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Improved Sample Complexity Estimates for Statistical Learning Control of Uncertain Systems

V. Koltchinskii, C. T. Abdallah, M. Ariola, P. Dorato, and D. Panchenko

***Abstract*—Recently, probabilistic methods and statistical learning theory have been shown to provide approximate solutions to "difficult" control problems. Unfortunately, the number of samples required in order to guarantee stringent performance levels may be prohibitively large. This paper introduces bootstrap learning methods and the concept of stopping times to drastically reduce the bound on the number of samples required to achieve a performance level. We then apply these results to obtain more efficient algorithms which probabilistically guarantee stability and robustness levels when designing controllers for uncertain systems.**

***Index Terms*—Decidability theory, $\mathcal{NP}$-hard problems, Radamacher bootstrap, robust control, sample complexity, statistical learning.**

I. Introduction

It has recently become clear that many control problems are too difficult to admit analytic solutions [1], [2]. New results have also emerged to show that the computational complexity of some "solved" control problems is prohibitive [3], [4]. Many of these (linear and nonlinear) control problems can be reduced to decidability problems or to optimization questions, both of which can then be reduced to the question of finding a real vector satisfying a set of (polynomial) inequalities. Even though such questions may be too difficult to answer analytically, or may not be answered exactly given a reasonable amount of computational resources, researchers have shown that we can "approximately" answer these questions "most of the time," and have "high confidence" in the correctness of the answers. Many authors have recently advanced the notion of probabilistic methods in control analysis and design. These methods build on the standard Monte Carlo approach (with justifications based on Chernoff Bounds, Hoeffding inequality, and other elementary probabilistic tools [5], [6]) with ideas advanced during the 1960s and 1970s [7] on the theory of empirical processes and statistical learning. In control theory, some of the original (Monte Carlo) ideas have already been used by Lee and Poolla [8], Ray and Stengel [9], Tempo *et al.* [10], [11], Barmish *et al.* [12], Chen and Zhou [13], and by Khargonakar and Tikku [14], to solve *robust analysis* problems while Vidyasagar used learning theory to solve *robust design* problems [15], [16].

Unfortunately, and as acknowledged by the various authors, probabilistic methods, while more efficient than gridding techniques (which suffer from the curse of dimensionality), still require a large number of samples in order to guarantee accurate designs. As an example, Vidyasagar in calculates that more than two million plant samples are needed in order to probabilistically guarantee a certain performance level in a robust control design problem. On the other hand, it was conjectured and verified experimentally that much smaller bounds on the number of plant samples may be sufficient (tens of thousands instead of millions) to guarantee a certain level of performance [16]. In fact, Vidyasagar in [16] uses 200 plants instead of the millions implied by his bounds, while acknowledging that the theoretical guarantees of accuracy and confidence no longer hold. The question then becomes: what (if any) guarantees are obtained by the smaller number of samples, or more appropriately, is there a smaller bound on the number of samples of plants which can still guarantee the desired level of accuracy and confidence?

This paper answers the last question affirmatively, and does so by invoking different versions of *bootstrap sequential learning* algorithms. For these algorithms, the necessary number of samples (known as the sample complexity of learning) is a random variable whose value is not known in advance and is to be determined in the process of learning. This value is bounded below by the sample size at which the algorithm starts to work, and bounded above by conservative upper bounds of the sample complexity, which are of the same order as the bounds well known in statistical learning theory, used, for instance, by Vidyasagar [6]. This will also lead to the notion of *efficient learning times* which is then used to present our results in a computationally attractive manner.

The mathematical justification of the methods of learning suggested in this paper relies heavily upon the methods of the empirical processes theory. This theory started in the seminal papers of Vapnik and Chervonenkis [17] and Dudley [18]. The exposition of more recent results on empirical processes can be found in [19] and [20], which also contain a number of deep applications of empirical processes in statistics. The applications of empirical processes to statistical learning problems are discussed in great detail in [21], [7], [2], and [6]. The major technical tools used in our paper are concentration inequalities for empirical and related processes. We are using in the current version of the results a relatively old form of these inequalities based on the extension of the classical Hoeffding-type bounds to the martingale differences. This extension is due to Azuma [23] and it was used very successfully by Yurinskii [24] in the problems of Probability in Banach Spaces.

The remainder of this paper is divided as follows: Section II contains the bootstrap learning method and its applications to control problems. Section III contains a numerical example illustrating our approach and contrasting it with earlier results, while Section IV contains conclusions and an outline for future research.

II. Sequential Learning Algortihms

In this section, we present sequential algorithms for a general problem of empirical risk minimization. They are designed to overcome some of the difficulties encountered with the standard learning methods [15], [16]. These algorithms do not depend on the explicit calculation of the VC-dimension (see for instance [6] for a definition of VC-dimension), although its finiteness remains critical to the termination of the design algorithm, in the distribution-free learning case. The sequential algorithms chosen are based on *Rademacher bootstrap* although other bootstrap techniques, developed in statistics (for instance, standard Efron bootstrap or various versions of weighted bootstrap), can also be adopted for our purposes. An important feature of our approach is the randomness of the sample size for which a given accuracy of learning is achieved with a guaranteed probability. Thus, the sample complexity of our method of learning is rather a random variable. Its value is not known in advance and is to be determined in the process of learning. The lower bound for this random variable is the value of the sample size which the sequential learning algorithm starts

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working with. The upper bounds for the random sample complexity are of the same order of magnitude as the standard conservative upper bounds for the sample complexity of empirical risk minimization algorithms. Thus, *in the worst case*, the sequential method of learning would take as much time (up to a numerical constant) as the standard methods do. We start with a brief overview of standard learning theory concepts.

Let $(S, \mathcal{A})$ be a measurable space and let $\{X_n\}_{n\geq 1}$ be a sequence of independent identically distributed (i.i.d.) observations in this space with common distribution P. We assume that this sequence is defined on a probability space $(\Omega, \Sigma, \mathbb{P})$. Denote by $\mathcal{P}(S) := \mathcal{P}(S, \mathcal{A})$ the set of all probability measures on $(S, \mathcal{A})$. Suppose $\mathcal{P} \subset \mathcal{P}(S)$ is a class of probability distributions such that $P \in \mathcal{P}$. One of the central problems of statistical learning theory is *the risk minimization problem*. Given a class $\mathcal{F}$ of $\mathcal{A}$-measurable functions f from S into [0, 1] (e.g., decision rules in a pattern recognition problem or performance indices in control problems), the risk functional is defined as

$$R_P(f) := P(f) := \int_S f dP := \mathbb{E}f(X), \qquad f \in \mathcal{F}.$$

The goal is to find a function f_P that minimizes R_P on $\mathcal{F}$. A method of *empirical risk minimization* is widely used in learning theory. Namely, the unknown distribution P is replaced by *the empirical measure* P_n, defined as

$$P_n(A) := \frac{1}{n}\sum_{k=1}^{n} I_A(X_k), \qquad A \in \mathcal{A}$$

where $I_A(x) = 1$ for $x \in A$ and $I_A(x) = 0$ for $x \notin A$. The risk functional R_P is replaced by the empirical risk R_{P_n}, defined by

$$R_{P_n}(f) := P_n(f) := \int_S f dP_n := \frac{1}{n}\sum_{k=1}^{n} f(X_k), \qquad f \in \mathcal{F}.$$

The problem is now to minimize the empirical risk R_{P_n} on $\mathcal{F}$.

Definition 1: Let $\{\Sigma_n\}_{n\geq 1}$ consist of the events that occur by time n (in particular, the value of random variable X_n is known by time n). A random variable τ, taking positive integer values, will be called a stopping time if and only if (iff), for all $n \geq 1$, we have $\{\tau = n\} \in \Sigma_n$. In other words, the decision whether $\tau \leq n$, or not, depends only on the information available by time n.

Given $\varepsilon > 0$ and $\delta \in (0, 1)$, let $\overline{n}(\varepsilon, \delta)$ denote the initial sample size of our learning algorithms. We assume that $\overline{n}$ is a nonincreasing function in both ε and δ. Denote by $\mathcal{T}(\varepsilon, \delta) := \mathcal{T}_{\mathcal{F}, \mathcal{P}}(\varepsilon, \delta)$ the set of all stopping times τ such that $\tau \geq \overline{n}(\varepsilon; \delta)$ and

$$\sup_{P\in\mathcal{P}} \mathbb{P}\{\|P_\tau - P\|_{\mathcal{F}} \geq \varepsilon\} \leq \delta. \tag{1}$$

If now $\tau \in \mathcal{T}(\varepsilon, \delta)$ and $\hat{f} := f_{P_\tau}$ is a function that minimizes the empirical risk based on the sample $(X_1, \ldots, X_\tau)$ then a bound similar to (1) immediately implies that

$$\sup_{P\in\mathcal{P}} \mathbb{P}\left\{R_P(f_{P_\tau}) \geq \inf_{f\in\mathcal{F}} R_P(f) + 2\varepsilon\right\} \leq \delta.$$

The questions, though, are how to construct a stopping time from the set $\mathcal{T}(\varepsilon, \delta)$, based only on the available data (without using the knowledge of P) and which of the stopping times from this set is best used in the learning algorithms. The following definition will be useful in this connection.

Definition 2: A parametric family of stopping times $\{\nu(\varepsilon, \delta)\colon \varepsilon > 0, \delta \in (0, 1)\}$ is called strongly (statistically) efficient for the class $\mathcal{F}$ with respect to $\mathcal{P}$ iff there exist constants $K_1 \geq 1$, $K_2 \geq 1$, and $K_3 \geq 1$ such that for all $\varepsilon > 0$ and $\delta \in (0, 1)$

$$\nu(\varepsilon, \delta) \in \mathcal{T}(K_1\varepsilon, \delta)$$

and for all $\tau \in \mathcal{T}(\varepsilon, \delta)$

$$\sup_{P\nu\mathcal{P}} \mathbb{P}\{\nu(K_2\varepsilon, \delta) > \tau\} \leq K_3\delta.$$

Thus, using strongly efficient stopping time $\nu(\varepsilon; \delta)$ allows one to solve the problem of empirical approximation with confidence $1 - \delta$ and accuracy $K_1\varepsilon$. With probability at least $1 - K_3\delta$, the time required by this algorithm is less than the time needed for *any* sequential algorithm of empirical approximation with accuracy ε/K_2 and confidence $1 - \delta$.

Definition 3: We call a family of stopping times $\{\nu(\varepsilon, \delta)\colon \varepsilon > 0, \delta \in (0, 1)\}$ weakly (statistically) efficient for the class $\mathcal{F}$ with respect to $\mathcal{P}$ iff there exist constants $K_1 \geq 1$, $K_2 \geq 1$, and $K_3 \geq 1$ such that for all $\varepsilon > 0$ and $\delta \in (0, 1)$

$$\nu(\varepsilon, \delta) \in \mathcal{T}(K_1\varepsilon, \delta)$$

and

$$\sup_{P\in\mathcal{P}} \mathbb{P}\{\nu(K_2\varepsilon, \delta) > N(\varepsilon; \delta)\}\} \leq K_3\delta.$$

Using weakly efficient stopping time $\nu(\varepsilon; \delta)$ also allows one to solve the problem of empirical approximation with accuracy $K_1\varepsilon$ and confidence $1 - \delta$. With probability at least $1 - K_3\delta$, the time required by this algorithm, is less than the sample complexity of empirical approximation with accuracy ε/K_2 and confidence $1 - \delta$. Note that, under the assumption $N(\varepsilon; \delta) \geq \overline{n}(\varepsilon; \delta)$, we have $N(\varepsilon, \delta) \in \mathcal{T}(\varepsilon, \delta)$. Hence, any strongly efficient family of stopping times is also weakly efficient. The converse to this statement is not true [25]. We show below how to construct efficient stopping times for empirical risk minimization problems. The construction is based on a version of bootstrap. Let $\{r_n\}_{n\geq 1}$ be *a Rademacher sequence* (i.e., a sequence of i.i.d. random variables taking values $+1$ and -1 with probability 1/2 each). We assume, in addition, that this sequence is independent of the observations $\{X_n\}_{n\geq 1}$. Suppose that (with $\lfloor\cdot\rfloor$ denoting the floor of the argument)

$$\overline{n}(\varepsilon, \delta) \geq \left\lfloor \frac{4}{\varepsilon^2} \log\left(\frac{2}{\delta(1 - e^{-\varepsilon^2/4})}\right)\right\rfloor + 1.$$

Let

$$\nu(\varepsilon, \delta) := \nu_{\mathcal{F}}(\varepsilon, \delta) := \min\left\{n \geq \overline{n}(\varepsilon, \delta)\colon \left\|n^{-1}\sum_{j=1}^{n} r_j\delta_{X_j}\right\|_{\mathcal{F}} \leq \varepsilon\right\}$$

where $\delta_x(f) := f(x)$.Note that for all $\varepsilon > 0$ and for all $\delta \in (0, 1)$, $\nu(\varepsilon, \delta)$, is a stopping time and it can be computed by Monte Carlo simulation of the sequence $\{r_j\}_{j\geq 1}$. The finiteness with probability one of the stopping time $\nu(\varepsilon; \delta)$ (and other stopping times, defined below) can be shown to follow from the Glivenko–Cantelli property for the class $\mathcal{F}$ (also referred to as UCEM property [6]). Define

$$\nu(\varepsilon, \delta) := \nu_{\mathcal{F}}(\varepsilon, \delta) := \min\left\{n\colon \left\|n^{-1}\sum_{j=1}^{n} r_j\delta_{X_j}\right\|_{\mathcal{F}} \leq \varepsilon, \quad n := n_k := 2^k\overline{n}(\varepsilon, \delta), \quad k = 0, 1, \ldots\right\}.$$

Theorem 1: Suppose that

$$\overline{n}(\varepsilon, \delta) \geq \left\lfloor \frac{4}{\varepsilon^2} \log\left(\frac{4}{\delta}\right) \right\rfloor + 1.$$

Then, for all $\varepsilon > 0$, $\delta \in (0, 1)$ we have the following.

1) $\nu(\varepsilon; \delta) \in \mathcal{T}(K_1\varepsilon; \delta)$ with $K_1 = 5$.
2) Moreover, suppose that

$$N(\varepsilon, \delta) \geq \overline{n}(\varepsilon, \delta) \geq \left\lfloor \frac{4}{\varepsilon^2} \log\left(\frac{4}{\delta}\right) \right\rfloor + 1.$$

Then $\{\nu_{\mathcal{F}}(\varepsilon, \delta)\colon \varepsilon > 0, \delta \in (0, 1/2)\}$ is a weakly efficient family of stopping times for any class $\mathcal{F}$ of measurable functions from S into [0, 1] with respect to the set $\mathcal{P}(S)$ of all probability distributions on S.

Proof: See the Appendix.

The result in Theorem 1 can be used to find a probably approximate near minimum of a stochastic process R with confidence $1 - \delta$, level α and accuracy ε as defined next (see also [15]).

Definition 4: Suppose that $R\colon \mathcal{Y} \to \mathbb{R}$ is a stochastic process, that Q is a given probability measure on $\mathcal{Y}$, and that $\alpha \in (0, 1)$, $\delta \in (0, 1)$, and $\varepsilon > 0$ are given. A number $\hat{R}$ is a *probably approximate near minimum of R with confidence $1 - \delta$, level α and accuracy ε*, if

$$\mathbb{P}\left\{ \inf_{Y \in \mathcal{Y}} R(Y) - \varepsilon \leq \hat{R} \leq \inf_{Y \in \mathcal{Y} \setminus S} R(Y) + \varepsilon \right\} \geq 1 - \delta$$

with some measurable set $\mathcal{S} \subseteq \mathcal{Y}$ such that $Q(\mathcal{S}) \leq \alpha$.

An interpretation of Definition 4 is that we are not searching for the minimum over all of the set $\mathcal{Y}$ but only over its subset $\mathcal{Y} \setminus \mathcal{S}$, where $\mathcal{S}$ has a small measure (at most α). Unless the actual infimum R^* is attained in the exceptional set $\mathcal{S}$, $\hat{R}$ is within ε from the actual infimum with confidence $1-\delta$. Although using Monte Carlo-type minimization, it is unlikely to obtain a better estimate of R^* than $\hat{R}$ (since the chances of getting into the set $\mathcal{S}$ are small), nothing can be said in practice about the size of the difference $\hat{R} - R^*$.

Based on Theorem 1, a probably approximate near minimum of f with confidence $1 - \delta$, level α and accuracy ε, can be found with the following algorithm.

Algorithm 1: Given:

- sets $\mathcal{X}$ and $\mathcal{Y}$;
- probability measures P on $\mathcal{X}$ and Q on $\mathcal{Y}$;
- a measurable function $f\colon \mathcal{X} \times \mathcal{Y} \to [0, 1]$;
- an accuracy parameter $\varepsilon \in (0, 1)$, a level parameter $\alpha \in (0, 1)$, and a confidence parameter $\delta \in (0, 1)$.

Let $R_P(\cdot) = \mathbb{E}_P[f(X, \cdot)]$ and $R_{P_n}(\cdot) = (1/n)\sum_{j=1}^{n} f(X_j, \cdot)$. Then we have the following.

1) Choose integers m and n

$$m \geq \frac{\log(2/\delta)}{\log[1/(1-\alpha)]} \qquad n = \left\lfloor \frac{100}{\varepsilon^2} \log\left(\frac{8}{\delta}\right) \right\rfloor + 1.$$

2) Generate m independent samples according to distribution Q and n independent samples according to distribution P.
3) Evaluate the stopping variable

$$\gamma = \max_{1 \leq i \leq m} \left| \frac{1}{n} \sum_{j=1}^{n} r_j f(X_j, Y_i) \right|$$

where r_j are *Rademacher* random variables, i.e., independent identically distributed random variables (also independent of the plant sample) taking values $+1$ and -1 with probability 1/2 each. If $\gamma > \varepsilon/5$, add n more independent samples generate according to distribution P, set $n := 2n$ and repeat Step 3.
4) Let $\hat{R} = \min_{1 \leq i \leq m} R_{P_n}$. Then with confidence at least $1 - \delta$, $\hat{R}$ is a minimum of R_P to a level α and accuracy ε.

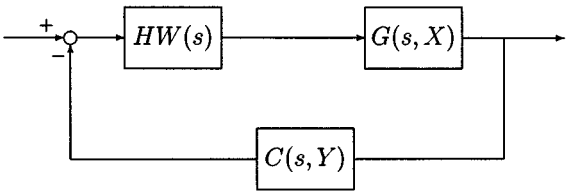

Fig. 1. The closed-loop system.

Comparison with Earlier Algorithms: In order to solve the problem of designing robust controllers, in [15] two other algorithms were proposed. The first of them is based on the Hoeffding's inequality, whereas the second one is obtained from VC theory. In all the cases, since the minimization is carried out in a Monte Carlo fashion, the number of controllers evaluates to

$$m \geq \frac{\log(2/\delta)}{\log[1/(1-\alpha)]}$$

as in our Algorithm 1 (see also Section III). On the other hand, using the Hoeffding's inequality the number of plants that are needed is

$$n \geq \frac{1}{2\varepsilon^2} \log \frac{4m}{\delta}$$

whereas based on VC theory n evaluates to

$$n \geq \max\left\{ \frac{16}{\varepsilon^2} \log \frac{4}{\delta}, \frac{32d}{\varepsilon^2} \log \frac{32e}{\varepsilon^2} \right\}$$

where d is an upper bound of the VC-dimension. We have already discussed at the beginning of Section II the advantages of our method over the methods which are based on the VC theory (see also the example in Section III). On the other hand, it is possible to show that, even though m and n are *coupled* in the bounds based on the Hoeffding's inequalities, unless one chooses an *extremely* small α, Hoeffding's bounds result to be more computationally efficient. Moreover in the multidimensional situation, the simple Monte Carlo scheme of minimization can be very misleading and the empirical minimum can be much larger than the true minimum with probability practically equal to one [25]. In these cases, one has to choose α extremely small such that the computational efficiency of the algorithm based on the Hoeffding's bounds disappears. In such situations, more efficient methods of minimization [26] should be used and their justification would heavily rely on statistical learning theory. Therefore in these cases the Hoeffding's bounds could not be used anymore.

III. Applications to Control Design

In this example we consider the control problem presented by Vidyasagar in [16] and solved via randomized algorithms. This will allow us to illustrate our method and to compare it to the ones proposed in [16]. The example concerns the design of an inner-loop controller for the longitudinal axis of an aircraft. The problem is to minimize the weighted sensitivity function over a certain set of uncertain plants, given some constraints on the nominal plant.

The closed-loop system is shown in Fig. 1. The plant $G(s, X)$ is in the form

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

where

$$A = \begin{bmatrix} Z_\alpha & 1 - Z_q \\ M_\alpha & M_q \end{bmatrix}, \quad B = \begin{bmatrix} Z_{\delta e} \\ M_{\delta e} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The parameters of the matrices have Gaussian distribution with means and standard deviations as in Table I. In the following, we let $X = [Z_\alpha \;\; Z_q \;\; M_\alpha \;\; M_q \;\; Z_{\delta e} \;\; M_{\delta e}]^T$.

TABLE I
PARAMETERS FOR THE AIRCRAFT MODEL

Parameter	Mean	Standard Deviation
Z_α	−0.9381	0.0736
Z_q	0.0424	0.0035
M_α	1.6630	0.1385
M_q	−0.8120	0.0676
$Z_{\delta e}$	−0.3765	0.0314
$M_{\delta e}$	−10.8791	3.4695

The transfer function $HW(s)$ models the different hardware components, such as the sensors, the actuators, the structural filters, etc. It is given by

$$HW(s) = \frac{0.000697s^2 - 0.0397s + 1}{0.000867s^2 + 0.0591s + 1}.$$

We will denote by $G_0(s)$ the nominal plant and by $\hat{G}(s, X)$, [respectively, $\hat{G}_0(s)$] the series connection $G(s, X)HW(s)$ [respectively, $G_0(s)HW(s)$].

We choose the controller to have the following structure:

$$C(s, Y) = \begin{bmatrix} -K_a & -K_q \dfrac{(1 + s\tau_1)}{(1 + s\tau_2)} \end{bmatrix}$$

where the four parameters K_a, K_q, τ_1, and τ_2 have uniform distributions in the ranges

$$K_a \in [0, 2], \quad K_q \in [0, 1] \quad \tau_1 \in [0.01, 0.1], \quad \tau_2 \in [0.01, 0.1].$$

We thus let $Y = [K_a \quad K_q \quad \tau_1 \quad \tau_2]^T$. Our objective is to find the controller which solves the following problem:

$$\min \left\| W\left(I + \hat{G}C\right)^{-1} \right\|_\infty \text{ subject to } \left\| \frac{0.75C\hat{G}_0}{1 + 1.25C\hat{G}_0} \right\|_\infty \leq 1$$

where the weighting function $W(s)$ is given by

$$W(s) = \begin{bmatrix} \dfrac{2.8 * 6.28 * 31.4}{(s + 6.28)(s + 31.4)} & 0 \\ 0 & \dfrac{2.8 * 6.28 * 3.14}{(s + 6.28)(s + 31.4)} \end{bmatrix}.$$

In order to adopt a randomized algorithm solution, in [16], this problem has been reformulated in the following way. Let us define a cost function

$$\Psi(Y) = \max\{\psi_1(Y), \psi_2(Y)\}$$

where

$$\psi_1(Y) = \begin{cases} 1, & \text{if } \left\| \dfrac{0.75C\hat{G}_0}{1 + 1.25C\hat{G}_0} \right\|_\infty > 1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$\psi_2(Y) = E_P(\zeta(X, Y))$$

with

$$\zeta(X, Y) = \begin{cases} 1, & \text{if } \left(\hat{G}(X), C(Y)\right) \text{ is unstable} \\ \dfrac{\|W\left(I + \hat{G}(X)C(Y)\right)^{-1}\|_\infty}{1 + \|W\left(I + \hat{G}(X)C(Y)\right)^{-1}\|_\infty}, & \text{otherwise.} \end{cases}$$

In our example, and for $\delta = 0.01$, $\alpha = 0.1$, and $\varepsilon = 0.1$, m evaluated to 51 controllers and n evaluated to 66 848 plants and the procedure outlined in Algorithm 1 stopped after one iteration, i.e., $k = 1$. The parameters of the *statistically optimal* controller are

$$K_a = 1.7826, \quad K_q = 0.7621, \quad \tau_1 = 0.0511, \quad \tau_2 = 0.0117$$

and the corresponding value of the cost function is $\hat{\Psi}(Y_{\text{opt}}) = 0.7149$, which compares favorably with the results of [16], where 2 619 047 plants were needed for the same ε, α, and δ.

IV. CONCLUSIONS

In this paper we have drastically reduced the number of plant samples needed in order to obtain performance guarantees in robust control synthesis problems. This reduction is achieved by introducing sequential bootstrapping algorithms and exploiting the fact that the sample complexity is itself a random variable. This has allowed us to present Algorithm 1 as an efficient design methodology for fixed-order robust control design problems [27]. Recall for example that the static output feedback (SOF) was shown in [1] to be NP-hard when the gains of the feedback matrix were bounded, but that Algorithm 1, is well suited to address the SOF problem exactly under those conditions.

It should be noted that the methodology presented in this paper can be used in many other application areas: one only needs to have an efficient analysis tool in order to convert it to an efficient design methodology. This is due to the fact that the design problem is converted to a sequence of analysis or verification problems after sampling more plants and controllers than the minimum number required by Algorithm 1. It should also be noted that the computational complexity or the undecidability of the problems studied are not eliminated but only avoided by relaxing the design requirements from absolute (hard) to probabilistic (soft) ones.

The randomized algorithms approach may be applied to design fixed-structure controllers for nonlinear systems and to building software systems for practical control design problems. Our future research is concentrating at the theoretical level in obtaining better optimization algorithms and at the application level in designing software modules for linear and nonlinear control design.

APPENDIX

The proof of Theorem 1 needs some preliminary lemmas.

Lemma 1: For all $\varepsilon > 0$

$$\mathbb{P}\{\|P_n - P\|_{\mathcal{F}} \geq \mathbb{E}\|P_n - P\|_{\mathcal{F}} + \varepsilon\} \leq \exp\{-\varepsilon^2 n/2\}$$

and

$$\mathbb{P}\{\mathbb{E}\|P_n - P\|_{\mathcal{F}} \geq \|P_n - P\|_{\mathcal{F}} + \varepsilon\} \leq \exp\{-\varepsilon^2 n/2\}.$$

Lemma 2: For all $\varepsilon > 0$

$$\mathbb{P}\left\{\mathbb{E}\left\|n^{-1}\sum_{j=1}^{n} r_j \delta_{X_j}\right\|_{\mathcal{F}} \geq \left\|n^{-1}\sum_{j=1}^{n} r_j \delta_{X_j}\right\|_{\mathcal{F}} + \varepsilon\right\} \leq \exp\{-\varepsilon^2 n/4\}$$

and

$$\mathbb{P}\left\{\left\|n^{-1}\sum_{j=1}^{n} r_j \delta_{X_j}\right\|_{\mathcal{F}} \geq \mathbb{E}\left\|n^{-1}\sum_{j=1}^{n} r_j \delta_{X_j}\right\|_{\mathcal{F}} + \varepsilon\right\} \leq \exp\{-\varepsilon^2 n/4\}.$$

Lemma 3: The following inequality holds:

$$\tfrac{1}{2}\mathbb{E}\left\|n^{-1}\sum_{j=1}^{n} r_j(\delta_{X_j} - P)\right\|_{\mathcal{F}} \leq \mathbb{E}\|P_n - P\|_{\mathcal{F}} \leq 2\mathbb{E}\left\|n^{-1}\sum_{j=1}^{n} r_j \delta_{X_j}\right\|_{\mathcal{F}}.$$

The proofs of Lemmas 1, 2 follow from the well known and widely used concentration inequalities for martingale difference sequences (see, e.g., Ledoux and Talagrand [28, Lemma 1.5]). See also [21, Ths. 9.1, 9.2]. The proof of Lemma 3 can be found, for instance, in [20].

Lemma 4: Suppose Z_1, Z_2 are independent stochastic processes in $\ell^\infty(\mathcal{F})$. Then for all $t > 0$, $c > 0$

$$\mathbb{P}\{\|Z_1\|_{\mathcal{F}} \geq t + c\} \leq \frac{\mathbb{P}\{\|Z_1 - Z_2\|_{\mathcal{F}} \geq t\}}{\inf_{f \in \mathcal{F}} \mathbb{P}\{|Z_2(f)| \leq c\}}.$$

Proof of Theorem 1: We set $\overline{n} := \overline{n}(\varepsilon; \delta)$, we then have here

$$\begin{aligned}\mathbb{P}&\left(\bigcap_{n \in \{2^k \overline{n}:\, k=0,\,1,\,\ldots\}} \{\|P_n - P\|_{\mathcal{F}} \leq \mathbb{E}\|P_n - P\|_{\mathcal{F}} + \varepsilon\}\right)\\ &\geq 1 - \sum_{k=0}^{\infty} \exp\{-\varepsilon^2 \overline{n} 2^k/4\}\\ &\geq 1 - 2\exp\{-\varepsilon^2 \overline{n}/4\} \geq 1 - \delta/2\end{aligned}$$

where we have used the fact that for any $\alpha \geq 1$ we have

$$\begin{aligned}\sum_{k=1}^{\infty} \exp\{-\alpha(2^k - 1)\} &\leq \sum_{k=1}^{\infty} \exp\{-(2^k - 1)\}\\ &\leq \sum_{k=1}^{\infty} e^{-k} = (e-1)^{-1} < 1\end{aligned}$$

and hence

$$\sum_{k=0}^{\infty} \exp\{-\alpha 2^k\} \leq 2e^{-\alpha}.$$

To prove the second property in the definition of the weakly efficient stopping times, let $N := N(\varepsilon; \delta)$, let $n_k := 2^k \overline{n}(24\varepsilon; \delta)$ and choose k such that $n_k \leq N < n_{k+1}$. Then

$$\mathbb{P}\{\nu(24\varepsilon; \delta) > N\} \leq \mathbb{P}\{\nu(24\varepsilon; \delta) > n_k\}.$$

If $\nu(24\varepsilon; \delta) > n_k$, then for $n = n_k$

$$\left\| n^{-1} \sum_{j=1}^{n} r_j \delta_{X_j} \right\|_{\mathcal{F}} > 24\varepsilon.$$

Since, by the assumptions, $N \geq \overline{n}$, we get $n_k \geq \overline{n}/2$. Then we obtain that with probability $\geq 1 - \delta$

$$\|P_n - P\|_{\mathcal{F}} \geq \frac{1}{2} \left\| n^{-1} \sum_{j=1}^{n} r_j \delta_{X_j} \right\|_{\mathcal{F}} - \frac{1}{2\sqrt{n}} - 6\varepsilon$$

which implies that

$$\begin{aligned}\mathbb{P}\{\nu(24\varepsilon; \delta) > n_k\} &\leq \mathbb{P}\{\|P_{n_k} - P\|_{\mathcal{F}} \geq 4\varepsilon\} + \delta\\ &= \mathbb{P}\{\|S_{n_k}\|_{\mathcal{F}} \geq 4\varepsilon n_k\} + \delta\\ &\leq \mathbb{P}\{\|S_{n_k}\|_{\mathcal{F}} \geq 2\varepsilon N\} + \delta\end{aligned}$$

where

$$S_n(f) := \sum_{j=1}^{n} [f(X_j) - P(f)], \qquad f \in \mathcal{F}.$$

Next we use Lemma 4

$$\mathbb{P}\{\|S_{n_k}\|_{\mathcal{F}} \geq 2\varepsilon N\} \leq \frac{\mathbb{P}\{\|S_N\|_{\mathcal{F}} \geq \varepsilon N\}}{\inf_{f \in \mathcal{F}} \mathbb{P}\{|(S_N - S_{n_k})(f)| \leq \varepsilon N\}}$$

and by Hoeffding's inequality [6]

$$\begin{aligned}&\inf_{f \in \mathcal{F}} \mathbb{P}\{|(S_N - S_{n_k})(f)| \leq \varepsilon N\}\\ &\quad = 1 - \sup_{f \in \mathcal{F}} \mathbb{P}\{|(S_N - S_{n_k})(f)| > \varepsilon N\}\\ &\quad \geq 1 - 2\exp\{-\varepsilon^2 N/2\} \geq 1 - \delta\end{aligned}$$

we get

$$\begin{aligned}&\mathbb{P}\{\|S_{n_k}\|_{\mathcal{F}} \geq 2\varepsilon N\}\\ &\quad \leq (1-\delta)^{-1} \mathbb{P}\{\|P_N - P\|_{\mathcal{F}} \geq \varepsilon\} \leq \delta(1-\delta)^{-1}.\end{aligned}$$

Hence, we get

$$\mathbb{P}\{\nu(24\varepsilon; \delta) > n_k\} \leq \delta(1-\delta)^{-1} + \delta \leq 3\delta$$

for $\delta < 1/4$, which implies weak efficiency with $K_1 = 5$, $K_2 = 24$, and $K_3 = 3$.

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REFERENCES

[1] V. Blondel and J. Tsitsiklis, "NP-hardness of some linear control design problems," *SIAM J. Contr. Optimiz.*, vol. 35, pp. 2118–2127, 1997.
[2] ——, "Three problems on the decidability and complexity of stability," in *Open Problems in Mathematical Systems and Control Theory*, V. Blondel, E. Sontag, M. Vidyasagar, and J. Willems, Eds. London: Springer-Verlag, 1999.
[3] ——, "A survey of computational complexity results in systems and control," *Automatica*, vol. 36, pp. 1249–1274, Sept. 2000.
[4] O. Toker and H. Özbay, "On the NP-hardness of solving bilinear matrix inequalities and simultaneous stabilization with static output feedback," in *Proc. American Control Conf.*, Seattle, WA, June 1995, pp. 2056–2064.
[5] H. Chernoff, "A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations," *Ann. Math. Statist.*, pp. 493–507, 1952.
[6] M. Vidyasagar, *A Theory of Learning and Generalization with Applications to Neural Networks and Control Systems*. Berlin, Germany: Springer-Verlag, 1996.
[7] V. Vapnik, *Estimation of Dependences Based on Empirical Data*. New York, NY: Springer-Verlag, 1982.
[8] L. H. Lee and K. Poolla, "Statistical validation for uncertain models," in *Feedback Control, Nonlinear Systems, and Complexity*, B. Francis and A. R. Tannenbaum, Eds. London: Springer-Verlag, 1995.
[9] L. Ray and R. Stengel, "A Monte Carlo approach to the analysis of control system robustness," *Automatica*, vol. 29, pp. 229–236, 1993.
[10] E. Bai, R. Tempo, and M. Fu, "Worst case properties of the uniform distribution and randomized algorithms for robustness analysis," in *Proc. American Control Conf.*, Albuquerque, NM, 1997, pp. 861–865.
[11] R. Tempo, E. Bai, and F. Dabbene, "Probabilistic robustness analysis: Explicit bounds for the minimum number of samples," *Syst. Contr. Lett.*, vol. 30, pp. 237–242, 1997.
[12] B. Barmish and C. Lagoa, "The uniform distribution: A rigorous justification for its use in robustness analysis," *Math. Contr., Signals, Syst.*, vol. 10, pp. 203–222, 1997.
[13] X. Chen and K. Zhou, "A probabilistic approach for robust control," in *Proc. 36th Conf. Decision Contr.*, San Diego, CA, 1997, pp. 4894–4895.
[14] P. P. Khargonekar and A. Tikku, "Randomized algorithms for robust control analysis and synthesis have polynomial complexity," in *Proc. 35th Conf. Decision Contr.*, Kobe, Japan, 1996, pp. 3470–3475.
[15] M. Vidyasagar, "Statistical learning theory and its applications to randomized algorithms for robust controller synthesis," in *European Control Conf. (ECC97)*, vol. Plenary Lectures and Mini-Courses, Brussels, Belguim, 1997, pp. 162–190.
[16] ——, "Statistical learning theory and randomized algorithms for control," *IEEE Contr. Syst. Mag.*, vol. 18, pp. 69–85, 1998.
[17] V. Vapnik and A. Chervonenkis, "On the uniform convergence of relative frequencies of events to their probabilities," *Theory Probability Appl.*, vol. 16, pp. 264–280, 1971.
[18] R. Dudley, "Central limit theorems for empirical measures," *Ann. Probability*, vol. 6, pp. 899–929, 1978.
[19] ——, *Uniform Central Limit Theorems*. Cambridge: Cambridge Univ. Press, 1999.
[20] A. van der Vaart and J. Wellner, *Weak Convergence and Empirical Processes*. Berlin, Germany: Springer-Verlag, 1996.
[21] L. Devroy, L. Gyorfi, and G. Lugosi, *A Probabilistic Theory of Pattern Recognition*. Berlin, Germany: Springer-Verlag, 1996.
[22] V. Vapnik, *Statistical Learning Theory*. New York: Wiley, 1998.

[23] K. Azuma, "Weighted sums of certain dependent random variables," *Tokuku Math. J.*, vol. 19, pp. 357–367, 1967.
[24] V. V. Yurinski, "Exponential bound for large deviations," *Theory Probability Appl.*, vol. 19, pp. 154–155, 1974.
[25] V. Koltchinskii, C. T. Abdallah, M. Ariola, P. Dorato, and D. Panchenko, "Statistical learning control of uncertain systems: It is better than it seems," Tech. Rep. EECE 99-001, EECE Dept., The University of New Mexico, Feb. 1999.
[26] V. Koltchinskii, "Rademacher penalties and structural risk minimization," *IEEE Trans. Inform. Theory*, 2000.
[27] V. Koltchinskii, M. Ariola, C. T. Abdallah, and P. Dorato, "Statistical controller design for the linear benchmark problem," in *Proc. 38th Conf. Decision Contr.*, Phoenix, AZ, 1999, pp. 507–509.
[28] M. Ledoux and M. Talagrand, *Probability in Banach Spaces*. Berlin, Germany: Springer-Verlag, 1991.

Input Output Linearization Approach to State Observer Design for Nonlinear System

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***Abstract*—In this paper, we present a state observer for a class of nonlinear systems based on the input output linearization. While the previous result presented state observers for nonlinear systems of full relative degree, we proposed a procedure for the design of nonlinear state observers which do not require the hypothesis of full relative degree. Assuming that there exists a global state observer for internal dynamics and that some functions are globally Lipschitz, we can design a globally convergent state observer. It is also shown that if the zero dynamics are locally exponentially stable, then there exists a local state observer. An example is given to illustrate the proposed design of nonlinear state observers.**

***Index Terms*—Coordinate change, nonlinear system, normal form, state observer.**

I. Introduction

The problem of observing the states of a nonlinear system has been considered in the literature. Some sufficient conditions for the existence of an observer have been established, and computational algorithms for construction of the observers have been presented. The first contributions to the nonlinear observer design were made by, for instance, [10] and [2]. Krener and Isidori [10] proposed the Lie-algebraic conditions under which nonlinear observers with linearizable error dynamics can be designed. Bestle and Zeitz [2] introduced a nonlinear observer canonical form in which system nonlinearities depend only on the input and output of the original system. To broaden the class of nonlinear systems for which a state observer exists, Keller [8] presented an observer design based on a transformation into a generalized observer canonical form (GOCF) that depends on the first n time derivatives of the input variables. Since afore-mentioned approaches require quite restrictive conditions on coordinate transformation, the problem of deriving approximate observers has been also studied in the literature [1], [11], [16].

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Apart from observers which are based on the state transformation into canonical forms, alternative methods for the design of nonlinear observers have been examined. Walcott and Zak [15] investigated an observer design technique utilizing theory of variable-structure systems (VSS) and Slotine *et al.* [12] discussed the potential use of sliding surfaces for observer design. Tsinias [14] provided a sufficient Lyapunov-like condition for the existence of a nonlinear observer and showed that it is equivalent to detectability condition for linear case. However, in general, the construction of this Lyapunov function is quite difficult. Gauthier *et al.* [5] showed that if a nonlinear system is uniformly observable for any inputs and some functions are globally Lipschitz, then there exists a nonlinear observer whose gain depends on the solution of some Lyapunov-like equation. Ciccarella *et al.* [4] proposed a nice extension of the Luengerger-like observer for nonlinear systems of full relative degree under the global Hölder condition for certain functions. However, if nonlinear system has relative degree less than system order, their technique requires additional assumption that some time derivatives of the input should be zero almost everywhere.

In this paper, we propose a global nonlinear observer that guarantees the estimation error to converge to zero asymptotically. Our scheme is based on the input output linearization technique and utilizes the state transformation into the normal form [7]. In contrast to [4], we do not require the hypothesis of full relative degree. Our main assumption is concerned with the existence of nonlinear observer for internal dynamics. Thus, the proposed technique can be regarded as a dual of stabilization problem via input–output linearization, since the latter is solvable if the zero dynamics of nonlinear system have a globally asymptotically stable equilibrium at the origin [3]. Moreover, as far as the local observation problem is concerned, the proposed condition is reduced to that the zero dynamics have a locally exponentially stable equilibrium at the origin.

Our paper is organized as follows. Section II states the problem formulation and motivation which comes from comparison between some result of the linear case and the previous work on nonlinear observer. Section III provides sufficient conditions for the existence of the proposed observer and the main theorem is given in Section IV. An illustrative example is given in Section V and finally some conclusions are given in Section VI.

Before we begin, some notations used in the paper are to be specified.

- A *Hurwitz matrix* will be a matrix with all eigenvalues λ such that $\mathrm{Re}(\lambda) < 0$.
- For any integer r, I_r denote the $r \times r$ identity matrix.
- For any integer r, 0_r denote the $r \times r$ zero matrix.
- $\|\cdot\|$ stands for the Euclidean norm of a vector in some Euclidean space.
- $\|x\|_\infty$ is defined by $\|x\|_\infty := \sup\{\|x(t)\|: t \geq 0\}$.
- Finally, the Jacobian matrices of $f(x_1, x_2) \in C^1$ with respect to its first and second argument at (x_1, x_2) are denoted by $D_1 f(x_1, x_2)$, $D_2 f(x_1, x_2)$, respectively.

II. Problem Formulation and Motivation

In this paper, we will consider the following single-input single-output (SISO) nonlinear systems:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{1}$$