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IMPROVED ALTERNATING-DIRECTION IMPLICIT METHOD FOR SOLVING TRANSIENT THREE-DIMENSIONAL HEAT DIFFUSION PROBLEMS

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IMPROVED ALTERNATING-DIRECTION IMPLICIT METHOD FOR SOLVING TRANSIENT THREE-DIMENSIONAL HEAT DIFFUSION PROBLEMS

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The conventional three-dimensional alternating-direction implicit (ADI) method is modified by introducing an f factor ($0 < f < 1$). This modification allows the time step limit to be increased by a factor of $1/f$ with the solutions remaining stable and high accuracy being retained. This new method is tested for two different boundary conditions: a constant heat flux and a sudden heating of the surface to a constant temperature. In addition, it is compared with the popular Brian and Douglas methods, the results showing that the new ADI method has higher accuracy and requires less computer storage than those methods.

INTRODUCTION

The diffusion of heat in solids has numerous applications in various branches of science and engineering. Generally, there are two different approaches to deal with this type of problem: analytical and numerical. The analytical methods are usually only applicable to linear problems with simple geometries. On the contrary, the numerical methods are useful for handling practical problems involving nonlinearities, complex geometries, and/or complicated boundary conditions.

Thibault [1] compared nine numerical schemes for the solution of the three-dimensional heat diffusion equation. Considering the relative accuracy, the computation time, and the computer core storage requirement, he recommended alternating-direction implicit (ADI) finite difference methods as being among the most preferred methods. The conventional two-dimensional ADI method was introduced by Peaceman and Rachford [2] in 1955. The advantage of the ADI method is that only tridiagonal matrices need to be solved. However, when extended to three dimensions, the conventional ADI method is conditionally stable, and very small time steps are required to ensure convergence and stability. Other forms of the ADI method include the Douglas method [3] and the Brian method [4]. These two ADI methods are unconditionally stable and possess the advantages of the implicit scheme with no limitation on size of the time step. However, Thibault [1] pointed out that these two unconditionally stable ADI methods cannot retain accuracy if the time step is more than 2 times larger than the time step limit required for the conventional ADI method.

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NOMENCLATURE

E	truncation error function, defined in Eq. (27)
I, J, K	number of nodal points in x, y, and z directions
k	thermal conductivity
L	half length of parallelepiped
L'	dimensionless half length of parallelepiped
L_c	characteristic length
q	surface heat flux
$\bar{q}$	dimensionless surface heat flux
t	time
T	temperature
U, V	temperature at intermediate time steps
x, y, z	space coordinates
X, Y, Z	dimensionless space coordinates
α	thermal diffusivity
δ^2	central difference operator
ϵ	average temperature error
θ	dimensionless temperature
λ	stability parameter, defined in Eq. (8)
ξ	amplification factor of truncation error function
τ	dimensionless time, or Fourier number

Subscripts

a	analytical solution
i, j, k	mesh point indices in x, y, and z directions
w	wall surface of parallelepiped
x, y, z	indicate x, y, and z directions
0	initial
1, 2, 3	indicate x, y, and z directions

Superscript

n	time index

In this paper, the conventional three-dimensional ADI method is modified by an f factor ($0 < f < 1$). A very important characteristic of this modification is that it is consistent with physical considerations and is not just based on mathematical manipulations. This modification allows the time step limit to be increased by approximately a factor of $1/f$ without compromising significantly on the accuracy of the numerical solution. This new ADI method is presented and compared with the Brian and Douglas ADI methods for two cases where analytical solutions are available. Compared with the Brian and Douglas methods, this new ADI method has higher accuracy when large time steps are used. Also, the present method requires less computer storage.

MATHEMATICAL FORMULATION

First, we will look at the formulations of existing three-dimensional ADI methods: the conventional, the Brian, and the Douglas methods. Then the proposed ADI method designed to overcome the shortcomings of these existing ADI methods will be introduced.

The differential equations for the three-dimensional heat diffusion equation can be written as

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \tag{1}$$

Introducing dimensionless parameters,

$$X = \frac{x}{L_c} \qquad Y = \frac{y}{L_c} \qquad Z = \frac{z}{L_c}$$

$$\theta = \frac{T}{T_0} \qquad \tau = \frac{\alpha t}{L_c^2}$$

Eq. (1) becomes

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial X^2} + \frac{\partial^2 \theta}{\partial Y^2} + \frac{\partial^2 \theta}{\partial Z^2} \tag{2}$$

Conventional ADI Method

In the conventional ADI method, the heat diffusion equation is solved implicitly in turn in the three coordinate directions for 1/3 of the time increment each [5]. The basic finite difference equations for each of the three 1/3 time steps can be expressed as

$$\frac{U_{i,j,k} - \theta^n_{i,j,k}}{\Delta\tau/3} = \delta_x^2 U_{i,j,k} + \delta_y^2 \theta^n_{i,j,k} + \delta_z^2 \theta^n_{i,j,k} \tag{3}$$

$$\frac{V_{i,j,k} - U_{i,j,k}}{\Delta\tau/3} = \delta_x^2 U_{i,j,k} + \delta_y^2 V_{i,j,k} + \delta_z^2 U_{i,j,k} \tag{4}$$

$$\frac{\theta^{n+1}_{i,j,k} - V_{i,j,k}}{\Delta\tau/3} = \delta_x^2 V_{i,j,k} + \delta_y^2 V_{i,j,k} + \delta_z^2 \theta^{n+1}_{i,j,k} \tag{5}$$

For convenience of analysis, we let $\Delta X = \Delta Y = \Delta Z$. After rearranging, Eq. (3) becomes

$$-U_{i-1,j,k} + \left[\frac{3(\Delta X)^2}{\Delta\tau} + 2\right] U_{i,j,k} - U_{i+1,j,k} = \left[\frac{3(\Delta X)^2}{\Delta\tau} - 4\right] \theta^n_{i,j,k} + \theta^n_{i,j-1,k} + \theta^n_{i,j+1,k} + \theta^n_{i,j,k-1} + \theta^n_{i,j,k+1} \tag{6}$$

Similar equations can be easily derived from Eqs. (4) and (5) for the y and z directions. Physically, an increase in the central nodal temperature or an increase in any one of the neighboring nodal temperatures at the old time step should, with other conditions remaining unchanged, lead to an increase in the central nodal temperature at the next 1/3 time step. This implies that all the coefficients on the righthand side of Eq. (6) must have the same sign (positive) as the coefficient of $U_{i,j,k}$. In other words, negative coefficients on the righthand side of Eq. (6) make the equations physically unrealistic and may lead to low accuracy [6]. Similar statements can be made regarding the equations for the y and z directions.

On the righthand side of Eq. (6), only the coefficient for $\theta^n_{i,j,k}$ could be negative if the time step $\Delta\tau$ is large. In order to have a positive coefficient for $\theta^n_{i,j,k}$, it is required that

$$\frac{\Delta\tau}{(\Delta X)^2} < 0.75 \tag{7}$$

Since $\Delta X = \Delta Y = \Delta Z$, the equations for the y and z directions require the same condition as in Eq. (7) to hold. The other important problem to consider is stability. We define the stability parameter λ as

$$\lambda = \frac{\Delta\tau}{(\Delta X)^2} \tag{8}$$

The stability criterion for the conventional three-dimensional ADI method is [5]

$$\lambda \leq 1.5 \tag{9}$$

Equation (7) is the condition for the solution of the conventional three-dimensional ADI method to be physically realistic and have good accuracy. Equation (9) is the criterion for the solution to be stable. The main disadvantage of the conventional ADI method is that it is conditionally stable and a very small time step is required.

Brian's ADI Method

The method proposed by Brian [4] is similar to the conventional ADI method. However, the successive approximations of temperature are calculated at the half time step. The basic equations of Brian's ADI method are given as

$$\frac{U_{i,j,k} - \theta^n_{i,j,k}}{\Delta\tau/2} = \delta_x^2 U_{i,j,k} + \delta_y^2 \theta^n_{i,j,k} + \delta_z^2 \theta^n_{i,j,k} \tag{10}$$

$$\frac{V_{i,j,k} - \theta^n_{i,j,k}}{\Delta\tau/2} = \delta_x^2 U_{i,j,k} + \delta_y^2 V_{i,j,k} + \delta_z^2 \theta^n_{i,j,k} \tag{11}$$

$$\frac{\theta^{n+1}_{i,j,k} - V_{i,j,k}}{\Delta\tau/2} = \delta_x^2 U_{i,j,k} + \delta_y^2 V_{i,j,k} + \delta_z^2 \theta^{n+1}_{i,j,k} \tag{12}$$

Subtracting Eq. (10) from Eq. (11), we have

$$\frac{V_{i,j,k} - U_{i,j,k}}{\Delta\tau/2} = \delta_y^2 V_{i,j,k} - \delta_y^2 \theta^n_{i,j,k} \tag{11'}$$

Subtracting Eq. (11) from Eq. (12), we have

$$\frac{\theta^{n+1}_{i,j,k} + \theta^n_{i,j,k} - 2V_{i,j,k}}{\Delta\tau/2} = \delta_z^2 \theta^{n+1}_{i,j,k} - \delta_z^2 \theta^n_{i,j,k} \tag{12'}$$

Equations (10), (11'), and (12') are the simplified equations suggested by Brian. After rearranging, the following equations can be obtained:

$$-U_{i-1,j,k} + \left[\frac{2(\Delta X)^2}{\Delta\tau} + 2\right] U_{i,j,k} - U_{i+1,j,k} = \left[\frac{2(\Delta X)^2}{\Delta\tau} - 4\right] \theta^n_{i,j,k} + \theta^n_{i,j-1,k} + \theta^n_{i,j+1,k} + \theta^n_{i,j,k-1} + \theta^n_{i,j,k+1} \tag{13}$$

$$-V_{i,j-1,k} + \left[\frac{2(\Delta Y)^2}{\Delta\tau} + 2\right]V_{i,j,k} - V_{i,j+1,k} = \left[\frac{2(\Delta Y)^2}{\Delta\tau}\right]U_{i,j,k} + 2\theta^n_{i,j,k} - \theta^n_{i,j-1,k} - \theta^n_{i,j+1,k} \tag{14}$$

$$-\theta^{n+1}_{i,j,k-1} + \left[\frac{2(\Delta Z)^2}{\Delta\tau} + 2\right]\theta^{n+1}_{i,j,k} - \theta^{n+1}_{i,j,k+1} = \left[\frac{4(\Delta Z)^2}{\Delta\tau}\right]V_{i,j,k} + \left[-\frac{2(\Delta Z)^2}{\Delta\tau} + 2\right]\theta^n_{i,j,k} - \theta^n_{i,j,k-1} - \theta^n_{i,j,k+1} \tag{15}$$

Brian showed that his scheme is unconditionally stable. However, there also exist negative coefficients on the righthand sides of the discretization equations, Eqs. (13)–(15). As we mentioned earlier, these negative coefficients are physically unrealistic.

Douglas's ADI Method

Another unconditionally stable three-dimensional ADI method was presented by Douglas [3]. The algorithm is given by the following three equations:

$$\frac{U_{i,j,k} - \theta^n_{i,j,k}}{\Delta\tau} = \frac{1}{2}\delta^2_x(U_{i,j,k} + \theta^n_{i,j,k}) + \delta^2_y\theta^n_{i,j,k} + \delta^2_z\theta^n_{i,j,k} \tag{16}$$

$$\frac{V_{i,j,k} - \theta^n_{i,j,k}}{\Delta\tau} = \frac{1}{2}\delta^2_x(U_{i,j,k} + \theta^n_{i,j,k}) + \frac{1}{2}\delta^2_y(V_{i,j,k} + \theta^n_{i,j,k}) + \delta^2_z\theta^n_{i,j,k} \tag{17}$$

$$\frac{\theta^{n+1}_{i,j,k} - \theta^n_{i,j,k}}{\Delta\tau} = \frac{1}{2}\delta^2_x(U_{i,j,k} + \theta^n_{i,j,k}) + \frac{1}{2}\delta^2_y(V_{i,j,k} + \theta^n_{i,j,k}) + \frac{1}{2}\delta^2_z(\theta^{n+1}_{i,j,k} + \theta^n_{i,j,k}) \tag{18}$$

Subtracting Eq. (16) from Eq. (17), we have

$$\frac{V_{i,j,k} - U_{i,j,k}}{\Delta\tau} = \frac{1}{2}\delta^2_y(V_{i,j,k} - \theta^n_{i,j,k}) \tag{17'}$$

Subtracting Eq. (17) from Eq. (18), we have

$$\frac{\theta^{n+1}_{i,j,k} - V_{i,j,k}}{\Delta\tau} = \frac{1}{2}\delta^2_z(\theta^{n+1}_{i,j,k} - \theta^n_{i,j,k}) \tag{18'}$$

Equations (16), (17') and (18') are the simplified equations and can be rearranged as

$$-0.5U_{i-1,j,k} + \left[\frac{(\Delta X)^2}{\Delta\tau} + 1\right]U_{i,j,k} - 0.5U_{i+1,j,k}$$
$$= \left[\frac{(\Delta X)^2}{\Delta\tau} - 5\right]\theta^n_{i,j,k} + 0.5\theta^n_{i-1,j,k} + 0.5\theta^n_{i+1,j,k}$$
$$+ \theta^n_{i,j-1,k} + \theta^n_{i,j+1,k} + \theta^n_{i,j,k-1} + \theta^n_{i,j,k+1} \tag{19}$$

$$-0.5V_{i,j-1,k} + \left[\frac{(\Delta Y)^2}{\Delta\tau} + 1\right]V_{i,j,k} - 0.5V_{i,j+1,k}$$
$$= \left[\frac{(\Delta Y)^2}{\Delta\tau}\right]U_{i,j,k} + \theta^n_{i,j,k} - 0.5\theta^n_{i,j-1,k} - 0.5\theta^n_{i,j+1,k} \tag{20}$$

$$-0.5\theta^{n+1}_{i,j,k-1} + \left[\frac{(\Delta Z)^2}{\Delta\tau} + 1\right]\theta^{n+1}_{i,j,k} - 0.5\theta^{n+1}_{i,j,k+1}$$
$$= \left[\frac{(\Delta Z)^2}{\Delta\tau}\right]V_{i,j,k} + \theta^n_{i,j,k} - 0.5\theta^n_{i,j,k-1} - 0.5\theta^n_{i,j,k+1} \tag{21}$$

The unconditional stability of this algorithm was proven by Douglas [3]. However, as in Brian's ADI method, Douglas's ADI method has negative coefficients on the righthand sides of the discretization equations, Eqs. (19)–(21).

New ADI Method

As we have seen above, the three existing ADI methods all have shortcomings. The conventional ADI method is conditionally stable, and very small time steps are required to satisfy the stability criterion. All three ADI methods have a common problem: negative coefficients in their discretization equations that are physically unrealistic.

In light of the above observation, an improved ADI method is proposed. The conventional three-dimensional ADI method is modified by introducing an f factor ($0 < f < 1$). Consider a control volume as shown in Fig. 1: the heat fluxes from the directions in which the equation is implicit are multiplied by a factor $(3 - 2f)$ and the heat fluxes from the remaining four directions are multiplied by a factor f. As we can see, the total heat flux counted in each direction through a full time step remains unchanged. The finite difference equations, Eqs. (3)–(5), of the conventional ADI method are modified by an f factor and become

$$\frac{U_{i,j,k} - \theta^n_{i,j,k}}{\Delta\tau/3} = (3 - 2f)\delta_x^2 U_{i,j,k} + f\delta_y^2\theta^n_{i,j,k} + f\delta_z^2\theta^n_{i,j,k} \tag{22}$$

$$\frac{V_{i,j,k} - U_{i,j,k}}{\Delta\tau/3} = f\delta_x^2 U_{i,j,k} + (3 - 2f)\delta_y^2 V_{i,j,k} + f\delta_z^2 U_{i,j,k} \tag{23}$$

$$\frac{\theta^{n+1}_{i,j,k} - V_{i,j,k}}{\Delta\tau/3} = f\delta_x^2 V_{i,j,k} + f\delta_y^2 V_{i,j,k} + (3 - 2f)\delta_z^2\theta^{n+1}_{i,j,k} \tag{24}$$

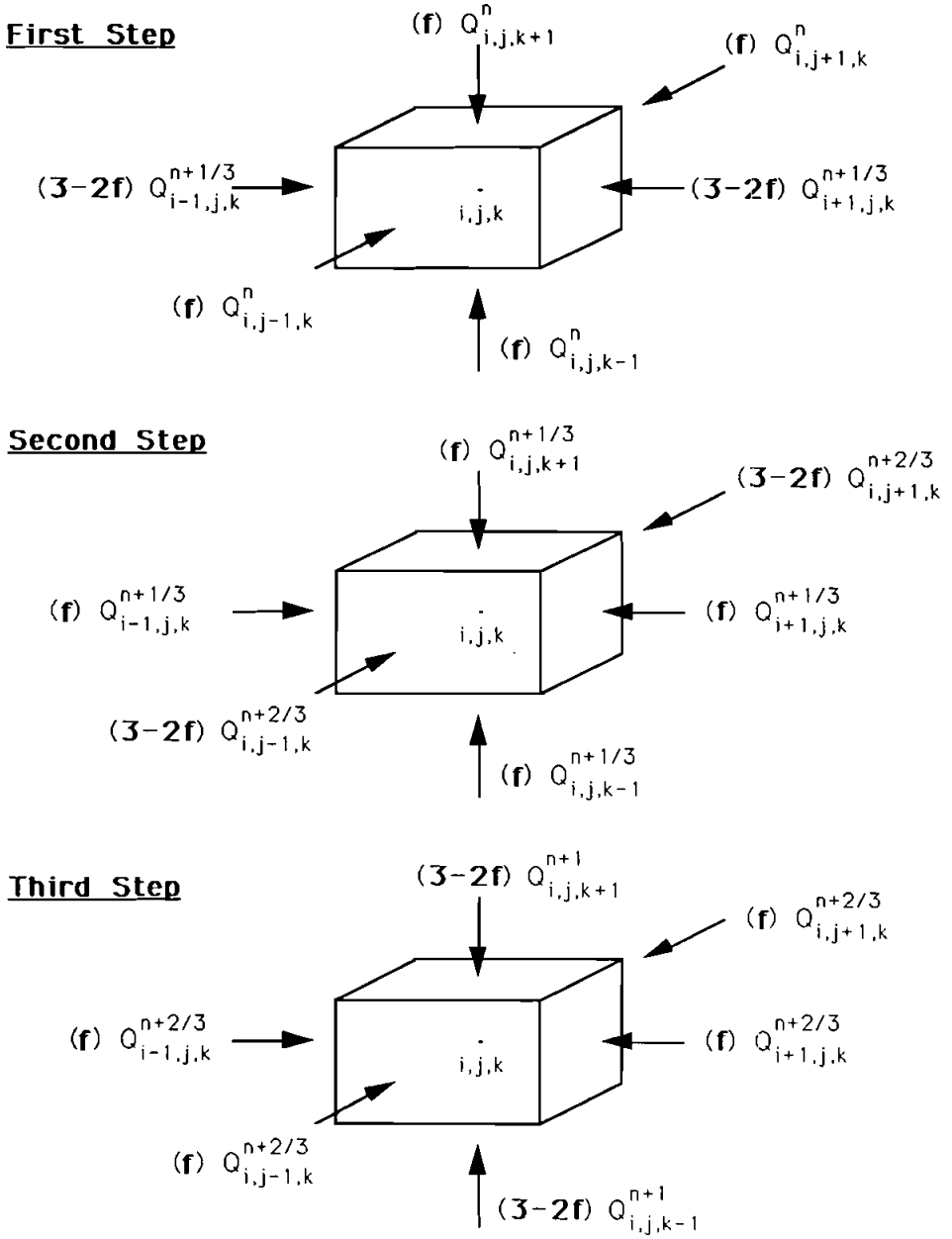

Fig. 1 The f factor modified ADI method.

After rearranging Eq. (22), the following discretization equation can be obtained:

$$-(3 - 2f)U_{i-1,j,k} + \left[\frac{3(\Delta X)^2}{\Delta\tau} + 2(3 - 2f)\right]U_{i,j,k} - (3 - 2f)U_{i+1,j,k}$$
$$= \left[\frac{3(\Delta X)^2}{\Delta\tau} - 4f\right]\theta^n_{i,j,k}$$
$$+ f\theta^n_{i,j-1,k} + f\theta^n_{i,j+1,k} + f\theta^n_{i,j,k-1} + f\theta^n_{i,j,k+1} \tag{25}$$

Similar equations can be easily derived from Eqs. (23) and (24) for the y and z directions. On the righthand side of Eq. (25), only the coefficient for central nodal temperatures at a previous time step could be negative. To avoid a negative coefficient, we require

$$\frac{\Delta\tau}{(\Delta X)^2} < \frac{0.75}{f} \tag{26}$$

The stability criterion can be determined by von Neumann's method. Assuming that there exists an error function $E_{p,q,r,n}$ at each nodal point in the following form [7],

$$E_{p,q,r,n} = \exp(i\beta_1 p\Delta X)\exp(i\beta_2 q\Delta Y)\exp(i\beta_3 r\Delta Z)\xi^n \tag{27}$$

where the parameter ξ is the amplification factor and $n = \tau/\Delta\tau$, the error will be bounded, provided that

$$|\xi| \leq 1$$

This is the condition for the solution to be stable. It can be shown for these linear problems with constant coefficients that the error function $E_{p,q,r,n}$ also satisfies the finite difference equation, Eq. (25), and two similar equations for the y and z directions. With $\Delta X = \Delta Y = \Delta Z$, substitution of $E_{p,q,r,n}$ from Eq. (27) into these equations gives

$$\xi_1 = \frac{3/\lambda - 4(f)\sin^2(\beta_2\Delta X/2) - 4(f)\sin^2(\beta_3\Delta X/2)}{3/\lambda + 4(g)\sin^2(\beta_1\Delta X/2)} \tag{28}$$

$$\xi_2 = \frac{3/\lambda - 4(f)\sin^2(\beta_1\Delta X/2) - 4(f)\sin^2(\beta_3\Delta X/2)}{3/\lambda + 4(g)\sin^2(\beta_2\Delta X/2)} \tag{29}$$

$$\xi_3 = \frac{3/\lambda - 4(f)\sin^2(\beta_1\Delta X/2) - 4(f)\sin^2(\beta_2\Delta X/2)}{3/\lambda + 4(g)\sin^2(\beta_3\Delta X/2)} \tag{30}$$

where $g = 3 - 2f$.

Here, ξ_1, ξ_2, and ξ_3 are the amplification factors for the finite difference equations for the x, y, and z directions, respectively. Since these equations are used alternately, the stability condition should be

$$|\xi_1\xi_2\xi_3| \leq 1$$

Rearranging $\xi_1\xi_2\xi_3$ as follows,

$$\xi_1\xi_2\xi_3 = \left[\frac{3/\lambda - 4(f)\sin^2(\beta_2\Delta X/2) - 4(f)\sin^2(\beta_3\Delta X/2)}{3/\lambda + 4(g)\sin^2(\beta_2\Delta X/2)}\right] \times \left[\frac{3/\lambda - 4(f)\sin^2(\beta_1\Delta X/2) - 4(f)\sin^2(\beta_3\Delta X/2)}{3/\lambda + 4(g)\sin^2(\beta_3\Delta X/2)}\right] \times \left[\frac{3/\lambda - 4(f)\sin^2(\beta_1\Delta X/2) - 4(f)\sin^2(\beta_2\Delta X/2)}{3/\lambda + 4(g)\sin^2(\beta_1\Delta X/2)}\right]$$

$$= [a] \times [b] \times [c]$$

the stability condition can be written as

$$|a|\,|b|\,|c| \leq 1$$

The stability criterion can be obtained from either one of the following three conditions:

$$|a| \leq 1 \qquad |b| \leq 1 \qquad |c| \leq 1$$

For the condition $|a| \leq 1$, since the value of a is always less than unity, we need only consider the condition $a \geq -1$. This leads to

$$\lambda \leq \frac{1.5}{(f)\sin^2(\beta_3\Delta X/2) - (g - f)\sin^2(\beta_2\Delta X/2)}$$

It should be mentioned here that the parameter λ defined in Eq. (8) is always positive. The righthand side of the above equation has a minimum value when $\sin^2(\beta_3\Delta X/2) = 1$ and $\sin^2(\beta_2\Delta X/2) = 0$. So the stability criterion becomes

$$\lambda \leq \frac{1.5}{f} \tag{31}$$

Comparing Eqs. (26) and (31) with Eqs. (7) and (9), the time step limit for the conventional ADI method can now be increased by a factor of $1/f$ by using this new ADI method. The computational results, which will be discussed later, show that this modification allows the time step limit to be increased by 2 orders of magnitude with $f = 0.01$, and the solutions still remain stable with high accuracy.

Also, it should be mentioned, this new ADI method only requires two-thirds of the computer storage compared with the Brian or Douglas methods. This is because only the temperatures at one intermediate time step need to be stored.

RESULTS AND DISCUSSION

To validate the new ADI method, the finite difference solutions obtained are tested for a simple geometry with two different boundary conditions: a constant surface heat

flux and a sudden heating of the surface to a constant temperature. In addition, it is compared with the Brian and Douglas methods.

Consider a parallelepiped ($-L_1 \le x \le L_1$, $-L_2 \le y \le L_2$, $-L_3 \le z \le L_3$), shown in Fig. 2, having constant thermophysical properties and initially at a uniform temperature $\theta_0 = 1.0$. At time $\tau > 0$, the parallelepiped is allowed to have heat flow through its boundaries. To obtain the temperature distribution within the parallelepiped, Eq. (2) must be solved with the following initial conditions:

At $\tau = 0$

$$\theta = 1 \qquad \text{for } -1 \le X \le 1, \quad -L_2' \le Y \le L_2', \quad -L_3' \le Z \le L_3' \tag{32}$$

where L_1 is chosen as the characteristic length L_c, $L_2' = L_2/L_1$, and $L_3' = L_3/L_1$.

Because of symmetry, only the regions $0 \le X \le 1$, $0 \le Y \le L_2'$, and $0 \le Z \le L_3'$ need to be solved. The boundary conditions are

At $\tau > 0$

$$\left.\frac{\partial\theta}{\partial X}\right|_{X=0} = \left.\frac{\partial\theta}{\partial Y}\right|_{Y=0} = \left.\frac{\partial\theta}{\partial Z}\right|_{Z=0} = 0 \tag{33a}$$

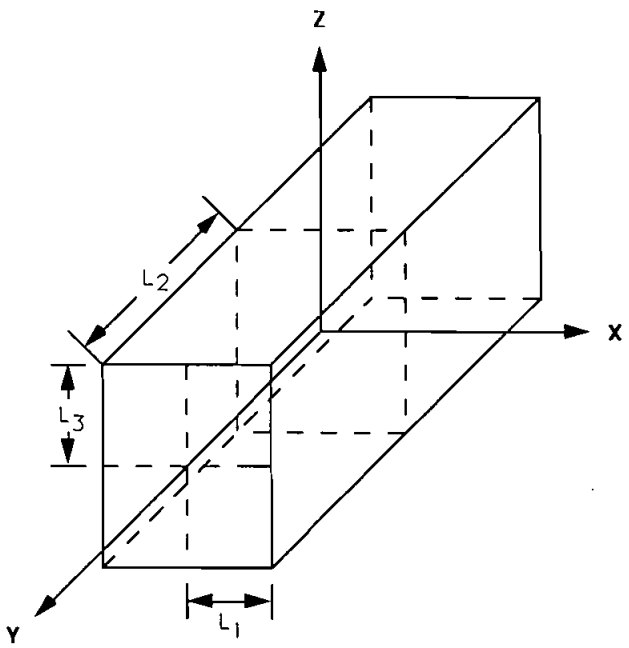

Fig. 2 Coordinate system: parallelpiped.

$$\left.\frac{\partial\theta}{\partial X}\right|_{X=1} = \left.\frac{\partial\theta}{\partial Y}\right|_{Y=L_2'} = \left.\frac{\partial\theta}{\partial Z}\right|_{Z=L_3'} = \bar{q}_w \tag{33b}$$

or

$$\theta|_{X=1} = \theta|_{Y=L_2'} = \theta|_{Z=L_3'} = \theta_w \tag{33c}$$

where $\bar{q}_w = q_w L_1/kT_0$ is dimensionless surface heat flux.

In this paper, each numerical method will be used to solve the three-dimensional heat diffusion equation for the two different boundary conditions. To evaluate the accuracy of the various methods, an average temperature error is used. It is defined as the square root of the average of the squares of the error between the predicted temperature and the analytical temperature. It is given by

$$\epsilon = \left(\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}[\theta_{i,j,k} - \theta_a]^2/\mathrm{IJK}\right)^{1/2} \tag{34}$$

where θ_a is the analytical dimensionless temperature.

Case 1: Constant Surface Heat Flux

Consider a parallelepiped initially at a uniform temperature $\theta_0 = 1.0$. At time $\tau = 0$, all faces of the parallelepiped are exposed to a constant surface heat flux $\bar{q}_w = 0.5$. For a parallelepiped exposed to a constant surface heat flux, the temperature distribution as a function of time can be represented by the summation of three one-dimensional solutions [1]:

$$\begin{aligned}\theta(X, Y, Z, \tau) = \theta_0 + 2\bar{q}_w\sqrt{\tau}\Bigg\{&\sum_{m=0}^{\infty}\left[i\,\mathrm{erfc}\left(\frac{(2m+1)+X}{2\sqrt{\tau}}\right) + i\,\mathrm{erfc}\left(\frac{(2m+1)-X}{2\sqrt{\tau}}\right)\right]\\ &+\sum_{m=0}^{\infty}\left[i\,\mathrm{erfc}\left(\frac{(2m+1)L_2'+Y}{2\sqrt{\tau}}\right) + i\,\mathrm{erfc}\left(\frac{(2m+1)L_2'-Y}{2\sqrt{\tau}}\right)\right]\\ &+\sum_{m=0}^{\infty}\left[i\,\mathrm{erfc}\left(\frac{(2m+1)L_3'+Z}{2\sqrt{\tau}}\right) + i\,\mathrm{erfc}\left(\frac{(2m+1)L_3'-Z}{2\sqrt{\tau}}\right)\right]\Bigg\}\end{aligned} \tag{35}$$

Presented in Fig. 3 are the results obtained for a cube exposed to a constant surface heat flux $\bar{q}_w = 0.5$ at time $\tau = 2.0$. Twenty nodal points are used in each direction for this calculation. According to Eq. (7), the time step limit required for the conventional ADI method ($f = 1.0$) is 0.001875. In Fig. 3 the solutions from the conventional ADI method show good accuracy with time step 0.002 but become unstable as the time step is increased. The Brian and Douglas methods are unconditionally stable, but the negative coefficients in the discretization equations cause their solutions to be physically unrealistic. The results show that their solutions have good accuracy if the time step is smaller than 0.2 but become more and more inaccurate if the time step is increased. On the contrary, the proposed ADI method with $f = 0.1$ and $f = 0.01$ is accurate even when a time step of 2.0 is used. The average temperature error is less than 0.007. It can be seen from Eqs. (26) and (31) that this f factor ADI method has a much higher time step limit than the conventional ADI method.

Shown in Fig. 4 are the results at time $\tau = 10.0$ for a cube with the same boundary condition as a constant surface heat flux $\bar{q}_w = 0.5$. For very small time steps, every method yields poor accuracy. This is due to the amount of calculating involved and the accumulation of round-off errors. For time steps greater than 0.01, the Brian and Douglas methods are always stable but yield poor accuracy with average temperature errors up to about 0.15. The new ADI method with $f = 0.01$ predicts the results exceptionally well; the average temperature errors are always less than 0.02 for time steps larger than 0.01. However, the new ADI method with $f = 0.1$ only predicts well up to a time step of 0.5 because of the lower time step limit compared with that using $f = 0.01$.

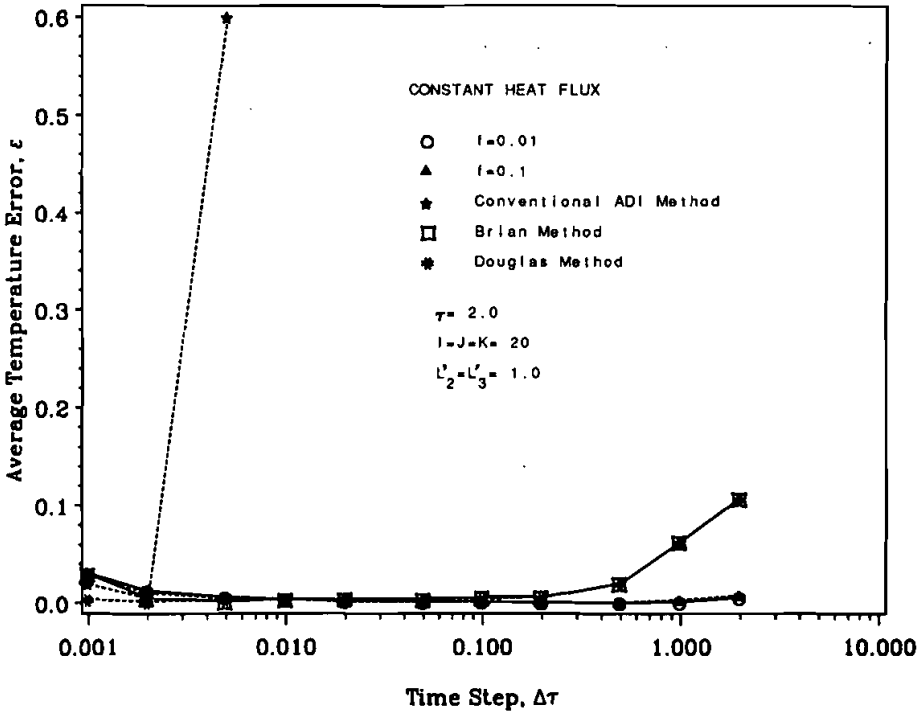

Fig. 3 Average temperature error for a cube with constant wall heat flux, $\tau = 2$.

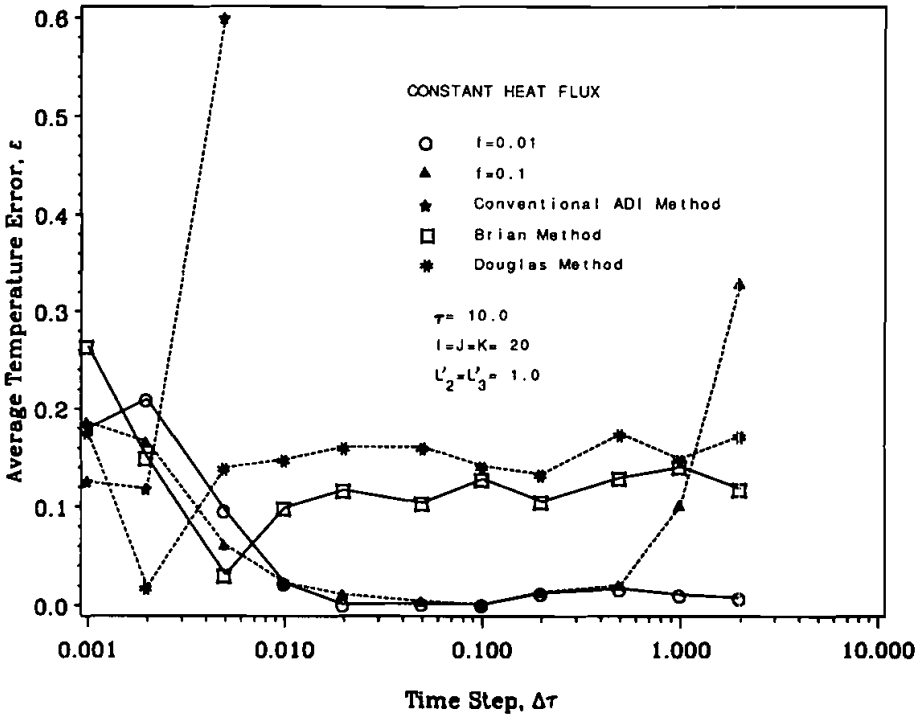

Fig. 4 Average temperature error for a cube with constant wall heat flux, $\tau = 10$.

Figure 5 shows the variation of the average temperature error with the f factor at $\tau = 10.0$ for a cube with the same boundary condition as a constant surface heat flux $\bar{q}_w = 0.5$. It can be seen, as long as the solutions do not diverge, that temperature errors remain almost the same with different values of the f factor. In other words, the value of f we chose does not influence the numerical results as long as the solutions remain stable. The results for a very small time step, $\Delta\tau = 0.001$, always have larger errors due to the accumulation of round-off errors mentioned earlier. Also, we can see that the solutions are more stable with smaller values of the f factor in the sense that much larger $\Delta\tau$ can be used.

Case 2: Constant Wall Temperature

In this case, the parallelepiped, initially at a uniform temperature $\theta_0 = 1.0$, has its surface temperatures suddenly increased and maintained at a constant temperature $\theta_w = 2.0$. The analytical temperature can be easily obtained by using the method of separation of variables [8]:

$$\theta(X, Y, Z, \tau) = \theta_w + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{l=1}^{\infty} a_{mnl} \exp\left[-\kappa_{mnl}^2 \tau\right] \times \cos\left[\frac{(2m-1)\pi X}{2}\right] \cos\left[\frac{(2n-1)\pi Y}{2L_2'}\right] \cos\left[\frac{(2l-1)\pi Z}{2L_3'}\right] \quad (36)$$

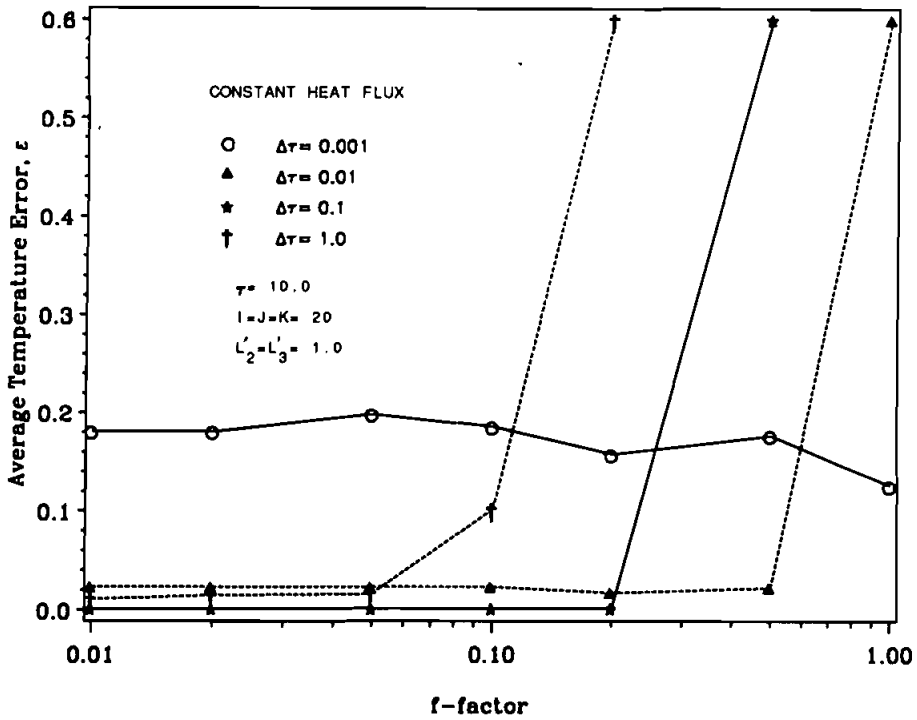

Fig. 5 Variation of average temperature error with f factor for a cube with constant wall heat flux, $\tau = 10$.

where

$$a_{mnl} = \frac{64(\theta_0 - \theta_w)}{\pi^3 (2m-1)(2n-1)(2l-1)} \sin\frac{(2m-1)\pi}{2} \sin\frac{(2n-1)\pi}{2} \sin\frac{(2l-1)\pi}{2}$$

$$\kappa^2_{mnl} = \left[\frac{(2m-1)\pi}{2}\right]^2 + \left[\frac{(2n-1)\pi}{2L'_2}\right]^2 + \left[\frac{(2l-1)\pi}{2L'_3}\right]^2$$

Presented in Fig. 6 are the results obtained for a cube at time $\tau = 0.2$. At this time, the temperature field is still undergoing transient development. Similar to case 1 with constant surface heat flux, the conventional ADI method becomes unstable if the time step is greater than 0.002. The Brian and Douglas methods predict the temperature field accurately only with a time step less than 0.02 and become inaccurate if the time step is increased beyond 0.02. The new ADI method with both $f = 0.1$ and $f = 0.01$ always yields better accuracy than the other methods; the average temperature error increases only slightly with the time step and is about 0.03 with a time step of 0.1.

Shown in Fig. 7 are the results for a cube at time $\tau = 1.0$. At this time, the temperature field has already reached steady state. The Brian and Douglas methods predict the steady state temperature field rather poorly if the time step is greater than 0.1. The average temperature error is about 0.5 with a time step of 1.0. On the contrary, the new

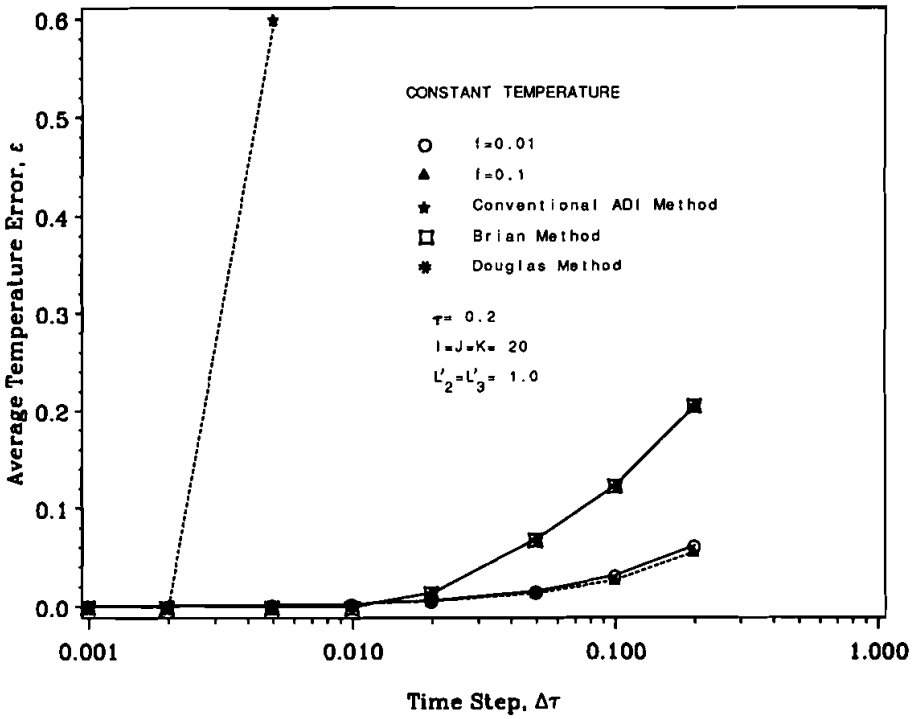

Fig. 6 Average temperature error for a cube with constant wall temperature, $\tau = 0.2$.

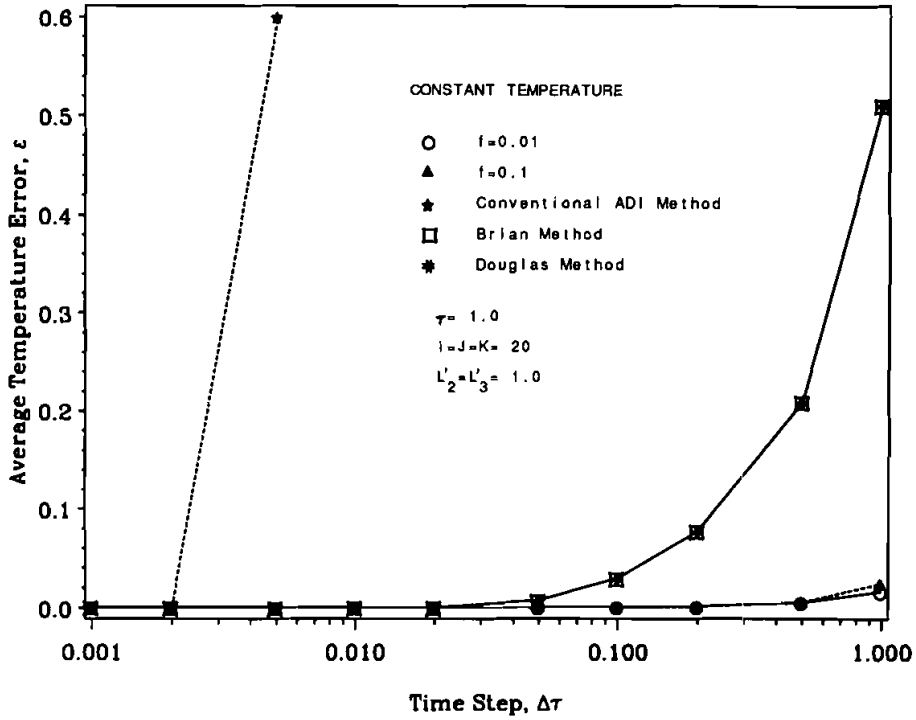

Fig. 7 Average temperature error for a cube with constant wall temperature, $\tau = 1$.

ADI method predicts the steady state results very well. With a time step of 1.0, the new ADI method yields solutions with an average temperature error about 0.024 for $f = 0.1$ and about 0.016 for $f = 0.01$.

CONCLUSIONS

In this paper, an f factor ADI method for solving transient three-dimensional heat diffusion problems is introduced. An important characteristic of this new ADI method is that the resulting finite difference equations are consistent with physical considerations. Compared with the conventional ADI method, this modification allows the time step to be increased by about a factor of $1/f$ without compromising the accuracy of the numerical solution. Compared with the conventional ADI method and the Brian and Douglas ADI methods, this new ADI method yields higher accuracy and requires less computer storage.

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