Improved constrained scheme for the Einstein equations: An approach to the uniqueness issue

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based on collaboration with
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J.L. Jaramillo and É. Gourgoulhon.

Cordero-Carrión *et al. Phys. Rev. D* ***79****, 024017 (2009)*

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Plan

1. Introduction
2. CFC and FCF
3. Non-uniqueness problem
4. A cure in CFC
5. New constrained formulation

Plan

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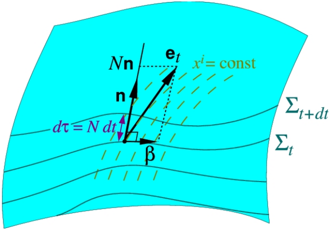
3+1 FORMALISM

Decomposition of spacetime and of Einstein equations



EVOLUTION EQUATIONS:

$$\frac{\partial K_{ij}}{\partial t} - \mathcal{L}_{\beta} K_{ij} =$$
$$-D_i D_j N + N R_{ij} - 2N K_{ik} K^k_{\ j} +$$
$$N \left[K K_{ij} + 4\pi((S - E)\gamma_{ij} - 2S_{ij}) \right]$$
$$K^{ij} = \frac{1}{2N} \left(\frac{\partial \gamma^{ij}}{\partial t} + D^i \beta^j + D^j \beta^i \right).$$

CONSTRAINT EQUATIONS:

$$R + K^2 - K_{ij} K^{ij} = 16\pi E,$$
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$$g_{\mu\nu}\, dx^\mu\, dx^\nu = -N^2\, dt^2 + \gamma_{ij}\, (dx^i + \beta^i dt)\, (dx^j + \beta^j dt)$$

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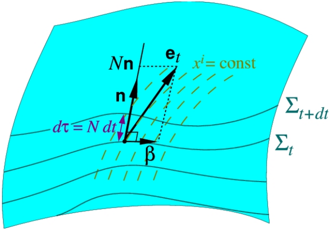

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3+1 formalism

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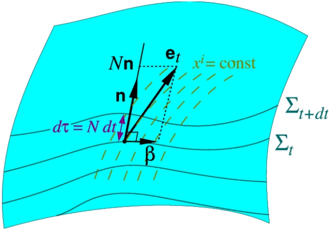

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Free vs. contrained formulations

As in electromagnetism, if the constraints are satisfied initially, they remain so for a solution of the evolution equations.

Free evolution

- start with initial data verifying the constraints,
- solve only the 6 evolution equations,
- recover a solution of all Einstein equations.

⇒apparition of constraint violating modes from round-off errors. Considered cures:

- Using of constraint damping terms and adapted gauges (many groups).
- Solving the constraints at every time-step (efficient elliptic solver?).

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Conformal flatness condition (CFC)

and

Fully constrained formulation (FCF)

Conformal flatness condition

Within 3+1 formalism, one imposes that :

$$\gamma_{ij} = \psi^4 f_{ij}$$

with f_{ij} the flat metric and $\psi(t, x^1, x^2, x^3)$ the conformal factor. First devised by Isenberg in 1978 as a waveless approximation to GR, it has been widely used for generating initial data, ...

Set of 5 non-linear elliptic PDEs ($K = 0$)

$$\Delta\psi = -2\pi\psi^{-1}\left(E^* + \frac{\psi^6 K_{ij}K^{ij}}{16\pi}\right),$$

$$\Delta(N\psi) = 2\pi N\psi^{-1}\left(E^* + 2S^* + \frac{7\psi^6 K_{ij}K^{ij}}{16\pi}\right),$$

$$\Delta\beta^i + \frac{1}{3}\nabla^i\nabla_j\beta^j = 16\pi N\psi^{-2}(S^*)^i + 2\psi^{10}K^{ij}\nabla_j\frac{N}{\psi^6}.$$

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Fully constrained formulation

Bonazzola *et al.* (2004)

With no approximation: $\tilde{\gamma}^{ij} = \psi^4 \gamma^{ij}$ and the choice of generalized Dirac gauge (and maximal slicing)

$$\nabla_j \tilde{\gamma}^{ij} = \nabla_j h^{ij} = 0. \qquad (\tilde{\gamma}^{ij} = f^{ij} + h^{ij})$$

$\Rightarrow$very similar equations to the CFC system + evolution equations for $\tilde{\gamma}^{ij}$:

$$\frac{\partial K^{ij}}{\partial t} - \mathcal{L}_{\boldsymbol{\beta}} K^{ij} = N D_k D^k h^{ij} - D^i D^j N + \mathcal{S}^{ij},$$
$$\frac{\partial h^{ij}}{\partial t} - \mathcal{L}_{\boldsymbol{\beta}} h^{ij} = 2 N K^{ij}.$$

When combined, reduce to a wave-like (strongly hyperbolic) operator on h^{ij}, with no incoming characteristics from a black hole excision boundary (Cordero-Carrión *et al.* (2008)).

Fully constrained formulation

Motivations for the FCF:

- Easy to use CFC initial data for an evolution using the constrained formulation,
- Evolution of two scalar fields: the rest of the tensor h^{ij} can be reconstructed using the gauge conditions.
  $\iff$ dynamical degrees of freedom of the gravitational field.
- Elliptic systems have good stability properties (what about uniqueness?).
- Constraints are verified!

+ the generalized Dirac gauge gives the property that h^{ij} is asymptotically transverse-traceless
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Non-uniqueness problem

Spherical collapse of matter

We consider the case of the collapse of an unstable relativistic star, governed by the equations for the hydrodynamics

$$\frac{1}{\sqrt{-g}}\left[\frac{\partial\sqrt{\gamma}\boldsymbol{U}}{\partial t}+\frac{\partial\sqrt{-g}\boldsymbol{F}^i}{\partial x^i}\right]=\boldsymbol{Q},$$

with $\boldsymbol{U} = (\rho W, \rho h W^2 v_i, \rho h W^2 - P - D)$.
At every time-step, we solve the equations of the CFC system (elliptic)
⇒exact in spherical symmetry! (isotropic gauge)

- During the collapse, when the star becomes very compact, the elliptic system would no longer converge, or give a wrong solution (wrong ADM mass).
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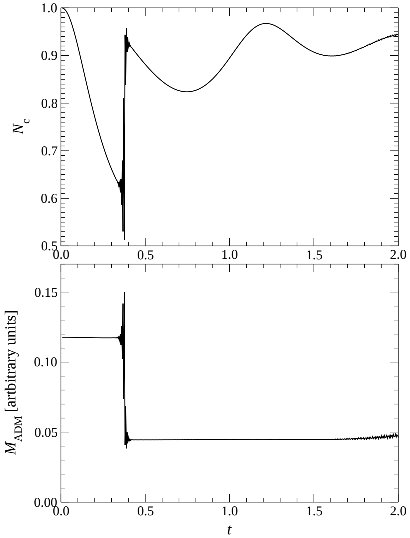
Collapse of gravitational waves

Using FCF (full 3D Einstein equations), the same phenomenon is observed for the collapse of a gravitational wave packet.



- Initial data: vacuum spacetime with Gaussian gravitational wave packet,
- if the initial amplitude is sufficiently large, the waves collapse to a black hole.
- As in the fluid-CFC case, the elliptic system of the FCF suddenly starts to converge to a wrong solution.

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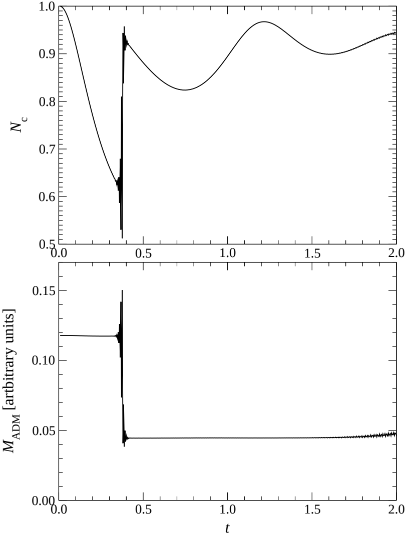

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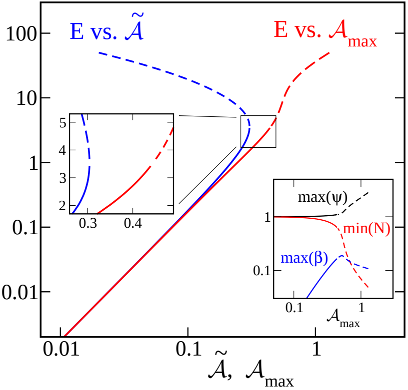
Other studies

- In the *extended conformal thin sandwich* approach for initial data, the system of PDEs is the same as in CFC.
- Pfeiffer & York (2005) have numerically oberved a parabolic branching in the solutions of this system for perturbation of Minkowski spacetime.
- Some analytical studies have been performed by Baumgarte *et al.* (2007), which have shown the genericity of the non-uniqueness behavior.



from Pfeiffer & York *(2005)*

A cure in the CFC case

Origin of the problem

In the simplified non-linear scalar-field case, of unknown function u

$$\Delta u = \alpha u^p + s.$$

Local uniqueness of solutions can be proven using a maximum principle:

> if α and p have the same sign, the solution is locally unique.

In the CFC system (or elliptic part of FCF), the case appears for the Hamiltonian constraint:

$$\Delta \psi = -2\pi \psi^5 E - \frac{1}{8}\psi^5 K_{ij}K^{ij};$$

Both terms (matter and gravitational field) on the r.h.s. have wrong signs.

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Approximate CFC

also in Saijo (2004)

$$\text{Let } L,\ V^i \mapsto (LV)^{ij} = \nabla^i V^j + \nabla^j V^i - \frac{2}{3} f^{ij} \nabla_k V^k.$$

$$\text{In CFC, } K^{ij} = \psi^{-4} \tilde{A}^{ij}, \text{ with } \tilde{A}^{ij} = \frac{1}{2N} (L\beta)^{ij},$$

$$\text{here } K^{ij} = \psi^{-10} \hat{A}^{ij}, \text{ with } \hat{A}^{ij} = (LX)^{ij} + \hat{A}^{ij}_{\mathrm{TT}}.$$

Neglecting $\hat{A}^{ij}_{\mathrm{TT}}$, we can solve in a hierarchical way:

1. Momentum constraints ⇒linear equation for X^i from the actually computed hydrodynamic quantity $S_j^* = \psi^6 S_j$,
2. Hamiltonian constraint ⇒$\Delta\psi = -2\pi\psi^{-1}E^* - \psi^{-7}\hat{A}^{ij}\hat{A}_{ij}/8$,
3. linear equation for $N\psi$,
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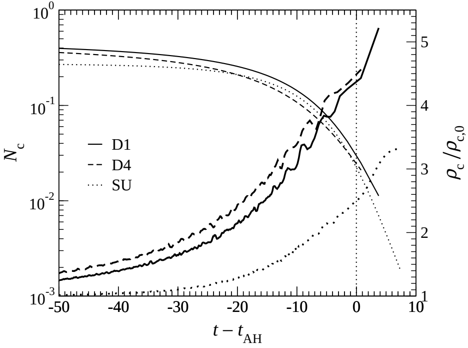
It can be shown that the error made neglecting $\hat{A}^{ij}_{\mathrm{TT}}$ falls within the error of CFC approximation.

Using the code CoCoNuT combining Godunov-type methods for the solution of hydrodynamic equations and spectral methods for the gravitational fields.

- Unstable rotating neutron star initial data, with polytropic equation of state,
- approximate CFC equations are solved every time-step.
- Collapse proceeds beyond the formation of an apparent horizon;
- Results compare well with those of Baoitti *et al.* (2005) in GR, although in approximate CFC.



Cordero-Carrión *et al.* (2009)

Other test: migration of unstable neutron star toward the stable branch.

New constrained formulation

In the general case, one cannot neglect the TT-part of $\hat{A}^{ij}$ and one must therefore evolve it numerically.

sym. tensor	longitudinal part	transverse part
$\hat{A}^{ij} =$	$(LX)^{ij}$	$+\hat{A}^{ij}_{\mathrm{TT}}$
$h^{ij} =$	0 (gauge)	$+h^{ij}$

The evolution equations are written only for the transverse parts:

$$\frac{\partial \hat{A}^{ij}_{\mathrm{TT}}}{\partial t} = \left[\mathcal{L}_\beta \hat{A}^{ij} + N\psi^2 \Delta h^{ij} + \mathcal{S}^{ij}\right]^{\mathrm{TT}},$$

$$\frac{\partial h^{ij}}{\partial t} = \left[\mathcal{L}_\beta h^{ij} + 2N\psi^{-6}\hat{A}^{ij} - (L\beta)^{ij}\right]^{\mathrm{TT}}.$$

New constrained formulation

If all metric and matter quantities are supposed known at a given time-step.

1. Advance hydrodynamic quantities to new time-step,
2. advance the TT-parts of $\hat{A}^{ij}$ and h^{ij},
3. obtain the logitudinal part of $\hat{A}^{ij}$ from the momentum constraint, solving a vector Poisson-like equation for X^i (the Δ^i_{jk}'s are obtained from h^{ij}):

$$\Delta X^i + \frac{1}{3}\nabla^i\nabla_j X^j = 8\pi (S^*)^i - \Delta^i_{jk}\hat{A}^{jk},$$

4. recover $\hat{A}^{ij}$ and solve the Hamiltonian constraint to obtain ψ at new time-step,
5. solve for $N\psi$ and recover β^i.

Summary - Perspectives

- We have presented, implemented and tested an approach to cure the uniqueness problem in the elliptic part of Einstein equations;
- This problem was appearing in the CFC approximation to GR and in the constrained formulation;
- Based on previous works (e.g. by Saijo (2004)) in the CFC case, it has been generalized to the fully constrained case (full GR).

⇒ the accuracy has been checked: the additional approximation does not introduce any new errors.

The numerical codes are present in the LORENE library: `http://lorene.obspm.fr`, publicly available under GPL.

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