Improved Inverse Theorems in Weighted Lebesgue and Smirnov Spaces

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Abstract

The improvement of the inverse estimation of approximation theory by trigonometric polynomials in the weighted Lebesgue spaces was obtained and its application in the weighted Smirnov spaces was considered.

1 Introduction and the main results

Let $\mathbb{T}$ be the interval $[-\pi,\pi]$ or the unit circle $|z|=1$ of the complex plane $\mathbb{C}$. A measurable $2\pi-$periodic function $\omega:\mathbb{T}\to[0,\infty]$ is said to be a weight function if $\omega^{-1}(\{0,\infty\})$ has measure zero. With any given weight ω, we associate the $\omega-$*weighted Lebesgue space* $L_p(\mathbb{T},\omega)$, $1\le p<\infty$, consisting of all measurable $2\pi-$periodic functions f on $\mathbb{T}$ such that

$$\|f\|_{L_p(\mathbb{T},\omega)} := \left(\int_{\mathbb{T}} |f(x)|^p \,\omega(x)\,dx\right)^{1/p} < \infty.$$

Let $1<p<\infty$. A weight function ω belongs to the *Muckenhoupt class* $A_p(\mathbb{T})$ if

$$\left(\frac{1}{|J|}\int_J \omega(x)\,dx\right)\left(\frac{1}{|J|}\int_J [\omega(x)]^{-1/(p-1)}\,dx\right)^{p-1} \le C$$

with a finite constant C independent of J, where J is any subinterval of $[-\pi,\pi]$ and $|J|$ denotes the length of J.

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The detailed information about the classes $A_p(\mathbb{T})$can be found in [22] and [9].

Let $1 < p < \infty$ and $\omega \in A_p(\mathbb{T})$. Since $L_p(\mathbb{T}, \omega)$ is noninvariant with respect to the usual shift, for the definition of the modulus of smoothness we consider the following mean value function as a shift for $g \in L_p(\mathbb{T}, \omega)$:

$$\sigma_h(g)(x) := \frac{1}{2h} \int_{-h}^{h} g(x+t)\, dt, \quad 0 < h < \pi, \quad x \in \mathbb{T}.$$

It is known (see, [23]) that the operator σ_h is a bounded linear operator on $L_p(\mathbb{T}, \omega)$, $1 < p < \infty$, i. e.,

$$\|\sigma_h(g)\|_{L_p(\mathbb{T},\omega)} \leq c \|g\|_{L_p(\mathbb{T},\omega)}, \quad g \in L_p(\mathbb{T}, \omega),$$

holds with a constant $c > 0$ independent of g and h. The kth *modulus of smoothness* $\Omega_k(g, \cdot)_{p,\omega}$ of the function $g \in L_p(\mathbb{T}, \omega)$ is defined by

$$\Omega_k(g, \delta)_{p,\omega} = \sup_{0<h\leq\delta} \left\| T_h^k g \right\|_{L_p(\mathbb{T},\omega)}, \quad \delta > 0 \tag{1}$$

where

$$T_h g = T_h^1 g := g - \sigma_h(g), \quad T_h^k g := T_h\left(T_h^{k-1} g\right), \quad k = 1, 2, ... \, .$$

The modulus of smoothness $\Omega_k(g, \cdot)_{p,\omega}$ is nondecreasing, nonnegative, continuous function and

$$\lim_{\delta \to 0} \Omega_k(g, \delta)_{p,\omega} = 0, \quad \Omega_k(g_1 + g_2, \cdot)_{p,\omega} \leq \Omega_k(g_1, \cdot)_{p,\omega} + \Omega(g_2, \cdot)_{p,\omega}.$$

Let $\mathcal{T}_n$ $(n = 0, 1, 2, ...)$ be the set of trigonometric polynomials of order at most n. The best approximation to $g \in L_p(\mathbb{T}, \omega)$ in the class $\mathcal{T}_n$ is defined by

$$E_n(g)_{p,\omega} = \inf_{T_n \in \mathcal{T}_n} \|g - T_n\|_{L_p(\mathbb{T},\omega)}$$

for $n = 0, 1, 2, ...$.

The problems of the approximation theory by trigonometric polynomials in the space $L_p(\mathbb{T}, \omega)$, when the weight function satisfies the Muckenhoupt condition, were investigated by E. A. Haciyeva in [11]. Haciyeva obtained the direct and inverse estimates in terms of the modulus of smoothness (1). N. X. Ky, using a relevant modulus of smoothness, investigated the approximation problems in the weighted Lebesgue spaces with Muckenhoupt weights (see [19], [20]). For more general class of weights, namely for doubling weights, similar problems were investigated by G. Mastroianni and V. Totik in [21]. Also, M. C. De Bonis, G. Mastroianni and M. G. Russo gave results for some special weight functions in [6].

The following inverse theorem was proved in [11].

Theorem A. *Let $1 < p < \infty$ and $\omega \in A_p(\mathbb{T})$. Then, for $g \in L_p(\mathbb{T}, \omega)$ the inequality*

$$\Omega_k\left(g, \frac{1}{n}\right)_{p,\omega} \leq \frac{c}{n^{2k}} \left\{ E_0(g)_{p,\omega} + \sum_{\nu=1}^{n} \nu^{2k-1} E_\nu(g)_{p,\omega} \right\}, \quad n = 1, 2, ..., \tag{2}$$

holds with a constant $c > 0$ independent of n.

In this work we improve the estimate (2). We shall denote by c, the constants (in general, different in different relations) depending only on numbers that are not important for the questions of interest.

Our main result is the following.

Theorem 1. *Let $1 < p < \infty$ and $\omega \in A_p(\mathbb{T})$. Then, for $g \in L_p(\mathbb{T},\omega)$ the estimate*

$$\Omega_k\left(g,\frac{1}{n}\right)_{p,\omega} \le \frac{c}{n^{2k}}\left\{\sum_{\nu=1}^{n}\nu^{2\beta k-1}E_\nu^\beta(g)_{p,\omega}\right\}^{1/\beta}, \quad n=1,2,..., \tag{3}$$

holds with a constant $c>0$ independent of n, where $\beta=\min(p,2)$.

The estimate (3) is better than (2). Indeed, let

$$\begin{aligned} x \ &: = \frac{1}{2}\left[\sum_{\mu=1}^{\nu}\mu^{2k-1}E_\mu(g)_{p,\omega}+(\nu-1)\,\nu^{2k-1}E_\nu(g)_{p,\omega}\right] \\ &= \frac{1}{2}\left[\sum_{\mu=1}^{\nu-1}\mu^{2k-1}E_\mu(g)_{p,\omega}+\nu\nu^{2k-1}E_\nu(g)_{p,\omega}\right] \end{aligned}$$

and

$$\begin{aligned} x-h \ &: = (\nu-1)\,\nu^{2k-1}E_\nu(g)_{p,\omega}, \quad x+h := \sum_{\mu=1}^{\nu}\mu^{2k-1}E_\mu(g)_{p,\omega} \\ x-\delta \ &: = \nu\nu^{2k-1}E_\nu(g)_{p,\omega}, \quad x+\delta := \sum_{\mu=1}^{\nu-1}\mu^{2k-1}E_\mu(g)_{p,\omega} \end{aligned}$$

for $\nu=1,2,...$. Since the function x^β is convex for $\beta=\min(p,2)$, we obtain

$$\begin{aligned} &\left[\nu\nu^{2k-1}E_\nu(g)_{p,\omega}\right]^\beta-\left[(\nu-1)\,\nu^{2k-1}E_\nu(g)_{p,\omega}\right]^\beta \\ \le &\left[\sum_{\mu=1}^{\nu}\mu^{2k-1}E_\mu(g)_{p,\omega}\right]^\beta-\left[\sum_{\mu=1}^{\nu-1}\mu^{2k-1}E_\mu(g)_{p,\omega}\right]^\beta. \end{aligned}$$

After summation with respect to ν we have

$$\begin{aligned} &\sum_{\nu=1}^{n}\left\{\left[\nu\nu^{2k-1}E_\nu(g)_{p,\omega}\right]^\beta-\left[(\nu-1)\,\nu^{2k-1}E_\nu(g)_{p,\omega}\right]^\beta\right\} \\ \le &\sum_{\nu=1}^{n}\left\{\left[\sum_{\mu=1}^{\nu}\mu^{2k-1}E_\mu(g)_{p,\omega}\right]^\beta-\left[\sum_{\mu=1}^{\nu-1}\mu^{2k-1}E_\mu(g)_{p,\omega}\right]^\beta\right\}, \end{aligned}$$

and after simple computations we obtain

$$\left\{\sum_{\nu=1}^{n}\nu^{2\beta k-1}E_\nu^\beta(g)_{p,\omega}\right\}^{1/\beta} \le 2\sum_{\nu=1}^{n}\nu^{2k-1}E_\nu(g)_{p,\omega}.$$

Consequently the estimate (3) is never worse than (2). In addition, in some cases, it leads to a more precise result. For example, if

$$E_n(g)_{p,\omega} = \mathcal{O}\left(\frac{1}{n^{2k}}\right), \quad n = 1, 2, ...,$$

then we obtain from (2)

$$\Omega_k(g,\delta)_{p,\omega} = \mathcal{O}\left(\delta^{2k}\left(\log\frac{1}{\delta}\right)\right) \tag{4}$$

and from (3)

$$\Omega_k(g,\delta)_{p,\omega} = \mathcal{O}\left(\delta^{2k}\left(\log\frac{1}{\delta}\right)^{1/\beta}\right),$$

which is better than (4).

The analogue of Theorem 1 in nonweighted Lebesgue spaces, in terms of the usual modulus of smoothness, was proved by M. F. Timan in [26] (see also [25, p. 338]).

We also give an improvement of the appropriate inverse theorem in the weighted Smirnov spaces, obtained in [16]. For its formulation we have to give some auxiliary definitions and notations.

Let G be a finite domain in the complex plane, bounded by a rectifiable Jordan curve Γ, and let $G^- := Ext\Gamma$ be the exterior of Γ. Further let

$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}, \quad \mathbb{D}^- := \mathbb{C} \backslash \overline{\mathbb{D}}.$$

We denote by φ the conformal mapping of G^- onto $\mathbb{D}^-$ normalized by

$$\varphi(\infty) = \infty, \quad \lim_{z\to\infty} \frac{\varphi(z)}{z} > 0.$$

Let ψ be the inverse of φ. The functions φ and ψ have continuous extensions to Γ and $\mathbb{T}$, their derivatives $\varphi'(z)$ and $\psi'(w)$ have definite nontangential limit values on Γ and $\mathbb{T}$ a. e., and they are integrable with respect to Lebesgue measure on Γ and $\mathbb{T}$, respectively [10, pp. 419-438].

We denote by $E_p(G), 1 \le p < \infty$, the Smirnov class of analytic functions in G. Each function $f \in E_p(G)$ has a nontangential limit almost everywhere (a. e.) on Γ, and the nontangential limit of f, belongs to the Lebesgue space $L_p(\Gamma)$. The general information about $E_p(G)$ can be found in [8, pp. 168-185] and [10, pp. 438-453].

Let ω be a weight function on Γ and let $L_p(\Gamma,\omega)$ be the $\omega-$weighted Lebesgue space on Γ. The *$\omega-$weighted Smirnov space* $E_p(G,\omega)$ defined as

$$E_p(G,\omega) := \{f \in E_1(G) : f \in L_p(\Gamma,\omega)\}.$$

The approximation problems in $E_p(G,\omega)$ and $L_p(\Gamma,\omega), 1 < p < \infty$, was studied in [14] , [15] and [16]. The nonweighted case was considered in [1], [17], [2], [13] and [4].

Definition 1. A rectifiable Jordan curve Γ is called a *Carleson curve* if the condition

$$\sup_{z\in\Gamma} \sup_{\varepsilon>0} \frac{1}{\varepsilon} \, |\Gamma(z,\varepsilon)| < \infty$$

holds, where $\Gamma(z,\varepsilon)$ is the portion of Γ in the open disk of radius ε centered at z, and $|\Gamma(z,\varepsilon)|$ its length.

The Muckenhoupt class on the rectifiable Jordan curve Γ seems as follows:

Definition 2. Let $1 < p < \infty$. A weight function ω belongs to the *Muckenhoupt class* $A_p(\Gamma)$ if the condition

$$\sup_{z\in\Gamma}\sup_{\varepsilon>0}\left(\frac{1}{\varepsilon}\int\limits_{\Gamma(z,\varepsilon)}\omega(\tau)\,|d\tau|\right)\left(\frac{1}{\varepsilon}\int\limits_{\Gamma(z,\varepsilon)}[\omega(\tau)]^{-1/(p-1)}\,|d\tau|\right)^{p-1}<\infty$$

holds.

The Carleson curves and Muckenhoupt classes $A_p(\Gamma)$ were studied in details in [3].

Let Γ be a rectifiable Jordan curve and $f \in L_1(\Gamma)$. Then the function f^+ defined by

$$f^+(z) := \frac{1}{2\pi i}\int\limits_{\Gamma}\frac{f(\varsigma)}{\varsigma - z}d\varsigma, \quad z \in G \tag{5}$$

is analytic in G. Furthermore, if Γ is a Carleson curve and $\omega \in A_p(\Gamma)$, then $f^+ \in E_p(G,\omega)$ for $f \in L_p(\Gamma,\omega)$, $1 < p < \infty$ (see [14, Lemma 3]).

With every weight function ω on the rectifiable Jordan curve Γ, we associate another weight ω_0 on $\mathbb{T}$ defined by $\omega_0 := \omega \circ \psi$.

Let $\omega \in A_p(\Gamma)$ and $\omega_0 \in A_p(\mathbb{T})$, where $1 < p < \infty$. If $f \in L_p(\Gamma,\omega)$, then

$$f_0 := (f\circ\psi)\left(\psi'\right)^{1/p} \in L_p(\mathbb{T},\omega_0).$$

We define the kth *modulus of smoothness* of the function $f \in L_p(\Gamma,\omega)$ by

$$\Omega_k(f,\delta)_{\Gamma,p,\omega} := \Omega_k\left(f_0^+,\delta\right)_{p,\omega_0}, \quad \delta > 0. \tag{6}$$

The following theorem was proved in [16].

Theorem B. *Let Γ be a Carleson curve, $1 < p < \infty$, $\omega \in A_p(\Gamma)$ and $\omega_0 \in A_p(\mathbb{T})$. Then, for $f \in E_p(G,\omega)$ the estimate*

$$\Omega_k\left(f,\frac{1}{n}\right)_{\Gamma,p,\omega} \le \frac{c}{n^{2k}}\left\{E_0(f)_{\Gamma,p,\omega} + \sum_{\nu=1}^{n}\nu^{2k-1}E_\nu(f)_{\Gamma,p,\omega}\right\}, \quad n = 1,2,..., \tag{7}$$

holds with a constant $c > 0$ independent of n.

This theorem can be improved by the aim of Theorem 1 as follows:

Theorem 2. *Let Γ be a Carleson curve, $1 < p < \infty$, $\omega \in A_p(\Gamma)$ and $\omega_0 \in A_p(\mathbb{T})$. Then, for $f \in E_p(G,\omega)$ the estimate*

$$\Omega_k\left(f,\frac{1}{n}\right)_{\Gamma,p,\omega} \le \frac{c}{n^{2k}}\left\{\sum_{\nu=1}^{n}\nu^{2\beta k-1}E_\nu^\beta(f)_{\Gamma,p,\omega}\right\}^{1/\beta}, \quad n = 1,2,..., \tag{8}$$

holds with a constant $c > 0$ independent of n, where $\beta = \min(p,2)$.

2 Auxiliary results

Let Γ be a rectifiable Jordan curve and $f \in L_1(\Gamma)$. Then the limit

$$S_\Gamma(f)(z) := \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int\limits_{\Gamma \setminus \Gamma(z,\varepsilon)} \frac{f(\varsigma)}{\varsigma - z}\, d\varsigma \tag{9}$$

is exists and is finite for almost all $z \in \Gamma$ (see [3, pp. 117-144]). $S_\Gamma(f)(z)$ is called the *Cauchy singular integral* of f at $z \in \Gamma$.

For $f \in L_1(\Gamma)$ the function f^+ (defined in (5)) has nontangential limits and the formula

$$f^+(z) = S_\Gamma(f)(z) + \frac{1}{2} f(z) \tag{10}$$

holds a. e. on Γ [10, p. 431].

For $f \in L_1(\Gamma)$, we associate the function $S_\Gamma(f)$ taking the value $S_\Gamma(f)(z)$ a. e. on Γ. The linear operator S_Γ defined in such way is called the *Cauchy singular operator.* The following theorem, which is analogously deduced from David's theorem (see [5]), states the necessary and sufficient condition for boundedness of S_Γ in $L_p(\Gamma, \omega)$ (see also [3, pp. 117-144]).

Theorem 3. *Let Γ be a Carleson curve, $1 < p < \infty$, and let ω be a weight function on Γ. The inequality*

$$\|S_\Gamma(f)\|_{L_p(\Gamma,\omega)} \le c\, \|f\|_{L_p(\Gamma,\omega)}$$

holds for every $f \in L_p(\Gamma, \omega)$ if and only if $\omega \in A_p(\Gamma)$.

For $k = 0, 1, 2, ...,$ and $R > 1$ let

$$F_{k,p}(z) := \frac{1}{2\pi i} \int\limits_{|t|=R} \frac{t^k \left(\psi'(t)\right)^{1-1/p}}{\psi(t) - z}\, dt, \quad z \in G.$$

Obviously, $F_{k,p}$ is a polynomial of degree k. The polynomials $F_{k,p}$ are called the $p-$*Faber polynomials* for G (see [17] and [2]).

For detailed information about Faber polynomials and Faber series see [24, pp. 33-116].

Let $\mathcal{P}_n$ $(n = 0, 1, 2, ...)$ be the set of the complex polynomials of degree at most n, $\mathcal{P}$ be the set of all polynomials (with no restrictions on degrees), and let $\mathcal{P}(\mathbb{D})$ be the set of restrictions of the polynomials to $\mathbb{D}$. If we define an operator T_p on $\mathcal{P}(\mathbb{D})$ as

$$T_p(P)(z) := \frac{1}{2\pi i} \int\limits_{\mathbb{T}} \frac{P(t) \left(\psi'(t)\right)^{1-1/p}}{\psi(t) - z}\, dt, \quad z \in G,$$

then it is clear that

$$T_p\left(\sum_{k=0}^{n} a_k t^k\right) = \sum_{k=0}^{n} a_k F_{k,p}(z).$$

From (5), we have

$$T_p(P)\left(z'\right) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{P(\varphi(\varsigma)) \left(\varphi'(\varsigma)\right)^{1/p}}{\varsigma - z'}\, d\varsigma = \left[(P \circ \varphi) \left(\varphi'\right)^{1/p}\right]^+ \left(z'\right)$$

for $z^{'} \in G$. Taking the limit $z^{'} \to z \in \Gamma$, over all nontangential paths inside Γ, we obtain by (10)

$$T_p(P)(z) = S_\Gamma\left[(P\circ\varphi)\left(\varphi^{'}\right)^{1/p}\right](z) + \frac{1}{2}\left[(P\circ\varphi)\left(\varphi^{'}\right)^{1/p}\right](z)$$

for almost all $z \in \Gamma$.

We can state the following theorem as a corollary of Theorem 3.

Theorem 4. *Let Γ be a Carleson curve, $1 < p < \infty$, and let w be a weight function on Γ. If $\omega \in A_p(\Gamma)$ and $\omega_0 \in A_p(\mathbb{T})$, then the linear operator $T_p : P(\mathbb{D}) \to E_p(G,\omega)$ is bounded.*

Hence if $\omega \in A_p(\Gamma)$ and $\omega_0 \in A_p(\mathbb{T})$, then the operator T_p can be extended to the whole of $E_p(\mathbb{D},\omega_0)$ as a bounded linear operator and we have the representation

$$T_p(g)(z) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{g(t)\left(\psi^{'}(t)\right)^{1-1/p}}{\psi(t)-z}dt, \quad z \in G,$$

for all $g \in E_p(\mathbb{D},\omega_0)$.

Theorem 5 ([16])**.** *Let Γ be a Carleson curve, $1 < p < \infty$, and let ω be a weight function on Γ such that $\omega \in A_p(\Gamma)$ and $\omega_0 \in A_p(\mathbb{T})$. Then the operator $T_p : E_p(\mathbb{D},\omega_0) \to E_p(G,\omega)$ is one-to-one and onto. In fact, we have $T_p\left(f_0^+\right) = f$ for $f \in E_p(G,\omega)$.*

3 Proofs of the main results

Let $g \in L_p(\mathbb{T},\omega)$ has the Fourier series

$$g(x) \sim \frac{a_0}{2} + \sum_{\nu=1}^{\infty}(a_\nu \cos\nu x + b_\nu \sin\nu x).$$

We denote the nth partial sum of this series by $S_n(g,x)$. Let also

$$A_\nu(g,x) := a_\nu\cos\nu x + b_\nu \sin\nu x, \quad \nu = 1,2,...,$$

and

$$\Delta_\mu(g,x) := \sum_{\nu=2^{\mu-1}}^{2^\mu-1} A_\nu(g,x).$$

By a simple calculation, one can show that the kth difference T_h^k has the Fourier series

$$T_h^k g(x) \sim \sum_{\nu=1}^{\infty}\left(1-\frac{\sin\nu h}{\nu h}\right)^k A_\nu(g,x).$$

It is also known that (see [12]) the partial sums of the Fourier series are bounded in the space $L_p(\mathbb{T},\omega)$ and hence

$$\|g - S_n(g,\cdot)\|_{p,\omega} \le c\, E_n(g)_{p,\omega}, \quad n = 1,2,... . \tag{11}$$

Proof of Theorem 1. Let $h>0$ and let m be any natural number. It is clear that

$$\begin{aligned} T_h^k(g)(x) &= T_h^k(g)(x) - T_h^k\left(S_{2^{m-1}}(g,\cdot)\right)(x) + T_h^k\left(S_{2^{m-1}}(g,\cdot)\right)(x) \\ &= T_h^k\left(g - S_{2^{m-1}}(g,\cdot)\right)(x) + T_h^k\left(S_{2^{m-1}}(g,\cdot)\right)(x). \end{aligned}$$

Using (11) yields

$$\left\| T_h^k\left(g - S_{2^{m-1}}(g,\cdot)\right)\right\|_{p,\omega} \le c\ \left\| g - S_{2^{m-1}}(g,\cdot)\right\|_{p,\omega} \le c\ E_{2^{m-1}}(g)_{p,\omega}.$$

On the other hand, by Theorem 1 of [18],

$$\begin{aligned} \left\| T_h^k\left(S_{2^{m-1}}(g,\cdot)\right)\right\|_{p,w} &= \left\| \sum_{\nu=1}^{2^{m-1}} \left(1 - \frac{\sin \nu h}{\nu h}\right)^k A_\nu(g,\cdot)\right\|_{p,\omega} \\ &\le c \left\| \left(\sum_{\mu=1}^{m} \Delta_\mu^2(g,\cdot;k,h)\right)^2 \right\|_{p,\omega}, \end{aligned}$$

where

$$\Delta_\mu(g,x;k,h) := \sum_{\nu=2^{\mu-1}}^{2^\mu - 1} \left(1 - \frac{\sin \nu h}{\nu h}\right)^k A_\nu(g,x).$$

By simple calculations, we obtain

$$\left\| \left(\sum_{\mu=1}^{m} \Delta_\mu^2(g,\cdot;k,h)\right)^2 \right\|_{p,\omega} \le \left\{ \sum_{\mu=1}^{m} \left\|\Delta_\mu(g,\cdot;k,h)\right\|_{p,\omega}^2 \right\}^{1/2}$$

if $p>2$, and

$$\left\| \left(\sum_{\mu=1}^{m} \Delta_\mu^2(g,\cdot;k,h)\right)^2 \right\|_{p,\omega} \le \left\{ \sum_{\mu=1}^{m} \left\|\Delta_\mu(g,\cdot;k,h)\right\|_{p,\omega}^p \right\}^{1/p}$$

if $p \le 2$. Hence we have to estimate $\left\|\Delta_\mu(g,\cdot;k,h)\right\|_{p,\omega}$.

Let's assume that $k=1$. By Abel's transformation, we get

$$\begin{aligned} \Delta_\mu(g,x;1,h) &= \sum_{\nu=2^{\mu-1}}^{2^\mu-1} \left(1 - \frac{\sin \nu h}{\nu h}\right) A_\nu(g,x) \\ &= \sum_{\nu=2^{\mu-1}}^{2^\mu-2} \left\{ \left[\left(1 - \frac{\sin \nu h}{\nu h}\right) - \left(1 - \frac{\sin(\nu+1)h}{(\nu+1)h}\right)\right] \sum_{j=2^{\mu-1}}^{\nu} A_j(g,x) \right\} \\ &\quad + \left(1 - \frac{\sin(2^\mu - 1)h}{(2^\mu-1)h}\right) \left(\sum_{\nu=2^{\mu-1}}^{2^\mu-1} A_\nu(g,x)\right) \\ &= \sum_{\nu=2^{\mu-1}}^{2^\mu-2} \left(\frac{\sin(\nu+1)h}{(\nu+1)h} - \frac{\sin \nu h}{\nu h}\right) \left(\sum_{j=2^{\mu-1}}^{\nu} A_j(g,x)\right) \\ &\quad + \left(1 - \frac{\sin(2^\mu - 1)h}{(2^\mu-1)h}\right) \left(\sum_{\nu=2^{\mu-1}}^{2^\mu-1} A_\nu(g,x)\right). \end{aligned}$$

If we take the norm, we obtain

$$\begin{aligned}\left\|\Delta_{\mu}\left(g,\cdot;1,h\right)\right\|_{p,\omega} &\leq \sum_{\nu=2^{\mu-1}}^{2^{\mu}-2}\left(\frac{\sin\nu h}{\nu h}-\frac{\sin\left(\nu+1\right)h}{\left(\nu+1\right)h}\right)\left\|\sum_{j=2^{\mu-1}}^{\nu}A_{j}\left(g,\cdot\right)\right\|_{p,\omega}\\ &\quad+\left|1-\frac{\sin\left(2^{\mu}-1\right)h}{\left(2^{\mu}-1\right)h}\right|\left\|\sum_{\nu=2^{\mu-1}}^{2^{\mu}-1}A_{\nu}\left(g,\cdot\right)\right\|_{p,\omega}.\end{aligned}$$

Using (11) we get

$$\begin{aligned}\left\|\sum_{j=2^{\mu-1}}^{\nu}A_{j}\left(g,\cdot\right)\right\|_{p,\omega} &= \left\|\sum_{j=2^{\mu-1}}^{\infty}A_{j}\left(g,\cdot\right)-\sum_{j=\nu+1}^{\infty}A_{j}\left(g,\cdot\right)\right\|_{p,\omega}\\ &\leq \left\|\sum_{j=2^{\mu-1}}^{\infty}A_{j}\left(g,\cdot\right)\right\|_{p,\omega}+\left\|\sum_{j=\nu+1}^{\infty}A_{j}\left(g,\cdot\right)\right\|_{p,\omega}\\ &= \left\|g-S_{2^{\mu-1}-1}\left(g,\cdot\right)\right\|_{p,\omega}+\left\|g-S_{\nu}\left(g,\cdot\right)\right\|_{p,\omega}\\ &\leq c\ E_{2^{\mu-1}-1}\left(g\right)_{p,\omega},\end{aligned}$$

and similarly

$$\left\|\sum_{\nu=2^{\mu-1}}^{2^{\mu}-1}A_{\nu}\left(g,\cdot\right)\right\|_{p,\omega}\leq c\ E_{2^{\mu-1}-1}\left(g\right)_{p,\omega}.$$

Hence we have

$$\begin{aligned}\left\|\Delta_{\mu}\left(g,\cdot;1,h\right)\right\|_{p,\omega} &\leq c\ E_{2^{\mu-1}-1}\left(g\right)_{p,\omega}\sum_{\nu=2^{\mu-1}}^{2^{\mu}-2}\left(\frac{\sin\nu h}{\nu h}-\frac{\sin\left(\nu+1\right)h}{\left(\nu+1\right)h}\right)\\ &\quad+c\ E_{2^{\mu-1}-1}\left(g\right)_{p,\omega}\left|1-\frac{\sin\left(2^{\mu}-1\right)h}{\left(2^{\mu}-1\right)h}\right|\\ &\leq c\ E_{2^{\mu-1}-1}\left(g\right)_{p,\omega}2^{2\mu}h^{2}.\end{aligned}$$

By the same way, for $k>1$ we can obtain

$$\left\|\Delta_{\mu}\left(g,\cdot;k,h\right)\right\|_{p,\omega}\leq c\ E_{2^{\mu-1}-1}\left(g\right)_{p,\omega}2^{2k\mu}h^{2k}.$$

Thus we have

$$\begin{aligned}\left\|T_{h}^{k}\left(S_{2^{m-1}}\left(g,\cdot\right)\right)\right\|_{p,\omega} &\leq c\left\{\sum_{\mu=1}^{m}\left\|\Delta_{\mu}\left(g,\cdot;k,h\right)\right\|_{p,\omega}^{\beta}\right\}^{1/\beta}\\ &\leq c\left\{\sum_{\mu=1}^{m}E_{2^{\mu-1}-1}^{\beta}\left(g\right)_{p,\omega}2^{2k\mu\beta}h^{2\beta k}\right\}^{1/\beta},\end{aligned}$$

and hence

$$\left\|T_{h}^{k}\left(g\right)\right\|_{p,\omega}\leq c\left\{E_{2^{m-1}}\left(g\right)_{p,\omega}+h^{2k}\left[\sum_{\mu=1}^{m}2^{2k\mu\beta}E_{2^{\mu-1}-1}^{\beta}\left(g\right)_{p,\omega}\right]^{1/\beta}\right\}.$$

Choosing $h=1/n$ for a given n, by the definition of the modulus of smoothness we have

$$\Omega_k\left(g,\frac{1}{n}\right)_{p,\omega}\leq c\left\{E_{2^{m-1}}(g)_{p,\omega}+\frac{1}{n^{2k}}\left[\sum_{\mu=1}^{m}2^{2k\mu\beta}E^{\beta}_{2^{\mu-1}-1}(g)_{p,\omega}\right]^{1/\beta}\right\}.$$

If we use the inequality

$$E_{2^{m-1}}(g)_{p,\omega}\leq\frac{2^{4\beta k}}{2^{2m\beta k}}\sum_{\nu=2^{m-2}+1}^{2^{m-1}}\nu^{2\beta k-1}E^{\beta}_{\nu}(g)_{p,\omega}$$

and select m such that $2^m\leq n<2^{m+1}$, we obtain

$$\begin{aligned}\Omega_k\left(g,\frac{1}{n}\right)_{p,\omega}&\leq c\left\{\frac{2^{6k}}{n^{2k}}\left[\sum_{\nu=2^{m-2}+1}^{2^{m-1}}\nu^{2\beta k-1}E^{\beta}_{\nu}(g)_{p,\omega}\right]^{1/\beta}+\frac{2^{6k}}{n^{2k}}\left[\sum_{\nu=1}^{2^{m-2}}\nu^{2\beta k-1}E^{\beta}_{\nu}(g)_{p,\omega}\right]^{1/\beta}\right\}\\&\leq\frac{c}{n^{2k}}\left[\sum_{\nu=1}^{n}\nu^{2\beta k-1}E^{\beta}_{\nu}(g)_{p,\omega}\right]^{1/\beta},\end{aligned}$$

and the theorem is proved.

Proof of Theorem 2. Let $f\in E_p(G,\omega)$. Then by Theorem 5 we have $T_p\left(f_0^+\right)=f$, where

$$f_0(t)=f(\psi(t))\left(\psi'(t)\right)^{1/p},\quad t\in\mathbb{T}.$$

Since $T_p:E_p(\mathbb{D},\omega_0)\rightarrow E_p(G,\omega)$ is bounded, one to one and onto, the linear operator $T_p^{-1}:E_p(G,\omega)\rightarrow E_p(\mathbb{D},\omega_0)$ is also bounded.

Let $P_n^*\in\mathcal{P}_n$ $(n=0,1,2,...)$ be the polynomials of best approximation to f in $E_p(G,\omega)$, that is,

$$E_n(f)_{\Gamma,p,\omega}=\|f-P_n^*\|_{L_p(\Gamma,\omega)}.$$

The existence of such polynomials follows, for example, from Theorem 1.1 in [7, p. 59]. Since $T_p^{-1}(P_n^*)$ is a polynomial of degree n, by the boundedness of T_p^{-1} we get

$$\begin{aligned}E_n\left(f_0^+\right)_{p,\omega_0}&\leq\left\|f_0^+-T_p^{-1}(P_n^*)\right\|_{L_p(\mathbb{T},w_0)}\\&=\left\|T_p^{-1}(f)-T_p^{-1}(P_n^*)\right\|_{L_p(\mathbb{T},w_0)}\\&\leq\left\|T_p^{-1}\right\|\|f-P_n^*\|_{L_p(\Gamma,\omega)},\end{aligned}$$

and hence

$$E_n\left(f_0^+\right)_{p,\omega_0}\leq\left\|T_p^{-1}\right\|E_n(f)_{\Gamma,p,\omega}.\tag{12}$$

By (6), Theorem 1 and (12) we obtain

$$\begin{aligned}\Omega_k\left(f,\frac{1}{n}\right)_{\Gamma,p,\omega}&=\Omega_k\left(f_0^+,\frac{1}{n}\right)_{p,\omega_0}\leq\frac{c}{n^{2k}}\left\{\sum_{\nu=1}^{n}\nu^{2\beta k-1}E^{\beta}_{\nu}\left(f_0^+\right)_{p,\omega_0}\right\}^{1/\beta}\\&\leq\frac{c\left\|T_p^{-1}\right\|}{n^{2k}}\left\{\sum_{\nu=1}^{n}\nu^{2\beta k-1}E^{\beta}_{\nu}(f)_{\Gamma,p,\omega}\right\}^{1/\beta},\end{aligned}$$

which prove the theorem.

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