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IMPROVED FRIEDMANN MODELS*

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Summary

The improvement of the Friedmann models of the Universe with the inclusion of non-interacting radiation was first made by Lemaître in 1927. These 'improved Friedmann models' are discussed and it is shown that they require modification in the early Universe because of pair production at high temperatures.

1. *Introduction*. The cosmological equations (without the cosmological constant) for a uniform universe are

$$\left(\frac{dR}{dt}\right)^2 = \frac{8\pi}{3} G\rho R^2 - \kappa, \tag{1}$$

$$\frac{d}{dt}(\rho c^2 R^3) + p\frac{dR^3}{dt} = 0, \tag{2}$$

where $R(t)$ scales the comoving coordinates, $\rho(t)$ is the density, $p(t)$ the pressure, κ the curvature constant and t is cosmic time. The Friedmann (1922, 1924) models have the properties: $p = 0$, $\kappa = 0$, ± 1 ($\kappa = 0$ is in fact the Einstein–de Sitter (1932) model), and were improved and made more realistic by Lemaître (1927, 1930, 1931) who included non-interacting radiation.

Let the subscripts m and r denote respectively matter and radiation (neutrinos and photons). From equation (2), for non-interacting matter and radiation: $\rho_m \propto R^{-3}$, $\rho_r \propto R^{-4}$, for $p_m \ll \rho_m c^2$, $p_r = \frac{1}{3}\rho_r c^2$. Equation (1) becomes

$$(dR/dt)^2 = \alpha_r R^{-2} + \alpha_m R^{-1} - \kappa, \tag{3a}$$

$$\alpha_r = 8\pi G\rho_r R^4/3, \qquad \alpha_m = 8\pi G\rho_m R^3/3, \tag{3b, c}$$

and this is the Lemaître equation‡. The solution for $\kappa = 1$ is given by de Sitter (1930), Tolman (1934) and Alpher & Herman (1949). The discovery of the 3 °K universal radiation (Penzias & Wilson 1965; Dicke, Peebles, Roll & Wilkinson 1965), predicted by Alpher & Herman (1948) and Gamow (1953), has reawakened interest in the Lemaître equation and solutions for $\kappa = 0$, ± 1 are given by Chernin (1965), Cohen (1967), and for $\kappa = 0$ by Jacobs (1967).

For non-interacting matter and radiation: $\rho_r \propto T^4$ and hence $T \propto R^{-1}$; also $\rho_m \propto n \propto T^3$ where n is the mean nucleon density. The dimensionless quantity

$$\eta = n(\hbar c/kT)^3 = n_0(\hbar c/kT_0)^3 \sim 10^{-9}, \tag{4}$$

is therefore constant and is evaluated with $n_0 \sim 10^{-6}\text{cm}^{-3}$, $T_0 \simeq 3$ °K. The ratio of

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‡ Radiation in an Eddington–Lemaître model is considered by Roeder & Chambers (1967).

the heat capacities of matter and radiation is of the order η (Gamow 1956) and consequently radiation is only slightly affected by interaction with matter. Furthermore, $p_m \sim \eta p_r$ and the smallness of η also justifies the neglect of the pressure of matter. The Lemaître equation is therefore reasonably realistic and except for the earliest moments is valid throughout the lifetime of the Universe.

Let $\beta = \rho_{r0}/\rho_{m0}$ (zero subscript denotes present epoch); then because $\rho_r c^2 \sim kT(kT/\hbar c)^3$,

$$\beta \sim kT_0/\eta m_n c^2 \sim 10^{-3}, \tag{5}$$

where m_n is the nucleon mass. If subscript* c denotes the epoch where $\rho_r = \rho_m$, then

$$\alpha_r/\alpha_m = R_c = \beta R_0, \tag{6}$$

and $\rho_c = 2\rho_{m0}\beta^{-3} \sim 10^9 \rho_{m0}$. Also, $T_c = \beta^{-1}T \simeq 3000\,°\mathrm{K}$, and when $T > T_c$ the density of radiation exceeds that of matter. The pressure and density are related by the equation

$$3p/\rho c^2 = R_c/(R_c+R) = \beta/(\beta+y), \tag{7}$$

$y = R/R_0$. When $T > T_b \sim m_e c^2/k \simeq 5 \times 10^9\,°\mathrm{K}$ (m_e is the electron mass), the Lemaître equation breaks down because of lepton and hadron pair production. As we go back in time, the progressive population of states of higher mass creates an admixture of nonrelativistic particles and the pressure is then significantly less than one third the energy density. Let us assume that in the early Universe pressure is proportional to energy density and use Zeldovich's (1962) equation

$$p = (\nu - 1)\rho c^2. \tag{8}$$

Thus $\rho R^{3\nu}$ is constant according to equation (2) and equation (1) becomes

$$(dR/dt)^2 = \alpha_\nu R^{2-3\nu} - \kappa, \tag{9a}$$

$$\alpha_\nu = 8\pi G \rho R^{3\nu}/3. \tag{9b}$$

In the early Universe the curvature term is negligible and for $0 < t < t_b$

$$t = (6\pi G \rho \nu^2)^{-1/2}. \tag{10}$$

At $T = T_b$ we have

$$\rho_b \sim (kT_b/c^2)(kT_b/\hbar c)^3 \sim m_e(m_e c/\hbar)^3 \sim 10^4 \text{ g cm}^{-3},$$

and therefore $t_b \sim 10$ s. Equation (10) is for the early charge symmetric universe (Zeldovich & Novikov 1966) in which the ratios of lepton number to number of leptons and baryon number to number of baryons are small and of order η (Harrison 1967, 1968), whereas the Lemaître equation (3) is for the subsequent charge asymmetric universe in which the asymmetry is either local or global.

2. *Relativistic thermodynamics and cosmology*. In a comoving element of volume $V(t)$ of an ideally uniform universe the internal energy is $E = \rho c^2 V$ and for thermal equilibrium

$$dE = TdS - pdV, \tag{11}$$

where S is the entropy in V. Because $V \propto R^3$, it follows that $dS/dt = 0$ from equations (2) and (11), and entropy is conserved.

* Significant epochs are denoted by subscripts a, b, and c (Harrison 1968).

Let $\Phi = W - ST = \Sigma\mu_i N_i$ be the thermodynamic potential and $W = V(\rho c^2 + p)$ the heat function, where (Landau & Lifshitz 1958) N_i is the number of the ith kind of particles in V of chemical potential μ_i. Because the lepton and baryon numbers are relatively small and therefore negligible in the early Universe, equilibrium conditions correspond to $\mu_i = 0$ and (Harrison 1968)

$$S = \frac{W}{T} = V\frac{dp}{dT} = \text{constant}, \tag{12}$$

and also

$$\rho c^2 + p = T\frac{dp}{dT}. \tag{13}$$

From this last equation and $p = (\nu - 1)\rho c^2$ we find

$$\rho \propto R^{-3\nu} \propto T^{\nu/(\nu-1)}. \tag{14}$$

The leptons are relatively unimportant and can be neglected. To each mass m_i of the hadron spectrum there are $(2J_i+1)(2I_i+1)\epsilon$ states (J_i is the spin, I_i the isospin, $\epsilon = 1$ if particle = antiparticle, and $\epsilon = 2$ if particle $\neq$ antiparticle). When kT is less than $m_i c^2$ then W_i is exponentially small and when $kT > m_i c^2$ we have

$$\rho_i c^2 + p_i = a_i T^4, \tag{15}$$

and the a_i are determined by the number of states and the statistics. As a crude approximation we use the truncated summation

$$W = VT^4 \Sigma a_i, \qquad m_i c^2 \leqslant kT, \tag{16}$$

and find from equations (12) and (14) that

$$\Sigma a_i \propto T^{(4-3\nu)/(\nu-1)}. \tag{17}$$

If the mass spectrum terminates at m_j, then Σa_i is constant at $T > m_j c^2/k$ and $\nu = 4/3$; however if the mass spectrum continues indefinitely then $\nu < 4/3$. Let $f(m)dm$ be the number of states in the mass interval dm, then

$$\Sigma a_i \simeq \int_0^{kT/c^2} f\, dm. \tag{18}$$

The density of states increases with m (Rosenfeld *et al.* 1967) and assuming $f \propto m^q$, $q > 0$, we have from equations (17) and (18) that

$$\nu = (5+q)/(4+q) \tag{19}$$

and $1 < \nu < 5/4$.*

* Hagedorn (1965, 1967) argues that the density of states increases exponentially and compensates for the exponential smallness of the contributions to W of mass $m_i > kT/c^2$. He finds a limiting temperature of $T_\pi = m_\pi c^2/k \sim 10^{12}$ °K (m_π is the pion mass) and in the vicinity of T_π particle creation is so violent that it is impossible to exceed this temperature. In the early Universe of $t < 10^{-4}$ s, as $T \to T_\pi$, we have $dW = Vdp \to 0$ and therefore $p/\rho c^2 \to 0$. A limiting temperature implies that the early universe conforms to the Einstein–de Sitter model. It is interesting that the statistical models of Fermi (1950) and Bialas & Weisskopf (1965) of high energy collisions are relevant to the state and composition of the early Universe. Clearly, if Hagedorn's (1965, 1967) ideas are correct they are of far reaching consequence.

3. *Improved Friedmann models.* The solutions of equations (3) and (9) are, for $t > t_b$, $R > R_b$: $\kappa = 0$;

$$R = \alpha_r^{1/2}\tau + \tfrac{1}{4}\alpha_m\tau^2, \tag{20a}$$

$$t = \tfrac{1}{2}\alpha^{1/2}\tau^2 + \tfrac{1}{12}\alpha_m\tau^3, \tag{20b}$$

$\kappa = +1$:

$$R = \alpha^{1/2}\sin\tau + \alpha_m\sin^2\tfrac{1}{2}\tau, \tag{21a}$$

$$t = 2\alpha_r^{1/2}\sin^2\tfrac{1}{2}\tau + \tfrac{1}{2}\alpha_m(\tau - \sin\tau), \tag{21b}$$

$\kappa = -1$:

$$R = \alpha^{1/2}\sinh\tau + \alpha_m\sinh^2\tfrac{1}{2}\tau, \tag{22a}$$

$$t = 2\alpha_r^{1/2}\sinh^2\tfrac{1}{2}\tau + \tfrac{1}{2}\alpha_m(\sinh\tau - \tau), \tag{22a}$$

and

$$R_b = \frac{3\nu - 2}{2}\,\alpha_\nu^{1/2}\tau_b^{2/(3\nu-2)}, \tag{23}$$

$$t_b = \frac{2}{3\nu}\,\alpha_\nu^{1/(3\nu-2)}\left(\frac{3\nu-2}{2}\,\tau_b\right)^{3\nu/(3\nu-2)}. \tag{24}$$

Since R_b and t_b are small we neglect them in order to study the solutions of the Lemaître equation.

In terms of the Hubble parameter $H = dR/Rdt$, the deceleration variable $q = -d^2R/RH^2dt^2$ and the density variable $\sigma = 4\pi G\rho/3H^2$ (McVittie 1965) we have

$$\kappa = -R_0^2H_0^2\{1 - 2q_0(\beta+1)/(2\beta+1)\}, \tag{25}$$

$$q = \sigma(2\beta+y)/(\beta+1) = q_0(2\beta+y)/[2q_0(\beta+y)+y^2\{2\beta+1-2q_0(\beta+y)\}]. \tag{26}$$

The corresponding Friedmann values are found by setting β equal to zero. We notice that

$$\kappa \gtreqless 0: \qquad q \gtreqless (2\beta+y)/(2\beta+2y), \qquad \sigma \gtreqless \tfrac{1}{2}; \tag{27}$$

also

$$(q_{0\mathrm{L}} - q_{0\mathrm{F}})/q_{0\mathrm{L}} = 2\beta/(2\beta+1) \sim 10^{-3}, \tag{28}$$

where the subscripts L and F denote Lemaître and Friedmann solutions. In the case of $\kappa = 0$

$$t_\mathrm{F}/t_\mathrm{L} = y^{3/2}/\{(\beta+y)^{1/2} - \beta^{1/2}\}^2\{(\beta+y)^{1/2} + 2\beta^{1/2}\}, \tag{29}$$

and

$$\frac{t_{0\mathrm{L}} - t_{0\mathrm{F}}}{t_{0\mathrm{L}}} = \frac{\{(\beta+1)^{1/2} - \beta^{1/2}\}^2\{(\beta+1)^{1/2} + 2\beta^{1/2}\} - 1}{\{(\beta+1)^{1/2} - \beta^{1/2}\}^2\{(\beta+1)^{1/2} + 2\beta^{1/2}\}} \simeq -\tfrac{3}{2}\beta \tag{30}$$

and the inclusion of radiation reduces the age of the universe by approximately $3\beta t_{0\mathrm{L}}/2$.

Recently McIntosh (1968) has also discussed cosmological models containing both matter and radiation.

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