# IMPROVED BOHR'S PHENOMENON IN QUASI-SUBORDINATION CLASSES

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ABSTRACT. Recently the present authors established refined versions of Bohr's inequality in the case of bounded analytic functions. In this article, we state and prove a generalization of these results in a reformulated "distance form" version and thereby we extend the refined versions of the Bohr inequality for the class of the quasi-subordinations which contains both the classes of majorization and subordination as special cases. As a consequence, we obtain several new results.

### 1. INTRODUCTION AND TWO MAIN RESULTS

Let  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  denote the open unit disk, and  $\overline{\mathbb{D}} = \mathbb{D} \cup \partial \mathbb{D} = \{z : |z| \le 1\}$ . Then the classical Bohr inequality [8], compiled by Hardy in 1914 from his correspondence with Bohr, states the following.

**Theorem A.** If  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  is analytic in  $\mathbb{D}$  with values in  $\overline{\mathbb{D}}$ , then

(1) 
$$M_f(r) := \sum_{n=0}^{\infty} |a_n| r^n \le 1 \text{ for each } r \le 1/3$$

and the constant 1/3 cannot be improved.

Bohr originally proved the inequality (1) only for  $r \leq 1/6$  and the value 1/3 was obtained independently by M. Riesz, I. Schur and N. Wiener. Some other proofs of this inequality (1) were given by Sidon [28] and Tomić [29]. Several extensions of Theorem A may be obtained from [7–10, 14]. For a detailed account of literature on this topic, we refer to Abu-Muhanna et al. [2], Defant and Prengel [13], Garcia et al. [17]. See also recent works from [3,4,6,12,16,18–22] and the references therein. Surprisingly, in a recent paper, the present authors in [25] refined the Bohr inequality in the following improved form.

**Theorem B.** Suppose that  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  is an analytic function in  $\mathbb{D}$ ,  $|f(z)| \leq 1$  in  $\mathbb{D}$ ,  $f_0(z) = f(z) - a_0$ , and  $||f_0||_r$  denotes the quantity defined by

$$||f_0||_r = \sum_{n=1}^{\infty} |a_n|^2 r^{2n}.$$

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Then

(2) 
$$|a_0| + \sum_{n=1}^{\infty} |a_n| r^n + \left(\frac{1}{1+|a_0|} + \frac{r}{1-r}\right) \|f_0\|_r \le 1, \text{ for every } r \le \frac{1}{2+|a_0|}$$

and the numbers  $\frac{1}{2+|a_0|}$  and  $\frac{1}{1+|a_0|}$  cannot be improved. Moreover,

(3) 
$$|a_0|^2 + \sum_{n=1}^{\infty} |a_n| r^n + \left(\frac{1}{1+|a_0|} + \frac{r}{1-r}\right) \|f_0\|_r \le 1 \text{ for every } r \le \frac{1}{2}$$

and the numbers  $\frac{1}{2}$  and  $\frac{1}{1+|a_0|}$  cannot be improved.

It is important to point out that  $\frac{1}{3} \leq \frac{1}{2+|a_0|} \leq \frac{1}{2}$  and 1/3 is achieved when  $|a_0| = 1$ . In the case  $a_0 = 0$ , we have a sharp result in [25, Theorem 2].

**Remark 1.** If the constant term  $|a_0|$  in (2) is replaced by  $|a_0|^p$  for 0 , then it can be easily seen from the hypothesis of Theorem B that the sharp inequality

$$(4) \quad |a_0|^p + \sum_{n=1}^{\infty} |a_n| r^n + \left(\frac{1}{1+|a_0|} + \frac{r}{1-r}\right) \|f_0\|_r \le 1 \quad \text{for every} \quad r \le \frac{1-|a_0|^p}{2-|a_0|^2-|a_0|^p}$$

holds, where

$$\inf_{|a_0|<1} \left\{ \frac{1 - |a_0|^p}{2 - |a_0|^2 - |a_0|^p} \right\} = \frac{p}{2 + p}.$$

The cases p = 1, 2 obviously lead to (2) and (3), respectively. The inequality (4) follows from the proof of Theorem B in [25]. Indeed in the proof of [25, Theorem 1], we just need to consider

$$\Psi_p(r) = |a_0|^p + \frac{r}{1-r} \left(1 - |a_0|^2\right)$$

and observe that  $\Psi_p(r) \leq 1$  if and only if

$$r \le \varphi(x) = \frac{1 - x^p}{2 - x^2 - x^p}, \quad x = |a_0| \in [0, 1).$$

Moreover, for 0 , it is a simple exercise to see that

$$\varphi'(x) = \frac{x^{p-1}A(x)}{(2-x^2-x^p)^2}, \quad A(x) = -p - (2-p)x^2 + 2x^{2-p}$$

Because  $A'(x) = 2(2-p)x^{1-p}(1-x^p) \ge 0$  for  $0 and <math>x \in [0,1]$ , it follows that  $A(x) \le A(1) = 0$  and thus,  $\varphi$  is decreasing on [0,1). This gives

$$\varphi(x) \ge \lim_{x \to 1^-} \varphi(x) = \frac{p}{2+p}$$

For the sharpness of the radius in question in (4), we consider the function  $f = \varphi_a$  given by

$$\varphi_a(z) = \frac{a-z}{1-az} = a - (1-a^2) \sum_{k=1}^{\infty} a^{k-1} z^k, \quad z \in \mathbb{D},$$

where  $a \in (0, 1)$ . For this function as in [25], it follows that

$$-a + a^{p} + M_{\varphi_{a}}(r) + \left(\frac{1}{1+a} + \frac{r}{1-r}\right) \|\varphi_{a} - a\|_{r} = 1 - a + a^{p} + \frac{(1-a)[(2+a)r - 1]}{1-r}$$

which is bigger than 1 if and only if

$$r \le \varphi(a) = \frac{1 - a^p}{2 - a^2 - a^p}, \quad a \in [0, 1),$$

and allowing  $a \to 1^-$  one also gets the value p/(2+p), independent of a.

Our main concern in this article is to deal with few other related questions about the Bohr inequality. For example, it is well-known that the Bohr radius 1/3 continues to hold in Theorem A even if the assumption on f is replaced by the condition  $\operatorname{Re} f(z) < 1$  in  $\mathbb{D}$  and  $a_0 = f(0) \in [0, 1)$ . In fact, this condition implies that (see [15, Carathéodory's Lemma, p.41])  $|a_n| \leq 2(1 - a_0)$  for all  $n \geq 1$  and thus, we have the following sharp inequality as observed in [24]

$$M_f(r) = a_0 + \sum_{n=1}^{\infty} |a_n| r^n \le a_0 + 2(1 - a_0) \frac{r}{1 - r} \le 1 \text{ for each } r \le 1/3.$$

Therefore a natural question is to look for the analog of the refined version of it in the settings of Theorem B. We answer this question in the following statement whose proof will be given in Section 2.

**Theorem 1.** Let f(z) be an analytic function in  $\mathbb{D}$  such that  $f(z) = \sum_{n=0}^{\infty} a_n z^n$ ,  $a_0 \in (0,1)$ , and  $\operatorname{Re} f(z) < 1$  in  $\mathbb{D}$ . Then

$$\sum_{n=0}^{\infty} |a_n| r^n + \left(\frac{1}{1+a_0} + \frac{r}{1-r}\right) \sum_{n=1}^{\infty} |a_n|^2 r^{2n} \le 1$$

holds for all  $r \leq r_*$ , where  $r_* \approx 0.24683$  is the unique root of the equation  $3r^3 - 5r^2 - 3r + 1 = 0$  in the interval (0,1). Moreover, for any  $a_0 \in (0,1)$  there exists a uniquely defined  $r_0 = r_0(a_0) \in (r_*, \frac{1}{3})$  such that

$$\sum_{n=0}^{\infty} |a_n| r^n + \left(\frac{1}{1+a_0} + \frac{r}{1-r}\right) \sum_{n=1}^{\infty} |a_n|^2 r^{2n} \le 1$$

for  $r \in [0, r_0]$ . The radius  $r_0 = r_0(a_0)$  can be calculated as the solution of the equation

(5) 
$$\Phi(\lambda, r) = 4r^3\lambda^2 - (7r^3 + 3r^2 - 3r + 1)\lambda + 6r^3 - 2r^2 - 6r + 2 = 0,$$

where  $\lambda = 1 - a_0$ . The result is sharp.

In Section 2, we generalize Theorem B for a general class of quasi-subordinations which contains both subordination and majorization. Furthermore, we present few other important consequences including the proof of Theorem 1. In Section 3, we introduce Bohr's phenomenon in a refined formulation in a more general family of subordinations.

#### 2. QUASI-SUBORDINATION AND THE PROOF OF THEOREM 1

For any two analytic functions f and g in  $\mathbb{D}$ , we say that the function f is quasisubordinate to g (relative to  $\Phi$ ), denoted by  $f(z) \prec_q g(z)$  (relative to  $\Phi$ ) in  $\mathbb{D}$ , if there exist two functions  $\Phi$  and  $\omega$ , analytic in  $\mathbb{D}$ , satisfying  $\omega(0) = 0$ ,  $|\Phi(z)| \leq 1$  and  $|\omega(z)| \leq 1$ for |z| < 1 such that

(6) 
$$f(z) = \Phi(z)g(\omega(z)).$$

The case f is quasi-subordinate to g (relative to  $\Phi \equiv 1$ ) corresponds to subordination. That is  $f(z) \prec_q g(z)$  (relative to  $\Phi \equiv 1$ ) in  $\mathbb{D}$  is equivalent to saying that  $f(z) \prec g(z)$ , the usual subordination. Similarly, the case  $\omega(z) = z$  gives majorization, i.e. (6) reduces to the form  $f(z) = \Phi(z)g(z)$ . Thus, the notion of quasi-subordination includes both the concept of subordination and the principle of majorization. See [23, 26, 27] and the recent paper [4] in connection with Bohr's radius.

2.1. Bohr's phenomenon for the class of quasi-subordinations. For the proof of Theorem 1, we need some preparation.

**Lemma 1.** Let f(z) and g(z) be two analytic functions in  $\mathbb{D}$  with the Taylor series expansions  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  and  $g(z) = \sum_{n=0}^{\infty} b_n z^n$  for  $z \in \mathbb{D}$ . Suppose that  $f_0(z) = f(z) - a_0$ ,  $g_0(z) = g(z) - a_0$  and  $||f_0||_r$  is defined as in Theorem B. If  $f(z) \prec_q g(z)$  (relative to  $\Phi$ ) then

$$\sum_{n=0}^{\infty} |a_n| r^n + \left(\frac{1}{1+|a_0|} + \frac{r}{1-r}\right) \|f_0\|_r \le \sum_{n=0}^{\infty} |b_n| r^n + \left(\frac{1}{1+|b_0\Phi_0|} + \frac{r}{1-r}\right) (|b_0|^2 (1-|\Phi_0|^2) + \|g_0\|_r)$$

holds for all  $r \leq 1/3$ , where  $a_0 = \Phi_0 b_0$  with  $\Phi_0 = \Phi(0)$ .

Proof. We remark that this theorem was proved in [4] without the second term on both sides of the last inequality. Suppose that  $f \prec_q g$ . Then there exist two analytic functions  $\Phi$  and  $\omega$  satisfying  $\omega(0) = 0$ ,  $|\omega(z)| \leq 1$  and  $|\Phi(z)| \leq 1$  for all  $z \in \mathbb{D}$  such that

(7) 
$$f(z) = \Phi(z)g(\omega(z)).$$

Setting z = 0 in (7) gives that  $a_0 = \Phi_0 b_0$ . According to [4, Theorem 2.1], we obtain that

(8) 
$$M_f(r) = \sum_{n=0}^{\infty} |a_n| r^n \le M_g(r) = \sum_{n=0}^{\infty} |b_n| r^n \text{ for } r \le 1/3.$$

Finally, by (7), it follows that

$$|f(z)|^2 \le |g(\omega(z))|^2$$
 for  $z \in \mathbb{D}$ 

and thus, as in the proof of Rogosinski's Theorem [27], we can easily obtain that

(9) 
$$||f||_{r} = \sum_{n=0}^{\infty} |a_{n}|^{2} r^{2n} \le ||g||_{r} = \sum_{n=0}^{\infty} |b_{n}|^{2} r^{2n} \text{ for all } r \in [0,1)$$

and therefore, since  $a_0 = \Phi_0 b_0$ , we have

(10) 
$$||f_0||_r \le |b_0|^2 (1 - |\Phi_0|^2) + ||g_0||_r$$
 for all  $r \in [0, 1)$ .

The desired inequality follows from (8), (9) and (10).

The following result is regarded as a generalization of Theorem B and can be used to cover many situations. Because of its independent interest, we state it here.

**Lemma 2.** Let f(z) and g(z) be two analytic functions in  $\mathbb{D}$  with the Taylor series expansions  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  and  $g(z) = \sum_{n=0}^{\infty} b_n z^n$  for  $z \in \mathbb{D}$ . If  $f(z) \prec g(z)$  then

$$\sum_{n=0}^{\infty} |a_n| r^n + \left(\frac{1}{1+|a_0|} + \frac{r}{1-r}\right) \sum_{n=1}^{\infty} |a_n|^2 r^{2n} \le \sum_{n=0}^{\infty} |b_n| r^n + \left(\frac{1}{1+|a_0|} + \frac{r}{1-r}\right) \sum_{n=1}^{\infty} |b_n|^2 r^{2n}$$
holds for all  $r \le 1/3$ .

holds for all  $r \leq 1/3$ .

Proof. Set  $\Phi(z) \equiv 1$ . Then  $\Phi_0 = 1$  and  $a_0 = b_0$ .

## **Problem 1.** Determine sharp radii in Lemmas 1 and 2.

2.2. Proof of Theorem 1. Since  $\operatorname{Re} f(z) < 1$ , we may write the given condition as

$$f(z) \prec g(z), \quad g(z) = a_0 - 2(1 - a_0) \frac{z}{1 - z} = a_0 - 2(1 - a_0) \sum_{n=0}^{\infty} z^n.$$

Here g(z) is a univalent mapping of  $\mathbb{D}$  onto the left half-plane  $\{w : \operatorname{Re}(w) < 1\}$ . According to Lemma 2, with  $g(z) = \sum_{n=0}^{11} b_n z^n$ , it suffices to show that

$$S_g(r) := \sum_{n=0}^{\infty} |b_n| r^n + \left(\frac{1}{1+|b_0|} + \frac{r}{1-r}\right) \sum_{n=1}^{\infty} |b_n|^2 r^{2n} \le 1 \text{ for every } r \le r_* ,$$

where  $r_*$  is as in the statement. For convenience, we let  $1 - a_0 = \lambda$  so that  $a_0 = 1 - \lambda$ and  $b_n = -2\lambda$  for  $n \ge 1$ . This gives for  $\lambda \in [0, 1]$  and  $r \in (0, 1)$  that

$$S_{g}(r) = 1 - \lambda + 2\lambda \sum_{n=1}^{\infty} r^{n} + \left(\frac{1}{2-\lambda} + \frac{r}{1-r}\right) 4\lambda^{2} \sum_{n=1}^{\infty} r^{2n}$$
  
$$= 1 - \lambda \left[1 - \frac{2r}{1-r} - \left(\frac{1+r-\lambda r}{(2-\lambda)(1-r)}\right) \frac{4\lambda r^{2}}{1-r^{2}}\right]$$
  
$$= 1 - \lambda \left[\frac{1-3r}{1-r} - \frac{4\lambda r^{2} + 4\lambda r^{3} - 4\lambda^{2} r^{3}}{(2-\lambda)(1-r)(1-r^{2})}\right]$$
  
$$= 1 - \lambda \left[\frac{1-3r}{1-r} - \frac{4\lambda r^{2} \{1+(1-\lambda)r\}}{(2-\lambda)(1-r)(1-r^{2})}\right]$$
  
$$= 1 - \lambda \left[\frac{\Phi(\lambda, r)}{(2-\lambda)(1-r)(1-r^{2})}\right],$$

which shows that the left hand side is less than or equal to 1 whenever  $\Phi(\lambda, r) \ge 0$ , where

$$\Phi(\lambda, r) = 4r^3\lambda^2 - (7r^3 + 3r^2 - 3r + 1)\lambda + 6r^3 - 2r^2 - 6r + 2$$

Before we continue, we observe from the fourth equality in the above equalities that  $S_g(r) > 1$  for r > 1/3 and for each  $\lambda \in (0, 1]$ .

We claim that  $\Phi(\lambda, r) \ge 0$  for every  $r \le r_*$  and for  $\lambda \in (0, 1]$ . It follows that

$$\frac{\partial^2 \Phi(\lambda, r)}{\partial \lambda^2} \ge 0 \text{ for every } \lambda \in (0, 1]$$

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and thus,  $\frac{\partial \Phi}{\partial \lambda}$  is an increasing function of  $\lambda$ . This gives

$$\frac{\partial \Phi(\lambda, r)}{\partial \lambda} \le \frac{\partial \Phi}{\partial \lambda}(1, r) = r^3 - 3r^2 + 3r - 1 = -(1 - r)^3,$$

whence  $\Phi$  is a decreasing function of  $\lambda$  on (0, 1] so that

 $\Phi(\lambda, r) \ge \Phi(1, r) = 3r^3 - 5r^2 - 3r + 1,$ 

which is greater than or equal to 0 for all  $r \leq r_*$ , where  $r_*$  is the unique root of the equation  $3r^3 - 5r^2 - 3r + 1 = 0$ , which lies in (0, 1). It is easy to see that  $\Phi(1, r) = 3r^3 - 5r^2 - 3r + 1$  is an increasing function of r in [0, 1] and using Mathematica or by numerical computation by Cardano's formula, one can find that

$$r_* = \frac{5}{9} - \frac{2}{9}\sqrt{13}\cos\left[\frac{1}{3}\arctan\left(\frac{9\sqrt{303}}{103}\right)\right] + \frac{2}{3}\sqrt{\frac{13}{3}}\sin\left[\frac{1}{3}\arctan\left(\frac{9\sqrt{303}}{103}\right)\right]$$

which is approximatively 0.24683.

Since  $\Phi(0,r) = 2(1-3r)(1-r^2)$ , we have  $\Phi(0,r) \ge 0$  for  $r \le 1/3$  and  $\Phi(0,r) < 0$  for r < 1/3.

Furthermore,  $\Phi(1,r) = 3r^3 - 5r^2 - 3r + 1$  and  $\Phi'(1,r) = -9r(1-r) - (r+3) < 0$  imply  $\Phi(1,r) \ge 0$  for  $r \ge r_*$  and  $\Phi(1,r) < 0$  for  $r < r_*$ . According to the fact that  $\Phi(\lambda,r)$  is a monotonic decreasing function of  $\lambda$ , we see that for any  $r \in (r_*, 1/3)$  there is a uniquely defined  $\lambda(r) \in (0,1)$  such that  $\Phi(\lambda(r), r) = 0$ .

To prove the last assertion, we have to show that  $\frac{d\lambda(r)}{dr} < 0$ . Since

$$\frac{d\lambda(r)}{dr} = -\frac{\frac{\partial\Phi(\lambda(r),r)}{\partial r}}{\frac{\partial\Phi(\lambda(r),r)}{\partial\lambda}}$$

it is sufficient to prove that

$$\frac{\partial \Phi(\lambda(r),r)}{\partial r} < 0$$

for  $\lambda \in (0, 1)$  and  $r \in [r_*, \frac{1}{3}]$ , where

$$\begin{aligned} \frac{\partial \Phi(\lambda(r), r)}{\partial r} &= 12r^2\lambda^2 - (21r^2 + 6r - 3)\lambda + 18r^2 - 4r - 6\\ &= [12r^2\lambda^2 - (21r^2 + 6r)\lambda + 3\lambda - 3] + [18r^2 - 4r - 3]\\ &= -[12r^2\lambda(1 - \lambda) + 3r(3r + 2)\lambda + 3(1 - \lambda)] - [2(1 - 9r^2) + 4r + 1]. \end{aligned}$$

This is clearly negative for the intervals in question. This completes the proof of our theorem.  $\hfill \Box$ 

#### 3. Bohr's phenomenon for a family of subordinations

We now turn to a discussion of Bohr's phenomenon in a refined formulation in a more general family of subordinations. Let us first rewrite the refined version of the Bohr inequality (2) in an equivalent form

$$\sum_{n=1}^{\infty} |a_n| r^n + \left(\frac{1}{2 - (1 - |f(0)|)} + \frac{r}{1 - r}\right) \|f_0\|_r \le 1 - |a_0| = 1 - |f(0)|,$$

where  $||f_0||_r$  is defined as in Theorem B. We observe that the number 1 - |f(0)| is the distance from the point f(0) to the boundary  $\partial \mathbb{D}$  of the unit disk  $\mathbb{D}$  and thus, we use this "distance form" formulation to generalize the concept of the Bohr radius for the class of functions f analytic in  $\mathbb{D}$  which take values in a given simply connected domain  $\Omega$  (see also [1]).

Now for a given univalent function g, let  $S(g) = \{f : f \prec g\}, \Omega = g(\mathbb{D})$  and dist $(c, \partial\Omega)$  denote the Euclidean distance from a point  $c \in \Omega$  to the boundary  $\partial\Omega$ . We say that the family S(g) has a Bohr phenomenon in the refined formulation if there exists an  $r_g$ ,  $0 < r_g \leq 1$ , such that whenever  $f(z) = \sum_{n=0}^{\infty} a_n z^n \in S(g)$ , then

(11) 
$$T_f(r) := \sum_{n=1}^{\infty} |a_n| r^n + \left(\frac{1}{2-\lambda} + \frac{r}{1-r}\right) \|f_0\|_r \le \lambda$$

for  $|z| = r < r_g$ , and  $\lambda = \text{dist}(g(0), \partial g(\mathbb{D})) \leq 1$ . The largest such  $r_g$ ,  $f \in S(g)$ , is called the Bohr radius in the refined formulation (as described above).

From our earlier two results, we have obtained that Bohr phenomenon in refined formulation exists for the class of bounded analytic functions and also for the case of analytic functions with real part less than 1 in the unit disk. Hence the distance form allows us to extend Bohr's theorem in refined formulation to a variety of distances. We have the following result which extends Theorem 1 in a natural way. Note that f(0 = g(0) and,  $f \prec g$  if and only if

$$\frac{f(z) - f(0)}{g'(0)} \prec \frac{g(z) - g(0)}{g'(0)} = z + \frac{1}{g'(0)} \sum_{n=2}^{\infty} \frac{g^{(n)}(0)}{n!} z^n, \quad z \in \mathbb{D}.$$

and thus, if needed, it might be convenient to work with normalized superordinate function.

**Theorem 2.** Let  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  and g be analytic in  $\mathbb{D}$  such that g is univalent and convex in  $\mathbb{D}$ . Assume that  $f \in S(g)$  and  $\lambda = \operatorname{dist}(g(0), \partial g(\mathbb{D})) \leq 1$ . Then (11) holds for all  $r \leq r_*$ , where  $r_* \approx 0.24683$  as in Theorem 1. Moreover, for any  $\lambda \in (0, 1)$  there exists a uniquely defined  $r_0 \in (r_*, \frac{1}{3})$  such that  $T_f(r) \leq \lambda$  for  $r \in [0, r_0]$ . The radius  $r_0$  is as in Theorem 1 given by (5).

Proof. Let  $f \prec g$ , where  $g(z) = \sum_{n=0}^{\infty} b_n z^n$  is a univalent mapping of  $\mathbb{D}$  onto a convex domain  $\Omega = g(\mathbb{D})$ . Then it is well known from the growth estimate for convex functions and Rogosinksi's coefficient estimate that (see [11, 27])

$$\frac{1}{2}|g'(0)| \le \lambda \le |g'(0)|$$
, and  $|b_n| \le |g'(0)|$  for  $n \ge 1$ ,

where  $\lambda = \text{dist}(g(0), \partial \Omega)$ . It follows then that  $|b_n| \leq 2\lambda$  for  $n \geq 1$ . Because  $f \prec g$ , it follows that  $||f_0||_r \leq ||g_0||_r$  for each  $0 \leq r < 1$  and

$$\sum_{n=1}^{\infty} |a_n| r^n \le \sum_{n=1}^{\infty} |b_n| r^n \text{ for } r \le \frac{1}{3}.$$

Combining these two inequalities, we see that the desired conclusion follows if we can show the conclusion for  $T_g(r)$ , i.e.,

$$T_g(r) = \sum_{n=1}^{\infty} |b_n| r^n + \left(\frac{1}{2-\lambda} + \frac{r}{1-r}\right) \|g_0\|_r \le \lambda.$$

Finally, because  $|b_n| \leq 2\lambda$  for  $n \geq 1$ , we have

$$T_g(r) \leq 2\lambda \sum_{n=1}^{\infty} r^n + \left(\frac{1}{2-\lambda} + \frac{r}{1-r}\right) 4\lambda^2 \sum_{n=1}^{\infty} r^{2n}$$
$$= \lambda - \lambda \left[\frac{\Phi(\lambda, r)}{(2-\lambda)(1-r)(1-r^2)}\right],$$

where  $\Phi(\lambda, r)$  is as in the proof of Theorem 1. Thus,  $T_g(r) \leq \lambda$  holds whenever  $\Phi(\lambda, r) \geq 0$ . Remaining part of the proof follows from the argument in Theorem 1. The sharpness follows from a suitable half-plane mapping.

The idea of this section and Theorem 2 can be applied to many other situations. Another instance of this is when g is just univalent in  $\mathbb{D}$  (compare with [1] where it is shown that the sharp radius without the consideration of second term in the expression  $T_f(r)$  in (11) turns out to be  $3 - 2\sqrt{2} \approx 0.17157$ ).

**Theorem 3.** Let g be an analytic and univalent function in  $\mathbb{D}$ ,  $f \in S(g)$  and  $f(z) = \sum_{n=0}^{\infty} a_n z^n$ . Then the inequality

$$\sum_{n=1}^{\infty} |a_n| r^n + \left(\frac{1}{2-\lambda} + \frac{r}{1-r}\right) \|f_0\|_r \le \lambda$$

holds for  $|z| = r < r_g$ , where  $\lambda = \text{dist}(g(0), \partial g(\mathbb{D})) < 1$  and  $r_g \approx 0.128445$  is the unique root of the equation

$$(1 - 6r + r^2)(1 - r)^2(1 + r)^3 - 16r^2(1 + r^2) = 0$$

in the interval (0,1). The sharpness of  $r_g$  is shown by the Koebe function  $f(z) = z/(1-z)^2$ .

Proof. Let  $f \prec g$ , where  $g(z) = \sum_{n=0}^{\infty} b_n z^n$  is a univalent mapping of  $\mathbb{D}$  onto a simply connected domain  $\Omega = g(\mathbb{D})$ . Then it is well known from the Koebe estimate and Rogosinksi's coefficient estimate for univalent functions that (see [11, 27])

(12) 
$$\frac{1}{4}|g'(0)| \le \lambda \le |g'(0)|, \text{ and } |b_n| \le n|g'(0)| \text{ for } n \ge 1,$$

where  $\lambda = \text{dist}(g(0), \partial \Omega)$ . Also, the first inequality above gives  $|b_n| \leq 4n\lambda$  for  $n \geq 1$ . As in the proof of Theorem 2, we easily have

$$T_{g}(r) \leq 4\lambda \sum_{n=1}^{\infty} nr^{n} + \left(\frac{1}{2-\lambda} + \frac{r}{1-r}\right) 16\lambda^{2} \sum_{n=1}^{\infty} n^{2}r^{2n}$$
  
=  $\lambda - \lambda \left[\frac{(1-r)^{2} - 4r}{(1-r)^{2}} - \left(\frac{1}{2-\lambda} + \frac{r}{1-r}\right)\frac{16\lambda r^{2}(1+r^{2})}{(1-r^{2})^{3}}\right]$   
=  $\lambda - \lambda \left[\frac{\Psi(\lambda, r)}{(2-\lambda)(1-r)(1-r^{2})^{3}}\right],$ 

where the equality in the above inequality is attained when g(z) equals the Koebe function  $z/(1-z)^2$ , and

$$\Psi(\lambda, r) = (1 - 6r + r^2)(2 - \lambda)(1 - r)^2(1 + r)^3 - (1 + r(1 - \lambda))16\lambda r^2(1 + r^2)$$
  
=  $16\lambda^2 r^3(1 + r^2) - \lambda[(1 - 6r + r^2)(1 - r)^2(1 + r)^3 + 16r^2(1 + r)(1 + r^2)]$   
 $+ 2(1 - 6r + r^2)(1 - r)^2(1 + r)^3.$ 

We claim that  $\Psi(\lambda, r) \ge 0$  for every  $r \le r_g$  and for  $\lambda \in (0, 1]$ . Clearly,

$$\frac{\partial^2 \Psi(\lambda, r)}{\partial \lambda^2} \ge 0 \text{ for every } \lambda \in (0, 1]$$

which implies that

$$\frac{\partial \Psi(\lambda, r)}{\partial \lambda} \leq \frac{\partial \Psi}{\partial \lambda} (1, r) 
= -(1-r)[16r^2(1+r^2) + (1-6r+r^2)(1-r)(1+r)^3] 
= -(1-r)[1-4r+5r^2+27r^4+4r^5-r^6] 
= -(1-r)[r^5(1-r)+r^2+27r^4+3r^5+(2r-1)^2]$$

from which we obtain that  $\Psi$  is an decreasing function of  $\lambda$  on (0,1) so that

$$\Psi(\lambda, r) \ge \Psi(1, r) = (1 - 6r + r^2)(1 - r)^2(1 + r)^3 - 16r^2(1 + r^2)$$

which is greater than or equal to 0 for all  $r \leq r_g$ , where  $r_g$  is as in the statement. The sharpness of  $r_g$  can be easily shown by the Koebe function  $f(z) = z/(1-z)^2$ .

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