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IMPROVED ERDŐS-RÉNYI AND STRONG APPROXIMATION LAWS FOR INCREMENTS OF PARTIAL SUMS

BY M. CSÖRGŐ[1] AND J. STEINEBACH[2]

Carleton University and University of Düsseldorf

Let $X_1, X_2, \cdots$ be an i.i.d. sequence with $EX_1 = 0$, $EX_1^2 = 1$, $Ee^{tX_1} < \infty$ $(|t| < t_0)$, and partial sums $S_n = X_1 + \cdots + X_n$. Starting from some analogous results for the Wiener process, this paper studies the almost sure limiting behaviour of $\max_{0 \le n \le N - a_N} a_N^{-1/2}(S_{n+a_N} - S_n)$ as $N \to \infty$ under various conditions on the integer sequence a_N. Improvements of the Erdős-Rényi law of large numbers for partial sums are obtained as well as strong invariance principle-type versions via the Komlós-Major-Tusnády approximation. An appearing gap between these two results is also going to be closed.

1. Introduction and Results. Csörgő and Révész (1979) obtained the following results for the increments of a standard Wiener process $W(t)$ $(0 \le t < \infty)$:

THEOREM A. *Let $a_T (T \ge 0)$ be a nondecreasing function of T for which*

(i) $0 < a_T \le T \qquad (T \ge 0)$,

(ii) *a_T/T is nonincreasing,*

(iii) $\lim_{T \to \infty} \dfrac{\log(T/a_T)}{\log \log T} = \infty.$

Then,

$$\lim_{T \to \infty} \sup_{0 \le t \le T - a_T} \frac{W(t + a_T) - W(t)}{(2a_T \log(T/a_T))^{1/2}} = 1 \qquad \text{w.p.1}, \tag{1}$$

and

$$\lim_{T \to \infty} \sup_{0 \le t \le T - a_T} \sup_{0 \le s \le a_T} \frac{W(t + s) - W(t)}{(2a_T \log(T/a_T))^{1/2}} = 1 \qquad \text{w.p.1}. \tag{2}$$

Actually, Theorem A is more detailed in [4], but it will be used in the sequel only as quoted here.

By the strong invariance principle of Komlós, Major and Tusnády (1976) Theorem A immediately implies

THEOREM B. *Let $X_1, X_2, \cdots$ be a sequence of* i.i.d. rv's *satisfying the conditions*

(i) $EX_1 = 0, \qquad EX_1^2 = 1$,

(ii) *there exists a $t_0 > 0$ such that* $\phi(t) = Ee^{tX_1} < \infty$ if $|t| < t_0$.

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Then for the sums $S_n = X_1 + \cdots + X_n$ *it holds that*

(3) $$\lim_{N\to\infty}\max_{0\le n\le N-a_N}\frac{S_{n+a_N}-S_n}{(2a_N\log(N/a_N))^{1/2}} = 1 \quad \text{w.p.1},$$

and

(4) $$\lim_{N\to\infty}\max_{0\le n\le N-a_N}\max_{0\le k\le a_N}\frac{S_{n+k}-S_n}{(2a_N\log(N/a_N))^{1/2}} = 1 \quad \text{w.p.1},$$

provided that a_N *is an integer sequence satisfying conditions* (i)–(iii) *of Theorem A and* $a_N/\log N \to \infty$.

The case $a_N = [C\log N]$, $C > 0$, which appears to be a critical one, cannot be treated by invariance principles. Indeed the latter was the first one of these theorems, considered by Erdős and Rényi (1970), who proved

THEOREM C. *Let* $X_1, X_2, \cdots$ *be as defined above. Then, for* $\alpha \in \{\phi'(t)/\phi(t): t \in (0, t_0)\}$ *and* $C = C(\alpha)$ *such that* $\exp(-1/C) = \inf_t \phi(t)\exp(-t\alpha) = \rho(\alpha)$, *we have*

(5) $$\lim_{N\to\infty}\max_{0\le n\le N-[C\log N]}\frac{S_{n+[C\log N]}-S_n}{[C\log N]} = \alpha \quad \text{w.p.1}.$$

REMARK 1. a) In case of $a_N/N^\delta \to 0$ ($\forall\delta > 0$) the normalizing constants $(2a_N\log(N/a_N))^{1/2}$ and $(2a_N\log N)^{1/2}$ are equivalent. In particular, for $a_N = [C\log N]$, $(2a_N\log N)^{1/2} \sim (2/C)^{1/2}[C\log N]$. Hence, in the standard normal case, (5) is analogous to the version of (1) for $a_T = C\log T$. Simply use that $\rho(\alpha) = \exp(-\alpha^2/2)$, $\alpha > 0$, which implies $\alpha = (2/C)^{1/2}$ for the one-to-one correspondence $C = C(\alpha)$ or $\alpha = \alpha(C)$ in (5).

b) The statement

(6) $$\lim_{N\to\infty}\max_{0\le n\le N-[C\log N]}\max_{0\le k\le[C\log N]}\frac{S_{n+k}-S_n}{[C\log N]} = \alpha \quad \text{w.p.1},$$

which was not given by Erdős and Rényi (1970), can be obtained in a similar way (cf., e.g., Steinebach (1979)).

In a recent paper, Révész (1980) was able to improve also the assertions of Theorem A in the following sense:

THEOREM D. *Let* $W(t)$ $(0 \le t < \infty)$ *and* a_T $(T \ge 0)$ *be as in Theorem A, but instead of* (iii) *let us assume the stronger condition*

(iv) $$\lim_{T\to\infty}\frac{(\log(T/a_T))^{1/2}}{\log\log T} = \infty.$$

Then

(7) $$\lim_{T\to\infty}\left(\sup_{0\le t\le T-a_T}\frac{W(t+a_T)-W(t)}{a_T^{1/2}} - (2\log(T/a_T))^{1/2}\right) = 0 \quad \text{w.p.1},$$

and

(8) $$\lim_{T\to\infty}\left(\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}\frac{W(t+s)-W(t)}{a_T^{1/2}} - (2\log(T/a_T))^{1/2}\right) = 0 \quad \text{w.p.1}.$$

If (iv) *does not hold, then statements* (7) *and* (8) *also fail to hold.*

From Révész's (1980) paper it can also be seen that (7) and (8) which yield convergence rates of order $o((\log(T/a_T))^{-1/2})$ in (1) and (2), cannot be improved to getting better rates like $o((\log(T/a_T))^{-(1/2+\delta)})$, say, for any $\delta > 0$.

Theorem D raises the question whether similar improvements of Theorems B and C are also available for the partial sum sequence. Using the Komlós-Major-Tusnády (1976) approximation again, the following consequence is immediate:

THEOREM 1. *Let S_n be as in Theorem B and a_N satisfy conditions* (i), (ii), (iv) *of Theorem D and the following condition*:

(v) $a_N/(\log N)^2 \to \infty$.

Then

$$\lim_{N\to\infty}\left(\max_{0\le n\le N-a_N}\frac{S_{n+a_N}-S_n}{a_N^{1/2}}-(2\log(N/a_N))^{1/2}\right)=0 \qquad \text{w.p.1} \tag{9}$$

and

$$\lim_{N\to\infty}\left(\max_{0\le n\le N-a_N}\max_{0\le k\le a_N}\frac{S_{n+k}-S_n}{a_N^{1/2}}-(2\log(N/a_N))^{1/2}\right)=0 \qquad \text{w.p.1.} \tag{10}$$

But the cases $a_N/(\log N)^2 \not\to \infty$ cannot be treated by strong invariance principles. We therefore prove:

THEOREM 2. *Under the assumptions of Theorem C we have*

$$\lim_{N\to\infty}\left(\max_{0\le n\le N-[C\log N]}\frac{S_{n+[C\log N]}-S_n}{[C\log N]^{1/2}}-[C\log N]^{1/2}\alpha\right)=0 \qquad \text{w.p.1} \tag{11}$$

and

$$\lim_{N\to\infty}\left(\max_{0\le n\le N-[C\log N]}\max_{0\le k\le[C\log N]}\frac{S_{n+k}-S_n}{[C\log N]^{1/2}}-[C\log N]^{1/2}\alpha\right)=0 \text{ w.p.1.} \tag{12}$$

Moreover, the gap between Theorems 1 and 2 can be closed by

THEOREM 3. *Let S_n be as in Theorem B and a_N satisfy conditions* (i) *and* (ii) *of Theorem A and the following conditions*:

(vi) $a_N/(\log N)^p \to 0$ *for some* $p > 2$,

(vii) $a_N/\log N \nearrow \infty$.

For $\alpha \in \mathscr{R}$, set $\rho(\alpha) = \inf_t \phi(t)\exp(-t\alpha)$ (hence ρ is finite in a neighborhood of the origin). Then, if $\alpha_N > 0$ is the unique positive solution of the equation

(viii) $\rho^{a_N}(\alpha_N a_N^{-1/2}) = a_N/N$ *or, equivalently, that of*

$$-a_N\log\rho(\alpha_N a_N^{-1/2}) = \log(N/a_N)$$

(N sufficiently large), we have

$$\lim_{N\to\infty}\left(\max_{0\le n\le N-a_N}\frac{S_{n+a_N}-S_n}{a_N^{1/2}}-\alpha_N\right)=0 \qquad \text{w.p.1,} \tag{13}$$

and

$$\lim_{N\to\infty}\left(\max_{0\le n\le N-a_N}\max_{0\le k\le a_N}\frac{S_{n+k}-S_n}{a_N^{1/2}}-\alpha_N\right)=0 \qquad \text{w.p.1.} \tag{14}$$

REMARK 2. a) Let $\psi(t) = \log\phi(t)(|t| > t_0)$ denote the cumulant-generating function. Since $EX_1 = \psi'(0) = 0$, $\operatorname{Var}(X_1) = \psi''(0) = 1$, and ψ is analytic in a neighborhood of the origin, we have

$$\psi(t) = t^2/2 + O(t^3), \qquad \psi'(t) = t + O(t^2) \qquad (t\to 0).$$

Moreover, by use of convex analysis,

$$-\log\rho(\alpha) = \alpha^2/2 + O(\alpha^3) \qquad (\alpha\to 0).$$

b) For N sufficiently large, the solution α_N in (viii) exists by continuity of $-\log \rho(\alpha)$ and using the facts $-\log \rho(0) = 0$, $\log(N/a_N) \leq -a_N \log \rho(\alpha)(\rho(\alpha) < 1$, N large). Furthermore,

$$\alpha_N a_N^{-1/2} \to 0 \qquad (N \to \infty).$$

c) From equation (viii) and Remarks 2a) and 2b) we obtain

$$\alpha_N \sim (2 \log(N/a_N))^{1/2},$$

but, since

$$\alpha_N - (2 \log(N/a_N))^{1/2} = \alpha_N - (-2a_N \log \rho(\alpha_N a_N^{-1/2}))^{1/2} = O(\alpha_N^2 a_N^{-1/2}) = O(\log N/a_N^{1/2}),$$

in general,

$$\alpha_N - (2 \log(N/a_N))^{1/2} \to 0 \qquad (N \to \infty)$$

only if $a_N/(\log N)^2 \to \infty$.

The proof of the original Erdős-Rényi law of large numbers (Theorem C) can be based already on Chernoff's (1952) large deviation theorem instead of the Bahadur and Ranga Rao (1960) refinement of the latter, which was used by Erdős and Rényi (1970) (cf. e.g., S. Csörgő (1979), where further Erdős-Rényi laws are proved for such functions of moving blocks of i.i.d. rv's and those of empirical measures of these blocks, for which functions a first order large deviation theorem holds). The essential ingredients of the original proof then are the monotonicity of $\rho(\alpha)$ and exponential bounds for the large deviation probabilities, i.e.,

$$\rho_1^N \leq P(S_N \geq N\alpha) \leq \rho_2^N,$$

where $\rho_1 < \rho(\alpha) < \rho_2$ and N is sufficiently large.

The proofs of our Theorems 2 and 3 are similarly based on certain exponential bounds which will be given in the following section.

2. Some auxiliary lemmas. Theorem 2 is based on

LEMMA 1. *Under the assumptions of Theorem* 2, *for any* $\epsilon > 0$, *there exist constants* $\delta = \delta(\epsilon) > 0$ *and* $A = A(\epsilon) > 0$ *such that*

$$P(\max_{0 \leq k \leq K} S_k \geq K\alpha + K^{1/2}\epsilon) \leq K\rho^K(\alpha)e^{-\delta K^{1/2}}, \tag{15}$$

$$P(S_K > K\alpha - K^{1/2}\epsilon) \geq A\rho^K(\alpha)e^{\delta K^{1/2}} \tag{16}$$

where $K = [C \log N]$ *and* N *is sufficiently large.*

PROOF. a) Let $t = t(\alpha) > 0$ be such that $\psi'(t) = \phi'(t)/\phi(t) = \alpha$. Then

$$P(\max_{0 \leq k \leq K} S_k \geq K\alpha + K^{1/2}\epsilon) \leq \sum_{k=1}^K P(S_k \geq K\alpha + K^{1/2}\epsilon) \leq \sum_{k=1}^K E(e^{t(S_k - K\alpha - K^{1/2}\epsilon)})$$
$$\leq (\sum_{k=1}^K \phi^k(t))e^{-Kt\alpha - K^{1/2}t\epsilon} \leq K\phi^K(t)e^{-Kt\alpha}e^{-K^{1/2}t\epsilon} = K\rho^K(\alpha)e^{-K^{1/2}t\epsilon},$$

remembering that $EX_1 = 0$ implies $\phi(t) > 1$ for $t > 0$.

b) Using associated probability measures $P_{t,K}$, defined by

$$P_{t,K}(E) = \int_E e^{ts_K}/\phi^K(t)\, dP,$$

we have

$$P(S_K > K\alpha - K^{1/2}\epsilon) = \phi^K(t) \int_{\{S_K - K\alpha > -K^{1/2}\epsilon\}} e^{-tS_K}\, dP_{t,K}$$
$$\geq \phi^K(t)e^{-t(S_K - \frac{\epsilon}{2}K^{1/2})} P_{t,K}\left(-\epsilon < \frac{S_K - K\alpha}{K^{1/2}} \leq -\frac{\epsilon}{2}\right).$$

Noting that the probability on the right-hand side of this inequality tends to $\phi(-\epsilon/2) - \phi(-\epsilon) > 0$ $(K \to \infty)$ by the central limit theorem, the proof is complete.

The proof of Theorem 3 is similarly based on:

LEMMA 2. *Under the assumptions of Theorem* 3, *for any* $\epsilon > 0$ *there exist constants* $\delta = \delta(\epsilon) > 0$ *and* $A = A(\epsilon) > 0$ *such that*

$$P(\max_{0 \le k \le K} S_k \ge K^{1/2}(\alpha_N + \epsilon)) \le K\rho^K(\alpha_N K^{-1/2})e^{-\delta\alpha_N}, \tag{17}$$

$$P(S_K > K^{1/2}(\alpha_N - \epsilon)) \ge A\rho^K(\alpha_N K^{-1/2})e^{\delta\alpha_N}, \tag{18}$$

where $K = a_N$ *and* N *is sufficiently large.*

PROOF. a) Setting $t = t_N$ such that $\alpha_N K^{-1/2} = \phi'(t)/\phi(t)$, we again have $\rho(\alpha_N K^{-1/2}) = \phi(t)e^{-t\alpha_N K^{-1/2}}$. Remembering that, by Remark 2, $t_N \sim \alpha_N K^{-1/2} (N \to \infty)$, the same estimations used to prove (15) will yield (17) too.

b) Following the lines of Lemma 1, we also have $P(S_K > K^{1/2}(\alpha_N - \epsilon)) \ge \rho^K(\alpha_N K^{-1/2}) \cdot e^{K^{1/2}t\epsilon/2} P_{t,K}\left(-\epsilon < \frac{S_K - K\alpha_N}{K^{1/2}} \le -\frac{\epsilon}{2}\right)$. Taylor series expansion of the cumulant-generating function of $(S_K - K\alpha_N)/K^{1/2}$ shows that asymptotic normality still holds, which renders the proof complete.

3. Proofs of Theorems 2 and 3.

Let us introduce the notations

$$D_1(N, K) = \max_{0 \le n \le N-K} (S_{n+k} - S_n),$$

$$D_2(N, K) = \max_{0 \le n \le N-K} \max_{0 \le k \le K} (S_{n+k} - S_n).$$

PROOF OF THEOREM 2. a) For any $\epsilon > 0$, we first prove

$$\limsup_{N \to \infty} \left(\frac{D_2(N, [C \log N])}{[C \log N]^{1/2}} - [C \log N]^{1/2}\alpha \right) \le \epsilon \quad \text{w.p.1.} \tag{19}$$

From Lemma 1, (15), we have

$$P(D_2(N, K) \ge K\alpha + K^{1/2}\epsilon) \le NK\rho^K(\alpha)e^{-\delta K^{1/2}}.$$

Now, let N_j be the greatest integer such that $[C \log N_j] = j$. Then, remembering $\exp(-1/C) = \rho(\alpha)$, we get

$$\begin{aligned} P(D_2(N_j, [C \log N_j]) &\ge [C \log N_j]\alpha + [C \log N_j]^{1/2}\epsilon) \\ &\le jN_j\rho(\alpha)^{C \log N_j - 1}e^{-\delta j^{1/2}} = \rho^{-1}(\alpha)je^{-\delta j^{1/2}}. \end{aligned}$$

By integral test, $\sum_{j=1}^{\infty} je^{-\delta j^{1/2}} < \infty$. Hence, using the Borel-Cantelli lemma,

$$\limsup_{j \to \infty} \left(\frac{D_2(N_j, [C \log N_j])}{[C \log N_j]^{1/2}} - [C \log N_j]^{1/2}\alpha \right) \le \epsilon \quad \text{w.p.1.}$$

Since, by definition, for $N_{j-1} < N \le N_j$,

$$[C \log N] = j \quad \text{and} \quad D_2(N, [C \log N]) \le D_2(N_j, [C \log N_j]),$$

assertion (19) is proved.

b) To prove

$$\liminf_{N \to \infty} \left(\frac{D_1(N, [C \log N])}{[C \log N]^{1/2}} - [C \log N]^{1/2}\alpha \right) \le -\epsilon \quad \text{w.p.1,} \tag{20}$$

we estimate, using Lemma 1, (16),

$$P(D_1(N, K) \leq K\alpha - K^{1/2}\epsilon) \leq P(\max_{i=1,\ldots,[N/K]}(S_{iK} - S_{(i-1)K}) \leq K\alpha - K^{1/2}\epsilon)$$
$$\leq (1 - A\rho^K(\alpha)e^{\delta K^{1/2}})^{\left[\frac{N}{K}\right]} \leq \exp\left(-A\rho^K(\alpha)e^{\delta K^{1/2}}\left[\frac{N}{K}\right]\right).$$

Inserting $K = [C \log N]$ and remembering $\exp(-1/C) = \rho(\alpha)$, the right-hand side term can further be bounded by

$$\exp(-A' e^{\delta[C \log N]^{1/2}}[C \log N]^{-1}) = b_N$$

where $0 < A' < A$ and N is sufficiently large. Again integral test shows that $\sum_{N=1}^{\infty} b_N < \infty$. Hence, by the Borel-Cantelli lemma, relation (20) is also proved.

Since $D_1(N, K) \leq D_2(N, K)$, by combining (19) and (20) and letting $\epsilon \to 0$, the proof of Theorem 2 is complete.

The proof of Theorem 3 can be performed in a similar way, with only a few modifications to be introduced.

PROOF OF THEOREM 3. a) To prove

$$\limsup_{N\to\infty}(a_N^{-1/2}D_2(N, a_N) - \alpha_N) \leq \epsilon \quad \text{w.p.1}, \tag{21}$$

we estimate, using Lemma 2, (17), and the definition of α_N,

$$P(a_N^{-1/2}D_2(N, a_N) \geq \alpha_N + \epsilon) \leq Na_N\rho^{a_N}(\alpha_N a_N^{-1/2})e^{-\delta\alpha_N} = a_N^2 e^{-\delta\alpha_N},$$

for sufficiently large N. Since $\alpha_N \sim (2 \log (N/a_N))^{1/2}$ and $a_N = o((\log N)^p)$, we get, for any q,

$$\delta\alpha_N \geq q \log\log N \leq q \log(a_N^{1/p}) = \log a_N^{q/p},$$

if N is large enough. Choosing e.g. $q = 4p$, we thus get

$$P(a_N^{-1/2}D_2(N, a_N) \geq \alpha_N + \epsilon) \leq a_N^{-2}.$$

Defining N_j to be the greatest integer such that $a_{N_j} = j$, it follows that

$$\sum_{j=1}^{\infty} P(a_{N_j}^{-1/2} D_2(N_j, a_{N_j}) \geq \alpha_{N_j} + \epsilon) < \infty,$$

and, by Borel-Cantelli lemma,

$$\limsup_{j\to\infty}(a_{N_j}^{-1/2}D_2(N_j, a_{N_j}) - \alpha_{N_j}) \leq \epsilon \quad \text{w.p.1}. \tag{22}$$

Now, by definition of N_j and α_N, for $N_{j-1} < N \leq N_j$ we have $a_N = j$, $a_N^{-1/2}D_2(N, a_N) \leq a_{N_j}^{-1/2} D_2(N_j, a_{N_j})$, $\alpha_N \leq \alpha_{N_j}$ and, moreover,

$$\log\frac{N_j}{a_{N_j}} - \log\frac{N}{a_N} = j(-\log\rho(\alpha_{N_j}j^{-1/2}) - (-\log\rho(\alpha_N j^{-1/2})))$$
$$\geq j\delta' \frac{\alpha_N}{j^{1/2}}\left(\frac{\alpha_{N_j}}{j^{1/2}} - \frac{\alpha_N}{j^{1/2}}\right) \geq \delta(\log N)^{1/2}(\alpha_{N_j} - \alpha_n)$$

for some $0 < \delta < \delta' < 1$, making use of convexity of $-\log\rho(\alpha)$ and $(-\log\rho(\alpha))' \sim \alpha(\alpha \to 0)$. Hence

$$0 \leq \alpha_{N_j} - \alpha_N \leq \frac{\log N_j - \log N}{\delta(\log N)^{1/2}}. \tag{23}$$

Since $a_N/\log N \nearrow \infty$ or, more precisely, $a_N = [\tilde{a}_N]$, where $\tilde{a}_N/\log N \nearrow \infty$ $(N \to \infty)$, we further obtain

$$1 \leq \frac{\log N_j}{\log N} \leq \frac{\tilde{a}_{N_j}}{\tilde{a}_N} \leq \frac{j+1}{j}$$

and

(24) $$0 \leq \frac{\log N_j - \log N}{\delta(\log N)^{1/2}} \leq \frac{(\log N)^{1/2}}{\delta j} \leq \frac{\log N}{\delta a_N} \to 0,$$

for $N_{j-1} < N \leq N_j$ and $j \to \infty$.

Combining (22)-(24) we have (21).

b) The proof of

(25) $$\liminf_{N\to\infty} (a_N^{-1/2} D_1(N, a_N) - \alpha_N) \leq -\epsilon \qquad \text{w.p.1}$$

follows in the same vein as part b of Theorem 2. We estimate

$$P(K^{-1/2} D_1(N, K) \leq \alpha_N - \epsilon) \leq \exp\left(-A\rho^K(\alpha_N K^{-1/2}) e^{\delta\alpha_N}\left[\frac{N}{K}\right]\right)$$
$$\leq \exp(-A' e^{\delta'(\log N)^{1/2}}) = b'_N$$

for some $0 < A' < A$, $0 < \delta' < \delta$, since

$$K = a_N, \quad \rho^K(\alpha_N K^{-1/2}) = \frac{K}{N}, \quad \alpha_N \sim (2\log(N/a_N))^{1/2} \sim (2\log N)^{1/2} \text{ and } \left[\frac{N}{K}\right] \sim \frac{N}{K}.$$

Checking out that $\sum_{N=1}^{\infty} b'_N < \infty$, Borel-Cantelli lemma again yields the desired result, i.e. (25), which completes the proof.

4. Some further results. Csörgő and Révész (1979) pointed out that, using their methods of proof, further results on the increments of the Wiener process are available, e.g.

THEOREM E. *Let a_T satisfy conditions* (i) *and* (ii) *of Theorem A. Then*

(26) $$\limsup_{T\to\infty} \frac{W(T + a_T) - W(T)}{(2a_T[\log(T/a_T) + \log\log T])^{1/2}} = 1 \qquad \text{w.p.1},$$

(27) $$\limsup_{T\to\infty} \sup_{0\leq s\leq a_T} \frac{W(T + s) - W(T)}{(2a_T[\log(T/a_T) + \log\log T])^{1/2}} = 1 \qquad \text{w.p.1}.$$

This theorem and the Komlós-Major-Tusnády (1976) approximation again imply similar results for partial sums in the case $a_N/\log N \to \infty$. The version of the Erdős-Rényi law, as given by Rényi (1970) for the coin-tossing situation, indicates that such results are also available for the critical case $a_N \sim C \log N (C > 0)$.

Now, using Révész's (1980, Theorem 2) arguments and Feller's (1968, page 175) bounds for the normal distribution, Theorem E can be improved in the following way.

THEOREM F. *Under the assumptions of Theorem D we have*

(28) $$\limsup_{T\to\infty} \left(\frac{W(T + a_T) - W(T)}{a_T^{1/2}} - (2\log(T/a_T))^{1/2}\right) = 0 \qquad \text{w.p.1},$$

(29) $$\limsup_{T\to\infty} \left(\sup_{0\leq s\leq a_T} \frac{W(T + s) - W(T)}{a_T^{1/2}} - (2\log(T/a_T))^{1/2}\right) = 0 \qquad \text{w.p.1}.$$

Via the quoted strong invariance principle, from this we have immediately

THEOREM 4. *Let S_n and a_N be as in Theorem* 1. *Then*

(30) $$\limsup_{N\to\infty} \left(\frac{S_{N+a_N}}{a_N^{1/2}} - (2\log(N/a_N))^{1/2}\right) = 0 \qquad \text{w.p.1},$$

(31) $$\limsup_{N\to\infty} \left(\max_{0\leq k\leq a_N} \frac{S_{n+k} - S_N}{a_N^{1/2}} - (2\log(N/a_N))^{1/2}\right) = 0 \qquad \text{w.p.1}.$$

In the same vein our Lemmas 1 and 2 can be used to prove

THEOREM 5. *Let S_n, C and α be as in Theorem* 2. *Then*

$$\limsup_{N\to\infty}\left(\frac{S_{N+[C\log N]}-S_N}{[C\log N]^{1/2}}-[C\log N]^{1/2}\alpha\right)=0 \quad \text{w.p.1}, \tag{32}$$

$$\limsup_{N\to\infty}\left(\max_{0\le k\le[C\log N]}\frac{S_{N+k}-S_N}{[C\log N]^{1/2}}-[C\log N]^{1/2}\alpha\right)=0 \quad \text{w.p.1}. \tag{33}$$

THEOREM 6. *Let S_n, a_N and α_N be as in Theorem* 3. *Then*

$$\limsup_{N\to\infty}\left(\frac{S_{N+a_N}-S_N}{a_N^{1/2}}-\alpha_N\right)=0 \quad \text{w.p.1}, \tag{34}$$

$$\limsup_{N\to\infty}\left(\max_{0\le k\le a_N}\frac{S_{n+k}-S_N}{a_N^{1/2}}-\alpha_N\right)=0 \quad \text{w.p.1}. \tag{35}$$

OUTLINE OF PROOFS. Note that inequalities (15) and (17) of Lemmas 1 and 2 imply

$$P(\max_{0\le k\le[C\log N]}(S_{N+k}-S_N)\ge[C\log N]\alpha+[C\log N]^{1/2}\epsilon)=O\left(\frac{\log N}{N}e^{-\delta(\log N)^{1/2}}\right),$$

$$P(\max_{0\le k\le a_N}a_N^{-1/2}(S_{n+k}-S_N)\ge\alpha_N+\epsilon)=O\left(\frac{(\log N)^{2p}}{N}e^{-\delta(\log N)^{1/2}}\right)$$

for any $\epsilon>0$ and some $\delta=\delta(\epsilon)>0$, making also use of $a_N=o((\log N)^p)$ and $\alpha_N\sim(2\log N)^{1/2}$ in the last line. This, by Borel-Cantelli lemma, yields a.s. upper bounds ϵ for the left-hand side terms in (33) and (35). On the other hand, relations (16) and (18) imply

$$P(S_{N+[C\log N]}-S_N>[C\log N]\alpha-[C\log N]^{1/2}\epsilon)\ge A\frac{[C\log N]}{N},$$

$$P(a_N^{-1/2}(S_{N+a_N}-S_N)>\alpha_N-\epsilon)\ge A\frac{a_N}{N},$$

for N sufficiently large and $A=A(\epsilon)>0$. Defining an integer sequence N_j by $N_1=2$, $N_{j+1}=N_j+a_{N_j}$, for the case $a_N=[C\log N]$ as well as $a_N/\log N\to\infty$, we have

$$\frac{N_{j+1}}{a_{N_{j+1}}}\le\frac{N_j}{a_{N_j}}+1, \quad \text{and} \quad \frac{a_{N_j}}{N_j}\ge\frac{1}{a+j}\ (a>0).$$

Hence from the second part of the Borel-Cantelli lemma, we also get a.s. lower bounds $-\epsilon$ in (32) and (34). This renders the proofs complete.

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Note added in proof. S. A. Book has informed us that James D. Lynch of Pennsylvania State University has discovered a 1964 article by L. A. Shepp (A limit theorem concerning moving averages. *Ann. Math. Statist.* **35** 424-428), which anticipates the Erdős-Rényi (1970) law. However, it seems to have been overlooked by all of us writing on this subject.

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DEPARTMENT OF MATHEMATICS AND STATISTICS
CARLETON UNIVERSITY
OTTAWA, CANADA K1S 5B6
CANADA

FACHBEREICH MATHEMATIK DER
UNIVERSITÄT MARBURG
LAHNBERGE
3550 MARBURG/LAHN
FEDERAL REPUBLIC OF GERMANY