

## NOTES AND CORRESPONDENCE

## Improved Magnus Form Approximation of Saturation Vapor Pressure

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## ABSTRACT

Algorithms, based on Magnus's form equations, are described that minimize the difference between several relationships between temperature and water vapor pressure at saturation that are commonly used in archiving data. The work was initiated in connection with the development of a unified upper-air dataset that will use measurements gathered from the late 1930s to the present and archived in several data centers. The conversion of field measurements to archived humidity values within the databases that are being used have not been consistent and in some cases are unknown. A goal of this work was to develop a uniform and accurate method to convert these data to various humidity variables without regard to the equations used in archiving the original data. Archived temperature values are recorded to 0.1°C. This precision creates a temperature dependent range in uncertainty in saturation vapor pressure. A procedure was developed to take this into account when the error minimizing equations were derived.

## 1. Introduction

The National Climatic Data Center (NCDC) and the Russian Research Institute of Hydrometeorological information (RRIHMI) are conducting a joint project to produce a unified upper-air dataset called CARDS (see Eskridge et al. 1995). Data from the late 1930s to the present are being used. One of the problems faced is that input datasets contain different humidity variables (e.g., relative humidity, dewpoint, dewpoint depression, specific humidity). Also, different formulations have been used by data centers (e.g., NCDC and RRIHMI) in converting from temperature and relative humidity measurements to specific humidity and dewpoint values.

A number of computational changes have been made in the United States data. Prior to 1948, saturation vapor pressure was calculated with respect to ice. Since 1948, saturation vapor pressure has been calculated with respect to water. In 1969 and 1981 new humidity

algorithms were introduced. If the equations used to make the various moisture calculations are known, inverse calculations can be made with great accuracy. However, the algorithms used in the initial data reduction are not always those used when the data are processed at data centers. For example, the NCDC until recently used an approximation to the Goff and Gratch (1946) equation derived by Bosen (1958). At United States radiosonde stations, data have been reduced using procedures found in the Federal Meteorological Handbook. The Federal Meteorological Handbook (1981) includes tables (source not identified) for moisture calculations, whereas the Federal Meteorological Handbook (1991) use an approximation of Buck (1981). Hence, there is an inhomogeneity introduced in the data due to different procedures used to calculate moisture variables in the United States data. Datasets from other sources are expected to have similar inconsistencies.

Humidity data that exist in publications and/or databases frequently have been converted using formulations of saturation vapor pressure based on various fits to the equations of Wexler (1976), Sonntag (1990), and especially Goff and Gratch (1946). Although the formulations of Sonntag and Wexler are more accurate for temperature above 0°C, much more data have been produced using the Goff–Gratch equations or their approximations. One goal of this research was to find Magnus's form approximations of satura-

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tion vapor pressure that are close to these three formulations of saturation vapor pressure so that a uniform recalculation of different moisture variables can be made.

To optimize the equations, a new approach was developed to determine the "accuracy"<sup>1</sup> of an approximation. It is standard practice world wide to report temperature to one decimal place. Assuming accurate measurement and reportage there is an uncertainty of  $\pm 0.05^\circ\text{C}$  in each reported temperature. Therefore, there is a range of values around the value of saturation vapor pressure denoted as  $[e(t) - \epsilon_0, e(t) + \epsilon_1]$ , corresponding to the temperature interval  $[t - 0.05, t + 0.05]$ . An approximation that produces saturation vapor pressure values that lie within the range  $[e(t) - \epsilon_0, e(t) + \epsilon_1]$  for  $t$  in  $[-40, 50]$  is as "accurate" as can be judged for temperature data reported to one decimal place.

Upper-air temperature data range from about  $+50^\circ$  to  $-80^\circ\text{C}$ . This large variation requires a saturation vapor pressure equation to be accurate over a large temperature range.

The errors introduced by the use of relatively inaccurate conversion equations are smaller than the errors due to the instruments, dewpoint coding errors, and dewpoint conversion algorithms (Elliott and Gaffen 1993).

Elliott and Gaffen (1991) compared the relative errors between the saturation vapor pressure formulations of Tetens (1930), Goff and Gratch (1946), Wexler (1976), and Buck (1981). This paper extends the analysis of Elliott and Gaffen. Several new saturation vapor pressure formulas are derived and the various errors in these formulations are documented.

## 2. Basic formulations of vapor pressure

Common meteorological practice is to calculate humidity using the saturation vapor pressure over a surface of water or ice. The only independent variable is temperature. The undercooling of water below its melting point is common in clouds. The WMO (1966) recommendation that vapor pressure over a plane of water be used in calculating humidity variables at negative (Celsius) temperatures is followed.

The first accurate formula for saturation vapor pressure was developed by Goff and Gratch (1945) (denoted by GG45):

$$\begin{aligned} \log_{10} E_w = & 7.90298 \left( 1 - \frac{T_s}{T} \right) + 5.02808 \log_{10} \left( \frac{T_s}{T} \right) \\ & + 1.3816 \times 10^{-7} [1 - 10^{11.344(1-T/T_s)}] - 8.1328 \\ & \times 10^{-3} [1 - 10^{-3.49149(1-T_s/T)}] + \log_{10} E_{w0}. \quad (1) \end{aligned}$$

<sup>1</sup> For present purposes, an "accurate" value is a value that lies within the range  $e(t) - \epsilon_0, e(t) + \epsilon_1$ . The accuracy of an approximation is related to the deviation of the calculated value from the accurate value range.

This equation was soon modified by Goff and Gratch (1946) (this formulation is denoted by GG46):

$$\begin{aligned} \log_{10} E_w = & 10.79574 \left( 1 - \frac{T_1}{T} \right) \\ & - 5.02800 \log_{10} \left( \frac{T}{T_s} \right) + 1.50475 \\ & \times 10^{-4} [1 - 10^{8.2969(1-T/T_1)}] - 0.42873 \\ & \times 10^{-3} [1 - 10^{4.76955(1-T_1/T)}] + 0.78614. \quad (2) \end{aligned}$$

Values using (2) are notably different from (1) for temperatures between  $-40^\circ$  and  $50^\circ\text{C}$ , the difference being more than 5% with the largest differences near  $-40^\circ\text{C}$ . The Smithsonian meteorological tables (List 1949) are based on (2). In 1966, the WMO recommended its use in meteorological calculations and it is still in use in many countries.

Goff (1965) revised the equation, denoted as GG65, based on a new analysis, but this equation is not notably different from (2). The largest relative difference between the GG65 and the GG46 formulations is 0.008% at  $-40^\circ\text{C}$  for the temperature range of  $-40^\circ$  to  $+50^\circ\text{C}$ . The GG65 and GG46 equations are essentially identical.

In 1976, Wexler published a new saturation vapor pressure formulation that was based on new measurements. This formulation is denoted as WE76:

$$\begin{aligned} E_w = & 0.01 \exp(-2.9912729 \times 10^3 T^{-2} \\ & - 6.0170128 \times 10^3 T^{-1} + 1.887643845 \\ & \times 10^1 - 2.8354721 \times 10^{-2} T + 1.7838301 \\ & \times 10^{-5} T^2 - 8.4150417 \times 10^{-10} T^3 \\ & + 4.4412543 \times 10^{-13} T^4 + 2.858487 \ln T). \quad (3) \end{aligned}$$

The values of saturation vapor pressure from Wexler's formulation, (3), differ the most from the Goff and Gratch's (1946) equations at negative temperatures.

Sonntag and his colleague Heinze conducted several important investigations of the relationship between vapor pressure and temperature. In 1982, Sonntag and Heinze published equations for saturation vapor pressure (SO82):

$$\begin{aligned} E_w = & \exp(-6094.4642 T^{-1} + 16.519825 \\ & - 2.7245552 \times 10^{-2} T + 1.6853396 \\ & \times 10^{-5} T^2 + 2.433502 \ln T). \quad (4) \end{aligned}$$

In 1990, Sonntag published a formulation, denoted as SO90, of vapor pressure:

$$\begin{aligned} E_w = & \exp(-6096.9385 T^{-1} + 16.635794 \\ & - 2.711193 \times 10^{-2} T + 1.673952 \\ & \times 10^{-5} T^2 + 2.433502 \ln T). \quad (5) \end{aligned}$$

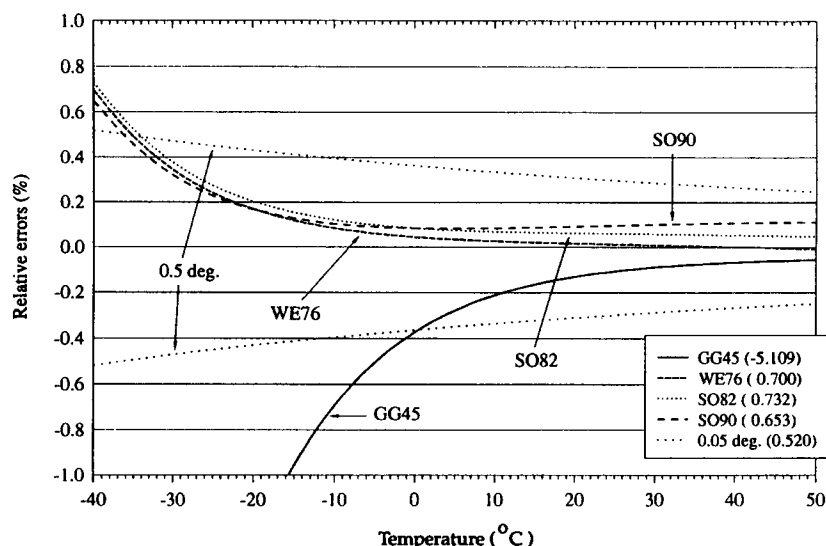


FIG. 1. Relative errors of the basic formulations with respect to the Goff and Gratch (1946), GG46. The relative error is the predicted value from one of the approximations minus the GG46 value divided by the GG46. The window shows the greatest error over the  $-40^{\circ}$  to  $50^{\circ}\text{C}$  range.

Saturation vapor pressures calculated from (4) and (5) are not significantly different from Wexler's formulation (3). The largest relative difference between

Wexler's formulation and Sonntag's 1982 formulation is 0.055% at  $50^{\circ}\text{C}$ , and the largest difference between the 1990 formulation and Wexler's is 0.120% at  $50^{\circ}\text{C}$ .

Figure 1 shows the relative differences between the SO82, SO90, GG45, and the WE76 equations with respect to the GG46 equation. The Goff–Gratch equation has been recommended by Gueymard (1993) for temperatures below  $0^{\circ}\text{C}$ . Hence, GG46 is used as the reference in Fig. 1.

We believe that the ideal saturation vapor pressure approximation will minimize the errors between it and the three basic formulations (GG46, WE76, and SO90). The ideal approximation will minimize systematic errors due to the use of a new equation when working with new and/or old data.

### 3. Approximations of saturation vapor pressure over water

Saturation vapor pressure formulations WE76, GG46, and SO90 are complex and inconvenient to use in making calculations. This complexity has led to various simplified and less accurate approximations of saturation vapor pressure. Reviews of these approximations can be found in Sargent (1980), Abbott and Tabony (1985), Alduchov (1988), and Elliott and Gaffen (1991). Some of the most widely employed approximations are shown in Table 1.

A simple computational form for the saturation vapor pressure [Eq. (8)] is due to Magnus (1844):

$$E_w = ce^{at/(b+t)}, \quad (6)$$

TABLE 1. Saturation vapor pressure approximations above a plane of water.

Author	Name	Formula
Magnus formulas		
Tetens (1930)	TE30	$e(t) = 6.11 \times 10^{7.5t/(237.3+t)}$
Matveev (1967)	MA67	$e(t) = 6.1078 \times 10^{7.63t/(241.9+t)}$
Buck (1981)	BU81	$e(t) = 6.1121e^{17.502t/(240.97+t)}$
Abbott-Tabony (1985)	AT85	$e(t) = 6.1070e^{17.38t/(239.0+t)}$
Alduchov (1988)	AL88	$e(t) = 6.107 \times 10^{7.665t/(243.33+t)}$
Sonntag (1990)	SA90	$e(t) = 6.112e^{17.62t/243.12+t}$
Non-Magnus formulas		
Tabata (1973)	TA73	$e(T) = 10^{8.42926609 - 1927.17843/T - 71208.271/T^2}$
Buck (1981)	BU-2	$e(t) = 6.1121e^{(18.729 - t/227.3)/(t+257.87)}$
Hooper (1986)	HO86	$e(t) = (1.3521 + 1.6369 \times 10^{-2} t + 3.1794 \times 10^{-5} t^2 - 1.4892 \times 10^{-7} t^3)^6$
Approximations developed this research for $t \in (-40^{\circ}, 50^{\circ}\text{C})$		
AEDK		$e(t) = 6.1102e^{17.621t/(242.97+t)}$
AEDG		$e(t) = 6.1105e^{17.546t/(241.81+t)}$
AEDW		$e(t) = 6.1128e^{17.610t/(242.89+t)}$
AEDS		$e(t) = 6.1152e^{17.616t/(242.91+t)}$
AERK		$e(t) = 6.1094e^{17.625t/(243.04+t)}$
AERG		$e(t) = 6.1072e^{17.578t/(242.25+t)}$
AERW		$e(t) = 6.1085e^{17.654t/(243.49+t)}$
AERS		$e(t) = 6.1107e^{17.660t/(243.51+t)}$

where  $a$ ,  $b$ , and  $c$  are constants and  $t$  is the temperature ( $^{\circ}\text{C}$ ).

The main reason to derive approximations to the basic formulations is to simplify the equations, so that with simple equations it is easy to calculate the saturation vapor pressure using only temperature. The requirements of simplicity and minimal error will eliminate many of the approximations. It is difficult to suggest a more convenient equation for the saturation vapor pressure than the Magnus form.

However, there are relatively large errors when the Magnus formula is used with published coefficients ( $a$ ,  $b$ ,  $c$ ). The motivation for this work was to see if different coefficients could be found to reduce these errors, and, as will be shown, coefficients  $a$ ,  $b$ , and  $c$  have been found for the Magnus equation and accurately approximate all three basic formulations of the saturation vapor pressure.

To determine the accuracy of a saturation vapor pressure approximation  $e(t)$ , a criterion of accuracy is needed. It is assumed that the observation error of the thermistor is zero,<sup>2</sup> and the temperature error is due only to roundoff error. It is standard meteorological practice to report temperature to  $0.1^{\circ}\text{C}$ . Therefore, the roundoff error in temperature will be less than or equal to  $0.05^{\circ}\text{C}$ . The error in the saturation vapor pressure due to this temperature roundoff error is defined to be

$$\delta E(t) = \max_{-0.05 \leq \tau \leq 0.05} |E(t) - E(t + \tau)|, \quad (7)$$

where  $E$  represents any basic formulation.

An approximation  $e(t)$  is said to be accurate if the difference between this approximation and a corresponding basic formulation  $E$  is not more than the error in saturation vapor pressure due to the temperature roundoff error. Hence, the measure of accuracy or goodness of fit for approximation  $e(t)$  is given by

$$d = \max_{-40 \leq t \leq 50} \left| \frac{e(t) - E(t)}{\delta E(t)} \right|. \quad (8)$$

If  $d \leq 1$  for an approximation, then one cannot distinguish between the calculated value and the actual value due to the  $\pm 0.05$  uncertainty in the reported value. Smaller values of  $d$  do imply a greater accuracy in the sense that if more accurate temperature data (more than one decimal place) were used, the approximation with a smaller  $d$  value will produce a more accurate estimate of the saturation vapor pressure.

This test,  $d \leq 1$ , is applied to a number of approximations from the literature. The approximation  $e(t)$  is

said to be accurate if  $d \leq 1$ . For example,  $d \leq 1$ , means that the relative error with respect to the GG46 formulation is less than or equal to 0.52% at  $-40^{\circ}\text{C}$  and 0.25% at  $50^{\circ}\text{C}$ . This definition of accuracy,  $d \leq 1$ , leads to similar limits of relative error for the other basic formulations.

The accuracy of each approximation  $e$  is determined with respect to the three basic formulations of vapor pressure (GG46, WE76, and SO90). The measures of accuracy from (8) are denoted as  $d_g$ ,  $d_w$ , and  $d_s$ , respectively. The measure of accuracy with respect to all three formulations,  $d_k$ , is defined as

$$d_k = \max(d_g, d_w, d_s). \quad (9)$$

The maximum relative errors for these approximations are denoted by  $r_g$ ,  $r_w$ ,  $r_s$ , respectively. Terms  $r$  and  $r_k$  are defined as

$$r = \max \left[ \frac{e(t) - E(t)}{E(t)} \right],$$

$$r_k = \max(r_g, r_w, r_s). \quad (10)$$

Using these definitions of error, coefficients  $a$ ,  $b$ , and  $c$  for the Magnus formula (6) have been found that minimize  $r_g$  (approximation denoted as AERG), minimize  $r_w$  for AERW, minimize  $r_s$  for AERS, and minimize  $r_k$  for AERK (see Table 2).

The coefficients  $a$ ,  $b$ , and  $c$  are found by an iterative process using (8) and (6). The simplest way to find coefficients  $a'$ ,  $b'$ , and  $c'$ , which describe the minimum  $d_k$ , is to iterate over a dense three-dimensional set of points ( $a$ ,  $b$ ,  $c$ ) calculating  $d_k$  and find the minimum ( $a'$ ,  $b'$ ,  $c'$ ).

The procedure will generate more than one set of coefficients as there will be many minima. The approximations found by this method are shown in Tables 1 and 3, with the measures of accuracy in Tables 2 and 4.

In order not to miss a real minimum, small increments must be used for each step in the iteration. We have chosen to use the minimum of the maximum deviation rather than, say, minimizing the total deviation over the range or a least squares fit in order to eliminate any large magnitude deviations from the basic formulation. This minimizing process yields approximations for which  $d \leq 1$  for all temperatures between  $-40^{\circ}$  and  $50^{\circ}\text{C}$ .

Several approximations have been found that do not minimize the relative errors but maximize accuracy. These approximations are AEDG ( $\text{mind}_g$ ), AEDW ( $\text{mind}_w$ ), AEDS ( $\text{mind}_s$ ), and AEDK ( $\text{mind}_k$ ). See Table 1 for the equations.

Table 2 shows the relative errors and measure of accuracy for the new approximations and several commonly used approximations. Note that the maximum relative errors usually occur at the end points of the temperature range  $-40^{\circ}$  to  $50^{\circ}\text{C}$ . Table 2 shows that only three of the approximations satisfy the criterion

<sup>2</sup> Observational requirements of the United States network for accuracy of temperature measurements are  $0.5^{\circ}\text{C}$  in the troposphere and  $1^{\circ}$ – $2^{\circ}\text{C}$  in the stratosphere.

TABLE 2. Maxima of relative errors  $r$  (%) and measures of accuracy  $d$  for various basic formulations and approximations of saturation vapor pressure over water. Temperature range is  $-40^{\circ}$  to  $50^{\circ}\text{C}$ . Temperature at the maximum is given in parentheses. The measure of accuracy  $d_k$  and the maximum relative error  $r_k$  are boldface.

	$r_g(t)$	$d_g(t)$	$r_w(t)$	$d_w(t)$	$r_s(t)$	$d_s(t)$
GG46	0.000 (—)	0.000 (—)	<b>0.695</b> (—40)	<b>1.343</b> (—40)	0.648 (—40)	1.252 (—40)
WE76	<b>0.700</b> (—40)	<b>1.346</b> (—40)	0.000 (—)	0.000 (—)	0.120 (50)	0.482 (50)
SO90	<b>0.653</b> (—40)	<b>1.255</b> (—40)	0.120 (50)	0.482 (50)	0.000 (—)	0.000 (—)
GG45	5.109 (—40)	9.825 (—40)	<b>5.768</b> (—40)	<b>11.145</b> (—40)	5.724 (—40)	11.057 (—40)
GG65	0.008 (—40)	0.016 (—40)	<b>0.703</b> (—40)	<b>1.359</b> (—40)	0.657 (—40)	1.269 (—40)
SO68	<b>0.732</b> (—40)	<b>1.407</b> (—40)	0.055 (50)	0.223 (50)	0.078 (—40)	0.260 (50)
TE30	2.537 (—40)	4.879 (—40)	<b>3.214</b> (—40)	<b>6.210</b> (—40)	3.169 (—40)	6.121 (—40)
MA67	0.558 (—40)	1.448 (50)	<b>1.249</b> (—40)	<b>2.413</b> (—40)	1.202 (—40)	2.323 (—40)
BU81	0.769 (—40)	1.478 (—40)	<b>1.458</b> (—40)	<b>2.817</b> (—40)	1.412 (—40)	2.727 (—40)
AT85	1.834 (—40)	3.527 (—40)	<b>2.516</b> (—40)	<b>4.862</b> (—40)	2.471 (—40)	4.772 (—40)
AL88	0.341 (—31)	1.018 (50)	<b>0.406</b> (—40)	<b>1.053</b> (50)	0.360 (—40)	0.832 (24)
SA90	<b>0.597</b> (—35)	<b>1.225</b> (—31)	0.292 (—23)	0.822 (28)	0.328 (28)	1.135 (30)
BU-2	<b>0.493</b> (—40)	<b>0.948</b> (—40)	0.205 (—40)	0.398 (50)	0.159 (—40)	0.306 (—40)
TA73	<b>1.322</b> (—40)	<b>2.542</b> (—40)	0.618 (—40)	1.716 (50)	0.665 (—40)	1.292 (—38)
HO86	0.140 (—28)	0.306 (—25)	0.592 (—40)	<b>1.143</b> (—40)	<b>0.545</b> (—40)	1.053 (—40)
Newly developed approximations						
AEDK	0.373 (—29)	0.804 (—26)	<b>0.412</b> (—40)	0.806 (50)	0.365 (—40)	0.804 (27)
AEDG	0.220 (—40)	0.430 (50)	<b>0.913</b> (—40)	<b>1.764</b> (—40)	0.866 (—40)	1.674 (—40)
AEDW	<b>0.477</b> (—31)	<b>1.018</b> (—27)	0.290 (—40)	0.561 (27)	0.252 (27)	0.862 (29)
AEDS	<b>0.456</b> (—29)	1.006 (50)	0.334 (—40)	<b>1.040</b> (50)	0.288 (—40)	0.563 (27)
AERK	<b>0.384</b> (—30)	0.822 (—27)	0.384 (—40)	0.742 (—40)	0.337 (—40)	<b>0.868</b> (27)
AERG	0.151 (24)	0.594 (50)	<b>0.844</b> (—40)	<b>1.630</b> (—40)	0.797 (—40)	1.540 (—40)
AERW	<b>0.508</b> (—36)	<b>1.033</b> (—32)	0.203 (—22)	0.812 (50)	0.278 (24)	0.932 (26)
AERS	<b>0.471</b> (—34)	1.253 (50)	0.320 (50)	<b>1.288</b> (50)	0.202 (—40)	0.804 (50)

for accuracy,  $d_k \leq 1$ . They are BU-2 (Buck 1981) and two approximations found in this research (AERK and AEDK). The AERK and AEDK approximations have the Magnus form. These two approximations maximize accuracy and minimize errors in that  $d_k$  and  $r_k$  are smallest for these approximations.

Figure 2 shows the behavior of several vapor pressure approximations with respect to the Wexler formulation for temperatures between  $-40^{\circ}$  and  $50^{\circ}\text{C}$ . Figure 2 shows that several of the approximations are excellent over the temperature range of  $0^{\circ}$  to  $40^{\circ}\text{C}$  and that the AERK Magnus formulation is the most accurate over the entire  $-40^{\circ}$  to  $50^{\circ}\text{C}$  temperature range.

When the lower limit of the temperature range was extended from  $-40^{\circ}$  to  $-80^{\circ}\text{C}$ , the differences between the basic formulations was found to increase. For example, the relative difference between the GG46 and WE76 formulations is more than 10% at  $-80^{\circ}\text{C}$  with respect to WE76. Obviously, it will be impossible to approximate all three formulations with a relative error of less than 5%. Approximations have been developed for each of the three basic formulation for the temperature range of  $-80^{\circ}$  to  $50^{\circ}\text{C}$ . The three approximations given in (11) were found by minimizing the relative error:

$$\begin{aligned}
 \text{AERG}_{50,-80}(t) &= 6.1037e^{17.641t/(243.27+t)}, \\
 &\quad (0.337/0.823) \\
 \text{AERW}_{50,-80}(t) &= 6.0612e^{18.102t/(249.52+t)}, \\
 &\quad (0.852/3.429) \\
 \text{AERS}_{50,-80}(t) &= 6.0620e^{18.112t/(249.59+t)}, \\
 &\quad (0.845/3.386).
 \end{aligned} \tag{11}$$

The maximum relative error in percent and the measure of accuracy are given in parentheses under each approximation in (11). Table 3 shows the relative errors and measure of accuracy for three approximations with respect to the three basic formulations for temperature range from  $50^{\circ}$  to  $-80^{\circ}\text{C}$ .

#### 4. Approximations of saturation vapor pressure over ice

There are three main approximations for calculating saturation vapor pressure over a plane surface of ice. These equations are valid for a temperature range of  $-80^{\circ}$  to  $0^{\circ}\text{C}$ . The first accurate algorithm (GG46i) was developed by Goff and Gratch (1946), and it is given by

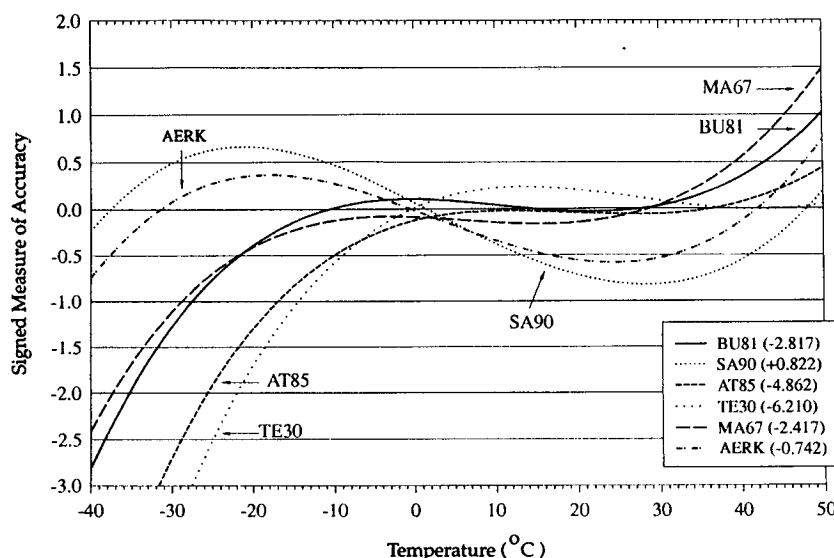


FIG. 2. Signed measure of accuracy from ten of several approximations of vapor pressure over water with respect to the Wexler (1976) formulation. The window shows the greatest error over the  $-40^{\circ}$  to  $50^{\circ}\text{C}$  range.

$$\log_{10} E_i = 9.09685 \left( 1 - \frac{T_1}{T} \right) - 3.56654 \log_{10} \left( \frac{T_1}{T} \right) + 0.87682 \left( 1 - \frac{T}{T_1} \right) + 0.78614. \quad (12)$$

Wexler's (1976) formulation (WE76i) is given by

$$E_i = 0.01 \exp(-5.8653696 \times 10^3 T^{-1} + 2.2241033 \times 10 + 1.3749042 \times 10^{-2} T - 3.4031775 \times 10^{-5} T^2 + 2.6967687 \times 10^{-7} T^3 + 0.6918651 \ln T), \quad (13)$$

and Sonntag's formulation (SO90i) is given by

$$E_i = \exp(-6024.5282^3 T^{-1} + 24.72219 - 1.0613868 \times 10^{-2} T + 1.3198825 \times 10^{-5} T^2 + 0.49382577 \ln T). \quad (14)$$

Coefficients  $a$ ,  $b$ , and  $c$  have been derived for the general Magnus formula (6), which minimizes  $d_{gi}$ ,  $d_{wi}$ ,

$d_{si}$ , and  $d_{ki}$  over the temperature range of  $-80^{\circ}$  to  $0^{\circ}\text{C}$  with respect to the basic formulations: GG46i, WE76i, and SO90i. Applying (8) and (9) to the basic formulations, the following measures of accuracy are defined:  $d_{gi}$ ,  $d_{wi}$ ,  $d_{si}$ , and  $d_{ki}$ . The new approximations are given in Table 4. New approximations of the basic formula-

TABLE 4. Saturation vapor pressure approximations above a plane of ice.

Author	Name	Formula
<b>Magnus formulas</b>		
Matveev (1967)	MA67i	$e(t) = 6.1078 \times 10^{9.5t/(265.5+t)}$
Buck-1 (1981)	BU81i	$e(t) = 6.1115e^{22.452t/(272.55+t)}$
Abbott-Tabony (1985)	AT85i	$e(t) = 6.1070e^{22.44t/(272.4+t)}$
Alduchov (1988)	AL88i	$e(t) = 6.106410^{9.800t/(273.66+t)}$
Sonnag (1990)	SA90i	$e(t) = 6.112e^{22.46t/(272.62)}$
<b>Non-Magnus formulas</b>		
Buck-2 (1981)	BU-2i	$e(t) = 6.1115e^{t(23.036 - t/333.7)/(t+279.82)}$
Hooper (1986)	HO86i	$e(t) = (1.2539 + 1.2924 \times 10^{-2} t + 1.9187 \times 10^{-5} t^2 - 9.41 \times 10^{-8} t^3)^8$
<b>Newly developed approximations</b>		
	AEDKi	$e(t) = 6.1128e^{22.571t/(273.71+t)}$
	AEDGi	$e(t) = 6.1093e^{22.570t/(273.69+t)}$
	AEDWi	$e(t) = 6.1150e^{22.561t/(273.64+t)}$
	AEDSi	$e(t) = 6.1148e^{22.574t/(273.72+t)}$
	AERKi	$e(t) = 6.1121e^{22.587t/(273.86+t)}$
	AERGi	$e(t) = 6.1084e^{22.575t/(273.74+t)}$
	AERWi	$e(t) = 6.1162e^{22.577t/(273.78+t)}$
	AERSi	$e(t) = 6.1140e^{22.579t/(273.81+t)}$

TABLE 3. Maximum relative errors (%) and measure of accuracy for the BU81, SA90, and AERK approximations of saturation vapor pressure with respect to the basic formulations for temperatures between  $-80^{\circ}$  and  $50^{\circ}\text{C}$ .

	GG46	WE76	SO90
BU81	4.850/6.005	14.423/18.499	14.312/18.355
SA90	0.707/1.252	9.425/10.089	9.308/11.937
AERK	0.539/0.822	10.067/12.067	9.951/12.767

tions of saturation vapor pressure over ice that maximize accuracy have been found. These new approximations are AEDGi, AEDWi, AEDSi, and AERKi. Table 4 gives the new approximations and Table 5 shows their relative error and measure of accuracy. Several approximations that are commonly used to calculate saturation vapor pressure over ice are also given in Table 4.

Magnus form equations were found with respect to a surface of ice by minimizing the relative error. These new equations, which are given in Table 4, are denoted by AERKi, AERGi, AERWi, and AERSi.

Table 5 shows the relative errors  $r$  and measures of accuracy  $d$  for several approximations of saturation vapor pressure over ice. The relative errors and measures of accuracy are given for the new relationships. The two most accurate approximations are AEDKi and AERKi. Table 5 shows that the relative error  $r$  for AEDKi and AERKi is 0.213% and 0.164%, respectively. The accuracy  $d$  of AEDKi and AERKi is 0.258% and 0.327%, respectively. Both of these formulations are more accurate than the BU81i formulation (Buck 1981).

## 5. Enhancement factor

The departure of the ideal gas law for a mixture of air and water vapor leads to errors that can be eliminated by an enhancement or correction factor. The correction factor is defined as the ratio of the saturation vapor pressure for moist air to that of pure water vapor over a plane of water

$$F_w(t, p) = \frac{E_{wa}(t, p)}{E_w(t)} \quad (15)$$

and over a plane surface of ice

$$F_i(t, p) = \frac{E_{ia}(t, p)}{E_i(t)}. \quad (16)$$

Relative errors (calculated minus observed divided by observed) in the saturation vapor pressure of moist air that can reach 0.596% with respect to water (at  $-40^\circ\text{C}$ , 1000 hPa) and 0.882% with respect to ice (at  $-80^\circ\text{C}$ , 1000 hPa) if the correction factor is neglected.

The following approximations to the data in Table 4.10 of the WMO (1966) report have been developed that describe the behavior of the enhancement factor over water and ice:

$$f_w(p) = 1.00071e^{0.0000045p} \quad (17)$$

and

$$f_i(p) = 0.99882e^{0.000008p}, \quad (18)$$

where  $p$  (hPa) is the pressure. These two approximations have maximum relative errors of 0.0773% (at  $0^\circ\text{C}$ , 900 hPa) over a plane surface of water and 0.209% relative error (at  $-80^\circ\text{C}$ , 1000 hPa) over a plane surface of ice, respectively.

Buck (1981) developed equations for the enhancement factor for water and ice. Buck equations are

$$f_w = 1.0007 + 3.46 \times 10^{-6}p \quad (19)$$

TABLE 5. Maxima of relative errors  $r$  (%) and measures of accuracy  $d$  for various approximations and basic formulations of saturation vapor pressure over ice for temperatures between  $-80^\circ$  and  $0^\circ\text{C}$ . Temperature at the maximum is given in parentheses. The measure of accuracy  $d_k$  and the maximum relative error  $r_k$  are boldface.

	$r_g(t)$	$d_g(t)$	$r_w(t)$	$d_w(t)$	$r_s(t)$	$d_s(t)$
GG46i	0.000 (—)	0.000 (—)	<b>0.230</b> (—80)	<b>0.294</b> (—48)	0.089 (—8)	0.207 (—3)
WE76i	<b>0.231</b> (—80)	<b>0.294</b> (—48)	0.000 (—)	0.000 (—)	0.156 (—80)	0.196 (—56)
SO90i	0.089 (—8)	<b>0.208</b> (—3)	<b>0.156</b> (—80)	0.196 (—56)	0.000 (—)	0.000 (—)
MA67i	10.672 (—80)	12.930 (—80)	<b>10.878</b> (—80)	<b>13.180</b> (—80)	10.738 (—80)	13.010 (—80)
BU81i	0.673 (—80)	0.816 (—80)	<b>0.902</b> (—80)	<b>1.093</b> (—80)	0.747 (—80)	0.905 (—80)
AT85i	0.973 (—80)	1.179 (—80)	<b>1.201</b> (—80)	<b>1.455</b> (—80)	1.047 (—80)	1.268 (—80)
AL88i	0.094 (—80)	0.189 (—17)	<b>0.324</b> (—80)	<b>0.456</b> (—19)	0.174 (—17)	0.375 (—14)
SA90i	0.672 (—80)	0.814 (—80)	<b>0.900</b> (—80)	<b>1.091</b> (—80)	0.745 (—80)	0.903 (—80)
B281i	<b>0.284</b> (—80)	<b>0.345</b> (—80)	0.054 (—80)	0.065 (—80)	0.210 (—80)	0.254 (—80)
HO86i	0.193 (—80)	0.234 (—80)	<b>0.423</b> (—80)	<b>0.512</b> (—80)	0.267 (—80)	0.324 (—80)
Newly developed approximations						
AEDKi	0.171 (—62)	0.256 (0)	<b>0.213</b> (—80)	<b>0.258</b> (—21)	0.106 (—61)	0.166 (—16)
AEDGi	0.095 (—80)	0.120 (—59)	<b>0.325</b> (—80)	<b>0.394</b> (—80)	0.169 (—80)	0.299 (—16)
AEDWi	<b>0.282</b> (—63)	<b>0.412</b> (—59)	0.101 (—80)	0.122 (—80)	0.216 (—62)	0.317 (—59)
AEDSi	0.148 (—60)	<b>0.335</b> (0)	<b>0.256</b> (—80)	0.310 (—80)	0.100 (—80)	0.129 (0)
AERKi	0.163 (—65)	0.231 (—63)	<b>0.164</b> (—80)	<b>0.327</b> (—22)	0.109 (—19)	0.230 (—17)
AERGi	0.076 (—20)	0.159 (—19)	<b>0.306</b> (—80)	<b>0.429</b> (—21)	0.160 (—19)	0.339 (—16)
AERWi	<b>0.272</b> (—65)	<b>0.391</b> (0)	0.069 (—63)	0.153 (0)	0.205 (—64)	0.295 (—61)
AERSi	<b>0.276</b> (—67)	<b>0.389</b> (—63)	0.098 (—22)	0.202 (—20)	0.207 (—66)	0.294 (—63)

and

$$f_i = 1.0003 + 4.18 \times 10^{-6} p. \quad (20)$$

Equations (19) and (20) have maximum relative errors of 0.183% and 0.438%, respectively. The new approximations (17) and (18) of the enhancement coefficient are more accurate than Bucks approximations.

## 6. Conclusions

To calculate the saturation vapor pressure of pure water vapor over a plane surface of water for the temperature range of  $-40^\circ$  to  $50^\circ\text{C}$  when working with standard surface and upper-air data, the approximation AERK is recommended:

$$e_w(t) = 6.1094 \times e^{17.625t/(243.04+t)}. \quad (21)$$

This equation has a maximum relative error of less than 0.384% and a maximum measure of accuracy of less than 0.868 with respect to any of the three basic formulations GG46, WE76, and SO90.

For moist air above a plane surface of water, the following is recommended:

$$e_{wa}(t) = 1.00071 e^{0.0000045p} e_w(t). \quad (22)$$

Equation (22) has a maximum relative error of less than 0.414% and a maximum measure of accuracy less than 0.932 with respect to the three basic formulations when using the WMO's formulation (1966) for the enhancement factor.

For saturation vapor pressure over a plane surface of ice ( $-80^\circ$  to  $0.0^\circ\text{C}$ ), the following two approximations are recommended. To calculate the saturation vapor pressure of pure water vapor over ice use AERKi

$$e_i(t) = 6.1121 e^{22.587t/(273.86+t)}, \quad (23)$$

which has a maximum relative error less than 0.213% and a maximum measure of accuracy of less than 0.258 with respect to any from three basic formulations GG46, WE76, or SO90. For moist air, use the following equation:

$$e_{ia}(t) = 0.99882 e^{0.000008p} e_i(t). \quad (24)$$

Equation (24) has a maximum relative error less than 0.397% and a maximum measure of accuracy less than 0.721 with respect to any from three basic formulations and the WMO (1966) enhancement factor.

To calculate the saturation vapor pressure with respect to water for the temperature range of  $-80^\circ$  to  $50^\circ\text{C}$ , any of the equations in (13) can be used. It is not possible to find a Magnus form approximation that fits all three basic formulations for this temperature range that does not have large errors because the basic formulations divergence below  $-40^\circ\text{C}$ .

Recently, Gueymard (1993) has proposed a new reference dataset for determining saturation vapor pressure. Gueymard (1993) has found several very accurate

approximations for the saturation vapor pressure using this dataset. However, as we have shown, to take advantage of the Gueymard equations requires temperature data reported to more than one decimal place.

Equation (21) for the temperature range of  $-80^\circ$  to  $50^\circ\text{C}$  closely matches the Goff-Gratch formulation (see Table 3) Gueymard (1993) recommended for temperatures below  $0^\circ\text{C}$ .

The errors discussed in this paper are much less than observational errors in humidity values due to the hygrometers. But the use of (relatively) inaccurate formulas can lead to the appearance of systematic errors in meteorological data.

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## APPENDIX

### Nomenclature

$e$	Represents any approximation to one of the basic formulations ( $E$ )
$E$	Represents any of the basic formulations of saturation vapor pressure such as Goff and Gratch (1946)
$E_i$	Saturation vapor pressure (hPa) of pure water vapor with respect to a plane of ice
$E_{ia}$	Saturation vapor pressure (hPa) of moist air with respect to a plane of ice
$E_w$	Saturation vapor pressure (hPa) of pure water vapor with respect to a plane surface of pure water
$E_{wa}$	Saturation vapor pressure (hPa) of moist air with respect to a plane of water
$E_{w0}$	Saturation vapor pressure (hPa) at the steam point temperature for a pressure of 1013.246 hPa.
$F$	Correction factor to account for the departure of the mixture of air and water vapor from the ideal gas laws— $F_w = E_{wa}/E_w$ and $F_i = E_{ia}/E_i$
$t$	Temperature in degrees Celsius
$T$	Temperature in kelvins
$T_s$	Temperature at the steam point ( $=373.16\text{ K}$ );
$T_1$	Temperature at the triple point of water ( $273.16\text{ K}$ )

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