axioms

MDPI

Article

Improved Chen's Inequalities for Submanifolds of Generalized Sasakian-Space-Forms

Yanlin Li [1,*,†], Mohan Khatri [2,†], Jay Prakash Singh [2,†] and Sudhakar K. Chaubey [3,†]

1 School of Mathematics, Hangzhou Normal University, Hangzhou 311121, China
2 Department of Mathematics and Computer Science, Mizoram University, Aizawl 796004, India; mohankhatri.official@gmail.com (M.K.); jpsmaths@gmail.com (J.P.S.)
3 Department of Information Technology, University of Technology and Applied Sciences, P.O. Box 77, Shinas 324, Oman; sudhakar.chaubey@shct.edu.om
* Correspondence: liyl@hznu.edu.cn
† These authors contributed equally to this work.

Abstract: In this article, we derive Chen's inequalities involving Chen's δ-invariant δ_M, Riemannian invariant $\delta(m_1,\ldots,m_k)$, Ricci curvature, Riemannian invariant $\Theta_k(2\le k\le m)$, the scalar curvature and the squared of the mean curvature for submanifolds of generalized Sasakian-space-forms endowed with a quarter-symmetric connection. As an application of the obtain inequality, we first derived the Chen inequality for the bi-slant submanifold of generalized Sasakian-space-forms.

Keywords: Chen inequalities; quarter-symmetric connection; generalized Sasakian-space-form; bi-slant; Riemannian invariants

MSC: 53B15; 53B25; 53D15

Citation: Li, Y.; Khatri, M.; Singh, J.P.; Chaubey, S.K. Improved Chen's Inequalities for Submanifolds of Generalized Sasakian-Space-Forms. *Axioms* **2022**, *11*, 324. https://doi.org/10.3390/axioms11070324

Academic Editor: Mica Stankovic

Received: 30 May 2022
Accepted: 29 June 2022
Published: 1 July 2022

1. Introduction

In submanifold theory, obtaining the relationship between an intrinsic invariant and an extrinsic invariant has been the primary goal of many geometers in recent decades. Chen invariants were introduced by B.Y. Chen [1] to tackle the question raised by Chen concerning the existence of minimal immersions into a Euclidean space of arbitrary dimension [2]. Chen's δ-invariant δ_M of a Riemannian manifold M introduced by Chen is

$$\delta_M(x)=\tau(x)-\inf\{K(\Pi)|\Pi \text{ is a plane section } \subset T_xM\}, \tag{1}$$

where τ is the scalar curvature of M.

In [1], Chen obtained an inequality for a Riemannian submanifold M^m of a real space form $\widetilde{M}$ with constant sectional curvature c as

$$\delta_M\le\frac{m^2(m-2)}{2(m-1)}\parallel H\parallel^2+\frac{1}{2}(m+1)(m-2)c, \tag{2}$$

where H is the mean curvature of the submanifold M^m. Equation (2) is known as the first Chen inequality.

Then in [3], Chen gave the inequality for a Riemannian submanifold M^m of complex-space-form $\widetilde{M}^n(4c)$ as follows:

$$\delta_M\le\frac{m^2(m-2)}{2(m-1)}\parallel H\parallel^2+\frac{1}{2}(m+1)(m-2)c+\frac{3}{2}\parallel P\parallel^2c-3\Theta(\pi)c. \tag{3}$$

Afterward, many authors obtained Chen's inequalities for different submanifolds in various ambient spaces, such as the Kenmotsu space form [4], the Sasakian-space-form [5], the Cosympletic space form [6], the Riemannian manifold of quasi-constant curvature [7], generalized space forms [8,9], Statistical manifolds [10–12], quaternionic space forms [13] and the GRW spacetime [14].

Qu and Wang [15] introduced the notion of a special type of quarter-symmetric connection as a generalization of a semi-symmetric metric connection [16] and a semi-symmetric non-metric connection [17]. They studied the Einstein warped product and multiple warped products with a quarter-symmetric connection [15]. In [18], the authors obtained Chen's inequalities for submanifolds of real space forms endowed with a quarter-symmetric connection. Mihai and Özgür [19] obtained the Chen inequalities for submanifolds of complex space forms and Sasakian-space-forms with a semi-symmetric metric connection. Wang [20] obtained Chen inequalities for submanifolds of complex space forms and Sasakian-space-forms with quarter-symmetric connections which improved the results of Mihai and Özgür [19]. Sular [21] obtained Chen inequalities for submanifolds of generalized space forms with a semi-symmetric metric connection. Al-Khaldi et al. [22] obtained the Chen–Ricci inequalities Lagrangian submanifold in generalized complex space form and a Legendrian submanifold in a generalized Sasakian-space-form endowed with the quarter-symmetric connection.

As a continuation of their studies, we obtained Chen inequalities for submanifolds of generalized Sasakian-space-form admitting a quarter-symmetric connection. The significance of this study is that it generalizes a large number of previously obtained results, some of which are [20,21]. The paper is organized as follows. In Section 2, we recall the properties of the quarter-symmetric connection. In Section 3, we establish the B.Y. Chen inequalities for submanifolds of a generalized Sasakian-space-form endowed with a quarter-symmetric connection. First, we prove the following inequality and also look at its equality case.

Theorem 1. *Let $M^m, m \geq 3$ be an m-dimensional submanifold of a $(2n+1)$-dimensional generalized Sasakian-space-form $\widetilde{M}(f_1, f_2, f_3)$ endowed with a quarter-symmetric connection $\overline{\nabla}$, then*

$$\begin{aligned}\tau(x) - K(\Pi) \leq (m-2)\Big(\frac{m^2}{2(m-1)} \parallel H \parallel^2 + (m+1)\frac{f_1}{2}\Big) \\ +\Big(3 \parallel \mathcal{T} \parallel^2 - 6\Theta^2(\Pi)\Big)\frac{f_2}{2} + \Big(\parallel \xi_\Pi \parallel^2 - (m-1) \parallel \xi^T \parallel^2 \Big) f_3 \\ +\frac{(\psi_1+\psi_2)}{2}\Big(tr(\beta_1 \mid_\Pi) - \lambda(m-1)\Big) + \frac{\psi_2(\psi_1-\psi_2)}{2}\Big(tr(\beta_2 \mid_\Pi) \\ -\mu(m-1)\Big) + \frac{(\psi_1-\psi_2)}{2}\Big(\Lambda(tr(h\mid_\Pi)) - m(m-1)\Lambda(H)\Big),\end{aligned}$$

where Π is a two-plane section T_xM, $x \in M$.

Next, we obtain bounds for the Riemannian invariant $\delta(m_1, \ldots, m_k)$ and a Ricci curvature in terms of the scalar curvature of the r-plane section L, squared mean curvature and some special functions. Among others, we obtain the inequality involving the Riemannian invariant $\Theta_k, 2 \leq k \leq m$, as follows:

$$\begin{aligned}\parallel H \parallel^2 (x) \geq \Theta_k(x) - f_1 - \frac{3f_2}{m(m-1)} \parallel \mathcal{T} \parallel^2 + \frac{2f_3}{m} \parallel \xi^T \parallel^2 \\ +\frac{\lambda}{m}(\psi_1+\psi_2) + \frac{\mu}{m}\psi_2(\psi_1-\psi_2) + (\psi_1-\psi_2)\Lambda(H).\end{aligned}$$

Using Theorem 1 in Section 4, we derive Chen inequalities for the bi-slant submanifold of generalized Sasakian-space-forms.

2. Preliminaries

Suppose that $\widetilde{M}^{m+p}$ is an $(m+p)$-dimensional Riemannian manifold with Riemannian metric g. A linear connection $\overline{\nabla}$ is known as a quarter-symmetric connection if its torsion tensor T is presented by

$$T(X_1, X_2) = \overline{\nabla}_{X_1} X_2 - \overline{\nabla}_{X_2} X_1 - [X_1, X_2]$$

satisfies

$$T(X_1, X_2) = \Lambda(X_2)\varphi X_1 - \Lambda(X_1)\varphi X_2,$$

where Λ is a 1-form, P is a vector field given by $\Lambda(X_1) = g(X_1, P)$, and φ is $(1,1)$-tensor. In [15], the authors introduced a special type of quarter-symmetric connection defined as:

$$\overline{\nabla}_{X_1} X_2 = \widehat{\widehat{\nabla}}_{X_1} X_2 + \psi_1 \Lambda(X_2) X_1 - \psi_2 g(X_1, X_2) P, \tag{4}$$

where $\widehat{\widehat{\nabla}}$ denote the Levi–Civita connection. It is easy to see that the quarter-symmetric connection $\overline{\nabla}$ includes the semi-symmetric metric connection ($\psi_1 = \psi_2 = 1$) and the semi-symmetric non-metric connection ($\psi_1 = 1, \psi_2 = 0$). Let the curvature tensor of $\overline{\nabla}$ be

$$\overline{R}(X_1, X_2)X_3 = \overline{\nabla}_{X_1} \overline{\nabla}_{X_2} X_3 - \overline{\nabla}_{X_2} \overline{\nabla}_{X_1} X_3 - \overline{\nabla}_{[X_1, X_2]} X_3.$$

Similarly, the curvature tensor $\widehat{\widehat{R}}$ of $\widehat{\widehat{\nabla}}$ can be defined as the same.

Let M^m be an m-dimensional submanifold of an $(m+p)$-dimensional Riemannian manifold $\widetilde{M}^{m+p}$ endowed with the quarter-symmetric connection $\overline{\nabla}$ and the Levi–Civita connection $\widehat{\widehat{\nabla}}$. Let ∇ and $\widehat{\nabla}$ denote the induced quarter-symmetric connection and the induced Levi–Civita connection on the submanifold M. The Gauss formula with respect to ∇ and $\widehat{\nabla}$ can be presented as

$$\begin{aligned} \overline{\nabla}_{X_1} X_2 &= \nabla_{X_1} X_2 + h(X_1, X_2), \quad X_1, X_2 \in \Gamma(TM) \\ \widehat{\widehat{\nabla}}_{X_1} X_2 &= \widehat{\nabla}_{X_1} X_2 + \widehat{h}(X_1, X_2), \quad X_1, X_2 \in \Gamma(TM) \end{aligned}$$

where h and $\widehat{h}$ are the second fundamental forms associated with the quarter-symmetric connection ∇ and the Levi-Civita connection $\widehat{\nabla}$, respectively, and are related as follows:

$$h(X_1, X_2) = \widehat{h}(X_1, X_2) - \psi_2 g(X_1, X_2) P^{\perp}, \tag{5}$$

where $P^{\perp}$ is the normal component of the vector field P on M. If P^T represents that tangent component of the vector field P on M, then $P = P^T + P^{\perp}$.

The curvature tensor $\overline{R}$ with respect to the quarter-symmetric connection $\overline{\nabla}$ on $\widetilde{M}^{m+p}$ can be expressed as [15]:

$$\begin{aligned} \overline{R}(X_1, X_2, X_3, X_4) &= \widehat{\widehat{R}}(X_1, X_2, X_3, X_4) + \psi_1 \beta_1(X_1, X_3) g(X_2, X_3) \\ -\psi_1 \beta_1(X_2, X_3) g(X_1, X_4) &+ \psi_2 g(X_1, X_3) \beta_1(X_2, X_4) - \psi_2 g(X_2, X_3) \beta_1(X_1, X_4) \\ &+ \psi_2(\psi_1 - \psi_2) g(X_1, X_3) \beta_2(X_2, X_4) - \psi_2(\psi_1 - \psi_2) g(X_2, X_3) \beta_2(X_1, X_4), \end{aligned} \tag{6}$$

where β_1 and β_2 are symmetric $(0,2)$-tensor fields defined as

$$\beta_1(X_1, X_2) = (\widehat{\widehat{\nabla}}_{X_1} \Lambda)(X_2) - \psi_1 \Lambda(X_1) \Lambda(X_2) + \frac{\psi_2}{2} g(X_1, X_2) \Lambda(P),$$

and

$$\beta_2(X_1, X_2) = \frac{\Lambda(P)}{2} g(X_1, X_2) + \Lambda(X_1) \Lambda(X_2).$$

Moreover, we assume that $tr(\beta_1) = \lambda$ and $tr(\beta_2) = \mu$.

Suppose that R and $\widehat{R}$ are the curvature tensors of ∇ and $\widehat{\nabla}$, respectively. Then the Gauss equation with respect to the quarter-symmetric connection is as follows [15]:

$$\begin{aligned}\overline{R}(X_1, X_2, X_3, X_4) &= R(X_1, X_2, X_3, X_4) - g(h(X_1, X_4), h(X_2, X_3)) \\ &+ g(h(X_2, X_4), h(X_1, X_3)) + (\psi_1 - \psi_2)g(h(X_2, X_3), P)g(X_1, X_4) \\ &+ (\psi_2 - \psi_1)g(h(X_1, X_3), P)g(X_2, X_4).\end{aligned} \tag{7}$$

Let $\{e_1, \ldots, e_m\}$ and $\{e_{m+1}, \ldots, e_{m+p}\}$ be an orthonormal frame of T_xM and $T_x^{\perp}M$ at the point $x \in M$. Then the mean curvature vector of M associated with ∇ is $H = \frac{1}{m}\sum_{i=1}^{m} h(e_i, e_j)$. Similarly, the mean curvature vector of M associated to $\widehat{\nabla}$ is $\widehat{H} = \frac{1}{m}\sum_{i=1}^{m} \widehat{h}(e_i, e_j)$. In addition, the squared length of h is $\| h \|^2 = \sum_{i,j=1}^{m} g(h(e_i, e_j), h(e_i, e_j))$.

Now, we recall some of the Riemannian invariants introduced by Chen [23] in a Riemannian manifold. Let L be an r-dimensional subspace of T_xM, $x \in M$, $r \geq 2$ and $\{e_1, \ldots, e_r\}$ an orthonormal basis of L. The scalar curvature τ of the r-plane section L is given by

$$\tau(L) = \sum_{1 \leq i < j \leq r} K_{ij}, \tag{8}$$

where K_{ij} is the sectional curvature of the plane section spanned by e_i and e_j at $x \in M$. Suppose that $\Pi \subset T_xM$ is a two-plane section and $K(\Pi)$ is the sectional curvature of M for a plane section Π in $T_xM, x \in M$. Then

$$K(\Pi) = \frac{1}{2}[R(e_1, e_2, e_2, e_1) - R(e_1, e_2, e_1, e_2)]. \tag{9}$$

The scalar curvature $\tau(x)$ of M at the point x is presented by

$$\tau(x) = \sum_{i<j} K_{ij}, \tag{10}$$

where $\{e_1, \ldots, e_m\}$ is an orthonormal basis for T_xM.

3. B. Y. Chen Inequalities

First, we recall the well-known lemma obtained by Chen [1], which is as follows:

Lemma 1. *If $a_1, \ldots, a_m, a_{m+1}$ are $m+1$ $(m \geq 2)$ real numbers such that*

$$\Big(\sum_{i=1}^{m} a_i\Big)^2 = (m-1)\Big(\sum_{i=1}^{m} a_i^2 + a_{m+1}\Big),$$

then $2a_1a_2 \geq a_{m+1}$, with equality holding if and only if $a_1 + a_2 = a_3 = \ldots = a_m$.

Now, let $\widetilde{M}$ be a $(2n+1)$-dimensional almost contact metric manifold with the structure (φ, η, g, ξ) where φ is a $(1,1)$-tensor, η is a 1-form which is dual to the Reeb vector field ξ, and g is a Riemannian metric on $\widetilde{M}$ which satisfies the follows [24]:

$$\varphi^2 = -I + \eta \otimes \xi, \;\; \eta(\xi) = 1, \;\; g(\varphi X_1, \varphi X_2) = g(X_1, X_2) - \eta(X_1)\eta(X_2).$$

Because of these conditions, we have

$$\varphi\xi = 0, \;\; \eta \cdot \varphi = 0, \quad \eta(X_1) = g(X_1, \xi),$$

for any vector fields $X_1, X_2 \in \Gamma(T\widetilde{M})$.

An almost contact metric manifold $(\widetilde{M}, \varphi, \eta, \xi, g)$ whose curvature tensor satisfies

$$\begin{aligned}\widehat{\widetilde{R}}(X_1, X_2)X_3 = f_1\{g(X_2, X_3)X_1 - g(X_1, X_3)X_2\} + f_2\{g(X_1, \varphi X_3)\varphi X_2 \\ -g(X_2, \varphi X_3)\varphi X_1 + 2g(X_1, \varphi X_2)\varphi X_3\} + f_3\{\eta(X_1)\eta(X_3)X_2 \\ -\eta(X_2)\eta(X_3)X_1 + g(X_1, X_3)\eta(X_2)\xi - g(X_2, X_3)\eta(X_1)\xi\},\end{aligned} \tag{11}$$

for any vector field $X_1, X_2, X_3 \in \Gamma(T\widetilde{M})$ and f_1, f_2, f_3 being differentiable functions on $\widetilde{M}$ is said to be a generalized Sasakian-space-form denoted by $\widetilde{M}(f_1, f_2, f_3)$. The notion of a generalized Sasakian-space-form $\widetilde{M}(f_1, f_2, f_3)$ was introduced by Alegre et al. [25], generalizing three important contact space forms, that is, the Sasakian-space-form $(f_1 = \frac{c+3}{4}, f_2 = f_3 = \frac{c-1}{4})$, the Kenmotsu space form $(f_1 = \frac{c-3}{4}, f_2 = f_3 = \frac{c+1}{4})$ and the Cosympletic space form $(f_1 = f_2 = f_3 = \frac{c}{4})$.

From (6) and (11), we obtain

$$\begin{aligned}\overline{R}(X_1, X_2, X_3, X_4) = f_1\{g(X_2, X_3)g(X_1, X_4) - g(X_1, X_3)g(X_2, X_4)\} \\ +f_2\{g(X_1, \varphi X_3)g(\varphi X_2, X_4) - g(X_2, \varphi X_3)g(\varphi X_1, X_4) \\ +2g(X_1, \varphi X_2)g(\varphi X_3, X_4)\} + f_3\{\eta(X_1)\eta(X_3)g(X_2, X_4) \\ -\eta(X_2)\eta(X_3)g(X_1, X_4) + g(X_1, X_3)\eta(X_2)\eta(X_4) \\ -g(X_2, X_3)\eta(X_1)\eta(X_4)\} + \psi_1\beta_1(X_1, X_3)g(X_2, X_3) \\ -\psi_1\beta_1(X_2, X_3)g(X_1, X_4) + \psi_2 g(X_1, X_3)\beta_1(X_2, X_4) \\ -\psi_2 g(X_2, X_3)\beta_1(X_1, X_4) + \psi_2(\psi_1 - \psi_2)g(X_1, X_3)\beta_2(X_2, X_4) \\ -\psi_2(\psi_1 - \psi_2)g(X_2, X_3)\beta_2(X_1, X_4),\end{aligned} \tag{12}$$

Let M^m be a submanifold of a generalized Sasakian-space-form $\widetilde{M}(f_1, f_2, f_3)$ of dimension $(2n+1)$. For any tangent vector field X_1 on M, we can write $\varphi X_1 = \mathcal{T}X_1 + \mathcal{F}X_1$, where $\mathcal{T}X_1$ is the tangential component, and $\mathcal{F}X_1$ is the normal component of φX_1. The squared norm of $\mathcal{T}$ at $x \in M$ is defined as

$$\parallel \mathcal{T} \parallel^2 = \sum_{i,j=1}^{m} g^2(\varphi e_i, e_j), \tag{13}$$

where $\{e_1, \ldots, e_m\}$ is any orthonormal basis of the tangent space T_xM and decomposing the structural vector field $\xi = \xi^T + \xi^\perp$, where ξ^T and $\xi^\perp$ denotes the tangential and normal components of ξ. Moreover, we set $\Theta^2(\Pi) = g^2(\mathcal{T}e_1, e_2) = g^2(\varphi e_1, e_2)$, where $\{e_1, e_2\}$ is the orthonormal basis of two-plane section Π.

Theorem 2. *Let $M^m, m \geq 3$ be an m-dimensional submanifold of a $(2n+1)$-dimensional generalized Sasakian-space-form $\widetilde{M}(f_1, f_2, f_3)$ endowed with a quarter-symmetric connection $\overline{\nabla}$, then*

$$\begin{aligned}\tau(x) - K(\Pi) \leq (m-2)\Big(\frac{m^2}{2(m-1)} \parallel H \parallel^2 + (m+1)\frac{f_1}{2}\Big) \\ +\Big(3 \parallel \mathcal{T} \parallel^2 - 6\Theta^2(\Pi)\Big)\frac{f_2}{2} + \Big(\parallel \xi_\Pi \parallel^2 - (m-1) \parallel \xi^T \parallel^2 \Big)f_3 \\ +\frac{(\psi_1 + \psi_2)}{2}\Big(tr(\beta_1 \mid_\Pi) - \lambda(m-1)\Big) + \frac{\psi_2(\psi_1 - \psi_2)}{2}\Big(tr(\beta_2 \mid_\Pi) \\ -\mu(m-1)\Big) + \frac{(\psi_1 - \psi_2)}{2}\Big(\Lambda(tr(h \mid_\Pi)) - m(m-1)\Lambda(H)\Big),\end{aligned}$$

where Π is a two-plane section T_xM, $x \in M$.

If in addition, P is a tangent vector field on M^m, then $H = \widehat{H}$ and the equality case holds at a point $x \in M$ if and only if there exists an orthonormal basis $\{e_1, \ldots, e_m\}$ of T_xM and an *orthonormal basis* $\{e_{m+1},\ldots,e_{2n+1}\}$ *of* $T_x^{\perp}M$ *such that the shape operators of* M *in* $\widetilde{M}(f_1,f_2,f_3)$ *at* x *have the following forms:*

$$A_{e_{m+1}}=\begin{pmatrix} h_{11}^{m+1} & 0 & 0 & \ldots & 0\\ 0 & h_{22}^{m+1} & 0 & \ldots & 0\\ 0 & 0 & h_{11}^{m+1}+h_{22}^{m+1} & \ldots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \ldots & h_{11}^{m+1}+h_{22}^{m+1}\end{pmatrix}$$

Theorem and

$$A_{e_r}=\begin{pmatrix} h_{11}^r & h_{12}^r & 0 & \ldots & 0\\ h_{12}^r & -h_{11}^r & 0 & \ldots & 0\\ 0 & 0 & 0 & \ldots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \ldots & 0\end{pmatrix}, m+2\le r\le 2n+1$$

Proof. Let $x\in M$ and $\{e_1,e_2,\ldots,e_m\},\{e_{m+1},\ldots,e_{2n+1}\}$ be an orthonormal basis of T_xM and $T_x^{\perp}M$, respectively, then from (7), (10) and (12) we obtain

$$\begin{aligned}2\tau(x)=&\ m^2\parallel H\parallel^2-\parallel h\parallel^2+m(m-1)f_1+3f_2\parallel\mathcal{T}\parallel^2-2(m-1)f_3\parallel\xi^T\parallel^2\\ &-(\psi_1+\psi_2)\lambda(m-1)-\psi_2(\psi_1-\psi_2)\mu(m-1)-m(m-1)(\psi_1-\psi_2)\Lambda(H).\end{aligned}\tag{14}$$

We set,

$$\begin{aligned}c=&\ 2\tau(x)-\frac{m^2(m-2)}{m-1}\parallel H\parallel^2-m(m-1)f_1-3f_2\parallel\mathcal{T}\parallel^2+2(m-1)f_3\parallel\xi^T\parallel^2\\ &+(\psi_1+\psi_2)\lambda(m-1)+\psi_2(\psi_1-\psi_2)\mu(m-1)+m(m-1)(\psi_1-\psi_2)\Lambda(H),\end{aligned}\tag{15}$$

then (14) becomes

$$m^2\parallel H\parallel^2=(m-1)\big(\parallel h\parallel^2+c\big).\tag{16}$$

For a chosen orthonormal basis, (16) can be written as:

$$\Big(\sum_{i=1}^{m}h_{ii}^{m+1}\Big)^2=(m-1)\Big[\sum_{i=1}^{m}(h_{ii}^{m+1})^2+\sum_{i\neq j}(h_{ij}^{m+1})^2+\sum_{r=m+2}^{2n+1}\sum_{i,j=1}^{m}(h_{ij}^r)^2+c\Big],$$

then using Lemma 1, we have

$$2h_{11}^{m+1}h_{22}^{m+1}\ge\sum_{i\neq j}(h_{ij}^{m+1})^2+\sum_{r=m+2}^{2n+1}\sum_{i,j=1}^{m}(h_{ij}^r)^2+c.\tag{17}$$

Now, let $\Pi=span\{e_1,e_2\}$, then from (7) and (12) we obtain

$$\begin{aligned}R(e_1,e_2,e_2,e_1)=&\sum_{r=m+1}^{2n+1}[h_{11}^rh_{22}^r-(h_{12}^r)^2]-(\psi_1-\psi_2)g(h(e_2,e_2),P)\\ &+f_1+3f_2g^2(\varphi e_1,e_2)-f_3(\eta^2(e_1)+\eta^2(e_2))\\ &-\psi_1\beta_1(e_2,e_2)-\psi_2\beta_1(e_1,e_1)-\psi_2(\psi_1-\psi_2)\beta_2(e_1,e_1).\end{aligned}\tag{18}$$

and

$$R(e_1,e_2,e_1,e_2) = \sum_{r=m+1}^{2n+1} [(h_{12}^r)^2 - h_{11}^r h_{22}^r] + (\psi_1 - \psi_2) g(h(e_1,e_1),P)$$
$$- f_1 - 3f_2 g^2(\varphi e_1, e_2) + f_3(\eta^2(e_1) + \eta^2(e_2))$$
$$+ \psi_1 \beta_1(e_1,e_1) + \psi_2 \beta_1(e_2,e_2) + \psi_2(\psi_1 - \psi_2)\beta_2(e_2,e_2). \quad (19)$$

Making use of (18) and (19) in (9), we obtain

$$K(\Pi) = \sum_{r=m+1}^{2n+1} [h_{11}^r h_{22}^r - (h_{12}^r)^2] - \frac{(\psi_1 - \psi_2)}{2} \Lambda(tr(h \mid_\Pi))$$
$$+ f_1 + 3f_2 \Theta^2(\Pi) - f_3(\| \xi_\Pi \|^2)$$
$$- \frac{\psi_1}{2} tr(\beta_1 \mid \Pi) - \frac{\psi_2}{2} tr(\beta_1 \mid_\Pi) - \frac{\psi_2}{2}(\psi_1 - \psi_2) tr(\beta_2 \mid_\Pi). \quad (20)$$

Combining (14) and (20) gives

$$\tau(x) - K(\Pi) = (m-2)\Big(\frac{m^2}{2(m-1)} \| H \|^2 + (m+1)\frac{f_1}{2}\Big)$$
$$+ \Big(3 \| \mathcal{T} \|^2 - 6\Theta^2(\Pi)\Big)\frac{f_2}{2} + \Big(\| \xi_\Pi \|^2 - (m-1) \| \xi^T \|^2 \Big) f_3$$
$$+ \frac{(\psi_1 + \psi_2)}{2}\Big(tr(\beta_1 \mid_\Pi) - \lambda(m-1)\Big) + \frac{\psi_2(\psi_1 - \psi_2)}{2}\Big(tr(\beta_2 \mid_\Pi)$$
$$- \mu(m-1)\Big) + \frac{(\psi_1 - \psi_2)}{2}\Big(\Lambda(tr(h \mid_\Pi)) - m(m-1)\Lambda(H)\Big)$$
$$+ \sum_{r=m+1}^{2n+1}\Big[\sum_{1 \le i < j \le m} h_{ii}^r h_{jj}^r - h_{11}^r h_{22}^r - \sum_{1 \le i < j \le m} (h_{ij}^r)^2 + (h_{12}^r)^2\Big]. \quad (21)$$

Making use of Lemma 2.4 [26], we have

$$\sum_{r=m+1}^{2n+1}\Big[\sum_{1 \le i < j \le m} h_{ii}^r h_{jj}^r - h_{11}^r h_{22}^r - \sum_{1 \le i < j \le m} (h_{ij}^r)^2 + (h_{12}^r)^2\Big] \le \frac{m^2(m-2)}{2(m-1)} \| H \|^2 . \quad (22)$$

In view of the last expression in (21), we obtain

$$\tau(x) - K(\Pi) \le (m-2)\Big(\frac{m^2}{2(m-1)} \| H \|^2 + (m+1)\frac{f_1}{2}\Big)$$
$$+ \Big(3 \| \mathcal{T} \|^2 - 6\Theta^2(\Pi)\Big)\frac{f_2}{2} + \Big(\| \xi_\Pi \|^2 - (m-1) \| \xi^T \|^2 \Big) f_3$$
$$+ \frac{(\psi_1 + \psi_2)}{2}\Big(tr(\beta_1 \mid_\Pi) - \lambda(m-1)\Big) + \frac{\psi_2(\psi_1 - \psi_2)}{2}\Big(tr(\beta_2 \mid_\Pi)$$
$$- \mu(m-1)\Big) + \frac{(\psi_1 - \psi_2)}{2}\Big(\Lambda(tr(h \mid_\Pi)) - m(m-1)\Lambda(H)\Big). \quad (23)$$

Now, if P is a tangent vector field on M, then (5) implies $h = \widehat{h}$ and $H = \widehat{H}$. If the equality case (23) holds at a point $x \in M$, then the equality cases of (17) and (22) hold, which gives

$$\begin{aligned} h_{11}^{m+1} = h_{22}^{m+1} = h_{33}^{m+1} = \cdots = h_{mm}^{m+1} \\ h_{1j}^{m+1} = h_{2j}^{m+1} = 0, j > 2 \\ h_{11}^{r} + h_{22}^{r} = 0, r = m+2, \ldots, 2n+1 \\ h_{ij}^{r} = 0, i \neq j, r = m+1, \ldots, 2n+1 \\ h_{ij}^{m+1} = 0, i \neq j, i, j > 2 \end{aligned}$$

Therefore, choosing a suitable orthonormal basis, the shape operators take the desired forms. □

Corollary 1. *Under the same arguments as in Theorem 2,*

1. *If the structure vector field ξ is tangent to M, we have*

$$\begin{aligned} \tau(x) - K(\Pi) \leq (m-2)\Big(\frac{m^2}{2(m-1)} \parallel H \parallel^2 + (m+1)\frac{f_1}{2}\Big) \\ + \Big(3 \parallel \mathcal{T} \parallel^2 - 6\Theta^2(\Pi)\Big)\frac{f_2}{2} + \Big(\parallel \xi_\Pi \parallel^2 - (m-1)\Big) f_3 \\ + \frac{(\psi_1+\psi_2)}{2}\Big(tr(\beta_1 \mid_\Pi) - \lambda(m-1)\Big) + \frac{\psi_2(\psi_1-\psi_2)}{2}\Big(tr(\beta_2 \mid_\Pi) \\ -\mu(m-1)\Big) + \frac{(\psi_1-\psi_2)}{2}\Big(\Lambda(tr(h\mid_\Pi)) - m(m-1)\Lambda(H)\Big). \end{aligned}$$

2. *If the structure vector field ξ is normal to M, we have*

$$\begin{aligned} \tau(x) - K(\Pi) \leq (m-2)\Big(\frac{m^2}{2(m-1)} \parallel H \parallel^2 + (m+1)\frac{f_1}{2}\Big) \\ + \Big(3 \parallel \mathcal{T} \parallel^2 - 6\Theta^2(\Pi)\Big)\frac{f_2}{2} + + \frac{(\psi_1+\psi_2)}{2}\Big(tr(\beta_1 \mid_\Pi) \\ -\lambda(m-1)\Big) + \frac{\psi_2(\psi_1-\psi_2)}{2}\Big(tr(\beta_2 \mid_\Pi) \\ -\mu(m-1)\Big) + \frac{(\psi_1-\psi_2)}{2}\Big(\Lambda(tr(h\mid_\Pi)) - m(m-1)\Lambda(H)\Big). \end{aligned}$$

Remark 1. *It should be noted that Theorem 2 generalizes the Theorem 6 obtained in [20]. Moreover, taking different values of $f_i, i = 1,2,3$, we can obtain similar inequalities as Theorem 1 for the Kenmotsu space form and the Cosympletic space form endowed with certain types of connections by restricting the values of $\psi_i, i = 1,2$.*

Remark 2. *If in Theorem 2, we take $\psi_1 = \psi_2 = 1$ then we obtain Theorem 5.1 [21].*

Corollary 2. *Let $M^m, m \geq 3$ be an m-dimensional submanifold of a $(2n+1)$-dimensional generalized Sasakian-space-form $\widetilde{M}(f_1, f_2, f_3)$ endowed with a semi-symmetric non-metric connection, then*

$$\begin{aligned} \tau(x) - K(\Pi) \leq (m-2)\Big(\frac{m^2}{2(m-1)} \parallel H \parallel^2 + (m+1)\frac{f_1}{2}\Big) \\ + \Big(3 \parallel \mathcal{T} \parallel^2 - 6\Theta^2(\Pi)\Big)\frac{f_2}{2} + \Big(\parallel \xi_\Pi \parallel^2 - (m-1) \parallel \xi^T \parallel^2 \Big) f_3 \\ + \frac{1}{2}\Big(tr(\beta_1 \mid_\Pi) - \lambda(m-1)\Big) + \frac{1}{2}\Big(\Lambda(tr(h\mid_\Pi)) - m(m-1)\Lambda(H)\Big), \end{aligned}$$

where Π is a two-plane section T_xM, $x \in M$.

For an integer $k \geq 0$, we denote by $S(m,k)$ the set of k-tuples $(m_1, \ldots, m_k)$ of integers ≥ 2 satisfying $m_1 < m$ and $m_1, \ldots, m_k \leq m$. In addition, let $S(m)$ be the set of unordered

k-tuples with $k \geq 0$ for a fixed m. Then, for each k-tuple $(m_1, \ldots, m_k) \in S(m)$, Chen introduced a Riemannian invariant $\delta(m_1, \ldots, m_k)$ as follows [23]

$$\delta(m_1, \ldots, m_k)(x) = \tau(x) - inf\{\tau(L_1) + \cdots + \tau(L_k)\}, \tag{24}$$

where $L_1, \ldots, L_k$ run over all k mutually orthogonal subspaces of T_xM such that $\dim L_j = m_j, j \in \{1, \ldots, k\}$. For simplicity, we set

$$\Psi_1(L_j) = \sum_{1\leq i<j\leq r} g^2(\mathcal{T}e_i, e_j), \;\; \Psi_2(L_j) = \sum_{1\leq i<j\leq r} [g(\xi^T, e_i)^2 + g(\xi^T, e_j)^2]$$
$$\Psi_3(L_j) = \sum_{1\leq i<j\leq r} [\beta_1(e_i, e_i) + \beta_1(e_j, e_j)], \;\; \Psi_4(L_j) = \sum_{1\leq i<j\leq r} [\beta_2(e_i, e_i) + \beta_2(e_j, e_j)]$$
$$\Psi_5(L_j) = \sum_{1\leq i<j\leq r} \Lambda(h(e_i, e_i) + h(e_j, e_j))$$

As the generalization of Theorem 2, we state and prove the following results using the methods used in [26].

Theorem 3. *Let $M^m, m \geq 3$ be an m-dimensional submanifold of a $(2n+1)$-dimensional generalized Sasakian-space-form $\widetilde{M}(f_1, f_2, f_3)$ endowed with a quarter-symmetric connection $\overline{\nabla}$, then*

$$\begin{aligned} \delta(m_1, \ldots, m_k) &\leq b(m_1, \ldots, m_k) \parallel H \parallel^2 + a(m_1, \ldots, m_k) f_1 \\ &+3f_2\Big(\frac{\parallel \mathcal{T} \parallel^2}{2} - \sum_{j=1}^{k} \Psi_1(L_j)\Big) - f_3\Big((m-1) \parallel \xi^T \parallel^2 - \sum_{j=1}^{k} \Psi_2(L_j)\Big) \\ &-\frac{(\psi_1+\psi_2)}{2}\Big((m-1)\lambda - \sum_{j=1}^{k} \Psi_3(L_j)\Big) - \frac{\psi_2}{2}(\psi_1 - \psi_2)\Big((m-1)\mu \\ &-\sum_{j=1}^{k} \Psi_4(L_j)\Big) + \frac{(\psi_1-\psi_2)}{2}\Big(m(m-1)\Lambda(H) - \sum_{j=1}^{k} \Psi_5(L_j)\Big), \end{aligned}$$

for any k-tuples $(m_1, \ldots, m_k) \in S(m)$. If P is a tangent vector field on M, the equality case holds at $x \in M^m$ if and only if there exists an orthonormal basis $\{e_1, \ldots, e_m\}$ of T_xM and an orthonormal basis $\{e_{m+1}, \ldots, e_{2n+1}\}$ of $T_x^{\perp}M$ such that the shape operators of M in $\widetilde{M}(f_1, f_2, f_3)$ at x have the following forms:

$$A_{e_{m+1}} = \begin{pmatrix} a_1 & 0 & \ldots & 0 \\ 0 & a_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_m \end{pmatrix}, A_{e_r} = \begin{pmatrix} A_1^r & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & A_k^r & 0 \\ 0 & \ldots & 0 & \varsigma_r I \end{pmatrix}, r = m+2, \ldots, 2n+1,$$

where $a_1, \ldots, a_m$ satisfy

$$a_1 + \cdots + a_{m_1} = \cdots = a_{m_1+\cdots+m_{k-1}+1} + \cdots + a_{m_1+\cdots+m_k+1} = \cdots = a_m$$

and each A_j^r is a symmetric $m_j \times m_j$ submatrix satisfying $tr(A_1^r) = \cdots = tr(A_k^r) = \varsigma_r$, I is an identity matrix.

Proof. Choose an orthonormal basis $\{e_1, \ldots, e_m\}$ of T_xM and an orthonormal basis $\{e_{m+1}, \ldots, e_{2n+1}\}$ of $T_x^\perp M$ such that mean curvature vector H is in the direction of the normal vector to e_{m+1}. We set

$$
\begin{aligned}
a_i &= h_{ii}^{m+1}, \quad i = 1, \ldots, m \\
b_1 = a_1, b_2 &= a_2 + \cdots + a_{m_1}, b_3 = a_{m_1+1} + \cdots + a_{m_1+m_2}, \ldots, \\
b_{k+1} &= a_{m_1+\cdots+m_{k-1}+1} + \cdots + a_{m_1+\cdots+m_{k-1}+m_k}, \ldots, b_{\gamma+1} = a_m,
\end{aligned}
$$

and consider the following sets

$$
\begin{aligned}
D_1 = \{1, \ldots, m_1\}, &\quad D_2 = \{m_1+1, \ldots, m_1+m_2\}, \ldots, \\
D_k = \{(m_1 + \cdots + m_{k-1}) &+ 1, \ldots, (m_1 + \cdots + m_{k-1}) + m_k\}.
\end{aligned}
$$

Let $L_1, \ldots, L_k$ be a mutually orthogonal subspace of T_xM with $\dim L_j = m_j$, defined by

$$
L_j = Span\{e_{m_1+\cdots+m_{j-1}+1}, \ldots, e_{m_1+\cdots+m_j}\}, \quad j = 1, \ldots, k.
$$

From (7), (8) and (12), we obtain

$$
\begin{aligned}
\tau(L_j) &= \frac{m_j(m_j-1)}{2} f_1 + 3f_2\Psi_1(L_j) - f_3\Psi_2(L_j) \\
-\frac{(\psi_1+\psi_2)}{2}\Psi_3(L_j) &- \frac{\psi_2}{2}(\psi_1-\psi_2)\Psi_4(L_j) - \frac{(\psi_1-\psi_2)}{2}\Psi_5(L_j) \\
&+ \sum_{r=m+1}^{2n+1} \sum_{\alpha_j<\beta_j} [h^r_{\alpha_j\alpha_j} h^r_{\beta_j\beta_j} - (h_{\alpha_j\beta_j})^2].
\end{aligned} \tag{25}
$$

We set

$$
\begin{aligned}
\varepsilon = 2\tau - 2b(m_1, \ldots, m_k) \parallel H \parallel^2 &- m(m-1)f_1 - 3f_2 \parallel \mathcal{T} \parallel^2 \\
+2(m-1)f_3 \parallel \xi^T \parallel^2 &+ (\psi_1+\psi_2)\lambda(m-1) \\
+\psi_2(\psi_1-\psi_2)\mu(m-1) &+ m(m-1)(\psi_1-\psi_2)\Lambda(H),
\end{aligned} \tag{26}
$$

where

$$
b(m_1, \ldots, m_k) = \frac{m^2\Big(m+k-1-\sum_{j=1}^{k} m_j\Big)}{2\Big(m+k-\sum_{j=1}^{k} m_j\Big)},
$$

for each $(m_1, \ldots, m_k) \in S(m)$.

In addition, let $\gamma = m + k - \sum_{j=1}^{k} m_j$. Then in view of this and (26), Equation (14) becomes

$$
m^2 \parallel H \parallel^2 = (\parallel h \parallel^2 + \varepsilon)\gamma,
$$

which can be written as

$$
\begin{aligned}
\Big(\sum_{i=1}^{\gamma+1} b_i\Big)^2 = \gamma\Big[\varepsilon &+ \sum_{i=1}^{\gamma+1} b_i^2 + \sum_{i\neq j}(h_{ij}^{m+1})^2 + \sum_{r=m+2}^{2n+1}\sum_{i,j=1}^{m}(h_{ij}^r)^2 \\
&-2\sum_{\alpha_1<\beta_1} a_{\alpha_1}a_{\beta_1} - \cdots - 2\sum_{\alpha_k<\beta_k} a_{\alpha_k}a_{\beta_k}\Big],
\end{aligned} \tag{27}
$$

where $\alpha_j, \beta_j \in D_j$ for all $j = 1, \ldots, k$.

Now applying Lemma 2.3 [26] in (27), we obtain

$$\sum_{\alpha_1<\beta_1} a_{\alpha_1}a_{\beta_1}+\cdots+\sum_{\alpha_k<\beta_k} a_{\alpha_k}a_{\beta_k}\geq$$
$$\frac{1}{2}\Big[\varepsilon+\sum_{i\neq j}(h_{ij}^{m+1})^2+\sum_{r=m+2}^{2n+1}\sum_{i,j=1}^{m}(h_{ij}^{r})^2\Big],$$

which further implies

$$\sum_{j=1}^{k}\sum_{r=m+1}^{2n+1}\sum_{\alpha_j<\beta_j}\Big[h^r_{\alpha_j\alpha_j}h^r_{\beta_j\beta_j}-(h^r_{\alpha_j\beta_j})^2\Big]\geq\frac{\varepsilon}{2}$$
$$+\frac{1}{2}\sum_{r=m+1}^{2n+1}\sum_{(\alpha,\beta)\notin D^2}(h^r_{\alpha\beta})^2+\sum_{r=m+2}^{2n+1}\sum_{\alpha_j\in D_j}(h^r_{\alpha_j\alpha_j})^2\leq\frac{\varepsilon}{2}, \tag{28}$$

where $D^2=(D_1\times D_1)\cup\cdots\cup(D_k\times D_k)$. Combining (14), (25) and (28) gives

$$\tau-\sum_{j=1}^{k}\tau(L_j)\leq b(m_1,\ldots,m_k)\parallel H\parallel^2+a(m_1,\ldots,m_k)f_1$$
$$+3f_2\Big(\frac{\parallel\mathcal{T}\parallel^2}{2}-\sum_{j=1}^{k}\Psi_1(L_j)\Big)-f_3\Big((m-1)\parallel\xi^T\parallel^2-\sum_{j=1}^{k}\Psi_2(L_j)\Big)$$
$$-\frac{(\psi_1+\psi_2)}{2}\Big((m-1)\lambda-\sum_{j=1}^{k}\Psi_3(L_j)\Big)-\frac{\psi_2}{2}(\psi_1-\psi_2)\Big((m-1)\mu$$
$$-\sum_{j=1}^{k}\Psi_4(L_j)\Big)+\frac{(\psi_1-\psi_2)}{2}\Big(m(m-1)\Lambda(H)-\sum_{j=1}^{k}\Psi_5(L_j)\Big), \tag{29}$$

where, $a(m_1,\ldots,m_k)=\frac{1}{2}\Big[m(m-1)-\sum_{j=1}^{k}m_j(m_j-1)\Big]$.

The equality case (29) at a point $x\in M$ holds if and only if all the previous inequalities hold; thus, the shape operators take the desired forms. □

Remark 3. *Restricting the values of f_i, $i=1,2,3$ and ψ_i for $i=1,2$, we can obtain similar bounds as Theorem 3 for certain contact space forms endowed with certain connections.*

Theorem 4. *Let M^m, $m\geq 3$ be an m-dimensional submanifold of a $(2n+1)$-dimensional generalized Sasakian-space-form $\widetilde{M}(f_1,f_2,f_3)$ endowed with a quarter-symmetric connection $\overline{\nabla}$, then*

(i) For each unit vector X_1 in T_xM, we have

$$Ric(X_1)\leq(m-1)f_1+3f_2\sum_{j=2}^{m}g^2(\varphi X_1,e_j)+f_3\big((2-m)\eta^2(X_1)-\parallel\xi^T\parallel^2\big)$$
$$+[\psi_1+(1-m)\psi_2]\beta_1(X_1,X_1)-\psi_1\lambda+\psi_2(\psi_1-\psi_2)(1-m)\beta_2(X_1,X_1)$$
$$-(\psi_1-\psi_2)[m\Lambda(H)-\Lambda(h(X_1,X_1))]+\frac{m^2}{4}\parallel H\parallel^2. \tag{30}$$

(ii) If $H(x)=0$, then a unit tangent vector X_1 at x satisfies the equality case of (30) if and only if $X_1\in\mathcal{M}(x)=\{X_1\in T_xM\mid h(X_1,X_2)=0,\forall X_2\in T_xM\}$.

(iii) The equality of (30) holds for all unit tangent vectors at x if and only if either

1. $m \neq 2, h^r_{ij} = 0, i, j = 1, 2 \ldots, m.r = m+1, \ldots, 2n+1$, *or*

2. $m = 2, h^r_{11} = h^r_{22}, h^r_{12} = 0, r = 3, \ldots, 2n+1.$

Proof. Choosing the orthonormal basis $\{e_1, \ldots, e_m\}$ such that $e_1 = X_1$, where $X_1 \in T_xM$ is a unit tangent vector at the point x on M. In view of (7) and (12), then proceeding similarly as the proof of Theorem 4 in [20], one can easily obtain the desire results. □

By choosing an orthonormal frame $\{e_1, \ldots, e_k\}$ of L such that $e_1 = X_1$, a unit tangent vector, Chen [23] defined the k-Ricci curvature of L at X_1 by

$$Ric_L(X_1) = K_{12} + K_{13} + \cdots + K_{1k}. \tag{31}$$

For an integer $k, 2 \leq k \leq m$, the Riemannian invariant Θ_k on M is defined by

$$\Theta_k(x) = \frac{1}{k-1} inf\{Ric_L(X_1) \mid L, X_1\}, x \in M$$

where L runs over all k-plane sections in T_xM and X_1 runs over all unit vectors in L. From [26], we have

$$\tau(x) \geq \frac{m(m-1)}{2}\Theta_k(x). \tag{32}$$

Let us choose $\{e_1, \ldots, e_m\}$ and $\{e_{m+1}, \ldots, e_{2n+1}\}$ as an orthonormal basis of T_xM and $T^\perp_x M, x \in M$, respectively, where e_{m+1} is parallel to the mean curvature vector H. In addition, let $\{e_1, \ldots, e_m\}$ diagonalize the shape operator $A_{e_{m+1}}$. Then,

$$A_{e_{m+1}} = \begin{pmatrix} a_1 & 0 & \ldots & 0 \\ 0 & a_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_m \end{pmatrix}$$

and

$$A_{e_r} = h^r_{ij}, \quad i, j = 1, \ldots, m, \quad r = m+2, \ldots, 2n+1, \quad trA_{e_r} = 0. \tag{33}$$

In consequence of the above assumptions, Equation (14) can be written as follows:

$$\begin{aligned} m^2 \parallel H \parallel^2 = 2\tau + \sum_{i=1}^m a_i^2 + \sum_{r=m+2}^{2n+1} \sum_{i,j=1}^m (h^r_{ij})^2 - m(m-1)f_1 \\ -3f_2 \parallel \mathcal{T} \parallel^2 + 2(m-1)f_3 \parallel \xi^T \parallel^2 + (\psi_1 + \psi_2)\lambda(m-1) \\ + \psi_2(\psi_1 - \psi_2)\mu(m-1) + m(m-1)(\psi_1 - \psi_2)\Lambda(H). \end{aligned} \tag{34}$$

Using the Cauchy–Schwartz inequality, we have

$$\sum_{i=1}^m a_i^2 \geq m \parallel H \parallel^2. \tag{35}$$

Combining (32) and (34), we can state the following:

Theorem 5. *Let $M^m, m \geq 3$ be an m-dimensional submanifold of a $(2n+1)$-dimensional generalized Sasakian-space-form $\widetilde{M}(f_1, f_2, f_3)$ endowed with a quarter-symmetric connection $\overline{\nabla}$. Then for any integer $k, 2 \leq k \leq m$ and any point $x \in M$, we have*

$$\| H \|^2 (x) \geq \Theta_k(x) - f_1 - \frac{3f_2}{m(m-1)} \| \mathcal{T} \|^2 + \frac{2f_3}{m} \| \xi^T \|^2$$
$$+ \frac{\lambda}{m}(\psi_1 + \psi_2) + \frac{\mu}{m}\psi_2(\psi_1 - \psi_2) + (\psi_1 - \psi_2)\Lambda(H).$$

As a particular case of Theorem 5, we obtained Theorem 6.2 [21] which is as follows:

Corollary 3 ([21])**.** *Let $M^m, m \geq 3$ be an m-dimensional submanifold of a $(2n+1)$-dimensional generalized Sasakian-space-form $\widetilde{M}(f_1, f_2, f_3)$ endowed with a semi-symmetric metric connection. Then for any integer $k, 2 \leq k \leq m$ and any point $x \in M$, we have*

$$\| H \|^2 (x) \geq \Theta_k(x) - f_1 - \frac{3f_2}{m(m-1)} \| \mathcal{T} \|^2 + \frac{2f_3}{m} \| \xi^T \|^2 + \frac{2\lambda}{m}.$$

Corollary 4. *Let $M^m, m \geq 3$ be an m-dimensional submanifold of a $(2n+1)$-dimensional generalized Sasakian-space-form $\widetilde{M}(f_1, f_2, f_3)$ endowed with a semi-symmetric non-metric connection. Then for any integer $k, 2 \leq k \leq m$ and any point $x \in M$, we have*

$$\| H \|^2 (x) \geq \Theta_k(x) - f_1 - \frac{3f_2}{m(m-1)} \| \mathcal{T} \|^2 + \frac{2f_3}{m} \| \xi^T \|^2 + \frac{\lambda}{m} + \Lambda(H).$$

Remark 4. *Restricting function $f_i, i = 1,2,3$, we can easily obtain similar inequality in the case of the Sasakian, Kenmotsu and Cosympletic space forms.*

4. Some Applications

The notion of slant submanifolds in almost contact geometry was introduced by Lotta [27]. A submanifold M of an almost contact metric manifold $(\widetilde{M}, \varphi, \xi, \eta, g)$ tangent to the structure vector field ξ is said to be a contact slant submanifold if, for any point $x \in M$ and any vector $X_1 \in T_xM$ linearly independent on ξ_x, the angle between the vector φX_1 and the tangent space T_xM is constant. This angle is known as the slant angle of M. The concept of slant submanifold is further generalized as follows:

Definition 1 ([28])**.** *A submanifold M of an almost contact metric manifold M is called a bi-slant submanifold, whenever we have*

1. $TM = \mathcal{D}_{\theta_1} \oplus \mathcal{D}_{\theta_2} \oplus \xi$
2. $\varphi\mathcal{D}_{\theta_1} \perp \mathcal{D}_{\theta_2}$ *and* $\varphi\mathcal{D}_{\theta_2} \perp \mathcal{D}_{\theta_1}$.
3. *For $i = 1,2$, the distribution D_i is slant with slant angle θ_i.*

Now, as a consequence of Theorem 2, we can state the following:

Theorem 6. *Let M be a $(m = 2d_1 + 2d_2 + 1)$-dimensional bi-slant submanifold of a $(2n+1)$-dimensional generalized Sasakian-space-form $\widetilde{M}(f_1, f_2, f_3)$ endowed with a quarter-symmetric connection $\overline{\nabla}$, then we have*

$$\tau(x) - K(\Pi) \leq (m-2)\Big(\frac{m^2}{2(m-1)} \| H \|^2 + (m+1)\frac{f_1}{2}\Big)$$
$$+ 3\Big((d_1 - 1)cos^2\theta_1 + d_2 cos^2\theta_2\Big)\frac{f_2}{2} - (m-1)f_3$$
$$+ \frac{(\psi_1 + \psi_2)}{2}\Big(tr(\beta_1 \mid_\Pi) - \lambda(m-1)\Big) + \frac{\psi_2(\psi_1 - \psi_2)}{2}\Big(tr(\beta_2 \mid_\Pi)$$
$$-\mu(m-1)\Big) + \frac{(\psi_1 - \psi_2)}{2}\Big(\Lambda(tr(h \mid_\Pi)) - m(m-1)\Lambda(H)\Big),$$

for any plane Π *invariant by* $\mathcal{T}$ *and tangent to slant distribution* $\mathcal{D}_{\theta_1}$ *and*

$$\tau(x) - K(\Pi) \leq (m-2)\Big(\frac{m^2}{2(m-1)} \parallel H \parallel^2 + (m+1)\frac{f_1}{2}\Big) + 3\Big(d_1 cos^2\theta_1 + (d_2-1)cos^2\theta_2\Big)\frac{f_2}{2} - (m-1)f_3 + \frac{(\psi_1+\psi_2)}{2}\Big(tr(\beta_1 |_{\Pi}) - \lambda(m-1)\Big) + \frac{\psi_2(\psi_1-\psi_2)}{2}\Big(tr(\beta_2 |_{\Pi}) - \mu(m-1)\Big) + \frac{(\psi_1-\psi_2)}{2}\Big(\Lambda(tr(h |_{\Pi})) - m(m-1)\Lambda(H)\Big),$$

for any plane Π *invariant by* $\mathcal{T}$ *and tangent to slant distribution* $\mathcal{D}_{\theta_2}$*. Moreover, the ideal case is the same as Theorem 2.*

Proof. Let M be a bi-slant submanifold of a generalized Sasakian-space-form $\widetilde{M}(f_1, f_2, f_3)$ of dimension $(m = 2d_1 + 2d_2 + 1)$ and let $\{e_1, \ldots, e_m = \xi\}$ be an orthonormal frame of tangent space T_xM at a point $x \in M$, such that

$$e_1, e_2 = sec\theta_1 \mathcal{T} e_1, \ldots, e_{2d_1-1}, e_{2d_1} = sec\theta_1 \mathcal{T} e_{2d_1-1}, e_{2d_1+1}, e_{2d_1+2} = sec\theta_2 \mathcal{T} e_{2d_1+1}, \ldots, e_{2d_1+2d_2-1}, e_{2d_1+2d_2} = sec\theta_2 \mathcal{T} e_{2d_1+2d_2-1}, e_{2d_1+2d_2+1} = \xi,$$

which gives

$$g^2(\varphi e_{i+1}, e_i) = \begin{cases} cos^2\theta_1, & \text{for } \ i = 1,2,\ldots,2d_1-1 \\ cos^2\theta_2, & \text{for } \ i = 2d_1+1,\ldots,2d_1+2d_2-1. \end{cases}$$

Thus we have

$$\parallel \mathcal{T} \parallel^2 = 2\{d_1 cos^2\theta_1 + d_2 cos^2\theta_2\}$$

Making use of the above facts in Theorem 2, the proof is straightforward. □

In a similar manner, Theorems 3, 4 and 5 can be stated for a bi-slant submanifold of a generalized Sasakian-space-form. Moreover, restricting the values of $\theta_i, i = 1,2$, similar results can be obtained for a large class of submanifolds such as slant, semi-slant, hemi-slant, semi-invariant submanifolds. Moreover, by taking different values of $f_i, i = 1,2,3$, we can derive similar inequalities for the Sasakian, Kenmotsu and Cosympletic space forms.

5. Conclusions and Future Work

In this article, we established the general form of Chen's inequalities are obtained for generalized Sasakian-space-forms endowed with a special type of quarter-symmetric connection. This work is in continuation of the previous works by Wang [20], Mihai and Özgür [19], Sular [21] and Wang and Zhang [18]. By using the obtained inequality, we derived the Chen inequality for the bi-slant submanifold of generalized Sasakian-space-forms. Recently, Chen inequality for lightlike hypersurfaces of GRW spacetime was obtained by Poyraz [14]. For future research, we would try to combine the methods and results in [29–52] to obtain the Chen inequalities for submanifolds of indefinite space forms such as spacelike and lightlike.

Author Contributions: Conceptualization, Y.L., M.K., J.P.S. and S.K.C.; methodology, Y.L., M.K., J.P.S. and S.K.C.; investigation, Y.L., M.K., J.P.S. and S.K.C.; writing—original draft preparation, Y.L., M.K., J.P.S. and S.K.C.; writing— review and editing, Y.L., M.K., J.P.S. and S.K.C. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by National Natural Science Foundation of China (Grant No. 12101168) and Zhejiang Provincial Natural Science Foundation of China (Grant No. LQ22A010014).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: We gratefully acknowledge the constructive comments from the editor and the anonymous referees. The second author is thankful to the DST, New Delhi, India (DST/INSPIRE Fellowship/2018/IF180830).

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Chen, B.Y. Some pinching and classification theorems for minimal submanifolds. *Arch. Math.* **1993**, *60*, 568–578. [CrossRef]
2. Chern, S.S. *Minimal Submanifolds in a Riemannian Manifold*; University of Kansas Press: Lawrence, KS, USA, 1968.
3. Chen, B.Y. A general inequality for submanifolds in complex-space-forms and its applications. *Arch. Math.* **1996**, *67*, 519–528. [CrossRef]
4. Kumar, P.K.; Gupta, R.S.; Sharfuddin, A. B. Y. Chen's inequalities for bi-slant submanifolds in Kenmotsu space forms. *Demonstr. Math.* **2010**, *4*, 887–898.
5. Cioroboiu, D. B. Y. Chen inequalities for semislant submanifolds in Sasakian-space-forms. *Int. J. Math. Math. Sci.* **2003**, *2003*, 354806. [CrossRef]
6. Gupta, R.S. B. Y. Chen's inequalities for Bi-slant submanifolds in cosympletic space forms. *Sarajevo J. Math.* **2013**, *9*, 117–128. [CrossRef]
7. Özgür, C. B. Y. Chen inequalities for submanifolds of a Riemannian manifold of a quasi-constant curvature. *Turk. J. Math.* **2011**, *35*, 501–509.
8. Alegre, P.; Carriazo, A.; Kim, Y.H.; Yoon, D.W. B.Y. Chen's inequality for submanifolds of generalized space forms. *Indian J. Pure Appl. Math.* **2007**, *38*, 185–201.
9. Mihai, A. B.Y. Chen inequalities for slant submanifolds in generalized complex space forms. *Rad. Math.* **2004**, *12*, 215–231.
10. Decu, S.; Haesen, S. Chen Inequalities for Spacelike Submanifolds in Statistical Manifolds of type Para-Kähler Space Forms. *Mathematics* **2022**, *10*, 330. [CrossRef]
11. Aytimur, H.; Kon, M.; Mihai, A.; Özgür, C.; Takano, K. Chen Inequalities for Statistical Submanifolds of Kähler-Like Statistical Manifolds. *Mathematics* **2019**, *7*, 1202. [CrossRef]
12. Lone, M.S.; Bahadir, O.; Park, C.; Hwang, I. Basic inequalities for statistical submanifolds in Golden-like statistical manifolds. *Open Math.* **2022**, *20*, 153–166. [CrossRef]
13. Vilcu, G.E. On Chen invariants and inequalities in quaternionic geometry. *J. Inequal. Appl.* **2013**, *2013*, 66. [CrossRef]
14. Poyraz, N.Ö. Chen inequalities on lightlike hypersurfaces of GRW spacetime. *Int. J. Geom. Methods Mod. Phys.* **2022**, *19*, 2250011. [CrossRef]
15. Qu, Q.; Wang, Y. Multiply warped products with a quarter-symmetric connection. *J. Math. Anal. Appl.* **2015**, *431*, 955–987. [CrossRef]
16. Hayden, H.A. Subspaces of a space with torsion. *Proc. London Math. Soc.* **1932**, *34*, 27–50. [CrossRef]
17. Agashe, N.S.; Chafle, M.R. A semi-symmetric non-metric connection on a Riemannian manifold. *Indian J. Pure Appl. Math.* **1992**, *23*, 399–409.
18. Wang, Y.; Zhang, P. Chen inequalities for submanifolds of real space forms with a quarter-symmetric connection. *Taiwanese J. Math.* **2010**, *14*, 1465–1477.
19. Mihai, A.; Özgür, C. Chen inequalities for submanifolds of complex space forms and Sasakian-space-forms endowed with semi-symmetric metric connections. *Rocky Mountain J. Math.* **2011**, *41*, 1653–1673. [CrossRef]
20. Wang, Y. Chen inequalities for submanifolds of complex space forms and Sasakian-space-forms with quarter-symmetric connections. *Int. J. Geom. Methods Modern Phys.* **2019**, *16*, 1950118. [CrossRef]
21. Sular, S. Chen inequalities for submanifolds of generalized space forms with a semi-symmetric metric connection. *Hacettepe J. Math. Stat.* **2016**, *45*, 811–825. [CrossRef]
22. Al-Khaldi, A.H.; Aquib, M.; Aslam, M.; Khan, M.A. Chen-Ricci Inequalities with a Quarter Symmetric Connection in Generalized Space Forms. *Adv. Math. Phys.* **2021**, *2021*, 3221643. [CrossRef]
23. Chen, B.Y. A Riemannian invariant and its applications to submanifold theorey. *Results Math.* **1995**, *27*, 17–26. [CrossRef]
24. Blair, D.E. *Riemannian Geometry of Contact and Sympletic Manifolds*; Birkhauser: Boston, MA, USA, 2002.
25. Alegre, P.; Blair, D.E.; Carriazo, A. Generalized Sasakian-space-forms. *Israel J. Math.* **2004**, *141*, 157–183. [CrossRef]
26. Zhang, P.; Zhang, L.; Song, W. Chen's inequalities for submanifolds of a Riemannian manifold of quasi-constant curvature with a semi-symmetric metric connection. *Taiwanese J. Math.* **2014**, *18*, 1841–1862. [CrossRef]
27. Lotta, A. Slant submanifolds in contact geometry. *Bull. Math. Soc. Roum.* **1996**, *39*, 183–198.
28. Alqahtani, L.S.; Stankovic, M.S.; Uddin, S. Warped product bi-slant submanifolds of cosympletic manifolds. *Filomat* **2017**, *17*, 5065–5071. [CrossRef]
29. Abedi, E.; Ziabari, R.B. Slant submanifolds of a conformal Sasakian manifold. *Acta Universitatis Apulensis* **2014**, *40*, 35–49.
30. Abedi, E.; Ziabari, R.B.; Tripathi, M.M. Ricci and scalar curvatures of submanifolds of a conformal Sasakian space form. *Arch. Math.* **2016** , *52*, 113–130. [CrossRef]

31. Akram, A.; Piscoran, L.I. Geometric classification of warped products isometrically immersed into Sasakian space form. *Math. Nachr.* **2019**, *292*, 234–251. [CrossRef]
32. Akram, A.; Piscoran, L.I. Geometry of warped product immersions of Kenmotsu space forms and its applications to slant immersions. *J. Geom. Phys.* **2017**, *114*, 276–290.
33. Akram, A.; Uddin, S.; Othman, W.A.M. Geometry of warped product pointwise semi-slant submanifolds of Kahler manifolds. *Filomat* **2017**, *12*, 3771–3778.
34. Al-Solamy, F.R.; Khan, V.A. Warped product semi-slant submanifolds of a Sasakian manifold. *Serdica Math. J.* **2008**, *34*, 597–606.
35. Li, Y.L.; Ali, A.; Ali, R. A general inequality for CR-warped products in generalized Sasakian-space-form and its applications. *Adv. Math. Phys.* **2021**, *2021*, 5777554. [CrossRef]
36. Li, Y.L.; Alkhaldi, A.H.; Ali, A.; Laurian-Ioan, P. On the Topology of Warped Product Pointwise Semi-Slant Submanifolds with Positive Curvature. *Mathematics* **2021**, *9*, 3156. [CrossRef]
37. Li, Y.L.; Abolarinwa, A.; Azami, S.; Ali, A. Yamabe constant evolution and monotonicity along the conformal Ricci flow. *AIMS Math.* **2022**, *7*, 12077–12090. [CrossRef]
38. Li, Y.L.; Ganguly, D.; Dey, S.; Bhattacharyya, A. Conformal η-Ricci solitons within the framework of indefinite Kenmotsu manifolds. *AIMS Math.* **2022**, *7*, 5408–5430. [CrossRef]
39. Li, Y.L.; Zhu, Y.S.; Sun, Q.Y. Singularities and dualities of pedal curves in pseudo-hyperbolic and de Sitter space. *Int. J. Geom. Methods Mod. Phys.* **2021**, *18*, 1–31. [CrossRef]
40. Li, Y.L.; Ali, A.; Mofarreh, F.; Alluhaibi, N. Homology groups in warped product submanifolds in hyperbolic spaces. *J. Math.* **2021**, *2021*, 8554738. [CrossRef]
41. Li, Y.L.; Lone, M.A.; Wani, U.A. Biharmonic submanifolds of Kähler product manifolds. *AIMS Math.* **2021**, *6*, 9309–9321. [CrossRef]
42. Li, Y.L.; Dey, S.; Pahan, S.; Ali, A. Geometry of conformal η-Ricci solitons and conformal η-Ricci almost solitons on Paracontact geometry. *Open Math.* **2022**, *20*, 1–20. [CrossRef]
43. Li, Y.L.; Ali, A.; Mofarreh, F.; Abolarinwa, A.; Ali, R. Some eigenvalues estimate for the ϕ-Laplace operator on slant submanifolds of Sasakian space forms. *J. Funct. Space* **2021**, *2021*, 6195939.
44. Yang, Z.C.; Li, Y.L.; Erdoğdub, M.; Zhu, Y.S. Evolving evolutoids and pedaloids from viewpoints of envelope and singularity theory in Minkowski plane. *J. Geom. Phys.* **2022**, *176*, 104513. [CrossRef]
45. Todorčević, V. Subharmonic behavior and quasiconformal mappings. *Anal. Math. Phys.* **2019**, *9*, 1211–1225. [CrossRef]
46. Todorčević, V. *Harmonic Quasiconformal Mappings and Hyperbolic Type Metrics*; Springer International Publishing: Berlin/Heidelberg, Germany, 2019.
47. Kojić, V. Quasi-nearly subharmonic functions and conformal mappings. *Filomat* **2007**, *21*, 243–249. [CrossRef]
48. Kojić, V.; Pavlović, M. Subharmonicity of $|f|^p$ for quasiregular harmonic functions, with applications. *J. Math. Anal. Appl.* **2008**, *342*, 742–746. [CrossRef]
49. Manojlović, V.; Vuorinen, M. On quasiconformal maps with identity boundary values. *Trans. Am. Math. Soc.* **2011**, *363*, 2367–2479. [CrossRef]
50. Manojlović, V. Bilipschitz mappings between sectors in planes and quasi-conformality. *Funct. Anal. Approx. Comput.* **2009**, *1*, 1–6.
51. Manojlović, V. On bilipschicity of quasiconformal harmonic mappings. *Novi Sad J. Math.* **2015**, *45*, 105–109. [CrossRef]
52. Manojlović, V. On conformally invariant extremal problems. *Appl. Anal. Discret. Math.* **2009**, *3*, 97–119. [CrossRef]