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IMPROVED INVARIANT CONFIDENCE INTERVALS FOR A NORMAL VARIANCE

BY CONSTANTINOS GOUTIS[1] AND GEORGE CASELLA[2]

University College London and Cornell University

Confidence intervals for the variance of a normal distribution with unknown mean are constructed which improve upon the usual shortest interval based on the sample variance alone. These intervals have guaranteed coverage probability uniformly greater than a predetermined value $1 - \alpha$ and have uniformly shorter length. Using information relating the size of the sample mean to that of the sample variance, we smoothly shift the usual minimum length interval closer to zero, simultaneously bringing the endpoints closer to each other. The gains in coverage probability and expected length are also investigated numerically. Lastly, we examine the posterior probabilities of the intervals, quantities which can be used as post-data confidence reports.

1. Introduction. In the problem of estimating the variance of the normal distribution, there are two possible cases, depending on whether the mean is known or unknown. When the mean is known, the structure of the problem is relatively simple, since by sufficiency the data can be reduced to the sum of squared deviations from the mean and every optimal point or interval estimator must be based on this sufficient statistic. Hodges and Lehmann (1951) proved that the point estimator that is a constant multiple of this sufficient statistic is admissible under squared-error loss. For interval estimators, Tate and Klett (1959) showed that the endpoints of the shortest $1 - \alpha$ confidence interval must be the sum of squared deviations from the mean multiplied by the appropriate constants.

A more complicated problem is that of constructing optimal estimators for the variance of the normal distribution when the mean is unknown, the history of which is given in Maatta and Casella (1990). This history can be traced back at least to Stein (1964), who showed that the usual point estimator for the variance can be improved by using information about the size of the sample mean relative to the sample variance. His estimation procedure can be thought of as first testing the null hypothesis that the population mean is zero, and, if accepted, pooling the sample mean and the sample variance. In this way, whenever the population mean seems to be small, another degree of freedom is gained and we are able to beat the usual estimator based on the sample variance alone. Brown (1968) extended Stein's results to more general loss functions and a larger class of distributions, considering estimation of

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general scale parameters when the location parameter is unknown. He uses the usual estimator s^2 for scale parameter whenever the estimate, y, of the location parameter seems large and a smaller multiple of s^2 whenever y seems small. The relative size of s^2 is measured by the statistic $t = y^2/s^2$.

Both Stein's and Brown's estimators are inadmissible, thus it is possible to improve upon these estimators. Brewster and Zidek (1974) were able to find better estimators by taking a finer partition of the set of possible values of t. Their estimator is smooth enough to be generalized Bayes and, under some conditions, admissible among scale invariant point estimators. Proskin (1985) showed later that it is admissible within the class of all estimators.

The problem of the interval estimation of variance is, in many ways, similar to the problem of point estimation. Tate and Klett (1959) calculated the endpoints of the shortest confidence intervals based on s^2 alone. Cohen (1972) was able to construct improved confidence intervals adapting Brown's (1968) techniques. Cohen's intervals keep the same length but, by shifting the endpoints toward zero whenever $t \leq K$, some fixed but arbitrary constant, he was able to dominate Tate and Klett's intervals in terms of coverage probability.

Shorrock (1990) further improved on Cohen's result. In a manner analogous to Brewster and Zidek, Shorrock was able to construct a smooth version of Cohen's interval. The resulting interval is a highest posterior density region with respect to an improper prior and dominates the usual interval based on s^2 alone. For both Shorrock- and Cohen-type intervals, the domination is only in terms of coverage probability since, by construction, the length is kept fixed and equal to the usual length. Furthermore, the confidence coefficient remains equal to $1 - \alpha$ since asymptotically, as the noncentrality parameter $\lambda = \mu^2/\sigma^2$ tends to infinity, the endpoints of the intervals coincide with the endpoints of the usual interval. Stein-type improvements of confidence intervals for a normal variance with unknown mean were also obtained by Nagata (1989) and a multivariate extension of Cohen's result is given by Sarkar (1989).

The problem considered in this paper is, in some sense, the dual problem. In Section 2 we construct intervals which improve upon the usual shortest interval based on s^2 alone, both in length and coverage probability. We keep the minimum coverage probability equal to a predetermined value $1 - \alpha$ and shift the interval closer to zero whenever the sample indicates that the mean is close to zero. By shifting we are able to bring the endpoints closer to each other producing shorter intervals. Using a method similar to that of Brewster and Zidek, we construct a family of smooth $(1 - \alpha)$ 100% intervals which are shorter than the usual interval and, consequently, Cohen- and Shorrock-type intervals. We also investigate the gain numerically. In Section 3, we investigate the bounds on posterior probabilities with respect to some priors and point out the use of these probabilities as frequentist confidence reports.

2. Construction of the interval. Let $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2)$ be a $(n + p) \times 1$ vector so that $\mathbf{X}_1 = (X_1, X_2, \dots, X_n)$ and $\mathbf{X}_2 = (X_{n+1}, \dots, X_{n+p})$. We assume that $\mathbf{X}$ is a random variable from a multivariate normal distribution with mean

$(\mathbf{0}, \boldsymbol{\mu})$, where $\mathbf{0}$ is a vector of order n, $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_p)$ is unknown and the covariance matrix is σ^2 times the identity matrix of order $n + p$. We are interested in estimating the unknown parameter σ^2.

Let $s^2 = \mathbf{X}_1'\mathbf{X}_1$, $y^2 = \mathbf{X}_2'\mathbf{X}_2$ and $t = y^2/s^2$. By sufficiency, the data can be reduced to $(s^2, \mathbf{X}_2)$. With the normality assumption we have that

$$\frac{s^2}{\sigma^2} \sim \chi_n^2 \quad \text{and} \quad \frac{y^2}{\sigma^2} \sim \chi_p^2(\lambda), \qquad \lambda = \frac{\boldsymbol{\mu}'\boldsymbol{\mu}}{\sigma^2}, \tag{2.1}$$

central and noncentral chi-squared random variables, the latter with noncentrality parameter λ; the latter density will be denoted by $f_p(x; \lambda)$. If $\lambda = 0$, we will omit λ from the notation and $f_p(x)$ will be the central chi-squared density. The respective cumulative distribution functions will be denoted by $F_p(x; \lambda)$ and $F_p(x)$.

We can think of the problem as the general linear hypothesis, where y^2 represents the model sum of squares and s^2 the error sum of squares in an analysis of variance table. A simple version of the problem is estimation of the variance from a sample $X_1, X_2, \ldots, X_N$ from a single normal population with unknown mean. Here we have $s^2 = \sum(X_i - \bar{X})^2$ and $y^2 = N\bar{X}^2$, where $\bar{X} = \sum X_i/N$ and $N - 1$ central and 1 noncentral degrees of freedom with noncentrality parameter μ^2/σ^2.

The minimum length intervals, based on s^2 alone were tabulated by Tate and Klett (1959) and have the form

$$C_U(s^2) = \left(\frac{1}{b_n}s^2, \frac{1}{a_n}s^2\right), \tag{2.2}$$

where a_n and b_n satisfy

$$\int_{a_n}^{b_n} f_n(x)\,dx = 1 - \alpha \quad \text{and} \quad f_{n+4}(a_n) = f_{n+4}(b_n). \tag{2.3}$$

Let K be a positive constant and $\tau(x)$ be an increasing continuous function defined on $(0, +\infty)$ such that $\tau(x) > x$ for every x. Define a confidence procedure as follows:

$$I_1(s^2, t, K) = \begin{cases} \left(\dfrac{1}{b_n}s^2, \dfrac{1}{a_n}s^2\right), & \text{if } t > K, \\ \left(\phi_1(K)s^2, \phi_2(K)s^2\right), & \text{if } t \le K, \end{cases} \tag{2.4}$$

where $\phi_1(K)$ and $\phi_2(K)$ are determined from the following equations:

$$\int_{\phi_1(K)}^{\phi_2(K)} f_{n+4}\left(\frac{1}{x}\right) F_p\left(\frac{K}{x}\right) dx = \int_{1/b_n}^{1/a_n} f_{n+4}\left(\frac{1}{x}\right) F_p\left(\frac{K}{x}\right) dx \tag{2.5}$$

and

$$f_{n+4}\left\{\frac{1}{\phi_1(K)}\right\} F_p\left\{\frac{\tau(K)}{\phi_1(K)}\right\} = f_{n+4}\left\{\frac{1}{\phi_2(K)}\right\} F_p\left\{\frac{\tau(K)}{\phi_2(K)}\right\}. \tag{2.6}$$

If $K = +\infty$, the interval coincides with the usual one since $\tau(+\infty) = +\infty$. Note that the procedure defines a class of confidence intervals rather than a single interval since the endpoints depend on an unspecified function τ. For any given K and τ, we can choose $\phi_1(K)$ and $\phi_2(K)$ in a unique way. For the most part, the values of K are only a means to an end, and for clarity of notation we will sometimes omit K from the notation if no confusion arises.

The construction of $I_1(s^2, t, K)$ is analogous to Brown's (1968) point estimator and Cohen's (1972) confidence interval. We partition the space of possible values of t and whenever t is smaller than a constant, we shift the endpoints towards zero. In our case we keep the coverage probability, under $\boldsymbol{\mu} = \mathbf{0}$, equal to $1 - \alpha$. The following theorems establish that the new interval improves upon $C_U(s^2)$.

THEOREM 2.1. *The coverage probability of the procedure $I_1(s^2, t, K)$ is greater than the coverage probability of $C_U(s^2)$. The probability is strictly greater if $\lambda > 0$.*

PROOF. Note that the intervals differ only when $t \le K$. Working with the joint probability it suffices to show

$$(2.7)\quad P\{\sigma^2 \in (\phi_1 s^2, \phi_2 s^2), t \le K\} \ge P\left\{\sigma^2 \in \left(\frac{1}{b_n}s^2, \frac{1}{a_n}s^2\right), t \le K\right\},$$

which is equivalent to showing

$$(2.8)\qquad \int_{\phi_1}^{\phi_2} f_{n+4}\left(\frac{1}{x}\right) F_p\left(\frac{K}{x};\lambda\right) dx \ge \int_{1/b_n}^{1/a_n} f_{n+4}\left(\frac{1}{x}\right) F_p\left(\frac{K}{x};\lambda\right) dx$$

because of (2.1) and the independence of y^2 and s^2. Expression (2.8) is an equality for $\lambda = 0$, and for $\lambda > 0$ we will show that we have strict inequality.

For fixed γ and λ, define the function $g_{\gamma,\lambda}(w)$ as the solution to

$$(2.9)\qquad \gamma = \int_w^{g_{\gamma,\lambda}(w)} f_{n+4}\left(\frac{1}{x}\right) F_p\left(\frac{K}{x};\lambda\right) dx,$$

and define γ_1 and γ_2 to satisfy $g_{\gamma_1,0}(1/b_n) = g_{\gamma_2,\lambda}(1/b_n) = 1/a_n$. Let $G(w) = g_{\gamma_1,0}(w) - g_{\gamma_2,\lambda}(w)$, and note that $G(\phi_1) = \phi_2 - g_{\gamma_2,\lambda}(\phi_1)$ and $G(1/b_n) = 0$. We can establish (2.8) by showing that G satisfies the assumptions of Lemma A.1, which implies $\phi_2 > g_{\gamma_2,\lambda}(\phi_1)$.

Let x_0 be a point such that $G(x_0) = 0$ and let $y_0 = g_{\gamma_1,0}(x_0) = g_{\gamma_2,\lambda}(x_0)$. Since γ_1 and γ_2 are fixed, differentiating and simplifying shows

$$(2.10)\qquad \left.\frac{dG(w)}{dw}\right|_{w=x_0} = \frac{f_{n+4}(1/x_0)}{f_{n+4}(1/y_0)}\left\{\frac{F_p(K/x_0)}{F_p(K/y_0)} - \frac{F_p(K/x_0;\lambda)}{F_p(K/y_0;\lambda)}\right\}.$$

Since the chi-squared distribution has monotone likelihood ratio in the noncentral parameter and $x_0 < y_0$, the term in braces in (2.10) is less than zero. From Lemma A.2, $\phi_1 < 1/b_n$, therefore $G(\phi_1)$ is positive and (2.8) is established. □

THEOREM 2.2. *The length of the interval $I_1(s^2, t, K)$ is, with positive probability, smaller than the length of the usual minimum length interval $C_U(s^2)$.*

PROOF. As before the intervals are the same if $t > K$. When $t \le K$, the length of the confidence interval, $(\phi_2 - \phi_1)s^2$, is equal to $(g_{\gamma_1,0}(w) - w)s^2$ by (2.5) and the definition of γ_1. Let ϕ_1^0 and ϕ_2^0 denote the numbers that satisfy (2.5) and (2.6) for $\tau(K) = K$ and

$$\frac{d[g_{\gamma_1,0}(w) - w]}{dw}\bigg|_{w=\phi_1^0} = \frac{f_{n+4}(1/\phi_1^0)F_p(K/\phi_1^0)}{f_{n+4}(1/\phi_2^0)F_p(K/\phi_2^0)} - 1 = 0. \tag{2.11}$$

Unimodality of $f_{n+4}(1/x)F_p(K/x)$ (Lemma A.4) implies that the length, as a function of w, has a unique minimum at ϕ_1^0. In order to prove that $\phi_2 - \phi_1$ is smaller than $(1/a_n) - (1/b_n)$ it would suffice to show that $\phi_1^0 < \phi_1 < 1/b_n$. From Lemma A.2, $\phi_1 < 1/b_n$, so the result follows if the derivative of length, evaluated at ϕ_1, is positive. Using the expression of the derivative and (2.6), it suffices to have

$$\frac{F_p\{K/\phi_1\}}{F_p\{\tau(K)/\phi_1\}} > \frac{F_p\{K/\phi_2\}}{F_p\{\tau(K)/\phi_2\}}, \tag{2.12}$$

which is true by applying Lemma A.5 with $x_1 = \tau(K)/\phi_1$, $x_2 = \tau(K)/\phi_2$ and $\beta = K/\tau(K)$. Note that β is smaller than 1 because we have assumed $\tau(K) > K$ and Lemma A.5 exploits the fact that gamma densities have monotone likelihood ratio in the scale parameter. □

The coverage probability of the procedure $I_1(s^2, t, K)$ depends on the unknown parameter λ and the length is a random variable depending on s^2 and t. The procedure can be further improved by taking a finer partition of the values of t, the technique implemented in the construction of a point estimator for the variance by Brewster and Zidek (1974) and in the confidence interval constructed by Shorrock (1990).

Given $\mathbf{K}_2 = (K_1, K_2)$, $K_2 < K_1$, define the confidence procedure

$$I_2(s^2, t, \mathbf{K}_2) = \begin{cases} \left(\dfrac{1}{b_n}s^2, \dfrac{1}{a_n}s^2\right), & \text{if } t > K_1, \\ (\phi_1(K_1)s^2, \phi_2(K_1)s^2), & \text{if } K_2 < t \le K_1, \\ (\phi_1(K_2)s^2, \phi_2(K_2)s^2), & \text{if } t \le K_2, \end{cases} \tag{2.13}$$

where, for $i = 1, 2$, ϕ_1 and ϕ_2 satisfy the equations

$$\int_{\phi_1(K_1)}^{\phi_2(K_1)} f_{n+4}\left(\frac{1}{x}\right)F_p\left(\frac{K_1}{x}\right)dx = \int_{1/b_n}^{1/a_n} f_{n+4}\left(\frac{1}{x}\right)F_p\left(\frac{K_1}{x}\right)dx, \tag{2.14}$$

$$\int_{\phi_1(K_2)}^{\phi_2(K_2)} f_{n+4}\left(\frac{1}{x}\right)F_p\left(\frac{K_2}{x}\right)dx = \int_{\phi_1(K_1)}^{\phi_2(K_1)} f_{n+4}\left(\frac{1}{x}\right)F_p\left(\frac{K_2}{x}\right)dx, \tag{2.15}$$

$$f_{n+4}\left\{\frac{1}{\phi_1(K_i)}\right\}F_p\left\{\frac{\tau(K_i)}{\phi_1(K_i)}\right\} = f_{n+4}\left\{\frac{1}{\phi_2(K_i)}\right\}F_p\left\{\frac{\tau(K_i)}{\phi_2(K_i)}\right\}. \tag{2.16}$$

The proofs of the following theorems are omitted, since they use arguments similar to those in the proofs of Theorems 2.1 and 2.2, together with the results of Lemma A.3 and Lemma A.5.

THEOREM 2.3. *The coverage probability of* $I_2(s^2, t, \mathbf{K}_2)$ *is greater than the coverage probability of* $I_1(s^2, t, K_1)$.

THEOREM 2.4. *If* $t \le K_2$, *the length of* $I_2(s^2, t, \mathbf{K}_2)$ *is smaller than the length of* $I_1(s^2, t, K_1)$.

We can easily generalize and improve on $I_2(s^2, t, \mathbf{K}_2)$ by taking three cutoff points and improve any interval based on a finite number of cutoff points by adding an extra cutoff point. Working as in Brewster and Zidek (1974) and Shorrock (1990) we can create a triangular array $\{\mathbf{K}_m\}$ that will fill up the interval $(0, +\infty)$ and take the confidence interval that is the limit of the confidence intervals based on $\mathbf{K}_m$, as m tends to $+\infty$. It is plausible that the limiting interval will be better in terms of length than the usual minimum length interval. However, the form of the limiting interval is not obvious, since as we can see in equations (2.14)–(2.16), the numbers $\phi_1(K_2)$ and $\phi_2(K_2)$ depend not only on K_2 and the function τ but also on K_1. Hence for a given t, the endpoints of the interval depend on all the cutoff points K_i that are greater than or equal to t.

We create a triangular array $\{\mathbf{K}_m\}$ array as follows: For each m, define $\mathbf{K}_m = (K_{m,1}, \dots, K_{m,m-1}, K_{m,m})$, where $0 < K_{m,1} < \cdots < K_{m,m-1} < K_{m,m} < +\infty$. Furthermore we require $\lim_{m\to\infty} K_{m,1} = 0$ and $\lim_{m\to\infty} K_{m,m} = +\infty$ and $\lim_{m\to\infty} \max_i (K_{m,i} - K_{m,i-1}) = 0$.

As $m \to \infty$, the endpoints of the intervals based on $\mathbf{K}_m$ tend to some functions $\phi_1(t)$ and $\phi_2(t)$. In order to determine $\phi_1(t)$ and $\phi_2(t)$, we define

$$K_{m,i(t)} = \inf\{K \in \mathbf{K}_m : K \ge t\}. \tag{2.17}$$

Then for given t and s, the confidence interval at the mth stage is $(\phi_1(K_{m,i(t)})s^2, \phi_2(K_{m,i(t)})s^2)$, where $\phi_1(K_{m,i(t)})$ and $\phi_2(K_{m,i(t)})$ satisfy

$$\begin{aligned} &\int_{\phi_1(K_{m,i(t)})}^{\phi_2(K_{m,i(t)})} f_{n+4}\left(\frac{1}{x}\right) F_p\left(\frac{K_{m,i(t)}}{x}\right) dx \\ &\quad = \int_{\phi_1(K_{m,i(t)+1})}^{\phi_2(K_{m,i(t)+1})} f_{n+4}\left(\frac{1}{x}\right) F_p\left(\frac{K_{m,i(t)}}{x}\right) dx \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} &f_{n+4}\left\{\frac{1}{\phi_1(K_{m,i(t)})}\right\} F_p\left\{\frac{\tau(K_{m,i(t)})}{\phi_1(K_{m,i(t)})}\right\} \\ &\quad = f_{n+4}\left\{\frac{1}{\phi_2(K_{m,i(t)})}\right\} F_p\left\{\frac{\tau(K_{m,i(t)})}{\phi_2(K_{m,i(t)})}\right\}. \end{aligned} \tag{2.19}$$

Now use Taylor's theorem and replace $F_p(K_{m,i(t)}/x)$ in the RHS of (2.18) by its series expansion around $K_{m,i(t)+1}$, keeping the first two terms. We obtain

$$\int_{\phi_1(K_{m,i(t)})}^{\phi_2(K_{m,i(t)})} f_{n+4}\left(\frac{1}{x}\right) F_p\left(\frac{K_{m,i(t)}}{x}\right) dx$$

$$(2.20)\quad = \int_{\phi_1(R_{m,i(t)+1})}^{\phi_2(K_{m,i(t)+1})} f_{n+4}\left(\frac{1}{x}\right)\left\{F_p\left(\frac{K_{m,i(t)+1}}{x}\right) + (K_{m,i(t)} - K_{m,i(t)+1})\frac{1}{x} f_p\left(\frac{K_{m,i(t)+1}}{x}\right) + \frac{(K_{m,i(t)} - K_{m,i(t)+1})^2}{2}\left(\frac{1}{x}\right)^2 f_p'\left(\frac{K_r}{x}\right)\right\} dx,$$

where K_r is some number in the interval $(K_{m,i(t)}, K_{m,i(t)+1})$. Bring the first term of the RHS of (2.20) to the left of the equality, divide both sides by $K_{m,i(t)} - K_{m,i(t)+1}$ and take the limit as m goes to $+\infty$. By the construction of the array the difference $K_{m,i(t)} - K_{m,i(t)+1}$ tends to zero and $K_{m,i(t)}$ tends to t. Hence equation (2.20) becomes

$$(2.21)\quad \frac{d}{dt}\left[\int_{\phi_1(t)}^{\phi_2(t)} f_{n+4}\left(\frac{1}{x}\right) F_p\left(\frac{t}{x}\right) dx\right] = \int_{\phi_1(t)}^{\phi_2(t)} f_{n+4}\left(\frac{1}{x}\right)\frac{1}{x} f_p\left(\frac{t}{x}\right) dx.$$

The limit of the RHS is justified because the remainder term disappears. Since the derivative f_p' is bounded on finite intervals, the integral

$$(2.22)\quad \int_{\phi_1(K_{m,i(t)+1})}^{\phi_2(K_{m,i(t)+1})} f_{n+4}\left(\frac{1}{x}\right)\left(\frac{1}{x}\right)^2 f_p'\left(\frac{K_r}{x}\right) dx$$

is also bounded. Hence $\lim_{m\to\infty}(K_{m,i(t)} - K_{m,i(t)+1}) = 0$ implies that the remainder tends to zero. Using Leibniz' formula for the differentiation of the integral, equation (2.21) can be written

$$(2.23)\quad \frac{d\phi_1(t)}{dt} f_{n+4}\left\{\frac{1}{\phi_1(t)}\right\} F_p\left\{\frac{t}{\phi_1(t)}\right\} = \frac{d\phi_2(t)}{dt} f_{n+4}\left\{\frac{1}{\phi_2(t)}\right\} F_p\left\{\frac{t}{\phi_2(t)}\right\}.$$

On the other hand, since we have assumed that the function τ is continuous, equation (2.19) becomes

$$(2.24)\quad f_{n+4}\left\{\frac{1}{\phi_1(t)}\right\} F_p\left\{\frac{\tau(t)}{\phi_1(t)}\right\} = f_{n+4}\left\{\frac{1}{\phi_2(t)}\right\} F_p\left\{\frac{\tau(t)}{\phi_2(t)}\right\}.$$

In order to solve equations (2.23) and (2.24) for ϕ_1 and ϕ_2, we need initial conditions which are given by the equalities

$$(2.25)\quad \lim_{t\to\infty} \phi_1(t) = \frac{1}{b_n} \quad \text{and} \quad \lim_{t\to\infty} \phi_2(t) = \frac{1}{a_n}.$$

It is obvious that for different forms of the function τ we have different confidence intervals. We will denote the intervals by $C_\tau(s^2, t)$. By the Lebesgue dominated convergence theorem, the confidence coefficient of any interval constructed in this way is $1 - \alpha$. For the length of the limiting intervals, we have the following result.

THEOREM 2.5. *For every $t < \infty$, the length $(\phi_2(t) - \phi_1(t))s^2$ is smaller than $((1/a_n) - (1/b_n))s^2$.*

PROOF. Rearranging equations (2.23) and (2.24) yields

$$\frac{d\phi_2(t)}{dt} \frac{F_p\{t/\phi_2(t)\}}{F_p\{\tau(t)/\phi_2(t)\}} = \frac{d\phi_1(t)}{dt} \frac{F_p\{t/\phi_1(t)\}}{F_p\{\tau(t)/\phi_1(t)\}}. \tag{2.26}$$

Using Lemma A.5 with $x_1 = \tau(t)/\phi_1(t)$, $x_2 = \tau(t)/\phi_2(t)$, $\beta = t/\tau(t)$, we have that $d[\phi_2(t) - \phi_1(t)]/dt > 0$, that is, the length is an increasing function of t. But we know that

$$\lim_{t\to\infty} [\phi_2(t) - \phi_1(t)] = \frac{1}{a_n} - \frac{1}{b_n} \tag{2.27}$$

so for any $t < +\infty$, the length is strictly smaller than $((1/a_n) - (1/b_n))s^2$. □

It is interesting to see how the endpoints of $C_\tau(s^2, t)$ degenerate in some special forms of the function $\tau(K)$. If $\tau(K) = \infty$, then equation (2.7) becomes

$$f_{n+4}\left\{\frac{1}{\phi_1(K)}\right\} = f_{n+4}\left\{\frac{1}{\phi_2(K)}\right\} \tag{2.28}$$

and, together with equations (2.5), implies that $\phi_1(K)$ and $\phi_2(K)$ coincide with $1/b_n$ and $1/a_n$, respectively. Therefore the interval based on one cutoff point is identical to $C_U(s^2)$. By taking more cutoff points we do not shift the endpoints, therefore the limiting interval coincides with the usual minimum length confidence interval based on s^2 alone.

On the other hand if we take $\tau(K) = K$ the endpoints after the first step are $\phi_1^0(K)$ and $\phi_2^0(K)$. It is tempting to choose such a function τ, since, if we do so, we maximize the gain in terms of length at the first step. However, by filling $(0, +\infty)$ with cutoff points, the defining limiting equations (2.23) and (2.24) imply

$$\frac{d\phi_2(t)}{dt} = \frac{d\phi_1(t)}{dt}, \tag{2.29}$$

hence the interval has constant length. Because of the initial conditions, we can conclude that

$$\phi_2(t) - \phi_1(t) = \frac{1}{a_n} - \frac{1}{b_n} = c_0. \tag{2.30}$$

Substituting $\phi_1(t) + c_0$ for $\phi_2(t)$, equation (2.24) becomes

$$f_{n+4}\left\{\frac{1}{\phi_1(t)}\right\}F_p\left\{\frac{t}{\phi_1(t)}\right\} = f_{n+4}\left\{\frac{1}{\phi_1(t)+c_0}\right\}F_p\left\{\frac{t}{\phi_1(t)+c_0}\right\}, \tag{2.31}$$

which, surprisingly enough, is the defining equation for Shorrock's interval. It is interesting to note that every interval, based on a finite number on cutoff points has length less than the usual or Shorrock's interval but the limiting length is equal to $c_0 s^2$.

The partitioning construction can be implemented to construct improved confidence intervals using the ratio of endpoints as a measure of volume. The minimum ratio intervals based on s^2 alone, tabulated by Tate and Klett (1959), have endpoints satisfying

$$\int_{a_n}^{b_n} f_n(x)\,dx = 1-\alpha \quad \text{and} \quad a_n f_n(a_n) = b_n f_n(b_n). \tag{2.32}$$

Building upon them in same way as for minimum length intervals, we arrive at a slightly different set of equations:

$$\frac{d\phi_1(t)}{dt} f_{n+4}\left\{\frac{1}{\phi_1(t)}\right\}F_p\left\{\frac{t}{\phi_1(t)}\right\} = \frac{d\phi_2(t)}{dt} f_{n+4}\left\{\frac{1}{\phi_2(t)}\right\}F_p\left\{\frac{t}{\phi_2(t)}\right\}, \tag{2.33}$$

$$f_{n+2}\left\{\frac{1}{\phi_1(t)}\right\}F_p\left\{\frac{\tau(t)}{\phi_1(t)}\right\} = f_{n+2}\left\{\frac{1}{\phi_2(t)}\right\}F_p\left\{\frac{\tau(t)}{\phi_2(t)}\right\}, \tag{2.34}$$

with initial conditions

$$\lim_{t\to\infty}\phi_1(t) = \frac{1}{b_n} \quad \text{and} \quad \lim_{t\to\infty}\phi_2(t) = \frac{1}{a_n}. \tag{2.35}$$

The ratio of the endpoints can be shown to be smaller than the ratio of Tate and Klett intervals while the converge probability is maintained above $1-\alpha$.

It is interesting to notice that the construction does not heavily depend on properties of the chi-squared densities. The most important property that was used is the monotone likelihood ratio of the noncentral chi-squared in the noncentrality parameter. Hence, extensions to the general case of estimating a scale parameter when the location parameter is unknown are straightforward. For a treatment of the general problem, see Goutis and Casella (1990).

We now investigate numerically the gains in coverage probability and expected length of the confidence intervals with endpoints given by equations (2.23) and (2.24). Previous relative risk calculations for the point estimator [Rukhin (1987)] and the numerical results of Shorrock (1990) suggest that the improvement is minimal for small values of p. However, the gains are substantial for small and moderate n and for p large relative to n.

The endpoints of the intervals depend on a rather arbitrary function $\tau(t)$ and the numerical results show that dependence of both coverage probability and expected length on $\tau(t)$ is rather strong. For different functional forms we have little feeling about what to expect, so we chose a wide variety of $\tau(t)$

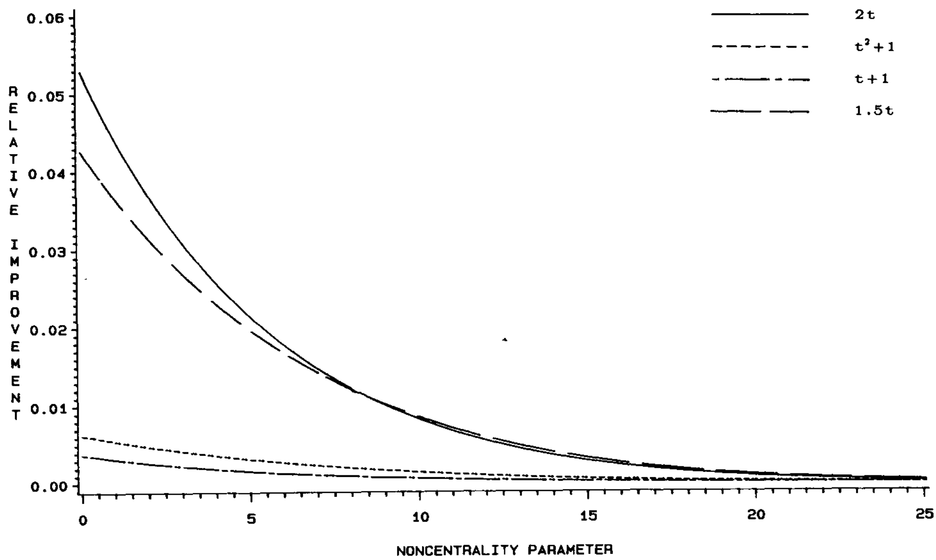

FIG. 1. *Expected relative improvement in length for* $n = 25$, $p = 10$ *and* $1 - \alpha = 0.95$.

forms. The functional forms of $\tau(t)$ which seem to be optimal are moderately or slowly increasing. Rapidly increasing $\tau(t)$ have an effect only when n is small. For large n only moderately increasing $\tau(t)$ can change the coverage probability and the length substantially.

In Figure 1 we see that the largest relative gain in length that we obtained was about 5.3%. Figure 2 shows that the largest difference between coverage probability and confidence coefficient was about 0.0033. The wide selection of $\tau(t)$ makes it difficult to find an optimal functional form and suggests that there may be other forms that perform better. However, among the intervals we computed, none dominates the others in both coverage probability and length.

The coverage probability and expected length were calculated by numerical integration. The computations were performed on the Purdue University Computing Center's IBM 3090-180E computer using FORTRAN programming language and IMSL subroutines. The graphs were produced on Cornell University's IBM 3090-200 computer using SAS/GRAPH.

3. Posterior probabilities. The Brewster–Zidek and Shorrock estimators are generalized Bayes rules, as are the intervals constructed here. The posterior probabilities of these intervals are a natural candidate for a post-data assessment of the confidence in these interval estimators. A question that arises is how these probabilities compare with the nominal value $1 - \alpha$, the predata coverage probability. We examine these comparisons and find that our improved interval can have a post-data probability higher than $1 - \alpha$, allowing a uniformly higher post-data confidence report.

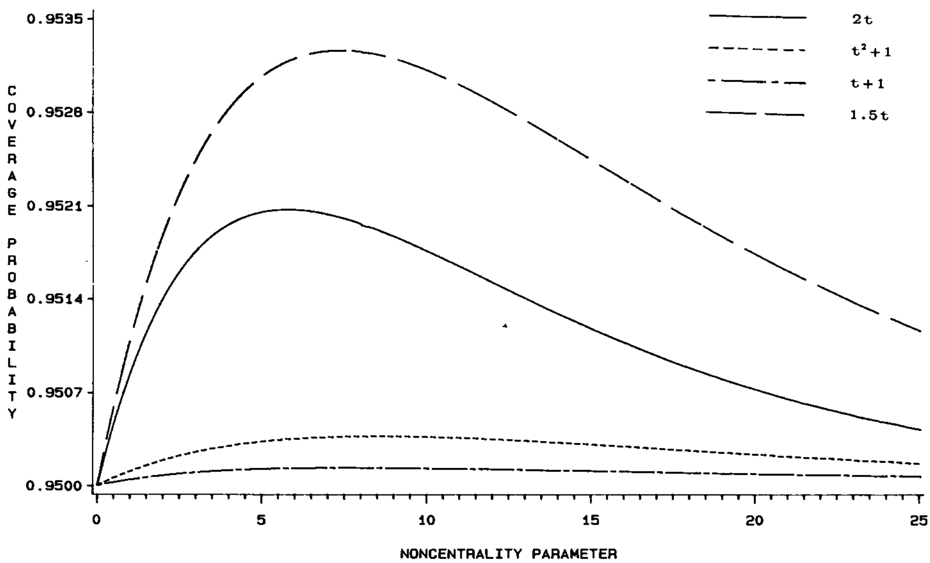

FIG. 2. *Coverage probability for $n = 25$, $p = 10$ and $1 - \alpha = 0.95$.*

Although the interval $C_\tau(s^2, t)$ is itself generalized Bayes, its prior is extremely difficult to deal with. Because of this we examine the posterior probabilities of $C_\tau(s^2, t)$ with respect to the Brewster–Zidek prior and with respect to the prior

$$\pi\left(\sigma^2, \mu_1, \mu_2, \dots, \mu_p\right) = \frac{1}{\sigma^2}, \tag{3.1}$$

whose highest prosterior density region is the usual minimum length interval.

The Brewster and Zidek point estimator and the interval constructed by Shorrock were shown to be Bayes rules with respect to the prior

$$\pi\left(\sigma^2, \mu_1, \mu_2, \dots, \mu_p\right) = \left(\frac{1}{\sigma^2}\right)^{p/2} \int_0^\infty \exp\left(-\frac{u \sum \mu_i^2}{2\sigma^2}\right) \frac{u^{(p/2)-1}}{u+1}\, du. \tag{3.2}$$

There are several versions of this prior depending on the setup and the parameterization of the problem, but we can represent the posterior density as

$$\pi(\sigma^2 | s^2, t) = \frac{f_{n+4}(s^2/\sigma^2) F_p(s^2 t/\sigma^2)}{(s^2/n(n+2)) F_{p,n}((n/p)t)}, \tag{3.3}$$

where $F_{p,n}$ denotes the F cumulative distribution function with p and n degrees of freedom.

THEOREM 3.1. *For $p \leq 2$, if $\gamma_\tau^N(s^2, t)$ is the posterior probability of $C_\tau(s^2, t)$ with respect to the prior given by* (3.2), *then $\gamma_\tau^N(s^2, t) \geq 1 - \alpha$.*

PROOF. The proof again uses the construction of $C_\tau(s^2, t)$ as a limit of intervals based on a finite number of cutoff points. First observe that the posterior probability of $I_1(s^2, t, K)$ of (2.4) with respect to the prior (3.2) is greater than the posterior probability of the interval $C_U(s^2)$ for every t, with strict inequality if $t < K$. This follows from reasoning as in Theorem 2.1 and Lemma A.5. Then we can show that the interval $I_2(s^2, t, \mathbf{K}_2)$ has posterior probability greater than that of $C_U(s^2)$. Now by applying the Lebesgue dominated convergence theorem we see that $\gamma_\tau^N(s^2, t)$ is greater than or equal to the posterior probability of $C_U(s^2)$. The latter will be denoted by $\gamma_U(s^2, t)$ and by change of variables it is equal to

$$\gamma_U(s^2, t) = \int_{a_n}^{b_n} \frac{f_n(x) F_p(tx)}{F_{p,n}((n/p)t)}\, dx. \tag{3.4}$$

We show next that $\gamma_U(s^2, t) \geq 1 - \alpha$ for $p \leq 2$. The proof is based on the observation that if a function has negative second derivative when the first derivative is zero, then the only possible interior extremum is a maximum. Checking the values of the function in the boundary points will give us the lower bound of the function.

Differentiating with respect to t, the first derivative of $\gamma_U(s^2, t)$ is

$$\begin{aligned} \frac{d}{dt}\left[\gamma_U(s^2, t)\right] &= \frac{(n/p) f_{p,n}((n/p)t)}{F_{p,n}((n/p)t)} \\ &\quad\times \left\{ \int_{a_n}^{b_n} \frac{f_n(x) x f_p(tx)}{(n/p) f_{p,n}((n/p)t)} - \frac{f_n(x) F_p(tx)}{F_{p,n}((n/p)t)}\, dx \right\}, \end{aligned} \tag{3.5}$$

where $f_{p,n}$ denotes the F density with p and n degrees of freedom. Using the explicit formulae for the chi-squared densities and simplifying, we can see that

$$\frac{f_n(x) x f_p(tx)}{(n/p) f_{p,n}((n/p)t)} = f_{n+p}\{(t+1)x\}. \tag{3.6}$$

Therefore substituting in (3.5) and making the transformation $w = (t+1)x$, the derivative becomes

$$\frac{(n/p) f_{p,n}((n/p)t)}{F_{p,n}((n/p)t)} \left\{ \int_{a_n(t+1)}^{b_n(t+1)} f_{n+p}(w)\, dw - \int_{a_n}^{b_n} \frac{f_n(x) F_p(tx)}{F_{p,n}((n/p)t)}\, dx \right\}. \tag{3.7}$$

The last expression is zero only when the term in braces is zero. Differentiating once more and ignoring the zero terms, the second derivative is negative if and only if

$$\frac{a_n f_{n+p}\{(t+1)a_n\}}{b_n f_{n+p}\{(t+1)b_n\}} > 1. \tag{3.8}$$

Using the condition $f_{n+4}(a_n) = f_{n+4}(b_n)$, after substituting the formula for the chi-squared density and simplifying we have

$$\frac{a_n f_{n+p}\{(t+1)a_n\}}{b_n f_{n+p}\{(t+1)b_n\}} = \frac{a_n^{(p/2)-1}e^{-(ta_n/2)}}{b_n^{(p/2)-1}e^{-(tb_n/2)}}, \tag{3.9}$$

which, since $a_n < b_n$, is greater than 1 for every positive t and $p \le 2$.

Next we check the boundary points. Using (3.4) we have

$$\lim_{t\to\infty} \gamma_U(s^2, t) = \int_{a_n}^{b_n} f_n(x)\,dx = 1 - \alpha. \tag{3.10}$$

At the other endpoint, using l'Hôpital's rule and (3.6),

$$\lim_{t\to 0} \gamma_U(s^2, t) = \int_{a_n}^{b_n} f_{n+p}(x)\,dx, \tag{3.11}$$

which for $p = 1, 2$ can be shown to be greater than $1 - \alpha$. If P_k denotes the probability $P\{a_n < \chi_k^2 < b_n\}$, where χ_k^2 is a chi-squared random variable with k degrees of freedom, integration by parts yields

$$P_k = P_{k+2} + 2\{f_{k+2}(b_n) - f_{k+2}(a_n)\}. \tag{3.12}$$

Since $f_{n+4}(b_n) = f_{n+4}(a_n)$, (3.12) implies $P_{n+2} > P_n = 1 - \alpha$. For $p = 1$, we use the variation-reducing properties of the chi-squared density [see Brown, Johnstone and MacGibbon (1981) for definitions and details]. Suppose that P_{n+1} were less than or equal to $1 - \alpha$. Since $\lim_{k\to\infty} P_k = 0$, for every $C \in [P_{n+1}, P_n]$, the maximum number of sign changes of the sequence $P_k - C$, counting zeros as either + or −, would be at least three. But

$$P_k - C = E\{\mathbf{I}_{(a_n, b_n)}(\chi_k^2) - C\} \tag{3.13}$$

and, since chi-squared densities belong to the exponential family, we know [Example 3.1 of Brown, Johnstone and MacGibbon (1981)] that the number of sign changes of $P_k - C$ as a function of k cannot exceed the number of sign changes of $\mathbf{I}_{(a_n, b_n)}(x) - C$ as a function of x. Hence we must have $P_{n+1} > 1 - \alpha$, which implies $\gamma_\tau^N(s^2, t) \ge \gamma_U(s^2, t) > 1 - \alpha$, proving the theorem. □

The proof of the Theorem fails for $p > 2$ because we cannot show inequality (3.8) to be true and also

$$\lim_{p\to\infty} \int_{a_n}^{b_n} f_{n+p}(x)\,dx = 0, \tag{3.14}$$

which implies that, for sufficiently large p, the limit cannot be greater than $1 - \alpha$. Hence, we know that the posterior probability of the interval $C_U(s^2)$ is below $1 - \alpha$ for small t and large p. However, we believe that the inequality $\gamma_\tau^N(s^2, t) \ge 1 - \alpha$ is always true because for small t and large p, the endpoints of $C_\tau(s^2, t)$ are far from the endpoints of $C_U(s^2)$ and we expect their respective posterior probabilities to differ substantially.

Using the usual generalized prior, we obtain a very different result for $C_\tau(s^2, t)$.

THEOREM 3.2. *If $\gamma_\tau^P(s^2, t)$ is the posterior probability of $C_\tau(s^2, t)$ with respect to the prior given by* (3.1), *then $\gamma_\tau^P(s^2, t) \le 1 - \alpha$.*

PROOF. The posterior density with respect to (3.1) after integrating out μ is

$$\pi(\sigma^2 | s^2, t) = \frac{n(n+2)}{s^2} f_{n+4}\left(\frac{s^2}{\sigma^2}\right), \tag{3.15}$$

hence, after a change of variables,

$$\gamma_\tau^P(s^2, t) = \int_{1/\phi_2(t)}^{1/\phi_1(t)} f_n(x)\, dx. \tag{3.16}$$

Now if we differentiate with respect to t, applying Leibniz' rule we get

$$\begin{aligned} \frac{d\gamma_\tau^P(s^2, t)}{dt} = {} & \frac{d\phi_2(t)}{dt}\left(\frac{1}{\phi_2(t)}\right)^2 f_n\left(\frac{1}{\phi_2(t)}\right) \\ & - \frac{d\phi_1(t)}{dt}\left(\frac{1}{\phi_1(t)}\right)^2 f_n\left(\frac{1}{\phi_1(t)}\right). \end{aligned} \tag{3.17}$$

Using equation (2.23) we can see that, since $F_p(t/\phi_2(t)) < F_p(t/\phi_1(t))$, the derivative is positive which implies that the function $\gamma_\tau^P(s^2, t)$ is increasing in t. But $\lim_{t\to\infty} \phi_1(t) = 1/b_n$ and $\lim_{t\to\infty} \phi_2(t) = 1/a_n$, so

$$\max_t \gamma_\tau^P(s^2, t) = \lim_{t\to\infty} \gamma_\tau^P(s^2, t) = \int_{a_n}^{b_n} f_n(x)\, dx = 1 - \alpha, \tag{3.18}$$

which completes the proof. □

A consequence of the above bounds on the posterior probabilities is that there are no *positively biased* or *negatively biased relevant betting strategies* [in the sense of Robinson (1979)] if we quote confidence $1 - \alpha$. Hence the confidence interval $C_\tau(s^2, t)$, with associated confidence $1 - \alpha$, has reasonable conditional properties. However, if we attach confidence $\gamma_\tau^N(s^2, t)$ to $C_\tau(s^2, t)$, we obtain a procedure with even stronger conditional properties. This latter procedure provides a frequentist inference that is more data-sensitive than the statement of $1 - \alpha$ confidence and exhibits good behavior against conditional criteria. For definitions and a formal introduction of the conditional properties using a betting based theory; see Robinson (1979) and for details about the conditional properties of $C_\tau(s^2, t)$, see Goutis, Casella and Maatta (1989).

APPENDIX

LEMMA A.1. *If a differentiable function $f(x)$ defined on the real line has $f'(x) < 0$ whenever $f(x) = 0$ and there is an x_0 such that $f(x_0) = 0$, then $f(x)$ is positive for $x < x_0$ and negative for $x > x_0$.*

LEMMA A.2. *If* $\phi_1(K)$ *is defined by* (2.5) *and* (2.6), *then* $\phi_1(K) < 1/b_n$ *for any choice of* τ *and* K.

PROOF. From equation (2.5), we see that neither of the sets (ϕ_1, ϕ_2) and $(1/b_n, 1/a_n)$ can be a proper subset of the other. Therefore we may have the following cases:

(i) $\phi_1 < 1/b_n < \phi_2 < 1/a_n$,
(ii) $\phi_1 < \phi_2 \le 1/b_n < 1/a_n$,
(iii) $1/b_n < \phi_1 < 1/a_n < \phi_2$,
(iv) $1/b_n < 1/a_n \le \phi_1 < \phi_2$,
(v) $1/b_n = \phi_1 < 1/a_n = \phi_2$.

We will show that cases (iii), (iv) and (v) are vacuous. Using the unimodality of $f_{n+4}(1/x)F_p(\tau(K)/x)$, from Lemma A.4 and equation (2.6), we have for every $x \in (\phi_1, \phi_2)$,

$$f_{n+4}\left(\frac{1}{x}\right)F_p\left(\frac{\tau(K)}{x}\right) > f_{n+4}\left(\frac{1}{\phi_1}\right)F_p\left(\frac{\tau(K)}{\phi_1}\right) \tag{A.1}$$

and for every $x \in (0, \phi_1)$,

$$f_{n+4}\left(\frac{1}{x}\right)F_p\left(\frac{\tau(K)}{x}\right) < f_{n+4}\left(\frac{1}{\phi_1}\right)F_p\left(\frac{\tau(K)}{\phi_1}\right) \tag{A.2}$$

and $f_{n+4}(1/x)F_p(\tau(K)/x)$ is increasing in x.

If $1/b_n < \phi_1 < 1/a_n$, then equations (A.1) and (A.2) imply

$$f_{n+4}(b_n)F_p(\tau(K)b_n) < f_{n+4}(a_n)F_p(\tau(K)a_n), \tag{A.3}$$

which contradicts $f_{n+4}(b_n) = f_{n+4}(a_n)$ since $F_p(\tau(K)x)$ is increasing in x and $b_n > a_n$. Hence case (iii) is not possible. By similar arguments we can show cases (iv) and (v) to contradict $f_{n+4}(b_n) = f_{n+4}(a_n)$, hence we must have $\phi_1(K) < 1/b_n$. □

LEMMA A.3. *If* $\phi_1(K_1)$ *and* $\phi_1(K_2)$ *are defined by equations* (2.14)–(2.16), *then* $\phi_1(K_1) > \phi_1(K_2)$.

PROOF. The proof is similar to that of Lemma A.2. There are five possible cases and using similar arguments, cases (iii), (iv) and (v) imply that

$$f_{n+4}\left\{\frac{1}{\phi_2(K_1)}\right\}F_p\left\{\frac{\tau(K_2)}{\phi_2(K_1)}\right\} \le f_{n+4}\left\{\frac{1}{\phi_1(K_1)}\right\}F_p\left\{\frac{\tau(K_2)}{\phi_1(K_1)}\right\}, \tag{A.4}$$

analogous to equation (A.3). Now, by using equation (2.16) with $i = 1$, (A.4) is equivalent to

$$\frac{F_p\{\tau(K_2)/\phi_1(K_1)\}}{F_p\{\tau(K_1)/\phi_1(K_1)\}} \le \frac{F_p\{\tau(K_2)/\phi_2(K_1)\}}{F_p\{\tau(K_1)/\phi_2(K_1)\}}. \tag{A.5}$$

Applying Lemma A.5 with $x_1 = \tau(K_1)/\phi_1(K_1)$, $x_2 = \tau(K_1)/\phi_2(K_1)$ and $\beta = \tau(K_2)/\tau(K_1)$ (note that $\beta < 1$ since the function τ is increasing), we obtain the necessary contradiction. □

LEMMA A.4. *If f_k and F_k denote, respectively, the chi-squared probability density and cumulative distribution function with k degrees of freedom, then for any integers n and p and positive constant M, the function $f_{n+4}(1/x)F_p(M/x)$ is a unimodal function of x.*

PROOF. Straightforward calculation. □

LEMMA A.5. *Let F_p be a chi-squared distribution function with $p \geq 1$ degrees of freedom. If $\beta < 1$ and $x_1 > x_2$, then*

$$\frac{F_p(\beta x_1)}{F_p(x_1)} > \frac{F_p(\beta x_2)}{F_p(x_2)}. \tag{A.6}$$

PROOF. It follows from the fact that the gamma densities have monotone likelihood ratio in the scale parameter [see also Lemma 4.2 of Cohen (1972)]. □

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DEPARTMENT OF STATISTICAL SCIENCE
UNIVERSITY COLLEGE LONDON
GOWER STREET
LONDON WC 1 E 6 BT
ENGLAND

BIOMETRICS UNIT
337 WARREN HALL
CORNELL UNIVERSITY
ITHACA, NEW YORK 14853-7801