_0^1\frac{\arcsin^{m-1}x}{x^{n-1}\sqrt{1-x^2}}dx=\frac{n-1}{m}\mathrm{S}(m,n)+\frac{1}{m}\left(\frac{\pi}{2}\right)^m. \tag{17}$$

By splitting $\mathfrak{S}(m,n)$ into two integrals, we have

$$\begin{aligned}\mathfrak{S}(m,n)&=\int_0^1\frac{\arcsin^{m-1}x}{x^{n-3}\sqrt{1-x^2}}dx+\int_0^1\frac{\sqrt{1-x^2}\arcsin^{m-1}x}{x^{n-1}}dx\\&=\mathfrak{S}(m,n-2)+\int_0^1\frac{\sqrt{1-x^2}\arcsin^{m-1}x}{x^{n-1}}dx.\end{aligned}$$

When $n \neq 2$, the last integral can be manipulated, again using integration by parts, as

$$\begin{aligned}\int_0^1 \frac{\sqrt{1-x^2}\arcsin^{m-1} x}{x^{n-1}}dx &= \frac{m-1}{n-2}\int_0^1 \frac{\arcsin^{m-2} x}{x^{n-2}}dx - \frac{1}{n-2}\int_0^1 \frac{\arcsin^{m-1} x}{x^{n-3}\sqrt{1-x^2}}dx \\ &= \frac{m-1}{n-2}\mathrm{S}(m-2,n-2) - \frac{1}{n-2}\mathfrak{S}(m,n-2),\end{aligned}$$

which leads us to the following expression

$$\mathfrak{S}(m,n) = \frac{n-3}{n-2}\mathfrak{S}(m,n-2) + \frac{m-1}{n-2}\mathrm{S}(m-2,n-2).$$

Substituting (17) into the above equation, we can simplify the resulting equation into the following three-term recurrence relation

$$(n-1)(n-2)\mathrm{S}(m,n) - (n-3)^2\mathrm{S}(m,n-2) - m(m-1)\mathrm{S}(m-2,n-2) + \left(\frac{\pi}{2}\right)^m = 0. \quad (18)$$

By making use of this recurrence relation, we can produce all the values of $\mathrm{S}(m,n)$ for $m \in \mathbb{N}_0$ and $n \in \mathbb{Z}$ subject to $m \geq n$ as long as the boundary values $\{\mathrm{S}(0,n), \mathrm{S}(1,n), \mathrm{S}(m,0), \mathrm{S}(m,1), \mathrm{S}(m,2)\}$ are explicitly determined.

4.1. $\mathrm{S}(0,n)$

For $n \leq 0$, it is routine to compute

$$\mathrm{S}(0,n) = \int_0^1 x^{-n}dx = \frac{1}{1-n}. \quad (19)$$

4.2. $\mathrm{S}(1,n)$

For $n < 0$, applying integration by parts yields

$$\mathrm{S}(1,n) = \int_0^1 x^{-n}\arcsin x dx = \frac{\pi}{2(1-n)} + \frac{1}{n-1}\int_0^1 \frac{x^{1-n}}{\sqrt{1-x^2}}dx. \quad (20)$$

The last integral can further be manipulated as follows:

$$\begin{aligned}\int_0^1 \frac{x^{1-n}}{\sqrt{1-x^2}}dx &= \int_0^1 \frac{x^{-1-n}}{\sqrt{1-x^2}}dx - \int_0^1 x^{-1-n}\sqrt{1-x^2}dx \\ &= \int_0^1 \frac{x^{-1-n}}{\sqrt{1-x^2}}dx + \frac{1}{n}\int_0^1 \frac{x^{1-n}}{\sqrt{1-x^2}}dx \\ &= \frac{n}{n-1}\int_0^1 \frac{x^{-1-n}}{\sqrt{1-x^2}}dx.\end{aligned}$$

Comparing the above relation with (20), we obtain the recurrence relation belew

$$\mathrm{S}(1,n) = \frac{n(n+1)}{(n-1)^2}\mathrm{S}(1,n+2) + \frac{\pi}{2(n-1)^2}.$$

By iterating this relation ℓ-times, we obtain the expression

$$\mathrm{S}(1,n) = \mathrm{S}(1,n+2\ell)\frac{(\frac{n}{2})_\ell(\frac{n+1}{2})_\ell}{(\frac{n-1}{2})_\ell^2} + \frac{\pi}{8}\sum_{k=0}^{\ell-1}\frac{(\frac{n}{2})_k(\frac{n+1}{2})_k}{(\frac{n-1}{2})_{k+1}^2}.$$

Letting $\ell = -\frac{n}{2}$ and $\ell = \frac{1-n}{2}$, respectively, for even n and odd n, we can make further simplifications

$$\boxed{n \equiv_2 0} \quad S(1,n) = S(1,0)\frac{(\frac{n}{2})_{-\frac{n}{2}}(\frac{n+1}{2})_{-\frac{n}{2}}}{(\frac{n-1}{2})^2_{-\frac{n}{2}}} + \frac{\pi}{8}\sum_{k=0}^{-\frac{n+2}{2}} \frac{(\frac{n}{2})_k(\frac{n+1}{2})_k}{(\frac{n-1}{2})^2_{k+1}}$$

$$= \frac{(1)_{-\frac{n}{2}}(2-\pi)}{(\frac{3}{2})_{-\frac{n}{2}}(2n-2)} + \frac{\pi}{8}\sum_{k=0}^{-\frac{n+2}{2}} \frac{(\frac{n}{2})_k(\frac{n+1}{2})_k}{(\frac{n-1}{2})^2_{k+1}},$$

$$\boxed{n \equiv_2 1} \quad S(1,n) = S(1,1)\frac{(\frac{n}{2})_{\frac{1-n}{2}}(\frac{n+1}{2})_{\frac{1-n}{2}}}{(\frac{n-1}{2})^2_{\frac{1-n}{2}}} + \frac{\pi}{8}\sum_{k=0}^{-\frac{n+1}{2}} \frac{(\frac{n}{2})_k(\frac{n+1}{2})_k}{(\frac{n-1}{2})^2_{k+1}}$$

$$= \frac{\pi}{8}\sum_{k=0}^{-\frac{n+1}{2}} \frac{(\frac{n}{2})_k(\frac{n+1}{2})_k}{(\frac{n-1}{2})^2_{k+1}};$$

where we have employed the initial values (see Entry (e) in Section 1)

$$S(1,1) = \frac{\pi}{2}\ln 2 \quad \text{and} \quad S(1,0) = \int_0^{\frac{\pi}{2}} y\cos y dy = \frac{\pi}{2} - 1.$$

Writing further

$$\frac{(\frac{n}{2})_k(\frac{n+1}{2})_k}{(\frac{n-1}{2})^2_{k+1}} = \frac{4}{(n-1)^2}\frac{(\frac{n}{2})_k}{(\frac{n+1}{2})_k} = \frac{4}{n-1}\left\{\frac{(\frac{n}{2})_{k+1}}{(\frac{n-1}{2})_{k+1}} - \frac{(\frac{n}{2})_k}{(\frac{n-1}{2})_k}\right\},$$

we can evaluate the partial sum by telescoping

$$\frac{\pi}{8}\sum_{k=0}^{\ell} \frac{(\frac{n}{2})_k(\frac{n+1}{2})_k}{(\frac{n-1}{2})^2_{k+1}} = \frac{\pi}{2n-2}\sum_{k=0}^{\ell}\left\{\frac{(\frac{n}{2})_{k+1}}{(\frac{n-1}{2})_{k+1}} - \frac{(\frac{n}{2})_k}{(\frac{n-1}{2})_k}\right\}$$

$$= \frac{\pi}{2-2n} - \frac{\pi}{2-2n}\frac{(\frac{n}{2})_{\ell+1}}{(\frac{n-1}{2})_{\ell+1}}.$$

From this, we derive the following closed formula, which is equivalent to those recorded in ([9], §4.523: Equations (1) and (2)):

$$S(1,n) = \frac{\pi}{2-2n} - \frac{\pi}{2-2n}\frac{(\frac{n}{2})_{-\lfloor\frac{n}{2}\rfloor}}{(\frac{n-1}{2})_{-\lfloor\frac{n}{2}\rfloor}} - \frac{(1)_{-\lfloor\frac{n}{2}\rfloor}(2-\pi)}{(\frac{3}{2})_{-\lfloor\frac{n}{2}\rfloor}(2-2n)}\chi(n \equiv_2 0). \tag{21}$$

4.3. $S(m,0)$

For $m \geq 2$, making the change in variable $x \to \sin y$ and then applying integration by parts, we can proceed with

$$S(m,0) = \int_0^1 \arcsin^m x dx = \int_0^{\frac{\pi}{2}} y^m \cos y dy$$

$$= \left(\frac{\pi}{2}\right)^m - m\int_0^{\frac{\pi}{2}} y^{m-1}\sin y dy$$

$$= \left(\frac{\pi}{2}\right)^m - m(m-1)\int_0^{\frac{\pi}{2}} y^{m-2}\cos y dy,$$

which can be restated as the following recurrence relation

$$S(m,0) = \left(\frac{\pi}{2}\right)^m - m(m-1)S(m-2,0).$$

Considering that

$$\mathrm{S}(0,0) = \int_0^1 dx = 1 \quad \text{and} \quad \mathrm{S}(1,0) = \frac{\pi}{2} - 1,$$

we can iterate the above recurrence ℓ-times

$$\mathrm{S}(m,0) = (-1)^\ell \langle m\rangle_{2\ell}\mathrm{S}(m-2\ell,0) + \sum_{k=0}^{\ell-1}(-1)^k\langle m\rangle_{2k}\Big(\frac{\pi}{2}\Big)^{m-2k}.$$

For $\ell = \lfloor \frac{m}{2} \rfloor$, the above expression becomes

$$\begin{aligned}
\mathrm{S}(m,0) &= (-1)^{\lfloor \frac{m}{2}\rfloor} m! S\Big(m - 2\lfloor\frac{m}{2}\rfloor, 0\Big) + \sum_{k=0}^{\lfloor \frac{m}{2}\rfloor -1}(-1)^k\langle m\rangle_{2k}\Big(\frac{\pi}{2}\Big)^{m-2k} \\
&= \sum_{k=0}^{\lfloor \frac{m}{2}\rfloor -1}(-1)^k\langle m\rangle_{2k}\Big(\frac{\pi}{2}\Big)^{m-2k} + (-1)^{\lfloor \frac{m}{2}\rfloor} m! \times \begin{cases} 1, & m \equiv_2 0; \\ \frac{\pi}{2} - 1, & m \equiv_2 1. \end{cases}
\end{aligned} \tag{22}$$

4.4. $\mathrm{S}(m,1)$

For $m \geq 1$, making the change in variable $x \to \sin y$ and then applying integration by parts, we can reformulate the integral

$$\begin{aligned}
\mathrm{S}(m,1) &= \int_0^1 \frac{\arcsin^m x}{x}dx = \int_0^{\frac{\pi}{2}} y^m \cot y dy \\
&= -\, m \int_0^{\frac{\pi}{2}} y^{m-1}\ln(\sin y)dy \qquad \boxed{y \to 2\theta} \\
&= -\, 2^m m \int_0^{\frac{\pi}{4}} \theta^{m-1}\ln(2\tan\theta\cos^2\theta)d\theta \\
&= -\, 2^m \ln 2\Big(\frac{\pi}{4}\Big)^m - m2^m \int_0^{\frac{\pi}{4}} \theta^{m-1}\ln(\tan\theta)d\theta \\
&\quad - m2^{m+1}\int_0^{\frac{\pi}{4}} \theta^{m-1}\ln(\cos\theta)d\theta.
\end{aligned}$$

Recalling Propositions 1 and 2, we find the following explicit formula

$$\mathrm{S}(m,1) = -2^m \ln 2\Big(\frac{\pi}{4}\Big)^m - m2^m\Lambda(m-1) - m2^{m+1}\Theta(m-1). \tag{23}$$

4.5. $\mathrm{S}(m,2)$

For $m \geq 2$, making the change in variable $x \to \sin y$ and then applying integration by parts twice, we can manipulate the integral

$$\begin{aligned}
\mathrm{S}(m,2) &= \int_0^1 \frac{\arcsin^m x}{x^2}dx = \int_0^{\frac{\pi}{2}} y^m \frac{\cos y}{\sin^2 y}dy \\
&= -\Big(\frac{\pi}{2}\Big)^m + m\int_0^{\frac{\pi}{2}} y^{m-1}\csc y dy \\
&= -\Big(\frac{\pi}{2}\Big)^m - m(m-1)\int_0^{\frac{\pi}{2}} y^{m-2}\ln\big(\tan\frac{y}{2}\big)dy \qquad \boxed{y \to 2x} \\
&= -\Big(\frac{\pi}{2}\Big)^m - m(m-1)2^{m-1}\int_0^{\frac{\pi}{4}} x^{m-2}\ln(\tan x)dx.
\end{aligned}$$

This leads us to the following formula

$$\mathrm{S}(m,2) = -\Big(\frac{\pi}{2}\Big)^m - m(m-1)2^{m-1}\Lambda(m-2), \tag{24}$$

where $\Lambda(m)$ is already evaluated by Proposition 2.

To summarise, we have shown the following general theorem.

Theorem 3 ($m \in \mathbb{N}_0$ and $n \in \mathbb{Z}$ with $m \geq n$)**.** *Let $\Theta(m)$ and $\Lambda(m)$ be as in Propositions 1 and 2, respectively. The integral values for $\mathrm{S}(m,n)$ are determined as follows:*

$$\boxed{n=0}\ \mathrm{S}(m,0) = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor -1} (-1)^k \langle m \rangle_{2k} \left(\frac{\pi}{2}\right)^{m-2k} + (-1)^{\lfloor \frac{m}{2} \rfloor} m! \times \begin{cases} 1, & m \equiv_2 0; \\ \frac{\pi}{2}-1, & m \equiv_2 1; \end{cases}$$

$$\boxed{n=1}\ \mathrm{S}(m,1) = -2^m \ln 2 \left(\tfrac{\pi}{4}\right)^m - m2^m \Lambda(m-1) - m2^{m+1}\Theta(m-1),\ m \geq 1;$$

$$\boxed{n=2}\ \mathrm{S}(m,2) = -\left(\tfrac{\pi}{2}\right)^m - m(m-1)2^{m-1}\Lambda(m-2),\ m \geq 2;$$

$$\boxed{n \geq 3}\ \mathrm{S}(m,n) = \frac{(n-3)^2 \mathrm{S}(m,n-2)}{(n-1)(n-2)} + \frac{m(m-1)\mathrm{S}(m-2,n-2)}{(n-1)(n-2)} - \frac{(\pi/2)^m}{(n-1)(n-2)},\ m \geq n;$$

$$\boxed{n<0}\ \mathrm{S}(m,n) = \begin{cases} \dfrac{1}{1-n}, & m=0; \\ \frac{\pi}{2-2n}\left\{1 - \frac{(\frac{n}{2})_{-\lfloor \frac{n}{2} \rfloor}}{(\frac{n-1}{2})_{-\lfloor \frac{n}{2} \rfloor}}\right\} - \frac{(1)_{-\lfloor \frac{n}{2} \rfloor}(2-\pi)}{(\frac{3}{2})_{-\lfloor \frac{n}{2} \rfloor}(2-2n)}\chi(n \equiv_2 0), & m=1; \\ \frac{n(n+1)}{(n-1)^2}\mathrm{S}(m,n+2) - \frac{m(m-1)}{(n-1)^2}\mathrm{S}(m-2,n) + \frac{(\pi/2)^m}{(n-1)^2}, & m \geq 2. \end{cases}$$

According to this theorem, $\mathrm{S}(m,n)$ can always be expressed in terms of π and $\ln 2$, as well as values of ζ-function and β-function. The initial values for $\mathrm{S}(m,n)$ with $1 \leq m \leq 4$ and $-4 \leq n \leq 4$ are recorded in Table 3.

Table 3. Values for $\mathrm{S}(m,n)$.

$n\backslash m$	1	2	3	4
-4	$\frac{\pi}{10}-\frac{8}{75}$	$\frac{\pi^2}{20}-\frac{298}{1125}$	$\frac{4144}{5625}-\frac{149\pi}{375}+\frac{\pi^3}{40}$	$\frac{254728}{84375}-\frac{149\pi^2}{375}+\frac{\pi^4}{80}$
-3	$\frac{5\pi}{64}$	$\frac{5\pi^2}{128}-\frac{1}{8}$	$\frac{5\pi^3}{256}-\frac{51\pi}{512}$	$\frac{3}{8}-\frac{51\pi^2}{512}+\frac{5\pi^4}{512}$
-2	$\frac{\pi}{6}-\frac{2}{9}$	$\frac{\pi^2}{12}-\frac{14}{27}$	$\frac{40}{27}-\frac{7\pi}{9}+\frac{\pi^3}{24}$	$\frac{488}{81}-\frac{7\pi^2}{9}+\frac{\pi^4}{48}$
-1	$\frac{\pi}{8}$	$\frac{\pi^2}{16}-\frac{1}{4}$	$\frac{\pi^3}{32}-\frac{3\pi}{16}$	$\frac{3}{4}-\frac{3\pi^2}{16}+\frac{\pi^4}{64}$
0	$\frac{\pi}{2}-1$	$\frac{\pi^2}{4}-2$	$6-3\pi+\frac{\pi^3}{8}$	$24-3\pi^2+\frac{\pi^4}{16}$
1	$\frac{\pi \ln 2}{2}$	$\frac{\pi^2 \ln 2}{4}-\frac{7\zeta(3)}{8}$	$\frac{\pi^3 \ln 2}{8}-\frac{9\pi\zeta(3)}{16}$	$\frac{\pi^4 \ln 2}{16}-\frac{9\pi^2\zeta(3)}{16}+\frac{93\zeta(5)}{32}$
2	★	$4G-\frac{\pi^2}{4}$	$6\pi G-\frac{21\zeta(3)}{2}-\frac{\pi^3}{8}$	$6\pi^2 G-48\beta(4)-\frac{\pi^4}{16}$
3	★	★	$\frac{3\pi \ln 2}{2}-\frac{\pi^3}{16}$	$\frac{3\pi^2 \ln 2}{2}-\frac{\pi^4}{32}-\frac{21\zeta(3)}{4}$
4	★	★	★	$8G+\pi^2 G-8\beta(4)-\frac{\pi^4}{48}-\frac{\pi^2}{2}$

(★ indicates that the corresponding integral diverges.)

5. Concluding Comments

By making the change of variable $x \to y^{-1}$, it is trivial to check

$$\mathrm{H}(m,n) = \int_1^\infty y^{n-2} \mathrm{arccoth}^m y dy \quad \text{and} \quad \mathrm{T}(m,n) = \int_1^\infty y^{n-2} \mathrm{arccot}^m y dy.$$

Therefore the two integrals on the right hand sides can easily be evaluated.

In this section, we shall further examine some integral variants that can be evaluated as consequences by the preceding Theorems and Propositions, as shown in this paper.

5.1. Evaluation of the Integral $\Phi(m)$

Firstly, for $m \in \mathbb{N}_0$, we examine

$$\Phi(m) := \int_0^{\frac{\pi}{4}} y^m \ln(\sin y) dy.$$

A similar integral of "$y^m \ln \sin \pi y$" over $[0,1]$ was treated by Espinosa and Moll [24]. This can easily be expressed as

$$\Phi(m) = \int_0^{\frac{\pi}{4}} y^m \ln(\sin y) dy = \int_0^{\frac{\pi}{4}} y^m \ln(\tan y \cos y) dy = \Theta(m) + \Lambda(m).$$

Hence, we can compute $\Phi(m)$ by employing Propositions 1 and 2. The initial values for $0 \leq m \leq 5$ are exemplified as follows (where $\Phi(0)$ can also be found in ([25], Equation 9.7.9)).

$$\begin{aligned}
\Phi(0) &= -\frac{G}{2} - \frac{\pi \ln 2}{4}, \\
\Phi(1) &= \frac{35\zeta(3)}{128} - \frac{\pi G}{8} - \frac{\pi^2 \ln 2}{32}, \\
\Phi(2) &= \frac{\beta(4)}{4} - \frac{\pi^2 G}{32} + \frac{3\pi\zeta(3)}{256} - \frac{\pi^3 \ln 2}{192}, \\
\Phi(3) &= \frac{3\pi\beta(4)}{16} - \frac{\pi^3 G}{128} - \frac{1581\zeta(5)}{4096} + \frac{9\pi^2\zeta(3)}{2048} - \frac{\pi^4 \ln 2}{1024}, \\
\Phi(4) &= \frac{3\pi^2\beta(4)}{32} - \frac{3\beta(6)}{4} - \frac{\pi^4 G}{512} - \frac{45\pi\zeta(5)}{4096} + \frac{3\pi^3\zeta(3)}{2048} - \frac{\pi^5 \ln 2}{5120}, \\
\Phi(5) &= \frac{5\pi^3\beta(4)}{128} - \frac{15\pi\beta(6)}{16} - \frac{\pi^5 G}{2048} + \frac{123825\zeta(7)}{65536} - \frac{225\pi^2\zeta(5)}{32768} + \frac{15\pi^4\zeta(3)}{32768} - \frac{\pi^6 \ln 2}{24576}.
\end{aligned}$$

5.2. Evaluation of the Integral $\Psi(m)$

For $m \in \mathbb{N}_0$, consider the integral

$$\Psi(m) := \int_0^{\infty} y^m \ln(\tanh y) dy.$$

Making the change of variable $y \to \operatorname{arctanh} x$ and then applying integration by parts, we obtain the transformation formula

$$\begin{aligned}
\Psi(m) &= \int_0^{\infty} y^m \ln(\tanh y) dy = \int_0^1 \frac{\ln x \operatorname{arctanh}^m x}{1-x^2} dx \\
&= \frac{-1}{m+1} \int_0^1 \frac{\operatorname{arctanh}^{m+1} x}{x} dx = \frac{-1}{m+1} \mathrm{H}(m+1,1).
\end{aligned}$$

In view of Theorem 1, the initial values for $0 \leq m \leq 5$ are exemplified as follows.

$$\begin{aligned}
\Psi(0) &= -\frac{\pi^2}{8}, & \Psi(3) &= -\frac{93\zeta(5)}{128}, \\
\Psi(1) &= -\frac{7\zeta(3)}{16}, & \Psi(4) &= -\frac{\pi^6}{640}, \\
\Psi(2) &= -\frac{\pi^4}{192}, & \Psi(5) &= -\frac{1905\zeta(7)}{512}.
\end{aligned}$$

5.3. Evaluation of the Integral $\mathcal{H}(m,n)$

For $m \in \mathbb{N}_0$ and $n \in \mathbb{Z}$ subject $m \geq n$, consider the integral

$$\mathcal{H}(m,n) := \int_0^{\infty} y^m \frac{\coth^n y}{\cosh^2 y} dy.$$

By making the change of variable $y \to \operatorname{arctanh} x$, we can derive the following transformation formula

$$\mathcal{H}(m,n) = \int_0^\infty y^m \frac{\coth^n y}{\cosh^2 y} dy = \int_0^1 \frac{\operatorname{arctanh}^m x}{x^n} dx = \mathrm{H}(m,n).$$

Therefore, $\mathcal{H}(m,n)$ can be computed by means of Theorem 1 with the initial values being given by the same Table 1.

Moll ([20], §6.5 and §6.6) evaluated two similar integrals

$$\int_0^\infty x \frac{\sinh x}{\cosh^2 x} dx = \frac{\pi}{2},$$
$$\int_0^\infty x^2 \frac{\sinh x}{\cosh^2 x} dx = 4G.$$

More hyperbolic integral identities can be found in ([9], §3.527).

5.4. *Evaluation of the Integral* $\mathcal{T}(m,n)$

Assuming $m \in \mathbb{N}_0$ and $n \in \mathbb{Z}$ with $m \geq n$, we examine the next integral

$$\mathcal{T}(m,n) := \int_0^{\frac{\pi}{4}} y^m \cot^n y dy.$$

When $n < 0$, Moll ([25], §11.5) recorded an explicit formula for $\mathcal{T}(0,n)$. Jameson and Lord [4] evaluated

$$\mathcal{T}(1,-1) = \frac{G}{2} - \frac{\pi}{8} \ln 2 \quad \text{and} \quad \int_0^{\frac{\pi}{2}} x \cot x dx = \frac{\pi}{2} \ln 2.$$

By making the change of variable $y \to \arctan x$, we can reformulate

$$\mathcal{T}(m,n) = \int_0^{\frac{\pi}{4}} y^m \cot^n y dy = \int_0^1 \frac{\arctan^m x}{x^n(1+x^2)} dx.$$

In view of (6), we find the expression

$$\mathcal{T}(m,n) = \frac{n}{m+1} \mathrm{T}(m+1,n+1) + \frac{1}{m+1} \left(\frac{\pi}{4}\right)^{m+1}.$$

According to Theorem 2, all the $\mathcal{T}(m,n)$ can be computed as consequences. For $0 \leq m \leq 3$ and $-3 \leq n \leq 3$, the corresponding $\mathcal{T}(m,n)$ values are given in Table 4.

Table 4. Values for $\mathcal{T}(m,n)$.

$n\backslash m$	0	1	2	3
-3	$\frac{1}{2}-\frac{\ln 2}{2}$	$\frac{\pi}{4}-\frac{1}{2}-\frac{G}{2}+\frac{\pi\ln 2}{8}$	$\frac{\pi^2}{16}-\frac{\pi}{4}-\frac{\pi G}{4}+\frac{\ln 2}{2}+\frac{\pi^2\ln 2}{32}+\frac{21\zeta(3)}{64}$	$\frac{\pi^3}{64}-\frac{3\pi^2}{32}-\frac{3\pi\ln 2}{8}+\frac{\pi^3\ln 2}{128}+\frac{3G}{2}-\frac{3\pi^2 G}{32}-\frac{9\pi\zeta(3)}{256}+\frac{3\beta(4)}{4}$
-2	$1-\frac{\pi}{4}$	$\frac{\pi}{4}-\frac{\pi^2}{32}-\frac{\ln 2}{2}$	$\frac{\pi^2}{16}+\frac{\pi\ln 2}{4}-\frac{\pi^3}{192}-G$	$\frac{\pi^3}{64}+\frac{3\pi^2\ln 2}{32}-\frac{\pi^4}{1024}-\frac{3\pi G}{4}+\frac{63\zeta(3)}{64}$
-1	$\frac{\ln 2}{2}$	$\frac{G}{2}-\frac{\pi\ln 2}{8}$	$\frac{\pi G}{4}-\frac{\pi^2\ln 2}{32}-\frac{21\zeta(3)}{64}$	$\frac{3\pi^2 G}{32}-\frac{3\beta(4)}{4}+\frac{9\pi\zeta(3)}{256}-\frac{\pi^3\ln 2}{128}$
0	$\frac{\pi}{4}$	$\frac{\pi^2}{32}$	$\frac{\pi^3}{192}$	$\frac{\pi^4}{1024}$
1	★	$\frac{G}{2}+\frac{\pi\ln 2}{8}$	$\frac{\pi G}{4}+\frac{\pi^2\ln 2}{32}-\frac{35\zeta(3)}{64}$	$\frac{3\pi^2 G}{32}-\frac{3\beta(4)}{4}-\frac{9\pi\zeta(3)}{256}+\frac{\pi^3\ln 2}{128}$
2	★	★	$G+\frac{\pi\ln 2}{4}-\frac{\pi^2}{16}-\frac{\pi^3}{192}$	$\frac{3\pi G}{4}-\frac{105\zeta(3)}{64}-\frac{\pi^4}{1024}-\frac{\pi^3}{64}+\frac{3\pi^2\ln 2}{32}$
3	★	★	★	$\frac{3G}{2}-\frac{3\pi^2 G}{32}-\frac{3\pi^2}{32}-\frac{\pi^3}{64}+\frac{3\pi\ln 2}{8}-\frac{\pi^3\ln 2}{128}+\frac{9\pi\zeta(3)}{256}+\frac{3\beta(4)}{4}$

(★ indicates that the corresponding integral diverges.)

5.5. Evaluation of the Integral $\mathcal{S}(m,n)$

Let $m\in\mathbb{N}_0$ and $n\in\mathbb{Z}$ satisfying the condition $m\geq n$. Define the integral $\mathcal{S}(m,n)$ by

$$\mathcal{S}(m,n):=\int_0^{\frac{\pi}{2}} y^m \csc^n y dy.$$

A similar integral $\int_0^{\frac{\pi}{2}} y^m\cos^n y dy$ was extensively examined by Moll ([25], §5.2).

By making the change of variable $y\to\arcsin x$, we can manipulate

$$\mathcal{S}(m,n)=\int_0^{\frac{\pi}{2}} y^m \csc^n y dy=\int_0^1 \frac{\arcsin^m x}{x^n\sqrt{1-x^2}}dx.$$

Recalling (17), we have the expression

$$\mathcal{S}(m,n)=\mathfrak{S}(m+1,n+1)=\frac{n}{m+1}\mathrm{S}(m+1,n+1)+\frac{1}{m+1}\left(\frac{\pi}{2}\right)^{m+1}.$$

Applying Theorem 3, we can evaluate $\mathcal{S}(m,n)$ as consequences. For $0\leq m\leq 4$ and $-3\leq n\leq 3$, the corresponding $\mathcal{S}(m,n)$ values are recorded in Table 5.

Table 5. Values for $\mathcal{S}(m,n)$.

$n\backslash m$	0	1	2	3	4
-3	$\frac{2}{3}$	$\frac{7}{9}$	$\frac{7\pi}{9}-\frac{40}{27}$	$\frac{7\pi^2}{12}-\frac{122}{27}$	$\frac{1456}{81}-\frac{244\pi}{27}+\frac{7\pi^3}{18}$
-2	$\frac{\pi}{4}$	$\frac{1}{4}+\frac{\pi^2}{16}$	$\frac{\pi}{8}+\frac{\pi^3}{48}$	$\frac{3\pi^2}{32}+\frac{\pi^4}{128}-\frac{3}{8}$	$\frac{\pi^3}{16}-\frac{3\pi}{8}+\frac{\pi^5}{320}$
-1	1	1	$\pi-2$	$\frac{3\pi^2}{4}-6$	$24-12\pi+\frac{\pi^3}{2}$
0	$\frac{\pi}{2}$	$\frac{\pi^2}{8}$	$\frac{\pi^3}{24}$	$\frac{\pi^4}{64}$	$\frac{\pi^5}{160}$
1	★	$2G$	$2\pi G-\frac{7\zeta(3)}{2}$	$\frac{3\pi^2 G}{2}-12\beta(4)$	$\pi^3 G-24\pi\beta(4)+\frac{93\zeta(5)}{2}$
2	★	★	$\pi\ln 2$	$\frac{3\pi^2\ln 2}{4}-\frac{21\zeta(3)}{8}$	$\frac{\pi^3\ln 2}{2}-\frac{9\pi\zeta(3)}{4}$
3	★	★	★	$\frac{3\pi^2 G}{4}-\frac{3\pi^2}{8}+6G-6\beta(4)$	$12\pi G-21\zeta(3)-12\pi\beta(4)-\frac{\pi^3}{4}+\frac{\pi^3 G}{2}+\frac{93\zeta(5)}{4}$

(★ indicates that the corresponding integral diverges.)

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