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Improved computation of ellipsoidal invariant sets for saturated control systems

T. Alamo, A. Cepeda and D. Limon

Abstract— **In this work, a new technique for the computation of ellipsoidal invariant sets for continuous-time linear systems controlled by a saturating linear control law is presented. New sufficient conditions to guarantee that an ellipsoid is a contractive invariant set for the closed-loop system is presented. The contractive nature of the invariant set ensures asymptotic stability of the controlled system. The main contributions of the paper are the following: the proposed sufficient condition is expressed in form of linear matrix inequalities constraints. The presented method includes (and consequently improves) previous results on this topic. The computational complexity of the proposed approach is analyzed. Illustrative examples are given.**

Keywords: Constrained nonlinear systems, domain of attraction, invariant sets, stability, saturation.

I. Introduction

Saturation is probably the most commonly encountered nonlinearity in control engineering because of the physical impossibility of applying unlimited control signals. It is well known that the input saturation is source of performance degeneration, limit cycles, different equilibrium points, and even instability. Hence, it is great the interest in the analysis and design of saturating control laws. See for instance [13], [9], [11], and references therein.

An important, and still open, topic of this field is the estimation of the domain of attraction of the closed-loop system. The estimation of the domain of attraction of linear systems subject to control saturation has received the attention of many authors in the last years (see, for example, [5], [9], [12], [14] and references therein).

One of the most relevant approaches to the analysis of saturated systems is based on a *linear differential inclusion* (LDI) of the saturation nonlinearity (see [3], [6], [14]). In the literature, invariant ellipsoids have been used to estimate the domain of attraction for nonlinear systems [1], [4], [7], [8]. The domain of attraction of a given saturated system can be approximated by means of an ellipsoid. In [14] and [9] a linear differential inclusion for a linear saturated system is presented. Based on that LDI, the authors propose how to choose simultaneously both the matrix H, that characterizes the LDI, and the greatest ellipsoid that is invariant under the corresponding LDI.

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The main contribution of this paper is a new sufficient condition for the contractiveness of a given ellipsoid. It is proved in the paper that the presented sufficient condition is less conservative than the one obtained when a linear differential inclusion approach is adopted.

The paper is organized as follows. In section II the problem statement is introduced. Some preliminary notation is given in section III. A novel sufficient condition for the contractiveness of a given ellipsoid is presented in section IV. It is shown in section V that the ellipsoidal estimation obtained by means of the results of the paper is less conservative than the ones obtained when a linear differential approach is adopted. The computational complexity is analyzed in section VI. Some illustrative examples are given in section VII. The paper draws to a close with a section of conclusions.

II. Problem Statement

Let us consider the following system

$$\dot{x} = Ax + B\sigma(Kx) \tag{1}$$

where $x \in \mathbb{R}^n$ denotes the state vector. The function $\sigma : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is the vector-valued standard saturation function defined as follows:

$$\sigma(u) = [\sigma(u_1) \quad \sigma(u_2) \quad \ldots \quad \sigma(u_m)]^\top,$$

where $\sigma(u_i) = sign(u_i)\min\{1, |u_i|\}$.

Denote $\mathcal{M} = \{1, 2, \ldots, m\}$. Denote also B_i, $i = 1, \ldots, m$ the columns of matrix B and K_i, $i = 1, \ldots, m$ the rows of matrix K. With this notation, system (1) can be rewritten as:

$$\dot{x} = Ax + \sum_{i=1}^{m} B_i\sigma(K_i x) = Ax + \sum_{i\in\mathcal{M}} B_i\sigma(K_i x)$$

The purpose of this paper is to present an LMI approach to the computation of ellipsoidal estimations of the domain of attraction for this class of saturated control systems.

III. Some preliminary notations

In order to present the main result of the paper, the following notations and preliminary results are introduced in this section.

Notation 1: Given a positive definite matrix P, and a positive scalar ρ, $\mathcal{E}(P, \rho)$ represents the following ellipsoid:

$$\mathcal{E}(P, \rho) = \{\ x\ :\ x^\top Px \leq \rho\ \}.$$

Definition 1: Given the set of integers $\mathcal{M}$, set $\mathcal{V}$ is the set of all subsets of $\mathcal{M}$. That is,

$$\mathcal{V} = \{\ S\ :\ S \subseteq \mathcal{M}\ \}$$

Example: If $m = 2$, then $\mathcal{M} = \{1, 2\}$ and $\mathcal{V} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. Note that the empty set $\emptyset$ belongs to $\mathcal{V}$.

Throughout this paper: S^c denotes the complementary of S in $\mathcal{M}$. That is, $S^c = \{\, i \in \mathcal{M} \;:\; i \notin S \,\}$.

IV. Main result

In this section, novel sufficient conditions for the contractiveness of a given ellipsoid are presented. The notion of contractiveness is given in the following definition:

Definition 2: An ellipsoidal set $\mathcal{E}(P, \rho)$ is said to be contractive for system $\dot{x} = Ax + B\sigma(Kx)$ if for every $x \in \mathcal{E}(P, \rho), x \neq 0$:

$$\frac{d}{dt}(x^\top Px) = 2x^\top P(Ax + B\sigma(Kx)) < 0$$

The main result of the paper is the following theorem, which, as it will be seen in section V, improves previous results from the literature.

Theorem 1: The ellipsoid $\mathcal{E}(W^{-1}, 1)$ is contractive if for every $S \in \mathcal{V}$ there exists $Y^S \in \mathbb{R}^{m\times n}$ such that

$$AW + \sum_{i\in S^c} B_iK_iW + \sum_{i\in S} B_iY_i^S +$$
$$(AW + \sum_{i\in S^c} B_iK_iW + \sum_{i\in S} B_iY_i^S)^\top < 0$$
$$\begin{bmatrix} 1 & Y_i^S \\ (Y_i^S)^\top & W \end{bmatrix} > 0, \forall i \in S$$

where Y_i^S denotes the i-th row of Y^S.

Proof:

Note that the assumptions of the theorem guarantee that for every $S \in \mathcal{V}$ there is Y^S and $\epsilon > 0$ such that:

$$AW + \sum_{i\in S^c} B_iK_iW + \sum_{i\in S} B_iY_i^S +$$
$$(AW + \sum_{i\in S^c} B_iK_iW + \sum_{i\in S} B_iY_i^S)^\top < -\epsilon \mathrm{I}$$

That is,

$$(A + \sum_{i\in S^c} B_iK_i)W + W(A + \sum_{i\in S^c} B_iK_i)^\top$$
$$+ \sum_{i\in S}(B_iY_i^S + (Y_i^S)^\top B_i^\top) < -\epsilon \mathrm{I} \qquad (2)$$

From the assumptions of the theorem:

$$\begin{bmatrix} 1 & Y_i^S \\ (Y_i^S)^\top & W \end{bmatrix} > 0, \forall i \in S$$

and property 1 (see Appendix A) it is inferred that for every $i \in S$:

$$B_iY_i^S + (Y_i^S)^\top B_i^\top \geq -\alpha_i W - \frac{B_iB_i^\top}{\alpha_i},\ \forall \alpha_i > 0$$

From the previous inequality and equation (2):

$$(A + \sum_{i\in S^c} B_iK_i)W + W(A + \sum_{i\in S^c} B_iK_i)^\top$$
$$- \sum_{i\in S}(\alpha_i W + \frac{B_iB_i^\top}{\alpha_i}) < -\epsilon \mathrm{I}, \forall \alpha > 0$$

where $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_m]^\top > 0$ denotes that each of the components of α is greater than zero. Denoting $P = W^{-1}$ and pre-multiplying and post-multiplying both sides of previous inequality by $x^\top P$ and Px respectively:

$$x^\top P(A + \sum_{i\in S^c} B_iK_i)x + x^\top(A + \sum_{i\in S^c} B_iK_i)^\top Px$$
$$- \sum_{i\in S}(\alpha_i x^\top Px + \frac{x^\top PB_iB_i^\top Px}{\alpha_i}) < -\epsilon x^\top P^2 x$$
$$\forall x \neq 0, \forall \alpha > 0$$

Taking into account that $x^\top Px \leq 1$ for every $x \in \mathcal{E}(P, 1) = \mathcal{E}(W^{-1}, 1)$:

$$x^\top P(A + \sum_{i\in S^c} B_iK_i)x + x^\top(A + \sum_{i\in S^c} B_iK_i)^\top Px$$
$$- \sum_{i\in S}(\alpha_i + \frac{x^\top PB_iB_i^\top Px}{\alpha_i}) < -\epsilon x^\top P^2 x$$
$$\forall x \in \mathcal{E}(W^{-1}, 1), x \neq 0, \forall \alpha > 0$$

Note that $\epsilon x^\top P^2 x \geq \sum\limits_{i\in S} \epsilon \frac{x^\top P^2 x}{m}$. Thus, denoting $\bar{\epsilon} = \epsilon \frac{x^\top P^2 x}{m}$:

$$x^\top P(A + \sum_{i\in S^c} B_iK_i)x + x^\top(A + \sum_{i\in S^c} B_iK_i)^\top Px$$
$$- \sum_{i\in S}(\alpha_i + \frac{x^\top PB_iB_i^\top Px}{\alpha_i} - \bar{\epsilon}) < 0$$
$$\forall x \in \mathcal{E}(W^{-1}, 1), x \neq 0, \forall \alpha > 0$$

Taking into account that the previous inequality is satisfied for every $\alpha > 0$:

$$x^\top P(A + \sum_{i\in S^c} B_iK_i)x + x^\top(A + \sum_{i\in S^c} B_iK_i)^\top Px$$
$$+ \sum_{i\in S} \sup_{\bar{\alpha}}(-\bar{\alpha} - \frac{x^\top PB_iB_i^\top Px}{\bar{\alpha}} + \bar{\epsilon}) < 0$$
$$\forall x \in \mathcal{E}(W^{-1}, 1), x \neq 0$$

Note that $\bar{\epsilon} = \epsilon \frac{x^\top P^2 x}{m} > 0$ for every $x \neq 0$. This and property 2 (see Appendix B) guarantees that:

$$x^\top P(A + \sum_{i\in S^c} B_iK_i)x + x^\top(A + \sum_{i\in S^c} B_iK_i)^\top Px$$
$$-2 \sum_{i\in S} |x^\top PB_i| < 0$$

$$\forall x \in \mathcal{E}(W^{-1},1), x < 0$$

Denote $z = B^\top Px \in \mathbb{R}^m$. With this notation, the i-th component of vector z is equal to $B_i^\top Px$. Using this notation, the previous inequality can be rewritten as:

$$2x^\top PAx + 2\sum_{i\in S^c} z_i K_i x - 2\sum_{i\in S} |z_i| < 0$$
$$\forall x \in \mathcal{E}(W^{-1},1), x \neq 0$$

This last inequality is satisfied for every $S \in \mathcal{V}$. Therefore:

$$2x^\top PAx + 2\max_{S\in\mathcal{V}} \left\{ \sum_{i\in S^c} z_i K_i x - \sum_{i\in S} |z_i| \right\} < 0$$
$$\forall x \in \mathcal{E}(W^{-1},1), x \neq 0$$

Taking into account property 3 (see Appendix C):

$$2x^\top PAx + 2z^\top \sigma(Kx) < 0$$
$$\forall x \in \mathcal{E}(W^{-1},1), x \neq 0$$

Recalling that $z = B^\top Px$:

$$2x^\top PAx + 2x^\top PB\sigma(Kx) = 2x^\top P\dot{x} = \frac{d}{dt}(x^\top Px) < 0$$
$$\forall x \in \mathcal{E}(W^{-1},1), x \neq 0$$

This proves the main result of the paper.

V. Comparison with the linear differential inclusion approach

One of the most efficient ways of computing ellipsoidal estimations of the domain of attraction of a saturated control systems relies in the use of a Linear Differential Inclusion (LDI) of the saturated system. In this section we show that theorem 1 yields less conservative ellipsoidal estimations than the ones provided by the LDI approach.

By means of the concept of Linear Differential Inclusion, the following sufficient condition for the contractiveness of a given ellipsoid is obtained (see [9] for a proof):

Theorem 2: The ellipsoid $\mathcal{E}(W^{-1},1)$ is contractive if there exists $Y \in \mathbb{R}^{m\times n}$ such that

$$AW + \sum_{i\in S^c} B_i K_i W + \sum_{i\in S} B_i Y_i +$$
$$(AW + \sum_{i\in S^c} B_i K_i W + \sum_{i\in S} B_i Y_i)^\top < 0$$
$$\begin{bmatrix} 1 & Y_i \\ Y_i^\top & W \end{bmatrix} > 0,\ i = 1,\ldots,m$$

where Y_i denotes the i-th row of Y.

The sufficient condition for an ellipsoid to be invariant provided by theorem (2) has been shown to be less conservative than the existing conditions resulting from the circle criterion or the vertex analysis [9], [14]. Moreover, as it is shown in [10], theorem (2) provides not only a sufficient but also a necessary condition for an ellipsoid to be invariant for the single input case ($m = 1$).

Note that the above result (theorem (2)) can be obtained directly from theorem (1). It suffices to make $Y = Y^S$ for every $S \in \mathcal{V}$. Therefore, we conclude that the results presented in this paper provide an alternative proof of theorem (2) (in this case without using the concept of Linear Difference Inclusion).

The sufficient conditions provided by the main result of this paper are less conservative than the ones corresponding to theorem (2) (this is due to the greater number of decision variables considered in theorem 1). We conclude then that the approach proposed in this paper improves the results obtained when a linear differential approach is adopted. The computational complexity of the ellipsoidal estimation of the domain of attraction presented in this paper is greater than the one corresponding to the linear differential approach. This is due to the greater number of matrices involved in theorem (1). See next section for an analysis of the computational complexity.

VI. Computational complexity

Theorem (1) can be applied to the computation of ellipsoidal estimation of the domain of attraction of a saturated system. However, the direct application of theorem (1) implies the solution of a convex optimization problem with $2^{(m+1)}$ constraints and $2^m + 1$ decision variables. Although the exponential number of constraints does not imply an excessive computational burden for practical values of m (there are convex algorithms in which the computational burden grows only linearly with the number of constraints), the same can not be affirmed for the number of decision variables: if m grows beyond a certain limit, the direct application of theorem (1) can be limited because of the exponential number of decision variables.

Fortunately, theorem 1 can be recast into an equivalent form in which the number of decision variables is reduced to only one: W. In this section it is proved that the main result of the paper (theorem (1)) can be applied to the estimation of the domain of attraction by means of the solution of a convex problem with a reduced number of variables. For that purpose, the following definition is introduced:

Definition 3: Given $W > 0$ and $S \in \mathcal{V}$, the function $\gamma_S(W)$ is defined as:

$$\gamma_S(W) = \min_{Y\in\mathbb{R}^{m\times n}} \bar{\lambda}\Big(AW + \sum_{i\in S^c} B_i K_i W + \sum_{i\in S} B_i Y_i +$$
$$(AW + \sum_{i\in S^c} B_i K_i W + \sum_{i\in S} B_i Y_i)^\top \Big)$$
$$\text{s.t.} \quad \begin{bmatrix} 1 & Y_i \\ Y_i^\top & W \end{bmatrix} > 0,\quad \forall i \in S$$

where Y_i denotes the i-th row of Y and $\bar{\lambda}(\cdot)$ denotes the matrix function greatest eigenvalue.

In what follow, it is shown that $\gamma_S(W)$ is a convex function on W for every $S \in \mathcal{V}$. It is clear that the function:

$$g(W,Y) = \bar{\lambda}\Bigg(AW + \sum_{i\in S^c} B_iK_iW + \sum_{i\in S} B_iY_i + (AW + \sum_{i\in S^c} B_iK_iW + \sum_{i\in S} B_iY_i)^\top\Bigg)$$

is a convex function on W and Y. Moreover, the constraint

$$\begin{bmatrix} 1 & Y_i \\ Y_i^\top & W \end{bmatrix} > 0, \;\; \forall i \in S$$

can be rewritten as

$$h(W,Y) = \max_{i\in S} \bar{\lambda}\left(-\begin{bmatrix} 1 & Y_i \\ Y_i^\top & W \end{bmatrix}\right) < 0$$

Therefore, $\gamma_S(W)$ can be rewritten as:

$$\gamma_S(W) = \min_{Y\in \mathbb{R}^{m\times n}} g(W,Y)$$
$$\text{s.t.} \;\; h(W,Y) < 0$$

As both $g(W,Y)$ and $h(W,Y)$ are jointly convex in W and Y, it is inferred that $\gamma_S(W)$ is convex with respect W (see [2]). For this class of optimization problems it is possible to find a subgradient of $\gamma_S(W)$ with respect W at any given W_0 (see also [2]).

Note that with the definition of $\gamma_S(W)$, theorem 1 can be rewritten as:

Theorem 3: The ellipsoid $\mathcal{E}(W^{-1},1)$ is contractive if

$$\gamma_S(W) < 0, \forall S \in \mathcal{V}$$

As the trace of W is equal to the sum of the semiaxis of ellipsoid $\mathcal{E}(W^{-1},1)$, the maximization problem:

$$\max_{W>0} \; \text{trace}\,(W)$$
$$\text{s.t.} \;\; \gamma_S(W) < 0, \;\; \forall S \in \mathcal{V}$$

yields to the maximization of the ellipsoidal estimation of the domain of attraction of the saturated system $\dot{x} = Ax + B\sigma(Kx)$. The proposed maximization problem has the following properties:

- It is a convex optimization problem. This convexity stems from the already proved fact that $\gamma_S(W)$ is convex on W.
- The evaluation of $\gamma_S(W)$ can be achieved solving an LMI problem with a unique decision variable: $Y \in \mathbb{R}^{n\times m}$.
- The computation of a subgradient of $\gamma_S(W)$ with respect W can also be done solving an LMI problem. This makes it possible the application of any cutting plane algorithm to the solution of the proposed optimization problem [2].

VII. Numerical Examples

In this section two different examples are presented. The first example shows the application of the presented approach to a two dimensional system. In the second example higher dimensional systems are considered.

A. Two dimensional system

Let us consider the system $\dot{x} = Ax + B\sigma(Kx)$ where

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix},$$

K is obtained as the solution of the LQR problem with $Q = I$ and $R = 0.1 \cdot I$. That is,

$$K = \begin{bmatrix} -2.0506 & -5.9715 \\ -3.1458 & 2.1906 \end{bmatrix}.$$

Figure 1 shows how the conservativeness in the computation of an ellipsoidal estimation of the domain of attraction is reduced by means of the main result of the paper. In that figure, three different contractive ellipsoids are drawn. The inner one represents the maximal contractive ellipsoid obtained when the ellipsoidal estimation is constrained to belong to the region of state space in which $u = Kx$ does not saturate (that is, $\sigma(Kx) = Kx$). The ellipsoid represented by means of a dotted line corresponds to the ellipsoid obtained when an LDI approach is adopted (theorem 2). The ellipsoid represented by a solid line corresponds to the application of the main result of the paper (theorem 1).

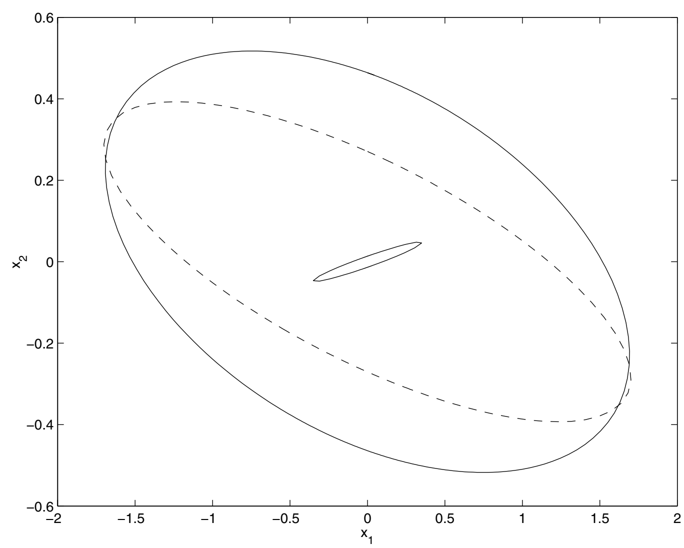

Fig. 1. Contractive ellipsoidal sets

It can be seen in the figure that the ellipsoid obtained by the sufficient condition presented in this paper is greater than the other two ones. This is not surprising because it has been proved in section V that theorem (1) provides less conservative results than theorem (2).

B. Higher dimensional systems

Let us consider the following family of linear systems:

$$y(s) = \frac{\sum_{i=1}^{n} s^{i-1}u_i(s)}{(s-1)^n}$$

where $u \in \mathbb{R}^m = \mathbb{R}^n$ is the saturated actuation and n is the dimension of the system. The control law is given by

$u = \sigma(Kx)$ where K corresponds to the solution of the LQR problem with $Q = I$ and $R = 0.1I$.

The following table shows the volume of the ellipsoid obtained using the novel approach presented in this paper ($Method\ A$) and the volume obtained by means of theorem (2)($Method\ B$).

Dimension	Method A	Method B	% of improv.
1	18.67	18.67	0
2	237.97	212.95	11.75
3	122.98	102.47	20.01
4	343.02	262.64	30.61
5	472.67	411.98	14.73
6	738.74	614.97	20.13

Dimension 1 shows the same result using both methods. In fact, the problem formulation is the same and therefore the results are identical.

The observed improvement depends on the dimension of the system. Although it usually increases with the dimension (more decision variables are considered in the proposed approach), it is not a fixed rule: dimension 5 has in this case a lower improvement than dimensions 3 or 4.

Computing time required in the computation of the ellipsoidal estimation of the domain of attraction is greater when using the proposed approach. Note that the computing time depends on the number of decision variables and constraints of the LMIs.

The following table shows the computing time corresponding to both methods (in seconds).

Dimension	Method A	Method B	Linear
1	0.016	0.016	0.011
2	0.087	0.266	0.077
3	0.161	0.131	0.039
4	0.752	0.418	0.084
5	5.520	2.036	0.194
6	81.61	9.606	0.206

The following table shows the decision variables corresponding to each of the estimation methods.

Dimension	Method A	Method B	Linear
1	2	2	1
2	5	3	1
3	13	4	1
4	33	5	1
5	81	6	1
6	193	7	1

VIII. Conclusions

In this paper, a novel approach to the estimation of the domain of attraction of a saturated linear system is presented. The main contribution of the paper is a new sufficient condition for the contractiveness of a given ellipsoid. It is shown that the proposed approach is less conservative than the one corresponding to the use of the concept of Linear Difference Inclusion. The computational complexity of the characterization of the proposed ellipsoidal estimation of the domain of attraction is analyzed. Some illustrative examples are given.

IX. Acknowledgements

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Appendix

Appendix A

Property 1: Suppose that $Y \in \mathbb{R}^{1\times n}$ and matrix $W = W^\top \in \mathbb{R}^{n\times n}$ are such that:

$$\begin{bmatrix} 1 & Y \\ Y^\top & W \end{bmatrix} > 0$$

then

$$\bar{B}Y + Y^\top \bar{B}^\top \geq -\alpha W - \frac{\bar{B}\bar{B}^\top}{\alpha}, \ \ \forall \alpha > 0, \forall \bar{B} \in \mathbb{R}^n$$

Proof:

Given $\bar{B} \in \mathbb{R}^n$ and $\alpha > 0$:

$$\begin{aligned} 0 &\leq (\sqrt{\alpha}Y^\top + \frac{\bar{B}}{\sqrt{\alpha}})(\sqrt{\alpha}Y^\top + \frac{\bar{B}}{\sqrt{\alpha}})^\top \\ &= \alpha Y^\top Y + \frac{\bar{B}\bar{B}^\top}{\alpha} + \bar{B}Y + Y^\top \bar{B}^\top \end{aligned}$$

It is then concluded that:

$$\bar{B}Y + Y^\top \bar{B}^\top \geq -\alpha Y^\top Y - \frac{\bar{B}\bar{B}^\top}{\alpha} \tag{3}$$

Applying Schur's complement to the assumption

$$\left[\begin{array}{cc} 1 & Y \\ Y^\top & W \end{array} \right] > 0$$

it results that $W > Y^\top Y$. From this and inequality (3) it is inferred that:

$$\bar{B}Y + Y^\top \bar{B}^\top \geq -\alpha W - \frac{\bar{B}\bar{B}^\top}{\alpha}, \ \ \forall \alpha > 0$$

Appendix B

Property 2: Suppose that $\bar{\epsilon} > 0$. Then, for every $a \in \mathbb{R}$:

$$\sup_{\bar{\alpha}>0} -\bar{\alpha} - \frac{a^2}{\bar{\alpha}} + \bar{\epsilon} > -2|a|$$

Proof:

Two cases must be taken into account

1) $a = 0$: In this case:

$$\sup_{\bar{\alpha}>0} -\bar{\alpha} - \frac{a^2}{\bar{\alpha}} + \bar{\epsilon} = \sup_{\bar{\alpha}>0} -\bar{\alpha} + \bar{\epsilon} > 0 = -2|a|$$

2) $a \neq 0$: It is clear that $-\bar{\alpha} - \frac{a^2}{\bar{\alpha}} + \bar{\epsilon}$ is a concave differentiable function on α in $\mathbb{R}^+$. Thus, at the supremum:

$$0 = \frac{d}{d\bar{\alpha}}(-\alpha - \frac{a^2}{\bar{\alpha}} + \bar{\epsilon}) = -1 + \frac{a^2}{\bar{\alpha}^2}$$

It is then concluded that the supremum is attained at $\bar{\alpha} = |a|$. Thus:

$$\sup_{\bar{\alpha}>0} -\bar{\alpha} - \frac{a^2}{\bar{\alpha}} + \bar{\epsilon} = -|a| - \frac{a^2}{|a|} + \bar{\epsilon} = -2|a| + \bar{\epsilon} > -2|a|$$

Appendix C

Property 3: Given $z \in \mathbb{R}^m$:

$$z^\top \sigma(Kx) \leq \max_{S\in\mathcal{V}} \{\sum_{i\in S^c} z_i K_i x - \sum_{i\in S} |z_i|\}$$

where z_i denotes the i-th component of vector z.

Proof:

Taking now into account property 4 in appendix D:

$$z^\top \sigma(Kx) = \sum_{i=1}^{m} z_i \sigma(K_i x) \leq$$

$$\sum_{i=1}^{m} \max \{z_i K_i x, -|z_i|\} =$$

$$\max_{S\in\mathcal{V}} \{\sum_{i\in S^c} z_i K_i x - \sum_{i\in S} |z_i|\}$$

Appendix D

Property 4: Given $a \in \mathbb{R}$ and $y \in \mathbb{R}$:

$$a\sigma(y) \leq \max \{ay, -|a|\}$$

Proof:

1) $|y| \leq 1$: $\max \{ay, -|a|\} = ay = a\sigma(y)$
2) $|y| > 1$ and $ay \geq 0$: $\max \{ay, -|a|\} = ay \geq a \text{ sign } (y) = a\sigma(y)$
3) $|y| > 1$ and $ay < 0$: $\max \{ay, -|a|\} = -|a| = a \text{ sign } (-a) = a \text{ sign } (y) = a\sigma(y)$