Improved Sobolev's inequalities, relative entropy and fast diffusion equations

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Abstract

The difference of the two terms in Sobolev's inequality (with optimal constant) measures a distance to the manifold of the optimal functions. We give an explicit estimate of the remainder term and establish an improved inequality, with explicit norms and fully detailed constants. Our approach is based on nonlinear evolution equations and improved entropy - entropy production estimates along the associated flow. Optimizing a relative entropy functional with respect to a scaling parameter, or handling properly second moment estimates, turns out to be the central technical issue. This is a new method in the theory of nonlinear evolution equations, which also applies to other interpolation inequalities of Gagliardo-Nirenberg-Sobolev type.

Keywords. Sobolev's inequality; Gagliardo-Nirenberg-Sobolev inequalities; improved inequalities; manifold of optimal functions; entropy - entropy production method; fast diffusion equation; Barenblatt solutions; second moment; intermediate asymptotics; sharp rates; optimal constants

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1. **Introduction and main results**

Sobolev's inequality on the euclidean space of dimension $d \geq 3$ can be written as

$$\int_{\mathbb{R}^d} |\nabla f|^2 \, dx - \mathsf{S}_d \left(\int_{\mathbb{R}^d} |f|^{\frac{2d}{d-2}} \, dx \right)^{\frac{d-2}{d}} \geq 0 \quad \forall \, f \in \mathcal{D}^{1,2}(\mathbb{R}^d) \tag{1}$$

where $\mathcal{D}^{1,2}(\mathbb{R}^d)$ is the completion with respect to the norm $\| \cdot \|$ defined by $\|f\|^2 = \|\nabla f\|_2^2 + \|f\|_{2d/(d-2)}^2$ of the set of smooth functions with compact support. Here $\|f\|_q = (\int_{\mathbb{R}^d} |f|^q \, dx)^{1/q}$ denotes the usual Lebesgue norm. When S_d is the optimal constant, it is known since T. Aubin and G. Talenti's papers [3, 27] that equality is achieved if and only if $f(x) = (1 + |x|^2)^{-(d-2)/2}$ for any

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$x \in \mathbb{R}^d$, up to multiplications by constants, translations and scalings. More precisely, the set of non-negative optimal functions is parametrized by three parameters, $M > 0$, $y \in \mathbb{R}^d$ and $\sigma > 0$ and these functions take the form

$$f_{M,y,\sigma}(x) = \frac{1}{\sigma^{\frac{d}{2}} \left(C_M + \frac{1}{\sigma}\,|x-y|^2\right)^{\frac{d-2}{2}}} \quad \forall\, x \in \mathbb{R}^d$$

where C_M is uniquely determined in terms of M by the condition that

$$\int_{\mathbb{R}^d} f_{M,y,\sigma}^{\frac{2d}{d-2}}\,dx = M\;.$$

Such a condition can be solved explicitly and it can be shown that

$$C_M := \left(\frac{M}{M_*}\right)^{-\frac{2}{d}} \quad \text{with} \quad M_* := \int_{\mathbb{R}^d} \left(1+|x|^2\right)^{-d} dx = \pi^{\frac{d}{2}}\,\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma(d)}\;.$$

With these observations in hand, it is straightforward to recover that

$$\mathsf{S}_d = \pi\,d\,(d-2)\left(\frac{\Gamma(d/2)}{\Gamma(d)}\right)^{\frac{2}{d}}\;.$$

We shall write that $(M,y,\sigma) \in \mathcal{M}_d := (0,\infty) \times \mathbb{R}^d \times (0,\infty)$ and define the manifold of the optimal functions as

$$\mathfrak{M}_d := \left\{f_{M,y,\sigma}\;:\;(M,y,\sigma) \in \mathcal{M}_d\right\}\;.$$

Our goal is to understand how the left hand side in (1) determines a measure of the distance of f to $\mathfrak{M}_d$, *with explicit estimates.* Consider the *relative entropy* functional

$$\mathcal{R}[f] := \inf_{g \in \mathfrak{M}_d} \int_{\mathbb{R}^d} \left[g^{-\frac{2}{d-2}}\left(|f|^{\frac{2d}{d-2}} - g^{\frac{2d}{d-2}}\right) - \tfrac{d}{d-1}\left(|f|^{2\frac{d-1}{d-2}} - g^{2\frac{d-1}{d-2}}\right)\right] dx\;.$$

Our first result goes as follows.

Theorem 1. *Let $d \ge 3$. For any $f \in \mathcal{D}^{1,2}(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} |x|^2 |f|^{\frac{2d}{d-2}}\,dx$ is finite, we have*

$$\int_{\mathbb{R}^d} |\nabla f|^2\,dx - \mathsf{S}_d \left(\int_{\mathbb{R}^d} |f|^{\frac{2d}{d-2}}\,dx\right)^{\frac{d-2}{d}} \ge \frac{1}{8}\,(d-2)^2\,\frac{(\mathcal{R}[f])^2}{\int_{\mathbb{R}^d} |x|^2 |f|^{\frac{2d}{d-2}}\,dx}\;.$$

Next, define the weighted norm $\|\cdot\|_{2,q}$ by

$$\|f\|_{2,q} := \left(\int_{\mathbb{R}^d} |x|^2\,|f|^q\,dx\right)^{\frac{1}{q}}$$

and denote by 2^* the exponent $\frac{2\,d}{d-2}$. The functional $\mathcal{R}[f]$ is a measure of the distance of f to $\mathfrak{M}_d$ and we shall see in Theorem 6 that

$$\mathcal{R}[f] \ge \mathsf{C}_{\rm CK}\,\frac{\|f\|_{2,2^*}^{\frac{d}{d-2}}}{\|f\|_{2^*}^{\frac{3\,d+2}{d-2}}}\,\inf_{g \in \mathfrak{M}_d} \left\||f|^{\frac{2d}{d-2}} - g^{\frac{2d}{d-2}}\right\|_1^2 \tag{2}$$

with $\mathsf{C}_{\mathrm{CK}} = \left(\frac{d-2}{d}\right)^{3/2} \frac{M_*^{1/d}}{16\,(d-1)}$. Putting this estimate together with the result of Theorem 1, with

$$\mathfrak{C}_d := \frac{1}{8}\,(d-2)^2\,{\mathsf{C}_{\mathrm{CK}}}^2\,,$$

we obtain the following estimate.

Corollary 2. *Let $d \ge 3$. For any $f \in \mathcal{D}^{1,2}(\mathbb{R}^d)$, we have*

$$\|f\|_2^2 - \mathsf{S}_d\,\|f\|_{2^*}^2 \ge \frac{\mathfrak{C}_d}{\|f\|_{2^*}^{2\,\frac{3\,d+2}{d-2}}}\,\inf_{g\in\mathfrak{M}_d}\left\||f|^{\frac{2d}{d-2}} - g^{\frac{2d}{d-2}}\right\|_1^4\,.$$

Actually, our method applies not only to Sobolev's inequality but also to the following sub-family of the Gagliardo-Nirenberg-Sobolev inequalities

$$\|f\|_{2\,p} \le \mathcal{C}_{p,d}^{\mathrm{GN}}\,\|\nabla f\|_2^\theta\,\|f\|_{p+1}^{1-\theta} \tag{3}$$

with $\theta = \theta(p) := \frac{p-1}{p}\,\frac{d}{d+2-p\,(d-2)}$, $1 < p \le \frac{d}{d-2}$ if $d \ge 3$ and $1 < p < \infty$ if $d = 2$. Such an inequality holds for any smooth function f with sufficient decay at infinity and, by density, for any function $f \in \mathrm{L}^{p+1}(\mathbb{R}^d)$ such that ∇f is square integrable. We shall assume that $\mathcal{C}_{p,d}^{\mathrm{GN}}$ is the best possible constant in (3). In [18], it has been established that equality holds in (3) if $f = F_p$ with

$$F_p(x) = (1+|x|^2)^{-\frac{1}{p-1}}\quad \forall\,x\in\mathbb{R}^d \tag{4}$$

and that all extremal functions are equal to F_p up to a multiplication by a constant, a translation and a scaling. See Appendix A for an expression of $\mathcal{C}_{p,d}^{\mathrm{GN}}$. If $d \ge 3$, the limit case $p = d/(d-2)$ corresponds to Sobolev's inequality and one recovers that $\mathcal{C}_{d/(d-2),d}^{\mathrm{GN}} = 1/\sqrt{\mathsf{S}_d}$. When $p \to 1$, the inequality becomes an equality, so that we may differentiate both sides with respect to p and recover the euclidean logarithmic Sobolev inequality in optimal scale invariant form (see [22, 29, 18] for details).

Define now

$$C_M := \left(\frac{M_*}{M}\right)^{-\frac{2\,(p-1)}{d-p\,(d-4)}},\quad M_* := \int_{\mathbb{R}^d}\left(1+|x|^2\right)^{-\frac{2\,p}{p-1}}\,dx = \pi^{\frac{d}{2}}\,\frac{\Gamma\!\left(\frac{d-p\,(d-4)}{2\,(p-1)}\right)}{\Gamma\!\left(\frac{2\,p}{p-1}\right)}$$

and observe that we recover the previous definitions of C_M and M^* when $p = d/(d-2)$. Consider next a generic, non-negative optimal function,

$$f_{M,y,\sigma}^{(p)}(x) := \sigma^{-\frac{d}{4\,p}}\left(C_M + \frac{1}{\sigma}\,|x-y|^2\right)^{-\frac{1}{p-1}}\quad \forall\,x\in\mathbb{R}^d\,.$$

As in the case of Sobolev's inequality, let us define the manifold of the optimal functions as

$$\mathfrak{M}_d^{(p)} := \left\{f_{M,y,\sigma}^{(p)}\ :\ (M,y,\sigma)\in\mathcal{M}_d\right\}$$

and consider the functional

$$\mathcal{R}^{(p)}[f] := \inf_{g \in \mathfrak{M}_d^{(p)}} \int_{\mathbb{R}^d} \left[g^{1-p} \left(|f|^{2\,p} - g^{2\,p} \right) - \tfrac{2\,p}{p+1} \left(|f|^{p+1} - g^{p+1} \right) \right] dx \,.$$

Our second result extends the one of Theorem 1.

Theorem 3. *Let $d \ge 2$, $p > 1$ and assume that $p \le d/(d-2)$ if $d \ge 3$. For any $f \in \mathrm{L}^{p+1} \cap \mathcal{D}^{1,2}(\mathbb{R}^d)$ such that $\|f\|_{2,2\,p}$ is finite, we have*

$$\left(\mathcal{C}_{p,d}^{\mathrm{GN}} \, \|\nabla f\|_2^\theta \, \|f\|_{p+1}^{1-\theta} \right)^{2\,\gamma\,p} - \|f\|_{2\,p}^{2\,\gamma\,p} \ge \mathsf{C}_{p,d} \, \frac{\left(\mathcal{R}^{(p)}[f] \right)^2}{\|f\|_{2,2\,p}^\alpha \, \|f\|_{2\,p}^{\beta\,\gamma}}$$

with $\theta = \frac{p-1}{p} \, \frac{d}{d+2-p\,(d-2)}$, $\alpha = d\,(p-1)$, $\beta = d - p\,(d-2)$ and $\gamma = \frac{d+2-p\,(d-2)}{d-p\,(d-4)}$.

The constant $\mathsf{C}_{p,d}$ is positive and explicit. See Appendix A for its expression. The space $\mathrm{L}^{p+1} \cap \mathcal{D}^{1,2}(\mathbb{R}^d)$ is the natural space for Gagliardo-Nirenberg inequalities as it can be characterized as the completion of the space of smooth functions with compact support with respect to the norm $\|\cdot\|$ such that $\|f\|^2 = \|\nabla f\|_2^2 + \|f\|_{p+1}^2$.

As in the case $p = \frac{d}{d-2}$, the functional $\mathcal{R}^{(p)}[f]$ is a measure of the distance of f to the manifold $\mathfrak{M}_d^{(p)}$ and we shall see in Theorem 6 that

$$\mathcal{R}^{(p)}[f] \ge \mathsf{C}_{\mathrm{CK}} \, \|f\|_{2,2\,p}^{\alpha/2} \, \|f\|_{2\,p}^{\delta/2} \inf_{g \in \mathfrak{M}_d^{(p)}} \left\| |f|^{2\,p} - g^{2\,p} \right\|_1^2 \tag{5}$$

with $\delta = d + 2 - p\,(d+6)$ for some constant C_{CK} whose expression is given in Section 3. Putting this estimate together with the result of Corollary 3, with

$$\mathfrak{C}_{p,d} = \mathsf{C}_{d,p} \, {\mathsf{C}_{\mathrm{CK}}}^2 \,,$$

we obtain the following estimate.

Corollary 4. *Let $d \ge 2$, $p > 1$ and assume that $p \le d/(d-2)$ if $d \ge 3$. With $\gamma = \frac{d+2-p\,(d-2)}{d-p\,(d-4)}$, for any $f \in \mathrm{L}^{p+1} \cap \mathcal{D}^{1,2}(\mathbb{R}^d)$, we have*

$$\left(\mathcal{C}_{p,d}^{\mathrm{GN}} \, \|\nabla f\|_2^\theta \, \|f\|_{p+1}^{1-\theta} \right)^{2\,\gamma\,p} - \|f\|_{2\,p}^{2\,\gamma\,p} \ge \mathfrak{C}_{p,d} \, \|f\|_{2\,p}^{2\,p\,(\gamma-4)} \inf_{g \in \mathfrak{M}_d(p)} \left\| |f|^{2\,p} - g^{2\,p} \right\|_1^4 \,.$$

We may notice that Theorem 1 and Corollary 2 are special cases of Theorem 3 and Corollary 4 respectively, corresponding to $p = d/(d-2)$, $\gamma = (d-2)/d$ and $\mathsf{C}_{p,d} = (d-2)^2/(8\,\mathsf{S}_d)$.

In [11, Question (c), p. 75], H. Brezis and E. Lieb asked the question of what is controlled by the difference of the two terms in the critical Sobolev inequality written with an optimal constant, that is, the left hand side in (1). Some partial answers have been provided over the years, of which we can list the following ones. First G. Bianchi and H. Egnell gave in [6] a result based on the concentration-compactness method, which determines a non-constructive estimate for a distance to the set of optimal functions. In [17], A. Cianchi, N. Fusco, F. Maggi and A. Pratelli established an improved inequality using symmetrization methods. Also see [16] for an overview of various results based on such methods. Recently another type of improvement, which relates Sobolev's inequality to the Hardy-Littlewood-Sobolev inequalities, has been established in [19], based on the flow of a nonlinear diffusion equation, in the regime of extinction in finite time. Theorems 1 and 3 provide an answer with fully explicit constants to the question asked by H. Brezis and E. Lieb twenty-five years ago. Our method of proof enlightens a new aspect of the problem. Indeed, Theorem 1 shows that the difference of the two terms in the critical Sobolev inequality provides a better control under the additional information that $\|f\|_{2,2^*}$ is finite. Such a condition disappears in the setting of Corollary 2.

In this paper, our goal is to establish an improvement of Sobolev's inequality based on the flow of the *fast diffusion equation* in the regime of convergence towards Barenblatt self-similar profiles, with an explicit measure of the distance to the set of optimal functions for the critical Sobolev inequality. Our approach is based on a *relative entropy* functional. The method relies on a recent paper, [21], which is itself based on a long series of studies on intermediate asymptotics of the fast diffusion equation, and on the entropy - entropy production method introduced in [4, 2] in the linear case and later extended to nonlinear diffusions: see [24, 25, 18, 14, 13]. In this setting, having a finite second moment is crucial. Let us give some explanations.

Consider the fast diffusion equation with exponent m given in terms of the exponent p of Theorem 3 by

$$p=\frac{1}{2\,m-1}\quad\Longleftrightarrow\quad m=\frac{p+1}{2\,p}\;. \tag{6}$$

More specifically, for $m\in(0,1)$, we shall consider the solutions of

$$\frac{\partial u}{\partial t}+\nabla\cdot\Big[u\,\Big(\eta\,\nabla u^{m-1}-2\,x\Big)\Big]=0\quad t>0\;,\quad x\in\mathbb{R}^d \tag{7}$$

with initial datum $u(t=0,\cdot)=u_0$. Here η is a positive parameter which does not depend on t. Let u_∞ be the unique stationary solution such that $M=\int_{\mathbb{R}^d}u\,dx=\int_{\mathbb{R}^d}u_\infty\,dx$. It is given by

$$u_\infty(x)=\Big(K+\frac{1}{\eta}\,|x|^2\Big)^{\frac{1}{m-1}}\quad\forall\,x\in\mathbb{R}^d$$

for some positive constant K which is uniquely determined by M. The following exponents are associated with the fast diffusion equation (7) and will be used all over this paper:

$$m_c := \frac{d-2}{d}\;, \quad m_1 := \frac{d-1}{d} \quad \text{and} \quad \widetilde{m}_1 := \frac{d}{d+2}\;.$$

To the critical exponent $2\,p = 2\,d/(d-2)$ for Sobolev's inequality, which appears in Theorem 1, corresponds the critical exponent m_1 for the fast diffusion equation. For $d \geq 3$, the condition $p \in (1, d/(d-2)]$ in Theorem 3 is equivalent to $m \in [m_1, 1)$ while for $d = 2$, $p \in (1, \infty)$ means $m \in (1/2, 1)$.

It has been established in [24, 25] that the *relative entropy* (or *free energy*)

$$\mathcal{F}[u|u_\infty] := \frac{1}{m-1}\int_{\mathbb{R}^d}\left[u^m - u_\infty^m - m\,u_\infty^{m-1}\,(u-u_\infty)\right]dx$$

decays according to

$$\frac{d}{dt}\mathcal{F}[u(\cdot,t)|u_\infty] = -\mathcal{I}[u(\cdot,t)|u_\infty]$$

if u is a solution of (7), where

$$\mathcal{I}[u(\cdot,t)|u_\infty] := \eta\,\frac{m}{1-m}\int_{\mathbb{R}^d} u\,\left|\nabla u^{m-1} - \nabla u_\infty^{m-1}\right|^2 dx$$

is the *entropy production term* or *relative Fisher information.* If $m \in [m_1, 1)$, according to [18], these two functionals are related by a Gagliardo-Nirenberg interpolation inequality, namely

$$\mathcal{F}[u|u_\infty] \leq \frac{1}{4}\,\mathcal{I}[u|u_\infty]\;. \tag{8}$$

We shall give a concise proof of this inequality in the next section (see Remark 1) based on the entropy - entropy production method, which amounts to relate $\frac{d}{dt}\mathcal{I}[u(\cdot,t)|u_\infty]$ and $\mathcal{I}[u(\cdot,t)|u_\infty]$. We shall later replace the diffusion parameter η in (7) by a time-dependent coefficient $\sigma(t)$, which is itself computed using the second moment of u, $\int_{\mathbb{R}^d}|x|^2\,u(x,t)\;dx$. By doing so, we are able to capture the *best matching* Barenblatt solution and get improved decay rates in the entropy - entropy production inequality. Elementary estimates allow to rephrase these improved rates into improved functional inequalities for f such that $|f|^{2\,p} = u$, with $p = d/(d-2)$ (Theorem 1) and any $p \in (1, d/(d-2)]$ (Theorem 3).

This paper is organized as follows. In Section 2, we apply the entropy - entropy production method to the fast diffusion equation as in [13]. The key computation, without justifications for the integrations by parts, is reproduced here since we need it later in Section 6, in the case of a time-dependent diffusion coefficient. Next, in Section 3, we establish a new estimate of Csiszár-Kullback type. By requiring a condition on the second moment, we are able to produce a new estimate which was not known before, namely to directly control the difference of the solution with a Barenblatt solution in $\mathrm{L}^1(\mathbb{R}^d)$.

Second moment estimates are the key of a recent paper and we shall primarily refer to [21] in which the asymptotic behaviour of the solutions of the fast diffusion equation was studied. In Section 4 we recall the main results that were proved in [21], and that are also needed in the present paper.

With these preliminaries in hand, an improved entropy - entropy production inequality is established in Section 5, which is at the core of our paper. It is known since [18] that entropy - entropy production inequalities amount to optimal Gagliardo-Nirenberg-Sobolev inequalities. Such a rephrasing of our result in a more standard form of functional inequalities is done in Section 6, which contains the proof of Theorems 1 and 3. Further observations have been collected in Section 7. One of the striking results of our approach is that all constants can be explicitly computed. This is somewhat technical although not really difficult. To make the reading easier, explicit computations have been collected in Appendix A.

2. **The entropy - entropy production method**

Consider a solution $u=u(x,t)$ of Eq. (7) and define

$$z(x,t):=\eta\,\nabla u^{m-1}-2\,x$$

so that Eq. (7) can be rewritten as

$$\frac{\partial u}{\partial t}+\nabla\cdot(u\,z)=0\;.$$

To keep notations compact, we shall use the following conventions. If $A=(A_{ij})_{i,j=1}^d$ and $B=(B_{ij})_{i,j=1}^d$ are two matrices, let $A:B=\sum_{i,j=1}^d A_{ij}\,B_{ij}$ and $|A|^2=A:A$. If a and b take values in $\mathbb{R}^d$, we adopt the definitions:

$$a\cdot b=\sum_{i=1}^d a_i\,b_i\;,\quad \nabla\cdot a=\sum_{i=1}^d\frac{\partial a_i}{\partial x_i}\;,\quad a\otimes b=(a_i\,b_j)_{i,j=1}^d\;,\quad \nabla\otimes a=\left(\frac{\partial a_j}{\partial x_i}\right)_{i,j=1}^d\;.$$

Later we will need a version of the entropy - entropy production method in case of a time-dependent diffusion coefficient. Before doing so, let us recall the key computation of the standard method. With the above notations, it is straightforward to check that

$$\frac{\partial z}{\partial t}=\eta\,(1-m)\nabla\Big(u^{m-2}\,\nabla\cdot(u\,z)\Big)\quad\text{and}\quad\nabla\otimes z=\eta\,\nabla\otimes\nabla u^{m-1}-2\,\mathrm{Id}\;.$$

With these definitions, the time-derivative of $\frac{1-m}{m}\,\eta\,\mathcal{I}[u|u_\infty]=\int_{\mathbb{R}^d}u\,|z|^2\;dx$ can be computed as

$$\frac{d}{dt}\int_{\mathbb{R}^d}u\,|z|^2\;dx=\int_{\mathbb{R}^d}\frac{\partial u}{\partial t}\,|z|^2\;dx+2\int_{\mathbb{R}^d}u\,z\cdot\frac{\partial z}{\partial t}\;dx\;.$$

The first term can be evaluated by

$$\begin{aligned}
&\int_{\mathbb{R}^d} \frac{\partial u}{\partial t}\,|z|^2 dx\\
&= -\int_{\mathbb{R}^d} \nabla\cdot(u\,z)\;|z|^2 dx\\
&= 2\int_{\mathbb{R}^d} u\,z\otimes z : \nabla\otimes z\;dx\\
&= 2\,\eta\int_{\mathbb{R}^d} u\,z\otimes z : \nabla\otimes\nabla u^{m-1}\;dx - \;4\int_{\mathbb{R}^d} u\,|z|^2 dx\\
&= 2\,\eta\,(1-m)\int_{\mathbb{R}^d} u^{m-2}\,\nabla u\otimes\nabla : (u\,z\otimes z)\;dx - \;4\int_{\mathbb{R}^d} u\,|z|^2 dx\\
&= 2\,\eta\,(1-m)\int_{\mathbb{R}^d} u^{m-2}\,(\nabla u\cdot z)^2 dx + \;2\,\eta\,(1-m)\int_{\mathbb{R}^d} u^{m-1}\,(\nabla u\cdot z)\,(\nabla\cdot z)\;dx\\
&\quad + 2\,\eta\,(1-m)\int_{\mathbb{R}^d} u^{m-1}\,(z\otimes\nabla u) : (\nabla\otimes z)\;dx - \;4\int_{\mathbb{R}^d} u\,|z|^2 dx\;.
\end{aligned}$$

The second term can be evaluated by

$$\begin{aligned}
&2\int_{\mathbb{R}^d} u\,z\cdot\frac{\partial z}{\partial t}\;dx\\
&= 2\,\eta\,(1-m)\int_{\mathbb{R}^d} (u\,z\cdot\nabla)\left(u^{m-2}\,\nabla\cdot(u\,z)\right)\,dx\\
&= -2\,\eta\,(1-m)\int_{\mathbb{R}^d} u^{m-2}\left(\nabla\cdot(u\,z)\right)^2 dx\\
&= -2\,\eta\,(1-m)\int_{\mathbb{R}^d} \left[u^m(\nabla\cdot z)^2 + 2\,u^{m-1}(\nabla u\cdot z)\,(\nabla\cdot z) + u^{m-2}(\nabla u\cdot z)^2\,\right]\,dx\;.
\end{aligned}$$

Summarizing, we have found that

$$\begin{aligned}
\int_{\mathbb{R}^d} \frac{\partial u}{\partial t}\,|z|^2 dx + 4\int_{\mathbb{R}^d} u\,|z|^2 dx
&= -2\,\eta\,(1-m)\int_{\mathbb{R}^d} u^{m-2}\Big[u^2(\nabla\cdot z)^2 + u\,(\nabla u\cdot z)\,(\nabla\cdot z)\\
&\qquad\qquad - u\,(z\otimes\nabla u) : (\nabla\otimes z)\Big]\,dx\;.
\end{aligned}$$

Using the fact that

$$\frac{\partial^2 z^j}{\partial x_i\,\partial x_j} = \frac{\partial^2 z^i}{\partial x_j^2}\;,$$

we obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^d} u^{m-1}\;(\nabla u\cdot z)\;(\nabla\cdot z)\;dx\\
&\qquad = -\frac{1}{m}\int_{\mathbb{R}^d} u^m\;(\nabla\cdot z)^2\;dx - \frac{1}{m}\int_{\mathbb{R}^d} u^m\sum_{i,j=1}^d z^i\,\frac{\partial^2 z^j}{\partial x_i\,\partial x_j}\;dx
\end{aligned}$$

and

$$-\int_{\mathbb{R}^d} u^{m-1}\,(z\otimes\nabla u):(\nabla\otimes z)\;dx$$
$$=\frac{1}{m}\int_{\mathbb{R}^d} u^m\,|\nabla z|^2\;dx+\frac{1}{m}\int_{\mathbb{R}^d} u^m\sum_{i,j=1}^d z^i\,\frac{\partial^2 z^i}{\partial x_j^2}\;dx$$

can be combined to give

$$\int_{\mathbb{R}^d} u^{m-2}\Big[u\;(\nabla u\cdot z)\;(\nabla\cdot z)-u\,\nabla u\otimes z:\nabla\otimes z\Big]\;dx$$
$$=-\frac{1}{m}\int_{\mathbb{R}^d} u^m\;(\nabla\cdot z)^2\;dx+\frac{1}{m}\int_{\mathbb{R}^d} u^m\,|\nabla z|^2\;dx\;.$$

This shows that

$$\frac{d}{dt}\int_{\mathbb{R}^d} u\,|z|^2\;dx+4\int_{\mathbb{R}^d} u\,|z|^2\;dx$$
$$=-2\,\eta\,\frac{1-m}{m}\int_{\mathbb{R}^d} u^m\left(|\nabla z|^2-(1-m)\;(\nabla\cdot z)^2\right)\;dx\;.$$

By the arithmetic geometric inequality, we know that

$$|\nabla z|^2-(1-m)\;(\nabla\cdot z)^2\geq 0$$

if $1-m\leq 1/d$, that is, if $m\geq m_1$. Altogether, we have formally established the following result.

Proposition 5. *Let $d\geq 1$, $m\in(m_1,1)$ and assume that u is a non-negative solution of* (7) *with initial datum u_0 in $\mathrm{L}^1(\mathbb{R}^d)$ such that u_0^m and $x\mapsto|x|^2\,u_0$ are both integrable on $\mathbb{R}^d$. With the above defined notations, we get that*

$$\frac{d}{dt}\mathcal{I}[u(\cdot,t)|u_\infty]\leq-\,4\,\mathcal{I}[u(\cdot,t)|u_\infty]\quad\forall\,t>0\;.$$

The proof of such a result requires to justify that all integrations by parts make sense. We refer to [14, 15] for a proof in the porous medium case ($m>1$) and to [13] for $m_1\leq m<1$.

Remark 1. Proposition 5 provides a proof of (8)*. Indeed, with a Gronwall estimate, we first get that*

$$\mathcal{I}[u(\cdot,t)|u_\infty]\leq\mathcal{I}[u_0|u_\infty]\,e^{-\,4\,t}\quad\forall\,t\geq 0$$

if $\mathcal{I}[u_0|u_\infty]$ is finite. Since $\mathcal{I}[u(\cdot,t)|u_\infty]$ is non-negative, we know that

$$\lim_{t\to\infty}\mathcal{I}[u(\cdot,t)|u_\infty]=0\;,$$

which proves the convergence of $u(\cdot,t)$ *to* u_∞ *as* $t\to\infty$. *As a consequence, we also have* $\lim_{t\to\infty}\mathcal{F}[u(\cdot,t)|u_\infty]=0$ *and since*

$$\frac{d}{dt}\Big(\mathcal{I}[u(\cdot,t)|u_\infty]-4\,\mathcal{F}[u(\cdot,t)|u_\infty]\Big)=\frac{d}{dt}\mathcal{I}[u(\cdot,t)|u_\infty]+4\,\mathcal{I}[u(\cdot,t)|u_\infty]\le 0\;,$$

an integration with respect to t *on* $(0,\infty)$ *shows that*

$$\mathcal{I}[u_0|u_\infty]-4\,\mathcal{F}[u_0|u_\infty]\ge 0\;,$$

which is precisely (8) *written for* $u=u_0$.

3. A Csiszár-Kullback inequality

Let $m\in(\widetilde{m}_1,1)$ with $\widetilde{m}_1=\frac{d}{d+2}$ and consider the relative entropy

$$\mathcal{F}_\sigma[u]:=\frac{1}{m-1}\int_{\mathbb{R}^d}\left[u^m-B_\sigma^m-m\,B_\sigma^{m-1}\,(u-B_\sigma)\right]\,dx$$

for some Barenblatt function

$$B_\sigma(x):=\sigma^{-\frac{d}{2}}\left(C_M+\tfrac{1}{\sigma}\,|x|^2\right)^{\frac{1}{m-1}}\quad\forall\,x\in\mathbb{R}^d \tag{9}$$

where σ is a positive constant and C_M is chosen such that $\|B_\sigma\|_1=M>0$. With p and m related by (6), the definition of C_M coincides with the one of Section 1. See details in Appendix A.

Theorem 6. *Let* $d\ge 1$, $m\in(\widetilde{m}_1,1)$ *and assume that* u *is a non-negative function in* $\mathrm{L}^1(\mathbb{R}^d)$ *such that* u^m *and* $x\mapsto|x|^2\,u$ *are both integrable on* $\mathbb{R}^d$. *If* $\|u\|_1=M$ *and* $\int_{\mathbb{R}^d}|x|^2\,u\,dx=\int_{\mathbb{R}^d}|x|^2\,B_\sigma\,dx$, *then*

$$\frac{\mathcal{F}_\sigma[u]}{\sigma^{\frac{d}{2}(1-m)}}\ge\frac{m}{8\int_{\mathbb{R}^d}B_1^m\,dx}\left(C_M\,\|u-B_\sigma\|_1+\frac{1}{\sigma}\int_{\mathbb{R}^d}|x|^2\,|u-B_\sigma|\,dx\right)^2\,.$$

Notice that the condition $\int_{\mathbb{R}^d}|x|^2\,u\;dx=\int_{\mathbb{R}^d}|x|^2\,B_\sigma\;dx$ is explicit and determines σ uniquely:

$$\sigma=\frac{1}{K_M}\int_{\mathbb{R}^d}|x|^2\,u\,dx\quad\text{with}\quad K_M:=\int_{\mathbb{R}^d}|x|^2\,B_1\,dx\;.$$

For further details, see Lemma 7 and (17) below, and Appendix A for detailed expressions of K_M and $\int_{\mathbb{R}^d}B_1^m\;dx$. With this choice of σ, since $B_\sigma^{m-1}=\sigma^{\frac{d}{2}(1-m)}\,C_M+\sigma^{\frac{d}{2}(m_c-m)}\,|x|^2$, we remark that $\int_{\mathbb{R}^d}B_\sigma^{m-1}\,(u-B_\sigma)\;dx=0$ so that the relative entropy reduces to

$$\mathcal{F}_\sigma[u]:=\frac{1}{m-1}\int_{\mathbb{R}^d}\left[u^m-B_\sigma^m\right]\,dx$$

Proof of Theorem 6. Let $v := u/B_\sigma$ and $d\mu_\sigma := B_\sigma^m\,dx$. With these notations, we observe that

$$\int_{\mathbb{R}^d}(v-1)\,d\mu_\sigma = \int_{\mathbb{R}^d} B_\sigma^{m-1}\,(u-B_\sigma)\,dx$$
$$= \sigma^{\frac{d}{2}\,(1-m)}\,C_M\int_{\mathbb{R}^d}(u-B_\sigma)\,dx + \sigma^{\frac{d}{2}(m_c-m)}\int_{\mathbb{R}^d}|x|^2\,(u-B_\sigma)\,dx = 0\;.$$

Thus

$$\int_{\mathbb{R}^d}(v-1)\,d\mu_\sigma = \int_{v>1}(v-1)\,d\mu_\sigma - \int_{v<1}(1-v)\,d\mu_\sigma = 0\;,$$

which, coupled with

$$\int_{v>1}(v-1)\,d\mu_\sigma + \int_{v<1}(1-v)\,d\mu_\sigma = \int_{\mathbb{R}^d}|v-1|\,d\mu_\sigma\;,$$

implies

$$\int_{\mathbb{R}^d}|u-B_\sigma|\,B_\sigma^{m-1}\,dx = \int_{\mathbb{R}^d}|v-1|\,d\mu_\sigma = 2\int_{v<1}|v-1|\,d\mu_\sigma\;.$$

On the other hand, a Taylor expansion shows that

$$\mathcal{F}_\sigma[u] = \frac{1}{m-1}\int_{\mathbb{R}^d}\Big[v^m - 1 - m\,(v-1)\Big]\,d\mu_\sigma = \frac{m}{2}\int_{\mathbb{R}^d}\xi^{m-2}\,|v-1|^2\,d\mu_\sigma$$

for some function ξ taking values in the interval $(\min\{1,v\},\max\{1,v\})$, thus giving the lower bound

$$\mathcal{F}_\sigma[u] \ge \frac{m}{2}\int_{v<1}\xi^{m-2}\,|v-1|^2\,d\mu_\sigma \ge \frac{m}{2}\int_{v<1}|v-1|^2\,d\mu_\sigma\;.$$

Using the Cauchy-Schwarz inequality, we get

$$\left(\int_{v<1}|v-1|\,d\mu_\sigma\right)^2 = \left(\int_{v<1}|v-1|\,B_\sigma^{\frac{m}{2}}\,B_\sigma^{\frac{m}{2}}\,dx\right)^2 \le \int_{v<1}|v-1|^2\,d\mu_\sigma\int_{\mathbb{R}^d}B_\sigma^m\,dx$$

and finally obtain that

$$\mathcal{F}_\sigma[u] \ge \frac{m}{2}\,\frac{\left(\int_{v<1}|v-1|\,d\mu_\sigma\right)^2}{\int_{\mathbb{R}^d}B_\sigma^m\,dx} = \frac{m}{8}\,\frac{\left(\int_{\mathbb{R}^d}|u-B_\sigma|\,B_\sigma^{m-1}\,dx\right)^2}{\int_{\mathbb{R}^d}B_\sigma^m\,dx}\;,$$

which concludes the proof. □

Notice that the inequality of Theorem 6 can be rewritten in terms of $|f|^{2\,p} = u$ and $g^{2\,p} = B_\sigma$ with $p = 1/(2\,m-1)$. See Appendix A for the computation of $\int_{\mathbb{R}^d}B_\sigma^m\,dx$, σ and C_M in terms of $\int_{\mathbb{R}^d}|x|^2\,u\,dx$ and M_*. Altogether we find

$$\tfrac{p+1}{p-1}\,\mathcal{R}^{(p)}[f]=\tfrac{2\,p}{p-1}\int_{\mathbb{R}^d}\left(g^{p+1}-|f|^{p+1}\right)dx$$
$$\ge\tfrac{d+2-p\,(d-2)}{32\,p}\,\left(\tfrac{d+2-p\,(d-2)}{d\,(p-1)}\,\|f\|_{2,2\,p}^{2\,p}\right)^{d\,\frac{p-1}{4\,p}}$$
$$\cdot\,M_*^{\frac{p-1}{2\,p}}\,\|f\|_{2\,p}^{\frac{1}{2}\,(d+2-p\,(d+6))}\,\Big\|\,|f|^{2\,p}-g^{2\,p}\Big\|_1^2\;.$$

This proves (5) with

$$\mathsf{C}_{\rm CK}=\tfrac{p-1}{p+1}\,\tfrac{d+2-p\,(d-2)}{32\,p}\,\left(\tfrac{d+2-p\,(d-2)}{d\,(p-1)}\right)^{d\,\frac{p-1}{4\,p}}\,M_*^{\frac{p-1}{2\,p}}$$

and (2) in the special case $p=d/(d-2)$.

Remark 2. Various other estimates can be derived, based on second order Taylor expansions. For instance, as in [18], *we can write that*

$$\mathcal{F}_\sigma[u]=\int_{\mathbb{R}^d}\left[\psi(v^m)-\psi(1)-\psi'(1)\,(v^m-1)\right]d\mu_\sigma$$

with $v:=u/B_\sigma$ *and* $\psi(s):=\frac{m}{1-m}\,s^{1/m}$, *and get*

$$\mathcal{F}_\sigma[u]\ge\frac{1}{m}\,2^{-2\,m}\,\frac{\|v^m-1\|^2_{\mathrm{L}^{1/m}(\mathbb{R}^d,d\mu_\sigma)}}{\max\left\{\|v^m\|_{\mathrm{L}^{1/m}(\mathbb{R}^d,d\mu_\sigma)},\|1\|_{\mathrm{L}^{1/m}(\mathbb{R}^d,d\mu_\sigma)}\right\}^{2-\frac{1}{m}}}\;.$$

Using $\|v^m\|_{\mathrm{L}^{1/m}(\mathbb{R}^d,d\mu_\sigma)}=\|1\|_{\mathrm{L}^{1/m}(\mathbb{R}^d,d\mu_\sigma)}=\|B_\sigma^m\|_1^m$ *and*

$$\int_{\mathbb{R}^d}|u^m-B_\sigma^m|\,dx=\int_{\mathbb{R}^d}|u^m-B_\sigma^m|\,B_\sigma^{m\,(m-1)}\,B_\sigma^{m\,(1-m)}\,dx$$
$$\le\|v^m-1\|_{\mathrm{L}^{1/m}(\mathbb{R}^d,d\mu_\sigma)}\,\|B_\sigma^m\|_1^{1-m}$$

by the Cauchy-Schwarz inequality, we find

$$\mathcal{F}_\sigma[u]\ge\frac{\|u^m-B_\sigma^m\|_1^2}{m\,2^{2\,m}\,\|B_\sigma^m\|_1}\;.$$

With $f=u^{m-\frac{1}{2}}$, *this also gives another estimate of Csiszár-Kullback type, namely*

$$\mathcal{R}^{(p)}[f]\ge\frac{\kappa_{p,d}}{\|f\|_{2,2\,p}^{\frac{d}{2}\,(p-1)}\,\|f\|_{2\,p}^{\frac{1}{2}\,(d+2-p\,(d-2))}}\,\inf_{g\in\mathfrak{M}_d^{(p)}}\Big\||f|^{p+1}-g^{p+1}\Big\|_1^2\;,$$

for some positive constant $\kappa_{p,d}$, *which is valid for any* $p\in(1,\infty)$ *if* $d=2$ *and any* $p\in(1,\frac{d}{d-2}]$ *if* $d\ge3$. *Also see* [28, 14, 12, 20] *for further results on Csiszár-Kullback type inequalities corresponding to entropies associated with porous media and fast diffusion equations.*

4. **Recent results on the optimal matching by Barenblatt solutions**

Consider on $\mathbb{R}^d$ the fast diffusion equation with harmonic confining potential given by

$$\frac{\partial u}{\partial t}+\nabla\cdot\left[u\,\left(\sigma^{\frac{d}{2}(m-m_c)}\,\nabla u^{m-1}-2\,x\right)\right]=0\quad t>0\;,\quad x\in\mathbb{R}^d\;, \tag{10}$$

with initial datum u_0. Here σ is a function of t. Let us summarize some results obtained in [21] and the strategy of their proofs.

Result 1. At any time $t>0$, we can choose the *best matching Barenblatt* as follows. Consider a given function u and optimize $\lambda\mapsto\mathcal{F}_\lambda[u]$.

LEMMA 7. *For any given $u\in\mathrm{L}^1_+(\mathbb{R}^d)$ such that u^m and $|x|^2\,u$ are both integrable, if $m\in(\widetilde{m}_1,1)$, there is a unique $\lambda=\lambda^*>0$ which minimizes $\lambda\mapsto\mathcal{F}_\lambda[u]$, and it is explicitly given by*

$$\lambda^*=\frac{1}{K_M}\int_{\mathbb{R}^d}|x|^2\,u\;dx$$

where $K_M=\int_{\mathbb{R}^d}|x|^2\,B_1\;dx$. For $\lambda=\lambda^*$, the Barenblatt profile B_λ satisfies

$$\int_{\mathbb{R}^d}|x|^2\,B_\lambda\;dx=\int_{\mathbb{R}^d}|x|^2\,u\;dx\;.$$

As a consequence, we know that

$$\frac{d}{d\lambda}\big(\mathcal{F}_\lambda[u]\big)_{\lambda=\lambda_*}=0\;.$$

Of course, if u is a solution of (10), the value of λ in Lemma 7 may depend on t. Now we choose $\sigma(t)=\lambda(t)$, *i.e.*,

$$\sigma(t)=\frac{1}{K_M}\int_{\mathbb{R}^d}|x|^2\,u(x,t)\;dx\quad\forall\,t\geq 0\;. \tag{11}$$

This makes (10) a non-local equation.

Result 2. With the above choice, if we consider a solution of (10) and compute the time derivative of the relative entropy, we find that

$$\frac{d}{dt}\mathcal{F}_{\sigma(t)}[u(\cdot,t)]=\sigma'(t)\left(\frac{d}{d\sigma}\,\mathcal{F}_\sigma[u]\right)_{|\sigma=\sigma(t)}+\frac{m}{m-1}\int_{\mathbb{R}^d}\left(u^{m-1}-B_{\sigma(t)}^{m-1}\right)\frac{\partial u}{\partial t}\;dx\;.$$

However, as a consequence of Lemma 7, we know that

$$\left(\frac{d}{d\sigma}\,\mathcal{F}_\sigma[u]\right)_{|\sigma=\sigma(t)}=0\;,$$

and we finally obtain

$$\frac{d}{dt}\mathcal{F}_{\sigma(t)}[u(\cdot,t)]=-\frac{m\,\sigma(t)^{\frac{d}{2}(m-m_c)}}{1-m}\int_{\mathbb{R}^d}u\left|\nabla\left[u^{m-1}-B_{\sigma(t)}^{m-1}\right]\right|^2\,dx\;. \tag{12}$$

From there on, the computation goes essentially as in [8, 10] (also see [24, 25, 18] for details). With our choice of σ, we gain an additional orthogonality condition which is useful for improving the rates of convergence (see [21, Theorem 1]) in the asymptotic regime $t \to \infty$, compared to the results of [10] (also see below).

Result 3. Now let us state one more result of [21] which is of interest for the present paper.

LEMMA 8. *With the above notations, if u and σ are defined respectively by* (10) *and* (11)*, then the function $t \mapsto \sigma(t)$ is positive, decreasing, with* $\lim_{t\to\infty} \sigma(t) =: \sigma_\infty > 0$ *and*

$$\sigma'(t) = -2\,d\,\frac{(1-m)^2}{m\,K_M}\,\sigma^{\frac{d}{2}(m-m_c)}\,\mathcal{F}_{\sigma(t)}[u(\cdot,t)] \le 0\;. \tag{13}$$

The main difficulty is to establish that σ_∞ is positive. This can be done with an appropriate change of variables which reduces (10) to the case where σ does not depend on t. The proof relies on the asymptotics which have been obtained in [18, 8, 7, 10].

Let us give some details. In [21], it has been established that the function v such that

$$v(\tau,y) = R^{-d}\,u(x,t)\;,\quad R = R(\tau)\;,\quad t = \tfrac{1}{2}\log R\;,\quad x = \frac{y}{R}$$

is a solution of

$$\frac{\partial v}{\partial \tau} + \nabla\cdot\left(v\,\nabla v^{m-1}\right) = 0 \tag{14}$$

with initial datum $v_0 = u_0 \in \mathrm{L}^1_+(\mathbb{R}^d)$ if R and σ are related by

$$2\,\sigma^{-\frac{d}{2}(m-m_c)} = R^{1-d\,(1-m)}\,\frac{dR}{d\tau}\;,\quad R(0) = 1\;.$$

Using this identity on the one hand, and the time-dependent change of variables which transforms v into u on the other hand, we obtain an ordinary differential equation for R in terms of the second moment of v, namely

$$\frac{d\log R}{d\tau} = 2\left(\frac{K_M}{\int_{\mathbb{R}^d}|y-x_0|^2\,v(\tau,y)\,dy}\right)^{\frac{d}{2}(m-m_c)}\;,\quad R(0) = 1\;.$$

Notice that Eq. (14) has no explicit dependence in τ, which is the key ingredient to establish that σ_∞ is positive. See [21] for details.

5. The scaled entropy - entropy production inequality

Consider the relative Fisher information

$$\mathcal{I}_\lambda[u] := \frac{m}{1-m}\int_{\mathbb{R}^d} u\,\left|\nabla u^{m-1} - \nabla B_\lambda^{m-1}\right|^2\,dx\;.$$

By applying (8) with $u_\infty = B_1$ and $\eta = 1$ to $x \mapsto \sigma^{d/2}\, u(\sqrt{\sigma}\, x)$ and using the fact that $B_1(x) = \sigma^{d/2}\, B_\sigma(\sqrt{\sigma}\, x)$, we get the inequality

$$\mathcal{F}_\sigma[u] \le \frac{1}{4}\, \mathcal{J}_\sigma[u] \quad \text{with} \quad \mathcal{J}_\sigma[u] := \sigma^{\frac{d}{2}(m-m_c)}\, \mathcal{I}_\sigma[u]\ .$$

Now, if σ is time-dependent as in Section 4, we have the following relations.

LEMMA 9. *If u is a solution of* (10) *with $\sigma(t) = \frac{1}{K_M} \int_{\mathbb{R}^d} |x|^2\, u(x,t)\, dx$, then σ satisfies* (13). *Moreover, for any $t \ge 0$, we have*

$$\frac{d}{dt}\mathcal{F}_{\sigma(t)}[u(\cdot,t)] = -\mathcal{J}_{\sigma(t)}[u(\cdot,t)] \tag{15}$$

and

$$\frac{d}{dt}\mathcal{J}_{\sigma(t)}[u(\cdot,t)] \le -\left[4 + \frac{d}{2}\,(m-m_c)\, \frac{|\sigma'(t)|}{\sigma(t)}\right] \mathcal{J}_{\sigma(t)}[u(\cdot,t)]\ . \tag{16}$$

Proof. Eq. (13) and (15) have already been stated respectively in Lemma 8 and in (12). They are recalled here only for the convenience of the reader. It remains to prove (16).

For any given $\sigma = \sigma(t)$, Proposition 5 gives

$$\begin{aligned}\frac{d}{dt}\mathcal{J}_{\sigma(t)}[u(\cdot,t)] &= \left(\frac{d}{dt}\mathcal{J}_\lambda[u(\cdot,t)]\right)_{|\lambda=\sigma(t)} + \sigma'(t)\left(\frac{d}{d\lambda}\,\mathcal{J}_\lambda[u]\right)_{|\lambda=\sigma(t)}\\ &\le -4\,\mathcal{J}_{\sigma(t)}[u(\cdot,t)] + \sigma'(t)\left(\frac{d}{d\lambda}\,\mathcal{J}_\lambda[u]\right)_{|\lambda=\sigma(t)}\ .\end{aligned}$$

Owing to the definition of $\mathcal{J}_\lambda$, we obtain

$$\begin{aligned}\frac{d}{d\lambda}\,\mathcal{J}_\lambda[u] &= \frac{d}{2}\,(m-m_c)\,\frac{1}{\lambda}\,\mathcal{J}_\lambda[u]\\ &\quad + \frac{m}{1-m}\,\lambda^{\frac{d}{2}(m-m_c)} \int_{\mathbb{R}^d} 2\,u\left(\nabla u^{m-1} - \nabla B_\lambda^{m-1}\right)\cdot \frac{d}{d\lambda}\left(\nabla B_\lambda^{m-1}\right) dx\ .\end{aligned}$$

By definition (9), $\nabla B_\lambda^{m-1}(x) = 2\,x\,\lambda^{-\frac{d}{2}(m-m_c)}$, which implies

$$\lambda^{\frac{d}{2}(m-m_c)}\,\frac{d}{d\lambda}\left(\nabla B_\lambda^{m-1}\right) = -\frac{d}{\lambda}\,(m-m_c)\,x\ .$$

Substituting this expression into the above computation and integrating by parts, we conclude with the equality

$$\begin{aligned}\frac{d}{d\lambda}\mathcal{J}_\lambda[u] &= \frac{d}{2}\,(m-m_c)\,\frac{1}{\lambda}\mathcal{J}_\lambda[u]\\ &\quad + \frac{2\,d}{\lambda}\,(m-m_c)\left[\frac{2\,m\,\lambda^{-\frac{d}{2}(m-m_c)}}{1-m}\int_{\mathbb{R}^d} |x|^2\,u\,dx - d\int_{\mathbb{R}^d} u^m\,dx\right].\end{aligned}$$

A simple computation shows that

$$d\int_{\mathbb{R}^d} B_1^m\,dx = -\int_{\mathbb{R}^d} x\cdot\nabla B_1^m\,dx = \frac{2\,m}{1-m}\int_{\mathbb{R}^d}|x|^2\,B_1\,dx = \frac{2\,m}{1-m}\,K_M \tag{17}$$

and, as a consequence, if $\lambda = \sigma = \frac{1}{K_M} \int_{\mathbb{R}^d} |x|^2 \, u \, dx$, then

$$\frac{2\, m\, \lambda^{-\frac{d}{2}(m-m_c)}}{1-m} \int_{\mathbb{R}^d} |x|^2 \, u \, dx = d \int_{\mathbb{R}^d} B_\lambda^m \, dx \,,$$

and finally

$$\frac{d}{d\lambda} \mathcal{J}_\lambda[u] = \frac{d}{2}\,(m-m_c)\,\frac{1}{\lambda} \mathcal{J}_\lambda[u] + \frac{2\,d^2}{\lambda}\,(m-m_c)\,(1-m)\,\mathcal{F}_\lambda[u]\,.$$

Altogether, we have found that

$$\begin{aligned}(18)\quad \frac{d}{dt} \mathcal{J}_{\sigma(t)}[u(\cdot,t)] \le -\,4\,\mathcal{J}_{\sigma(t)}[u(\cdot,t)] &+ \frac{d}{2}\,(m-m_c)\,\frac{\sigma'(t)}{\sigma(t)}\,\mathcal{J}_{\sigma(t)}[u(\cdot,t)] \\ &+ 2\,d^2\,(1-m)\,(m-m_c)\,\frac{\sigma'(t)}{\sigma(t)}\,\mathcal{F}_{\sigma(t)}[u(\cdot,t)]\,.\end{aligned}$$

The last term of the right hand side is non-positive because by (13) we know that $\sigma'(t) \le 0$. This implies (16). □

Multiplying both sides of (15) by $4 + \frac{d}{2}\,(m-m_c)\,|\sigma'(t)|/\sigma(t)$, which is a positive quantity, and using Inequality (16), we obtain

$$(19)\qquad \left[4 + \frac{d}{2}\,(m-m_c)\,\frac{|\sigma'(t)|}{\sigma(t)}\right] \frac{d}{dt}\left(\mathcal{F}_{\sigma(t)}[u(\cdot,t)]\right) \ge \frac{d}{dt}\left(\mathcal{J}_{\sigma(t)}[u(\cdot,t)]\right)\,.$$

Using (13) and the fact that $\sigma(t)$ is non-increasing in t, we get that

$$\begin{aligned}\frac{|\sigma'(t)|}{\sigma(t)} = 2\,d\,\frac{(1-m)^2}{m\,K_M}\,\sigma(t)^{-\frac{d}{2}\,(1-m)}\,\mathcal{F}_{\sigma(t)}[u(\cdot,t)] \\ \ge 2\,d\,\frac{(1-m)^2}{m\,K_M}\,\sigma_0^{-\frac{d}{2}\,(1-m)}\,\mathcal{F}_{\sigma(t)}[u(\cdot,t)]\end{aligned}$$

where we set $\sigma_0 := \sigma(t=0) = \frac{1}{K_M} \int_{\mathbb{R}^d} |x|^2 \, u_0(x) \, dx$.

Recalling that the derivatives on both sides of (19) are non-positive, we can use the above inequality into (19) to obtain

$$4\left[1 + 2\,C_{m,d}\,\frac{\mathcal{F}_{\sigma(t)}[u(\cdot,t)]}{K_M\,\sigma_0^{\frac{d}{2}\,(1-m)}}\right] \frac{d}{dt}\left(\mathcal{F}_{\sigma(t)}[u(\cdot,t)]\right) \ge \frac{d}{dt}\left(\mathcal{J}_{\sigma(t)}[u(\cdot,t)]\right)$$

where

$$(20)\qquad C_{m,d} := d^2\,\frac{(1-m)^2\,(m-m_c)}{8\,m}\,.$$

Integrating from 0 to ∞ with respect to t, we finally get the improved inequality

$$C_{m,d}\,\frac{\left(\mathcal{F}_{\sigma_0}[u_0]\right)^2}{K_M\,\sigma_0^{\frac{d}{2}\,(1-m)}} \le \frac{1}{4}\,\mathcal{J}_{\sigma_0}[u_0] - \mathcal{F}_{\sigma_0}[u_0]\,,$$

which holds for any admissible function u_0. Observing that

$$K_M\,\sigma_0^{\frac{d}{2}\,(1-m)}=\sigma_0^{-\frac{d}{2}\,(m-m_c)}\int_{\mathbb{R}^d}|x|^2\,u_0\,dx$$

and omitting the index 0, we have achieved our key estimate, which can be written as follows.

THEOREM 10. *Let $d\geq 1$, $m\in[m_1,1)$ and assume that u is a non-negative function in $\mathrm{L}^1(\mathbb{R}^d)$ such that u^m and $x\mapsto|x|^2\,u$ are both integrable on $\mathbb{R}^d$. Let $\sigma=\frac{1}{K_M}\int_{\mathbb{R}^d}|x|^2\,u(x)\,dx$ where $M=\int_{\mathbb{R}^d}u(x)\,dx$. Then the following inequality holds*

$$C_{m,d}\,\sigma^{\frac{d}{2}\,(m-m_c)}\,\frac{(\mathcal{F}_\sigma[u])^2}{\int_{\mathbb{R}^d}|x|^2\,u\,dx}\leq\frac{1}{4}\,\mathcal{J}_\sigma[u]-\mathcal{F}_\sigma[u]\tag{21}$$

where $C_{m,d}$ is defined by (20)*.*

Notice that (17) gives a straightforward proof of (13) using the definition of σ. Since σ depends on u, the left hand side of (21) can be rewritten to give

$$d^2\,\frac{(1-m)^2\,(m-m_c)}{8\,m\,K_1}\,\frac{(\mathcal{F}_\sigma[u])^2}{M^\gamma\,\sigma^{\frac{d}{2}\,(1-m)}}\leq\frac{1}{4}\,\mathcal{J}_\sigma[u]-\mathcal{F}_\sigma[u]$$

with

$$\gamma:=\frac{(d+2)\,m-d}{d\,(m-m_c)}\;.$$

See Appendix A for details. Notice that this definition of γ is compatible with the one of Corollary 3 if $p=1/(2\,m-1)$.

Remark 3. Our estimates are actually better. With

$$f(t):=\mathcal{F}_{\sigma(t)}[u(\cdot,t)]\;,\;j(t):=\mathcal{J}_{\sigma(t)}[u(\cdot,t)]\text{ and }\sigma(t)=\frac{1}{K_M}\int_{\mathbb{R}^d}|x|^2\,u(x,t)\,dx\;,$$

we can rewrite (13)*,* (15) *and* (18) *as the coupled system*

$$\begin{aligned}
&f'=-j\leq 0\;,\\
&\sigma'=-2\,d\,\frac{(1-m)^2}{m\,K_M}\,\sigma^{\frac{d}{2}(m-m_c)}\,f\leq 0\;,\\
&j'+4\,j\leq\frac{d}{2}\,(m-m_c)\,\Big[\,j+4\,d\,(1-m)\,f\Big]\,\frac{\sigma'}{\sigma}\;.
\end{aligned}$$

It is then clear that the estimates $\sigma\leq\sigma_0$ and

$$j'+4\,j\leq\frac{d}{2}\,(m-m_c)\,j\,\frac{\sigma'}{\sigma}\;,$$

which have been used for the proof of Theorem 10, are not optimal.

6. **Proofs**

Let us start by rephrasing Theorem 10 in terms of $f = u^{m-1/2}$. Assume that

$$M = \int_{\mathbb{R}^d} u\,dx = \int_{\mathbb{R}^d} |f|^{2\,p}\,dx \quad \text{and} \quad \sigma = \frac{1}{K_M} \int_{\mathbb{R}^d} |x|^2\,u\,dx = \int_{\mathbb{R}^d} |x|^2\,|f|^{2\,p}\,dx$$

where $p = 1/(2\,m-1)$ and consider the functional

$$\mathsf{R}^{(p)}[f] := -\tfrac{2\,p}{p+1} \int_{\mathbb{R}^d} \left[(|f|^{p+1} - \left(f^{(p)}_{M,0,\sigma}\right)^{p+1}\right] dx\;.$$

Corollary 11. *Let $d \geq 2$, $p > 1$ and assume that $p \leq d/(d-2)$ if $d \geq 3$. For any $f \in \mathrm{L}^{p+1} \cap \mathcal{D}^{1,2}(\mathbb{R}^d)$ such that $\|f\|_{2,2\,p}$ is finite, we have*

$$\left(\mathcal{C}^{\rm GN}_{p,d}\;\|\nabla f\|_2^\theta\;\|f\|_{p+1}^{1-\theta}\right)^{2\,\gamma\,p} - \;\|f\|_{2\,p}^{2\,\gamma\,p} \geq \mathsf{C}_{p,d}\,\frac{\left(\mathsf{R}^{(p)}[f]\right)^2}{\|f\|_{2,2\,p}^\alpha\;\|f\|_{2\,p}^{\beta\,\gamma}} \tag{22}$$

with $\theta = \frac{p-1}{p}\,\frac{d}{d+2-p\,(d-2)}$, $\alpha = d\,(p-1)$, $\beta = d - p\,(d-2)$ and $\gamma = \frac{d+2-p\,(d-2)}{d-p\,(d-4)}$.

The constant $\mathsf{C}_{p,d}$ is the same as in Theorem 3: see Appendix A for its expression. In preparation for the statements of Theorems 1 and 3, we distinguish the case $m = m_1$ and the case $m \in [m_1, 1)$ in the proof.

Proof. By expanding the square in $\mathcal{J}_\sigma[u]$ and collecting the terms with the ones of $\mathcal{F}_\sigma[u]$, we find that

$$\begin{aligned}\frac{1}{4}\,\mathcal{J}_\sigma[u] - \mathcal{F}_\sigma[u] = \tfrac{m\,(1-m)}{(2\,m-1)^2}\,\sigma^{\frac{d}{2}(m-m_c)} \int_{\mathbb{R}^d} |\nabla u^{m-\frac{1}{2}}|^2\,dx \\ + d\,\tfrac{m-m_1}{1-m} \int_{\mathbb{R}^d} u^m\,dx + \tfrac{1}{1-m}\left(m\,K_M\,\sigma^{\frac{d}{2}\,(1-m)} - \int_{\mathbb{R}^d} B_\sigma^m\,dx\right)\,.\end{aligned}$$

The last term of the right hand side can be rewritten as

$$\tfrac{1}{1-m}\left(m\,K_M\,\sigma^{-\frac{d}{2}\,(1-m)} - \int_{\mathbb{R}^d} B_\sigma^m\,dx\right) = -\tfrac{m}{1-m}\,\tfrac{d\,(m-m_c)}{(d+2)\,m-d}\,\sigma^{\frac{d}{2}\,(1-m)}\,C_1\,M^\gamma$$

with $\gamma = \frac{(d+2)\,m-d}{d\,(m-m_c)}$ (as in the previous Section) and $C_1 = M_*^{\frac{2\,(1-m)}{d\,(m-m_c)}}$ (see Appendix A for details). Consequently Inequality (21) can be equivalently rewritten as

$$\begin{aligned}\tfrac{m\,(1-m)}{(2\,m-1)^2}\,\sigma^{\frac{d}{2}(m-m_c)} \int_{\mathbb{R}^d} |\nabla u^{m-\frac{1}{2}}|^2\,dx \\ + d\,\tfrac{m-m_1}{1-m} \int_{\mathbb{R}^d} u^m\,dx - \tfrac{m}{1-m}\,\tfrac{d\,(m-m_c)}{(d+2)\,m-d}\,\sigma^{\frac{d}{2}\,(1-m)}\,C_1\,M^\gamma \\ \geq C_{m,d}\,\sigma^{\frac{d}{2}\,(m-m_c)}\,\frac{(\mathcal{F}_\sigma[u])^2}{\int_{\mathbb{R}^d} |x|^2\,u\,dx}\;.\end{aligned} \tag{23}$$

If $m=m_1$, we observe that $\frac{d}{2}\,(m-m_c)=\frac{d}{2}\,(1-m)=\frac{1}{2}$ so that the result of the inequality amounts to

$$\|\nabla f\|_2^2-d\,(d-2)\,C_1\;\|f\|_{2^*}^2\ge\frac{1}{8}\left(\frac{d-2}{d-1}\right)^2\;\frac{(\mathcal{F}_\sigma[u])^2}{\int_{\mathbb{R}^d}|x|^2\,u\;dx}$$

with $f=u^{m-\frac{1}{2}}$. Since $\mathcal{F}[u]=(d-1)\,\mathcal{R}[f]$ and $\mathsf{S}_d=d\,(d-2)\,C_1$, this concludes the proof of Corollary 11 when $m=m_1$.

If $m\in[m_1,1)$, we can rewrite Inequality (23) in terms of a function u_λ such that $\int_{\mathbb{R}^d}u_\lambda\;dx=M$ and $\int_{\mathbb{R}^d}|x|^2\,u_\lambda\;dx=\sigma_\lambda$ as

$$\begin{aligned}\tfrac{m\,(1-m)}{(2\,m-1)^2}\,\sigma_\lambda^{d\,(m-m_1)}\int_{\mathbb{R}^d}|\nabla u_\lambda^{m-\frac{1}{2}}|^2\;dx\\ +\,d\,\tfrac{m-m_1}{1-m}\,\sigma_\lambda^{-\frac{d}{2}\,(1-m)}\int_{\mathbb{R}^d}u_\lambda^m\;dx-\tfrac{m}{1-m}\,\tfrac{d\,(m-m_c)}{(d+2)\,m-d}\,C_1\,M^\gamma\\ \ge C_{m,d}\,\sigma_\lambda^{d\,(m-m_1)}\,\frac{(\mathcal{F}_{\sigma_\lambda}[u_\lambda])^2}{\int_{\mathbb{R}^d}|x|^2\,u_\lambda\;dx}\;.\end{aligned}$$

If we choose $u_\lambda(x)=\lambda^d\,u(\lambda\,x)$, then

$$\sigma_\lambda=\frac{\sigma}{\lambda^2}\;,\quad\int_{\mathbb{R}^d}|x|^2\,u_\lambda\;dx=\frac{1}{\lambda^2}\int_{\mathbb{R}^d}|x|^2\,u\;dx\;,\quad\mathcal{F}_{\sigma_\lambda}[u_\lambda]=\lambda^{d\,(m-1)}\,\mathcal{F}_\sigma[u]$$

and the above inequality becomes

$$\begin{aligned}\tfrac{m\,(1-m)}{(2\,m-1)^2}\,(\lambda^{-2}\,\sigma)^{d\,(m-m_1)}\int_{\mathbb{R}^d}|\nabla u^{m-\frac{1}{2}}|^2\;dx\\ +\,d\,\tfrac{m-m_1}{1-m}\,(\lambda^{-2}\,\sigma)^{-\frac{d}{2}\,(1-m)}\int_{\mathbb{R}^d}u^m\;dx-\tfrac{m}{1-m}\,\tfrac{d\,(m-m_c)}{(d+2)\,m-d}\,C_1\,M^\gamma\\ \ge C_{m,d}\,\frac{\sigma^{d\,(m-m_1)}\,(\mathcal{F}_\sigma[u])^2}{\int_{\mathbb{R}^d}|x|^2\,u\;dx}\;.\end{aligned}$$

Notice that the right hand side is independent of λ. By optimizing the left hand side with respect to $\lambda>0$ and replacing u by $f^{2\,p}$ with $p=\frac{1}{2\,m-1}$, we find

$$\left(\mathcal{C}_{p,d}^{\rm GN}\;\|\nabla f\|_2^\theta\;\|f\|_{p+1}^{1-\theta}\right)^{2\,\gamma\,p}-\;\|f\|_{2\,p}^{2\,\gamma\,p}\ge\tfrac{1-m}{m}\;\tfrac{(d+2)\,m-d}{d\,(m-m_c)}\;\frac{C_{m,d}}{C_1}\;\frac{\sigma^{d\,(m-m_1)}\,(\mathcal{F}_\sigma[u])^2}{\int_{\mathbb{R}^d}|x|^2\,u\;dx}$$

with the notations of Inequality (3). It is indeed clear that the left hand side has to vanish if $u=B_\sigma$, which guarantees that, after optimization, the constant is precisely equal to $\mathcal{C}_{p,d}^{\rm GN}$. Using

$$\sigma=\frac{1}{K_M}\;\|f\|_{2,2\,p}^{2\,p}\;,\quad K_M=\frac{d\,(1-m)}{(d+2)\,m-d}\,C_1\;\|f\|_{2\,p}^{2\,p\,\gamma}$$

(see (27) in Appendix A) and $\mathcal{F}_\sigma[u]=\frac{m}{1-m}\,\mathsf{R}^{(p)}[f]$ completes the proof of Corollary 11 when $m\in[m_1,1)$. See Appendix A for an expression of $\mathsf{C}_{p,d}$. □

Proof of Theorems 1 and 3. Theorem 1 is a special case of Theorem 3, which is itself a simple consequences of Corollary 11.

Let us consider the relative entropy with respect to a general Barenblatt function, not even normalized with respect to its mass. For a given function $u \in \mathrm{L}^1_+(\mathbb{R}^d)$ with $u^m \in \mathrm{L}^1(\mathbb{R}^d)$ and $|x|^2\, u \in \mathrm{L}^1(\mathbb{R}^d)$, we can consider on $(0,\infty)\times\mathbb{R}^d\times(0,\infty)$ the function h defined by

$$h(C,y,\sigma)=\frac{1}{m-1}\int_{\mathbb{R}^d}\left[u^m-B_{C,y,\sigma}^m-m\,B_{C,y,\sigma}^{m-1}\,(u-B_{C,y,\sigma})\right]dx$$

where $B_{C,y,\sigma}$ is a general Barenblatt function

$$B_{C,y,\sigma}(x):=\sigma^{-\frac{d}{2}}\left(C+\tfrac{1}{\sigma}\,|x-y|^2\right)^{\frac{1}{m-1}}\quad\forall\,x\in\mathbb{R}^d\,.$$

An elementary computation shows that

$$\begin{aligned}
\frac{\partial h}{\partial C}&=\frac{m\,\sigma^{\frac{d}{2}\,(1-m)}}{1-m}\int_{\mathbb{R}^d}(u-B_{C,y,\sigma})\;dx\;,\\
\nabla_y h&=\frac{2\,m\,\sigma^{-\frac{d}{2}\,(m-m_c)}}{1-m}\int_{\mathbb{R}^d}(x-y)\,(u-B_{C,y,\sigma})\;dx\;,\\
\frac{\partial h}{\partial\sigma}&=m\,\frac{d}{2}\,\sigma^{-\frac{d}{2}\,(m-m_c)}\Bigg[C\int_{\mathbb{R}^d}(u-B_{C,y,\sigma})\;dx\\
&\qquad\qquad-\frac{m-m_c}{1-m}\,\frac{1}{\sigma}\int_{\mathbb{R}^d}|x-y|^2\,(u-B_{C,y,\sigma})\;dx\Bigg]\,.
\end{aligned}$$

Optimizing with respect to C fixes $C=C_M$, with $M=\int_{\mathbb{R}^d}u\;dx$. Once $C=C_M$ is assumed, optimizing with respect to σ amounts to choose it such that $\int_{\mathbb{R}^d}|x|^2\,B_{C,y,\sigma}\;dx=\int_{\mathbb{R}^d}|x-y|^2\,u\;dx$ as it has been shown in Lemma 7.

This completes the proof of Theorem 3, since $\mathcal{R}^{(p)}[f]\geq\mathsf{R}^{(p)}[f]$ by definition of $\mathcal{R}^{(p)}$ (see Section 1). Notice that optimizing on y amounts to fix the center of mass of the Barenblatt function to be the same as the one of u. This is however not required neither in the proof of Corollary 11 nor in the one of Theorem 3. □

Proof of Corollary 4. It is a straightforward consequence of Theorem 3 and of the Csiszár-Kullback inequality (5) when $f\in\mathcal{D}^{1,2}(\mathbb{R}^d)$ is such that $\|f\|_{2,2\,p}$ is finite. However, $\|f\|_{2,2\,p}$ does not enter in the inequality. Since smooth functions with compact support (for which $\|f\|_{2,2\,p}$ is obviously finite) are dense $\mathcal{D}^{1,2}(\mathbb{R}^d)$, the inequality therefore holds without restriction, by density. Recall that Corollary 2 is a special case of Corollary 4. □

7. **Concluding remarks**

Let us conclude this paper with a few remarks. First of all, notice that Theorem 6 gives a stronger information than Theorem 3, as not only the

$L^1(\mathbb{R}^d, dx)$ norm is controlled, but also a stronger norm involving the second moment, properly scaled.

No condition is imposed on the location of the center of mass, which simply has to satisfy $\left(\int_{\mathbb{R}^d} x\, u\, dx\right)^2 \le \int_{\mathbb{R}^d} u\, dx \int_{\mathbb{R}^d} |x|^2\, u\, dx = \sigma\, M\, K_M$ according to the Cauchy-Schwarz inequality.

Hence in the definition of $\mathcal{R}[f]$ and $\mathcal{R}^{(p)}[f]$ (Theorems 1 and 3) as well as in Corollaries 2 and 4, the result holds without optimizing on $y \in \mathbb{R}^d$. In [10, 21], improved asymptotic rates were obtained by fixing the center of mass in order to kill the linear mode associated to the translation invariance of the Barenblatt functions. Here this is not required since, as $t \to \infty$, the squared relative entropy is simply higher order. Our improvement is better when the relative entropy is large, and is clearly not optimal for large values of t.

Our approach differs from the one of G. Bianchi and H. Egnell in [6] and the one of A. Cianchi, N. Fusco, F. Maggi and A. Pratelli, [17]. It gives fully explicit constants. The norms involved in the corrective term are not the same either. However, our estimates are not optimal, as it has been noticed in Remark 3, in the sense that the only functions for which we have equality are the Aubin-Talenti functions in case of Theorem 1 and the functions in $\mathfrak{M}_d^{(p)}$ in case of Theorem 3.

Non scale invariant forms of the improved Gagliardo-Nirenberg inequality of Theorem 3 are by themselves of interest. Starting from (23), it is for instance possible to scale out some numerical coefficients as follows without putting the inequality in scale invariant form. Let

$$f(x) := u^{m-\frac{1}{2}}(x/\lambda) \quad \forall\, x \in \mathbb{R}^d\,.$$

We have $M = \lambda^{-d} \int_{\mathbb{R}^d} |f|^{2\,p}\, dx$, and all other quantities are also changed:

$$\begin{aligned}
&\int_{\mathbb{R}^d} |\nabla u^{m-\frac{1}{2}}|^2\, dx = \lambda^{2-d} \int_{\mathbb{R}^d} |\nabla f|^2\, dx\;,\\
&K_M\, \sigma = \int_{\mathbb{R}^d} |x|^2\, u\, dx = \lambda^{-(d+2)} \int_{\mathbb{R}^d} |x|^2\, |f|^{2\,p}\, dx\;,\\
&\int_{\mathbb{R}^d} |u|^m\, dx = \lambda^{-d} \int_{\mathbb{R}^d} |f|^{p+1}\, dx\;.
\end{aligned}$$

With the choice $\lambda^d = \left(\frac{1}{4}\,(p-1)^2\,(p+1)\right)^{\frac{4\,p}{d+2-p\,(d-2)}}\;\|f\|_{2,p}^{2\,p\,\frac{d-p\,(d-4)}{d+2-p\,(d-2)}}$, we find that (23) can be rewritten as

$$\int_{\mathbb{R}^d} |\nabla f|^2\, dx + [d - p\,(d-2)] \int_{\mathbb{R}^d} |f|^{p+1}\, dx - \mathsf{K}_{p,d} \left(\int_{\mathbb{R}^d} |f|^{2\,p}\, dx\right)^{\gamma(p,d)} \ge \mathsf{K}_{p,d}\, \mathsf{C}_{p,d} \frac{\left(\mathsf{R}^{(p)}[f]\right)^2}{\|f\|_{2,2\,p}^{\alpha}\;\|f\|_{2\,p}^{\beta\,\gamma}} \tag{24}$$

where $\mathsf{C}_{p,d}$ and $\mathsf{R}^{(p)}[f]$ are defined as in Corolllary 11. See Appendix A for an explicit expression of $\mathsf{K}_{p,d}$ and $\mathsf{C}_{p,d}$.

If we optimize the left hand side of (24) under scaling, that is, if we rewrite it for $f_\lambda(x) := \lambda^{d/(2\,p)}\,f(\lambda\,x)$ for any $x\in\mathbb{R}^d$ and optimize with respect to $\lambda>0$, we find that it can be rewritten as

$$[d-p\,(d-4)]\,2^{-2\,\frac{d-p\,(d-2)}{d-p\,(d-4)}}\,[d\,(p-1)]^{-\frac{d\,(p-1)}{d-p\,(d-4)}}\left(\,\|\nabla f\|_2^\theta\,\|f\|_{p+1}^{1-\theta}\right)^{2\,\gamma\,p}\,,$$

which is of course consistent with the fact that

$$\mathsf{K}_{p,d}\left(\mathcal{C}_{p,d}^{\rm GN}\right)^{2\,\gamma\,p}=[d-p\,(d-4)]\,2^{-2\,\frac{d-p\,(d-2)}{d-p\,(d-4)}}\,[d\,(p-1)]^{-\frac{d\,(p-1)}{d-p\,(d-4)}}\,. \tag{25}$$

Altogether, we have recovered (22) with $\mathsf{R}^{(p)}[f]=-\frac{2\,p}{p+1}\int_{\mathbb{R}^d}\left[(|f|^{p+1}-g^{p+1}\right]\,dx$ and $g(x):=B_\sigma^{m-\frac12}(x/\lambda)=f^{(p)}_{M,0,\sigma}(x/\lambda)$.

Consider the scaling $\lambda\mapsto u_\lambda$ with $u_\lambda(x):=\lambda^d\,u(\lambda\,x)$ for any $x\in\mathbb{R}^d$. Then we have

$$\sigma_\lambda:=\frac{1}{K_M}\int_{\mathbb{R}^d}|x|^2\,u_\lambda\;dx=\frac{1}{\lambda^2}\,\frac{1}{K_M}\int_{\mathbb{R}^d}|x|^2\,u\;dx=\frac{\sigma}{\lambda^2}$$

and may observe that

$$B_{\sigma_\lambda}(x)=\lambda^d\,B_\sigma(\lambda\,x)\;.$$

Similarly notice that for any $m\in[\frac{d-1}{d},1)$, we have $C_M=C_1\,M^{-\frac{2\,(1-m)}{d\,(m-m_c)}}$ and $K_M=K_1\,M^{1-\frac{2\,(1-m)}{d\,(m-m_c)}}$. Let $u_\lambda:=\lambda\,u$ and denote by B_{σ_λ} the corresponding best matching Barenblatt function. Using the fact that $\|u_\lambda\|_1=\lambda\,M$ if $\|u\|_1=M$ and observing that

$$K_{\lambda M}=K_M\,\lambda^{1-\frac{2\,(1-m)}{d\,(m-m_c)}}\quad\text{and}\quad\int_{\mathbb{R}^d}|x|^2\,u_\lambda\;dx=\lambda\int_{\mathbb{R}^d}|x|^2\,u\;dx\;,$$

we find

$$\sigma_\lambda=\frac{1}{K_{\lambda M}}\int_{\mathbb{R}^d}|x|^2\,u_\lambda\;dx=\lambda^{\frac{2\,(1-m)}{d\,(m-m_c)}}\,\sigma\;.$$

Since $C_{\lambda M}=\lambda^{-\frac{2\,(1-m)}{d\,(m-m_c)}}\,C_M$, we find that

$$B_{\sigma_\lambda}(x)=\left(\lambda^{\frac{2\,(1-m)}{d\,(m-m_c)}}\,\sigma\right)^{-\frac d2}\left(\lambda^{-\frac{2\,(1-m)}{d\,(m-m_c)}}\,C_M+\frac{|x|^2}{\lambda^{\frac{2\,(1-m)}{d\,(m-m_c)}}\,\sigma}\right)^{\frac{1}{m-1}}=\lambda\,B_\sigma(x)\;.$$

Coming back to (22), the function g which appears in (22) has therefore the same homogeneity and scaling properties as the function f to which it corresponds. In terms of homogeneity, this means that g has to be replaced by $\lambda\,g$ if f is replaced by $\lambda\,f$. It is then straightforward to check that homogeneity in terms of f is the same on both sides of Inequality (22) since

$$2\,\gamma\,p=2\,(p+1)-\alpha-\beta\,\gamma\;.$$

Scalings are also consistent with Inequality (22): to $f_\lambda(x)=\lambda^{\frac{d}{2p}}\,f(\lambda\,x)$ corresponds $g_\lambda(x)=\lambda^{\frac{d}{2p}}\,g(\lambda\,x)$. A simple computation indeed shows that

$$\frac{\left[\int_{\mathbb{R}^d}\left(|f_\lambda|^{p+1}-g_\lambda^{p+1}\right)\,dx\right]^2}{\|f_\lambda\|_{2,2p}^{\alpha}\;\|f_\lambda\|_{2p}^{\beta\,\gamma}}=\frac{\left[\int_{\mathbb{R}^d}\left(|f|^{p+1}-g^{p+1}\right)\,dx\right]^2}{\|f\|_{2,2p}^{\alpha}\;\|f\|_{2p}^{\beta\,\gamma}}\quad\forall\,\lambda>0\,.$$

As $m\to 1$, which also corresponds to $p\to 1$, we observe that the constant $\mathsf{C}_{p,d}$ in Theorem 3 has a finite limit. Hence we get no improvement by dividing the improved Gagliardo-Nirenberg inequality by $(p-1)$ and passing to the limit $p\to 1_+$, since $\mathcal{R}^{(p)}[f]=O(p-1)$. By doing so, we simply recover the logarithmic Sobolev inequality as in [18].

This is consistent with the fact that, as $m\to 1_-$, we have $C_{m,d}\sim(1-m)^2$, $\sigma=O(K_M^{-1})=O(1-m)$ and, since

$$B_\sigma(x)\sim B_0(x):=M\left(\frac{d\,M}{2\,\pi\,\int_{\mathbb{R}^d}|x|^2\,u\,dx}\right)^{\frac{d}{2}}\exp\left(-\frac{d}{2}\,\frac{M}{\int_{\mathbb{R}^d}|x|^2\,u\,dx}\,|x|^2\right)\,,$$

we also get that $\mathcal{F}_\sigma[u]\sim\int_{\mathbb{R}^d}u\,\log\left(\frac{u}{B_0}\right)\,dx$. Hence, in Theorem 10, the additional term in (21) is of the order of $1-m$ and disappears when passing to the limit $m\to 1_-$.

Appendix A. Computation of the constants

Let us recall first some useful formulae. The surface of the $d-1$ dimensional unit sphere $\mathbb{S}^{d-1}$ is given by $|\mathbb{S}^{d-1}|=2\,\pi^{d/2}/\,\Gamma(d/2)$. Using the integral representation of Euler's Beta function (see [1, 6.2.1 p. 258]), we have

$$\int_{\mathbb{R}^d}\left(1+|x|^2\right)^{-a}\,dx=\pi^{\frac{d}{2}}\,\frac{\Gamma\left(a-\frac{d}{2}\right)}{\Gamma(a)}\,.$$

With this formula in hand, various quantities associated with *Barenblatt functions* can be computed. Applied to the function $B(x):=(1+|x|^2)^{\frac{1}{m-1}}$, $x\in\mathbb{R}^d$, we find that

$$M_*:=\int_{\mathbb{R}^d}B\,dx=\pi^{\frac{d}{2}}\,\frac{\Gamma\left(\frac{d\,(m-m_c)}{2\,(1-m)}\right)}{\Gamma\left(\frac{1}{1-m}\right)}\,.\tag{26}$$

Notice that when $M=M_*$, $B=B_1$ with the notation (9) of Section 3. As a consequence, for $B_1(x)=(C_M+|x|^2)^{\frac{1}{m-1}}$, a simple change of variables shows that

$$M:=\int_{\mathbb{R}^d}B_1\,dx=\int_{\mathbb{R}^d}\left(C_M+|x|^2\right)^{\frac{1}{m-1}}\,dx=M^*\,C_M^{-\frac{d\,(m-m_c)}{2\,(1-m)}}\,,$$

which determines the value of C_M, namely

$$C_M = \left(\frac{M_*}{M}\right)^{\frac{2\,(1-m)}{d\,(m-m_c)}} .$$

A useful equivalent formula is $C_M = C_1\, M^{-\frac{2\,(1-m)}{d\,(m-m_c)}}$ where $C_1 = M_*^{\frac{2\,(1-m)}{d\,(m-m_c)}}$.

By recalling (17) and observing that

$$\int_{\mathbb{R}^d} B_1^m\,dx = \int_{\mathbb{R}^d} B_1^{m-1}\,B_1\,dx = \int_{\mathbb{R}^d} (C_M + |x|^2)\,B_1\,dx = M\,C_M + K_M$$

where $K_M := \int_{\mathbb{R}^d} |x|^2\,B_1\,dx$, using $M\,C_M = C_1\,M^\gamma$ with $\gamma = \frac{(d+2)\,m-d}{d\,(m-m_c)}$, we find that

(27)
$$K_M = \frac{d\,(1-m)}{(d+2)\,m-d}\,C_1\,M^\gamma \quad\text{and}\quad \int_{\mathbb{R}^d} B_1^m\,dx = \frac{2\,m}{(d+2)\,m-d}\,C_1\,M^\gamma .$$

Concerning *best constants in Sobolev's inequality* (1) in $\mathbb{R}^d$, $d \geq 3$, equality is achieved by $f(x) = (1+|x|^2)^{-(d-2)/2}$, $x \in \mathbb{R}^d$, which provides the expression of S_d given in Section 1. According to the *duplication formula* of Legendre (see for instance [1, 6.1.18 p. 256]) for the Γ function, we know that $\Gamma(x)\,\Gamma\left(x+\frac{1}{2}\right) = 2^{1-2x}\,\sqrt{\pi}\,\Gamma(2\,x)$ for any $x > 0$. As a consequence, the best constant in Sobolev's inequality (1) can also be written as $\mathsf{S}_d = \frac{1}{4}\,d\,(d-2)\,|\mathbb{S}^d|^{2/d}$ (see for instance [5]; also see [26, 9, 23] for earlier related results).

Consider the sub-family of *Gagliardo-Nirenberg-Sobolev inequalities* (3). It has been established in [18, Theorem 1] that optimal functions are all given by (4), up to multiplications by a constant, translations and scalings. This allows to compute $\mathcal{C}^{\rm GN}_{p,d}$. All computations done, we find

$$\mathcal{C}^{\rm GN}_{p,d} = \left(\tfrac{(p-1)^{p+1}}{(p+1)^{d+1-p(d-1)}}\right)^\eta \left(\tfrac{d+2-p\,(d-2)}{2\,(p-1)}\right)^{\frac{1}{2\,p}} \left(\frac{\Gamma\left(\frac{p+1}{p-1}\right)}{(2\,\pi\,d)^{\frac{d}{2}}\,\Gamma\left(\frac{p+1}{p-1}-\frac{d}{2}\right)}\right)^{(p-1)\,\eta}$$

with $1/\eta = p\,(d+2-p\,(d-2))$. This expression of $\mathcal{C}^{\rm GN}_{p,d}$ will be recovered below by a different method.

Next, the computation of $\mathsf{C}_{p,d}$ in Theorem 3 goes as follows. With $p = \frac{1}{2\,m-1}$, that is, $m = \frac{p+1}{2\,p}$, and $\mathcal{F}[u] = \frac{m}{1-m}\,\mathcal{R}^{(p)}[f]$ with $u = f^{2\,p}$, we first get

$$\mathsf{C}_{p,d} = \tfrac{1-m}{m}\,\tfrac{(d+2)\,m-d}{d\,(m-m_c)}\,\frac{C_{m,d}}{C_1}\,K_1^{-d\,(m-m_1)}\left(\tfrac{m}{1-m}\right)^2 .$$

We can also rewrite (26) as

$$M_* = \pi^{\frac{d}{2}}\,\frac{\Gamma\left(\frac{d-p\,(d-4)}{2\,(p-1)}\right)}{\Gamma\left(\frac{2\,p}{p-1}\right)} .$$

With $C_1 = M_*^{\frac{2\,(p-1)}{d-p\,(d-4)}}$ and $K_1 = \frac{d\,(p-1)}{d+2-d\,(p-2)}\,C_1$, we finally obtain

$$\mathsf{C}_{p,d} = d\,\frac{p-1}{32\,p^2}\,\left(\frac{C_1}{d+2-p\,(d-2)}\right)^{\frac{d-p\,(d-4)}{2\,p}}\,(d\,(p-1))^{-\frac{d-p\,(d-2)}{2\,p}}\;.$$

Finally, we turn our attention to (24) and compute an explicit expression of $\mathsf{K}_{p,d}$, which is the best constant in the following *non-homogeneous Gagliardo-Nirenberg-Sobolev inequalities:* for any $f \in \mathrm{L}^{p+1} \cap \mathcal{D}^{1,2}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} |\nabla f|^2\,dx + [d-p\,(d-2)]\int_{\mathbb{R}^d} |f|^{p+1}\,dx \geq \mathsf{K}_{p,d}\left(\int_{\mathbb{R}^d} |f|^{2\,p}\,dx\right)^{\gamma(p,d)} \tag{28}$$

with $\gamma = \gamma(p,d) = \frac{d+2-p\,(d-2)}{d-p\,(d-4)}$. As observed in Section 7, By optimizing the left hand side of (28) written for $f_\lambda(x) := \lambda^{d/(2\,p)}\,f(\lambda\,x)$ for any $x \in \mathbb{R}^d$, with respect to $\lambda > 0$, one recovers that (28) and (3) are equivalent, with optimal constants related by (25).

Consider the radial function g defined by $g(x) = F_p(x) = (1+|x|^2)^{-\frac{1}{p-1}}$ for any $x \in \mathbb{R}^d$ (*c.f.* Eq. (4)), which solves the equation

$$-\Delta g + 2\,\frac{d-p\,(d-2)}{(p-1)^2}\,g^p - \frac{4\,p}{(p-1)^2}\,g^{2\,p-1} = 0\;.$$

With a rescaling, namely by considering $f(x) := \sigma^{-\frac{d}{4\,p}}\,g(x/\sqrt{\sigma})$, we find that f solves

$$-\Delta f + 2\,\frac{d-p\,(d-2)}{(p-1)^2}\,\sigma^{-\frac{d-p\,(d-4)}{4\,p}}\,f^p - \frac{4\,p}{(p-1)^2}\,\sigma^{-\frac{d-p\,(d-2)}{2\,p}}\,f^{2\,p-1} = 0\;.$$

Owing to the uniqueness of the radial solution as in [18], we identify f with the optimal function for (28). This can be done by identifying the coefficients of the Euler-Lagrange equation. Requiring that f solves

$$-2\,\Delta f + (p+1)\,[d-p\,(d-2)]\,f^p - 2\,\gamma\,p\,\mathsf{K}_{p,d}\left(\int_{\mathbb{R}^d} |f|^{2\,p}\,dx\right)^{\gamma-1} f^{2\,p-1} = 0$$

means that σ is such that

$$\frac{p+1}{2} = \frac{2}{(p-1)^2}\,\sigma^{-\frac{d-p\,(d-4)}{4\,p}}\;,$$

that is

$$\sigma = \left[(p+1)\left(\tfrac{p-1}{2}\right)^2\right]^{-\frac{4\,p}{d-p\,(d-4)}}$$

and allows to compute $\mathsf{K}_{p,d}$ by solving

$$\gamma\,p\,\mathsf{K}_{p,d}\left(\int_{\mathbb{R}^d} |f|^{2\,p}\,dx\right)^{\gamma-1} = \frac{4\,p}{(p-1)^2}\,\sigma^{-\frac{d-p\,(d-2)}{2\,p}}$$

$$= p\,(p+1)^{2\,\frac{d-p\,(d-2)}{d-p\,(d-4)}}\left(\tfrac{p-1}{2}\right)^{-\frac{2\,d\,(p-1)}{d-p\,(d-4)}}\;.$$

All computations done, we find that

$$\mathsf{K}_{p,d}=\tfrac{d-p\,(d-4)}{d+2-p\,(d-2)}\,(p+1)^{2\,\frac{d-p\,(d-2)}{d-p\,(d-4)}}\left(\tfrac{p-1}{2}\right)^{-\frac{2\,d\,(p-1)}{d-p\,(d-4)}}\left[\frac{\Gamma\left(\frac{d-p\,(d-4)}{2\,(p-1)}\right)}{\pi^{-\frac{d}{2}}\,\Gamma\left(\frac{2\,p}{p-1}\right)}\right]^{\frac{2\,(p-1)}{d-p\,(d-4)}}.$$

Using (25), this also justifies the expression of $\mathcal{C}^{\rm GN}_{p,d}$ which was stated earlier in this Appendix.

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