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Improved Technique for Design of Perfect Reconstruction FIR QMF Banks with Lossless Polyphase Matrices

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Abstract—This paper develops an improved technique for the design of analysis filters in an M channel maximally decimated FIR perfect reconstruction QMF bank, having a lossless polyphase-component matrix $E(z)$. As in earlier work, the aim is to optimize the parameters characterizing $E(z)$ until the sum of the stopband energies of the analysis filters is minimized. There are four new ingredients in the procedure reported here. The first is a technique for efficient initialization of one of the M analysis filters, as a spectral factor of an Mth band filter. This factorization itself is done without root-finding techniques, in an efficient manner using the *eigenfilters* approach. The second component is the initialization of the internal *parameters* which characterize $E(z)$, based on the above spectral factor. In earlier work, the parameters characterizing the lossless $E(z)$ were rotation angles, which resulted in slow convergence of the above-mentioned optimization. The third ingredient of the improved approach is a modified characterization, mostly free from rotation angles, of the lossless FIR $E(z)$. The fourth component of improvement is the incorporation of symmetry among the analysis filters, so as to minimize the number of unknown parameters being optimized. The resulting design procedure always gives better filter responses than earlier ones (for a given filter length), and converges much faster.

I. INTRODUCTION

IN this paper we consider the maximally decimated M-channel analysis/synthesis system shown in Fig. 1. The basic operational principles of this analysis/synthesis system and its applications are discussed in a number of references [1]–[13]. The analysis filters $H_k(z)$ split the signal $x(n)$ into M subband signals which are then decimated by M and encoded prior to transmission. At the synthesis end, the M subband signals are decoded, interpolated, and recombined using the synthesis filters $F_k(z)$. Ignoring the nonlinear coding/decoding error and quantization errors in the filter implementations, the signal $\hat{x}(n)$ suffers from three errors [5], viz., aliasing, amplitude distortion, and phase distortion. Several techniques have been discussed in the past for eliminating (partially or completely) some or all of these errors (see [1]–[13]). In this paper we shall restrict our attention to one such technique, and incorporate several improvements.
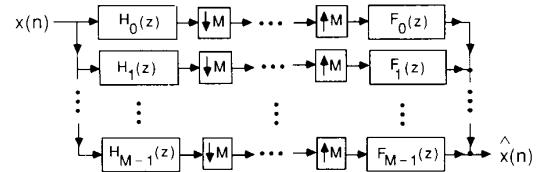
Fig. 1. The M-channel maximally decimated QMF bank.

In the literature, two-channel versions of the system of Fig. 1 have been referred to as the *quadrature mirror filter* (QMF) banks [1]–[4]. This is because the magnitude responses $|H_1(e^{j\omega})|$ and $|H_0(e^{j\omega})|$ are *images* of each other with respect to the frequency $\pi/2$ which is a *quarter* of the sampling frequency 2π. For the case of general M, the structure of Fig. 1 should not actually be called the QMF bank because the traditional 2-channel meaning does not hold. However, as the word QMF has been used in the past by other authors, we shall retain the same jargon.

The technique we shall discuss here is the one described in [12] (also see [13]). Here, all three distortions mentioned above are eliminated, so that $\hat{x}(n)$ is a delayed version of $x(n)$, i.e., $\hat{x}(n) = cx(n - n_0)$, $c \neq 0$. This is called the perfect reconstruction (abbreviated PR) property, and the QMF bank of Fig. 1 is then said to be a PR system.

The method in [12] is based on the following observation. Each analysis filter $H_k(z)$ can be written in the form $H_k(z) = \sum_{l=0}^{M-1} z^{-l} E_{kl}(z^M)$, and each synthesis filter $F_k(z)$ can be written in the form $F_k(z) = \sum_{l=0}^{M-1} z^{-(M-1-l)} R_{lk}(z^M)$. The quantities $E_{kl}(z)$, $0 \leq l \leq M - 1$ are the M polyphase components [14], [2] of the kth analysis filter $H_k(z)$. Similarly, $R_{lk}(z)$ are the M polyphase components of $F_k(z)$. These components can be used to define two $M \times M$ matrices $E(z) = [E_{kl}(z)]$ and $R(z) = [R_{lk}(z)]$, called the polyphase component matrices for the analysis bank and synthesis bank, respectively. With these definitions, the QMF bank of Fig. 1 can always be redrawn as in Fig. 2. The method described in [12] constrains the matrix $E(z)$ to be FIR and lossless (i.e., $E(e^{j\omega})$ to be unitary for all ω). Under this condition, if the matrix $R(z)$ is chosen as[1] $R(z) = \tilde{E}(z)$, then

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[1] See the end of this section for meanings of $A^\dagger$, $\tilde{A}$, etc.
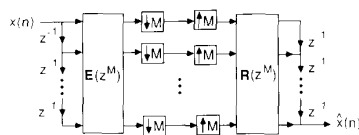
Fig. 2. Redrawing of Fig. 1 in terms of polyphase matrices $E(z)$ and $R(z)$.

the system of Fig. 1 is forced to be a PR system. With lossless $E(z)$, this choice of $R(z)$ is equivalent to choosing the synthesis filters as

$$F_k(z) = z^{-S} \tilde{H}_k(z) \qquad (1)$$

so that $f_k(n)$ are time-reversed (and conjugated) versions of $h_k(n)$. Thus, once the analysis filters $h_k(n)$, $0 \leq k \leq M - 1$ are known, we can find $f_k(n)$, $0 \leq k \leq M - 1$, of the PR system easily, without matrix inversions. Notice that, with this setup $\{F_k(z)\}$ are also FIR filters of the same lengths as $\{H_k(z)\}$'s.

The central problem in such a design of a PR QMF bank is therefore the design of analysis filters $H_k(z)$ under the constraint that $E(z)$ be lossless. With $E(z)$ constrained to be lossless, the aim is to minimize the sum of stopband energies

$$\sum_{k=0}^{M-1} \int_{\text{stopband}} \left| H_k(e^{j\omega}) \right|^2 d\omega \qquad (2)$$

by optimizing the parameters characterizing $E(z)$. The passbands of $H_k(z)$ automatically come out to be good for reasons mentioned in [12].

A. Summary of Past Designs

In the above optimization, the objective function (2) is highly nonlinear with respect to the parameters of $E(z)$. Moreover, no convenient *initialization point* for the parameters of $E(z)$ was available in the past. Finally, the parameters in $E(z)$ were rotation angles [12], which required the computation of several cosines and sines in order to evaluate (2) for a given parameter set. These facts resulted in very slow convergence of the optimization process. Because of the possibility of multiple minima, it was also necessary to perform optimization with several random initial points. For these reasons, the designs in [12] were restricted to low-order analysis filters. For example, the three channel design in [12] has analysis filters of length 15, each filter having a stopband attenuation of only about 20 dB.

B. Some Recent Improvements

It was soon observed [15] that, if the analysis filters were further constrained to have pairwise image property

$$\left| H_k(e^{j\omega}) \right| = \left| H_{M-1-k}(e^{j(\omega - \pi)}) \right| \qquad (3)$$

this reduces the number of degrees of freedom [i.e., the number of parameters in $E(z)$] for a given filter length, resulting in faster optimization and better filters. Such a design example for $M = 3$, with analysis filter lengths equal to 62, can be found in [15], providing stopband at-

tenuations of about 35 dB. The initialization of parameters in [15], however, continued to be random.

Subsequent to this, it was realized that the initialization of parameters can actually be done more judiciously [16]. This is based on several facts. First, if $E(z)$ is lossless, then each analysis filter $H_k(z)$ is a spectral factor of an Mth band filter (i.e., $\tilde{H}_k(z) H_k(z)$ are Mth band filters). Second, if any one of the analysis filters, say, $H_0(z)$, is fixed at a certain value, then the lossless constraint of $E(z)$ takes away most of the freedom available for the choice of $H_k(z)$, $1 \leq k \leq M - 1$. Finally, if $H_0(z)$ is somehow initialized, there exists a *synthesis procedure* to find (i.e., initialize) the majority of parameters of $E(z)$. This leads to a substantial improvement in the design speed, and in the filter performance upon convergence. The design example for $M = 3$, reported in [16], is based on this scheme, coupled with the imposition of the symmetry property (3). The analysis filters have length 56 and stopband attenuations exceeding 70 dB.

The final phase of improvement is based on a renewed characterization of an FIR lossless matrix $E(z)$. The earlier characterizations [17], [18] were in the form of a cascade of constant unitary-matrix building blocks, separated by delay elements. The unitary matrices are themselves characterized by rotation angles [17]. Even though this characterization is completely general [18] (in the sense that every lossless FIR matrix can be realized in this form) and canonic (i.e., has the minimum number of delay elements and parameters for a given filter length), the presence of angles makes it necessary to compute several cosines and sines, during each computation of (2). On most general purpose computers, the computation of a cosine (or sine) is about 20 times slower than a multiplication operation. This is, therefore, a major computational overhead, while carrying out the optimization on most machines. A second characterization of FIR lossless matrices was recently outlined [19], which again is general and canonic, but is mostly free from rotation angles. This, coupled with the initialization techniques, and the imposition of symmetry (3), has now emerged into a more efficient algorithm for the design of analysis filters $H_k(z)$ of an FIR PR system.

C. Outline of the Paper

The purpose of this paper is to give a comprehensive presentation of these results. As many of these improvements have been reported only in conference proceedings [16], [19], [20] (which are incomplete due to space constraints), our presentation here will be self-contained (with the exception that the results in the recent TRANSACTIONS paper [15] will be freely used). Section II describes our technique to initialize an analysis filter of the PR QMF bank. Section III develops a new minimal characterization for $M \times M$ FIR lossless systems and for $M \times 1$ FIR lossless vectors. Section IV derives the number of degrees of freedom available in the design of an $M \times M$ FIR lossless system, if one row is fixed (corresponding to fixing or *initializing* an analysis filter). Section V ties up these im-

provements with the imposition of pairwise symmetry of analysis filters [15]. A design example is presented in Section VI.

Notations: The downgoing and upgoing arrows in Fig. 1 represent decimators and interpolators as defined in any one of [2], [11]-[13]. Boldfaced italic letters denote matrices. The symbol I denotes the identity matrix, whose dimensions will be clear from the context (if not, a subscript will be used to indicate it). The notations A^T, A^*, and $A^\dagger$ denote transposition, conjugation, and transposed-conjugation, respectively. For an $M \times 1$ vector v, the norm $\sqrt{v^\dagger v}$ is denoted as $\|v\|$. A $p \times M$ matrix A ($p \geq M$) is said to be orthogonal if $A^T A = cI$, and unitary if $A^\dagger A = cI$, where $c \neq 0$ is scalar. Subscript $*$ denotes conjugation of coefficients only. Thus, if $E(z) = e_0 + e_1 z^{-1}$, then $E_*(z) = e_0^* + e_1^* z^{-1}$. Finally, $\tilde{H}(z)$ stands for $H_*^T(z^{-1})$. For real-coefficient systems, $\tilde{H}(z) = H^T(z^{-1})$. In any case, on the unit circle, we have $\tilde{H}(z) = H^\dagger(z)$. A $p \times M$ transfer matrix $E(z)$ is said to be lossless if a) its entries $E_{kl}(z)$ are stable and b) $E(e^{j\omega})$ is unitary for all ω. For $p = M$, this property is a discrete-time version of the well-known property of scattering matrices of LC multiports in electrical network theory [29], [30]. (A lossless matrix with real coefficients is said to be LBR.) Such a system automatically satisfies the property

$$\tilde{E}(z) E(z) = cI \quad \text{for all } z, \tag{4}$$

where $c \neq 0$ is a constant. For $E(z)$ to be lossless, we require $p \geq M$. If $p = M$, note that (4) is equivalent to $E(z) \tilde{E}(z) = cI$. Note that product of lossless matrices is lossless. The M components of an $M \times 1$ lossless vector are said to be power complementary (since the magnitude-squares add up to a constant for all ω). Note that every column of a lossless system is lossless (hence power complementary).

The degree of a $p \times M$ system (also called McMillan degree [22]) $E(z)$ is equal to the number of scalar delays (i.e., z^{-1} building blocks) required to implement it. Even though it is difficult to find the degree by inspection, it is simple to find in some cases. For example, the degree of an $M \times 1$ FIR transfer matrix $H(z) \triangleq \sum_{n=0}^K h(n) z^{-n}$ with $h(K) \neq 0$ is equal to K. The degree of an $M \times M$ system of the form $z^{-1} A$ is equal to the rank r of A because we can write $z^{-1} A = B[z^{-1} I_r] C$ where B is $M \times r$ and C is $r \times M$.

II. INITIALIZATION OF AN ANALYSIS FILTER

The set of M analysis filters $H_k(z)$, expressed in terms of the polyphase components $E_{kl}(z)$, can be represented as in Fig. 3(a). Since $H_k(z)$ are FIR, the entries of $E(z)$ are FIR. In our method, we constrain $E(z)$ to be lossless. The discussions in this section assume $H_k(z)$ [hence $E(z)$] to have real coefficients. A structural representation for $E(z)$, developed in [17] and [18], is shown in Fig. 3(b). This is a cascade of N (real) orthogonal matrices R_k, $0 \leq k \leq N - 1$ [where $N - 1$ is the degree of
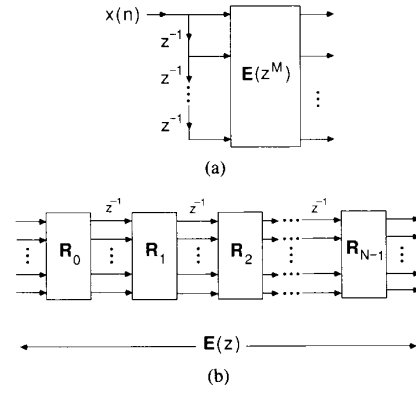

(a)

(b)

Fig. 3. Implementation of FIR lossless $E(z)$ as a cascade of unitary matrices separated by delays.
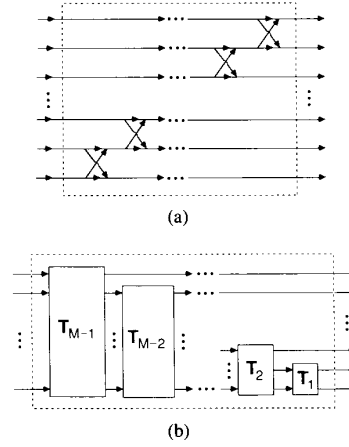

(a)

(b)

Fig. 4. Details of the building blocks in Fig. 3(b). In Fig. 4(b), T_k is a $(k + 1) \times (k + 1)$ matrix with appearance as in Fig. 4(a). The matrix T_k has k criss-crosses.

$E(z)$] separated by diagonal matrices of the form

$$\Lambda(z) = \begin{bmatrix} z^{-1} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}. \tag{5}$$

Recall (Appendix D) that an arbitrary $M \times M$ real orthogonal matrix requires $\binom{M}{2}$ real parameters (rotation angles) for complete characterization. The $N - 1$ matrices R_k, $0 \leq k \leq N - 2$ are special types of orthogonal matrices with $M - 1$ planar rotations, as shown in Fig. 4(a). In this figure, each criss-cross represents a planar rotation operation of the form

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

The rightmost matrix R_{N-1}, on the other hand, is a general orthogonal matrix characterized by $\binom{M}{2}$ planar rotations (see Appendix D) as in Fig. 4(b). It is shown in [18] that

every real-coefficient FIR lossless system $E(z)$ of degree $N - 1$ can be represented as in Fig. 3(b) with matrices as in Fig. 4. Conversely, the structure of Fig. 3(b) with matrices as in Fig. 4 necessarily represents a real-coefficient FIR lossless system. Thus, the angles in Fig. 4 form a complete set for characterizing real-coefficient FIR lossless $E(z)$. Moreover, the representation in Fig. 3(b) is canonic (i.e., minimal in number of delays, and in number of planar rotation parameters).

A. The Parameter Space and the Number of Degrees of Freedom

The total number of planar rotation angles in Fig. 3(b) is

$$N_p = (M - 1)(N - 1) + \binom{M}{2}. \tag{6}$$

In order to minimize the objective function (2), it is necessary to optimize these N_p parameters. From Fig. 3(b), the maximum length of an analysis filter with this setup is

$$L' = M(N - 1) + M. \tag{7}$$

Clearly, the number of parameters N_p grows linearly with respect to L' and quadratically with respect to M. For example, with $M = 3$ and analysis filters of length $L' = 57$, we have $N_p = 39$. We thus have a large parameter space, and a very nonlinear objective function. With no clue for initialization of angles in Fig. 3(b), the optimization task is formidable indeed.

In Section III we shall show that if we can make an initial guess of one of the length L' transfer functions, say $H_0(z)$, then almost all the N_p parameters can be initialized based only on $H_0(z)$. To be more specific, once $H_0(z)$ is fixed, only

$$N_f = \binom{M}{2} - (M - 1) \tag{8}$$

parameters are still undetermined in Fig. 3(b). Equation (8) therefore measures the number of degrees of freedom available for the design of $H_k(z)$, $1 \le k \le M - 1$, once $H_0(z)$ has been fixed. This is a consequence of the fact that losslessness of $E(z)$ puts a constraint on $H_k(z)$, $k \ne 0$, once $H_0(z)$ is fixed.

For example, with $M = 2$, we have $N_f = 0$ which shows that we have no freedom of choice of $H_1(z)$ once $H_0(z)$ is fixed. This is consistent with the earlier observations [5], [12] that, for the $M = 2$ case, losslessness of $E(z)$ completely constrains $H_1(z)$ to be $H_1(z) = z^{-(L'-1)}H_0(-z^{-1})$. For the $M = 3$ case, $N_f = 1$ so that, once $H_0(z)$ is fixed, only one degree of freedom can be exercised in choosing $H_1(z)$ and $H_2(z)$. It is important to realize that N_f is independent of the filter length L'. Consequently, if $H_0(z)$ is known, we have to optimize only the remaining N_f parameters, regardless of filter length, to obtain $H_k(z)$, $1 \le k \le M - 1$. (In practice, we have the option of reoptimizing all the parameters, after ini-

tialization of a subset of parameters based on the initial choice of $H_0(z)$.)

In order to exploit this to our advantage, it is first necessary to find an appropriate initialization for $H_0(z)$, which is the purpose of this section. We can mathematically express Fig. 3(a) as

$$h(z) = E(z^M) d(z) \tag{9}$$

where

$$h(z) = \left[H_0(z) \ H_1(z) \cdots H_{M-1}(z) \right]^T, \tag{10}$$

and

$$d(z) = \left[1 \ z^{-1} \cdots z^{-(M-1)} \right]^T. \tag{11}$$

From the fact that $E(z)$ is lossless, it can be proved (see Appendix A) that the function

$$G_k(z) \triangleq \tilde{H}_k(z) H_k(z) \tag{12}$$

is an Mth band filter [23], [32], i.e., it satisfies

$$\sum_{l=0}^{M-1} G_k(zW^l) = c \tag{13}$$

where $W = e^{-j2\pi/M}$ and c is a nonzero constant. Conversely, given an arbitrary FIR filter $H_0(z)$ such that $G_0(z)$ is an Mth band filter, there will exist FIR filters $H_k(z)$, $1 \le k \le M - 1$, such that the resulting $E(z)$ is lossless (as we shall see in Section IV).

Thus, one way to initialize $H_0(z)$ would be to first design a low-pass Mth band filter $G_0(z)$ and then compute a spectral factor $H_0(z)$ of $G_0(z)$. Of course, it is necessary to ensure that $G_0(e^{j\omega})$ is nonnegative for all ω, so that a spectral factor $H_0(z)$ exists. Such an Mth band filter can be obtained as follows: first design an Mth band equiripple zero-phase FIR filter $G(z)$ as in [23], by using the McClellan-Parks program [24]. Then define $G_0(z) = G(z) + \delta$ where δ is the peak stopband ripple of $G(e^{j\omega})$, so that $G_0(e^{j\omega})$ has all the desired properties. Finally, compute the coefficients of $H_0(z)$ such that $G_0(z) = H_0(z^{-1}) H_0(z)$ (this is the spectral factorization step).

If $H_0(z)$ is required to have a stopband attenuation of 70 dB (as an arbitrary example), then $G_0(z)$ has stopband attenuation 140 dB. This implies that $G(z)$ has large order, and has several zeros in the stopband. Under such conditions, the spectral factorization is an inaccurate process, and gives rise to numerical difficulties even when clever techniques such as [25] are employed. In this section we show how $H_0(z)$ with the above properties can be *directly* designed, thereby eliminating the spectral factorization step. According to our experience, this direct approach always gives rise to more accurate and faster designs for the filter $H_0(z)$.

B. The Eigenfilter Approach for Initialization of $H_0(z)$

Any FIR transfer function $H_0(z)$ can be written as

$$H_0(z) = H_{00}(z) H_{01}(z) \tag{14}$$

where $H_{01}(z)$ has all zeros on the unit circle, and $H_{00}(z)$ has none on the unit circle. If $H_0(z)$ is a spectral factor of $G_0(z)$, then we can write

$$G_0(z) = G_{00}(z) \, H_{01}^2(z) \tag{15}$$

where $G_{00}(z)$ has no zeros on the unit circle. The form (15) reflects the fact that zeros of $G_0(z)$ on the unit circle must be *double* so that $G_0(e^{j\omega})$ is nonnegative for all ω. Since $G_0(z)$ has zero phase, $G_{00}(z)$ has zeros occurring in reciprocal pairs.

Our aim is to design $H_0(z)$ such that its stopband energy

$$\int_{\omega_S}^{\pi} \left| H_{00}(e^{j\omega}) \right|^2 \left| H_{01}(e^{j\omega}) \right|^2 \, d\omega / 2\pi \tag{16}$$

is minimized under the constraint that $H_0(z^{-1}) \, H_0(z)$ is an Mth band filter. This is equivalent to minimizing the quantity

$$\mathcal{E} = \int_{\omega_S}^{\pi} \left| G_{00}(e^{j\omega}) \right| \left| H_{01}(e^{j\omega}) \right|^2 \, d\omega \tag{17}$$

under the constraint that $G_0(z)$ be an Mth band filter. The advantage of writing $G_0(z)$ in the form (15) and optimizing (17) is that the resulting design will automatically reveal the coefficients of $H_{01}(z)$ and $G_{00}(z)$. If we take $H_{00}(z)$ to be a spectral factor of $G_{00}(z)$, the design of $H_0(z)$ is complete. This involves finding only the spectral factor of $G_{00}(z)$ which is a much lower order polynomial than $G_0(z)$. In addition, $G_{00}(z)$ has *no zeros* on the unit circle. It is therefore easy to find a spectral factor $H_{00}(z)$ of $G_{00}(z)$.

In what follows, we shall denote the orders of $H_{00}(z)$ and $H_{01}(z)$ by l_0 and l_1, respectively. Accordingly, $G_{00}(z)$ has order $2l_0$ and $G_0(z)$ has order $2(l_0 + l_1)$. Notice that in the above setup $G_{00}(z)$ and $H_{01}(z)$ are linear-phase transfer functions. Assume, for a moment, that the coefficients of $G_{00}(z)$ are known. Given the function $G_{00}(z)$, this task of finding the linear-phase polynomial $H_{01}(z)$ to minimize (17) is precisely the *weighted eigenfilter design* problem described in [28] (to be elaborated below). While minimizing (17) we shall impose a constraint that avoids the trivial solution $H_{01}(z) = 0$.

Now assume that, for a given $G_{00}(z)$, we have found the coefficients of $H_{01}(z)$ using the eigenfilters approach. With $H_{01}(z)$ fixed at this value, we can recompute the coefficients of $G_{00}(z)$ such that the quantity $G_0(z)$ in (15) is indeed an Mth band filter. With this new $G_{00}(z)$, we can again solve for $H_{01}(z)$ to minimize (17). A few repetitions (typically four or five) result in an Mth band filter $G_0(z)$ with excellent stopband attenuation. Once $G_{00}(z)$ and $H_{01}(z)$ are found in this manner, it is only necessary to find a spectral factor $H_{00}(z)$ of $G_{00}(z)$. The analysis filter $H_0(z)$ is then obtained as in (14). Summarizing, the steps in the design of $H_0(z)$ are as follows.

1) Initialize $G_{00}(e^{j\omega})$ to be unity for all ω.

2) For this $G_{00}(z)$, find the linear phase transfer function $H_{01}(z)$ of order l_1 such that (17) is minimized (under appropriate constraints to be elaborated).

3) With $H_{01}(z)$ so fixed, find the linear phase transfer function $G_{00}(z)$ of order $2l_0$ such that the product (15) is an Mth band filter.

4) If the result $G_0(z)$ is not satisfactory, go to step 2.

Even though there is no formal proof that this approach converges, excellent designs for $G_0(z)$ could be obtained in every case attempted. At most, five iterations between steps 2 and 3 were necessary in all cases. The details of steps 2 and 3 are described next.

The Eigenfilter Approach [28] for Step 2: Let $H_{01}(z)$ be a real-coefficient linear-phase FIR filter [26] of the form

$$H_{01}(z) = \sum_{n=0}^{l_1} h_{01}(n) z^{-n} \tag{18}$$

with

$$h_{01}(n) = h_{01}(l_1 - n). \tag{19}$$

Assume first that l_1 is even and let $M_1 = l_1/2$. We can express the frequency response as [26], [27, p. 72]

$$H_{01}(e^{j\omega}) = e^{-jM_1\omega} \sum_{n=0}^{M_1} b_n \cos(\omega n). \tag{20}$$

The coefficients b_n are given by $b_0 = h(M_1)$ and $b_n = 2h(M_1 - n), n \neq 0$. If we define the vectors

$$b = [b_0 \;\; b_1 \;\; \cdots \;\; b_{M_1}]^T,$$

$$c(\omega) = [1 \;\; \cos(\omega) \;\; \cdots \;\; \cos(M_1\omega)]^T \tag{21}$$

we can express (20) as $H_{01}(e^{j\omega}) = e^{-jM_1\omega} b^T c(\omega)$. Now consider the quantity

$$\mathcal{E} = \int_{\omega_S}^{\pi} W(e^{j\omega}) \left| H_{01}(e^{j\omega}) \right|^2 \, d\omega \tag{22}$$

where $W(e^{j\omega})$ is a nonnegative real function of ω. Suppose we wish to minimize (22) under the constraint

$$b^T b = 1. \tag{23}$$

We can rewrite (22) as

$$\mathcal{E} = b^T \left[\int_{\omega_S}^{\pi} W(e^{j\omega}) \, c(\omega) \, c^T(\omega) \, d\omega \right] b. \tag{24}$$

The integral in (24) is an $(M_1 + 1) \times (M_1 + 1)$ matrix P. We can rewrite (24) as

$$\mathcal{E} = b^T P b. \tag{25}$$

Clearly, the matrix P, which is real and symmetric, is positive definite (except under trivial situations such as $\omega_S = \pi$, or $W(e^{j\omega}) |H_{01}(e^{j\omega})|^2 = 0$ for all ω). The vector b minimizing (25) under the constraint (23) is the eigenvector of P corresponding to its minimum eigenvalue. Once b is found in this way, the coefficients of $H_{01}(z)$ which minimize (22) can be obtained. For obvious reasons, $H_{01}(z)$, so found, will be called an eigenfilter. It is clear that with $W(e^{j\omega})$ taken as $|G_{00}(e^{j\omega})|$, we can find the optimal linear-phase $H_{01}(z)$ using this eigenfilter approach.

If the order l_1 is odd, the procedure differs in some minor details, which can be inferred from [28, p. 13].

Details of Step 3: Given a linear-phase transfer function $G_{01}(z) \triangleq H_{01}^2(z)$ of order $2l_1$, the task here is to find the coefficients of a linear phase filter $G_{00}(z)$ of order $2l_0$ such that the product $G_0(z)$ in (15) is an Mth band filter. It is well known [23] that the Mth band property (13) of $G_0(z)$ is equivalent to the condition

$$g_0(Mn) = \begin{cases} \dfrac{1}{M} & \text{for } n = 0 \\ 0 & \text{otherwise.} \end{cases} \quad (26)$$

With $g_{00}(n)$ and $g_{01}(n)$ denoting the coefficients of $G_{00}(z)$ and $G_{01}(z)$, respectively, we see that the coefficients $g_0(n)$ of $G_0(z)$ are obtained by convolving $g_{00}(n)$ with the known coefficients $g_{01}(n)$. As a result, the condition (26) gives rise to a set of linear equations from which we can solve for the coefficients $g_{00}(n)$ of $G_{00}(z)$.

To be more specific about the number of equations and variables, recall that the order of $G_0(z)$ is $2(l_0 + l_1)$. Express $(l_0 + l_1)$ in the form

$$(l_0 + l_1) = p_0 M + p_1 \quad (27a)$$

where $0 \leq p_1 \leq M - 1$. Then the Mth band constraint (26) of $G_0(z)$ gives rise to $p_0 + 1$ equations. This implies that we require $p_0 + 1$ independent coefficients in the linear-phase filter $G_{00}(z)$. From this, we deduce that the order of $G_{00}(z)$ should be $2p_0$. In other words, $l_0 = p_0$. This, along with (27a), yields

$$l_0 = \frac{l_1 - p_1}{M - 1}. \quad (27b)$$

Summarizing, given an arbitrary positive integer l_1, we pick p_1 in the range $0 \leq p_1 \leq M - 1$ such that the ratio (27b) is an integer. Clearly, p_1 and l_0 come out to be unique. This gives the smallest l_0 such that $G_0(z)$ satisfies (26). We then solve for the coefficients of $g_{00}(n)$ by inverting an approprite $(l_0 + 1) \times (l_0 + 1)$ matrix. The details are readily developable and can be found in the paper by Nguyen et al. [20].

III. New Characterizations for FIR Lossless Systems

Recall that $E(z)$ is an $M \times M$ matrix. Its McMillan degree (or simply the degree) is, by definition, the minimum number of delay elements (i.e., z^{-1} elements) required to implement $E(z)$. It is known [17] that any real-coefficient $M \times M$ FIR lossless matrix with degree $N - 1$ can be realized as in Fig. 3(b). As mentioned in Section II, the building blocks in Fig. 3(b) are orthogonal matrices, which in turn are combinations of planar rotations.

In this section we shall obtain a second type of characterization (or realization) of FIR lossless matrices, which are free from planar rotations. We find this characterization to be more convenient for optimization of (2). This realization shares the minimality property of Fig.

3(b) (both in terms of number of delays and number of parameters).

We shall obtain the results for the general case where $E(z)$ has complex coefficients, as it turns out to be no more complicated than the real case. Two crucial properties of FIR lossless systems are required in the developments, and are stated next.

Property 3.1: Let $H(z)$ be an $M \times M$ causal FIR lossless transfer matrix of degree $N - 1$. Then

$$\det H(z) = cz^{-(N-1)} \quad (28)$$

where c is a nonzero constant. As a result, the degree $N - 1$ of $H(z)$ is equal to the degree of the determinant of $H(z)$:

$$\deg H(z) = \deg \det H(z). \quad (29)$$

This property, which is not true for arbitrary linear time invariant systems, is true for FIR and IIR lossless systems with complex or real coefficients. For the FIR case with real coefficients, the proof can be seen by noting that the determinant of $E(z)$ is the product of the determinants of the $M \times M$ elements in the cascade of Fig. 3(b), which is clearly of the form $cz^{-(N-1)}$. For the FIR case with complex coefficients, a representation as in Fig. 3(b) is valid with minor modifications (with "complex rotators" taking the place of the real ones). The result (28) again follows. Even though IIR systems are not considered in this paper, we note for completion that in this case $\det H(z) = cA(z)$, where $A(z)$ is an allpass function with degree equal to that of $H(z)$.

Property 3.2: Let $H(z)$ be any FIR lossless system with

$$H(z) = \sum_{n=0}^{K} h(n) z^{-n} \quad (30)$$

with $h(0) \neq 0$ and $h(K) \neq 0$. Then $h^\dagger(0) h(K) = 0$, unless $K = 0$. In particular, for the $M \times M$ case, both $h(0)$ and $h(K)$ are singular, and $h(0) h^\dagger(K) = 0$ as well.

The proof follows immediately from the paraunitary property $\tilde{H}(z) H(z) = I$ by equating like powers of z. The first result we shall derive in this section is the following.

Lemma 3.1: An $M \times M$ causal FIR system $H(z)$ is lossless of degree one if and only if it can be represented in the form

$$H(z) = [I - vv^\dagger + z^{-1} vv^\dagger] R \quad (31)$$

where R is an arbitrary $M \times M$ unitary matrix and v is an $M \times 1$ column vector with unit norm.

We shall then show that any arbitrary $M \times M$ FIR lossless system of degree $N - 1$ can be expressed as a cascade of $N - 1$ degree-one systems. To be more specific, we shall prove the following.

Theorem 3.1: Let $H_{N-1}(z)$ be an $M \times M$ causal FIR system. Then, it is lossless of degree $N - 1$ if and only if it can be written in the form

$$H_{N-1}(z) = V_{N-1}(z) V_{N-2}(z) \cdots V_1(z) H_0 \quad (32)$$

where H_0 is a constant $M \times M$ unitary matrix, and $V_k(z)$ are $M \times M$ degree-one FIR lossless matrices of the form

$$V_k(z) = [I - v_k v_k^\dagger + v_k v_k^\dagger z^{-1}] \qquad (33)$$

and where v_k are $M \times 1$ column vectors of unit norm.

Notice that the parameters characterizing the lossless system now are the $N - 1$ unit-norm vectors v_k and the unitary matrix H_0. Matrices of the form $vv^\dagger$ appearing in (31) and (33) are *diadic* matrices. Because of their importance in many of our proofs, Appendix B reviews some properties of diadics. The proof of the above theorem will automatically give rise to a synthesis procedure, which can be used to compute the parameters v_k and H_0 for a given FIR lossless $H(z)$. For the case of lossless systems with real coefficients, we see that the number of parameters in the characterization of H_0 is $\binom{M}{2}$ since it is real orthogonal (Appendix D). Moreover, each of the unit-norm vectors v_k has $M - 1$ degrees of freedom. The total number of parameters therefore works out to be N_p in (6), so that (32) is a minimal characterization. In (32), only $\binom{M}{2}$ parameters are planar rotation angles, unlike in Fig. 3(b), where all N_p parameters are rotations. We shall use the above form for the polyphase matrix $E(z)$ of Fig. 2. We shall now prove the above Lemma and Theorem. One proof can be obtained by starting with some of the characterizations in [29] for continuous-time lossless scattering matrices, applying the bilinear transform, and setting all pole locations to $z = 0$ (because of the FIR nature). We shall, however, give a self-contained proof, working entirely in the z-domain.

Proof of Lemma 3.1: The "if" part can be proved by proving $H(z) \tilde{H}(z) = I$. This follows immediately from (31) by noting that product terms of the form $vv^\dagger vv^\dagger$ simplify to $vv^\dagger$ (because $v^\dagger v = 1$). To prove the "only if" part, note that any $M \times M$ causal FIR system of degree one can be written as

$$H(z) = h(0) + h(1)z^{-1} \qquad (34)$$

where $h(0)$ and $h(1)$ are $M \times M$ matrices, with $h(1) \neq 0$. Note, however, that (34) does not always represent a degree-one matrix. (For example, with $h(1) = I$, the degree of (34) is M.) The degree-one condition imposes further restrictions on $h(0)$ and $h(1)$. Thus, from Property 3.2, both $h(0)$ and $h(1)$ are singular. Moreover (see comment at the end of Section I), the rank of $h(1)$ is required to be equal to unity.

Since $H(z)$ is lossless, we know, in particular, that $H(z)$ is unitary for $z = 1$. So we can write

$$H(z) = (1 - z^{-1})S + R \qquad (35)$$

where R is a constant unitary matrix. Losslessness of (34) implies, in particular, the condition $h(0) h^\dagger(1) = 0$. In view of (35), this condition is equivalent to

$$RS^\dagger = -SS^\dagger. \qquad (36)$$

Substituting (36) in (35) and rearranging, we get the form

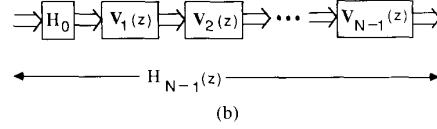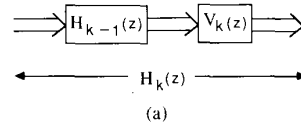$$H(z) = [I - SS^\dagger + z^{-1}SS^\dagger]R. \qquad (37)$$

Fig. 5. Implementation of $M \times M$ lossless FIR $H_{N-1}(z)$ as a cascade of degree-one lossless systems. (a) Degree-reduction step. (b) Overall cascade.

From (37) we see that $h(1) = SS^\dagger R$. Since $h(1)$ has rank one, and since R is unitary, $SS^\dagger$ has rank one. So we can rewrite

$$SS^\dagger = vv^\dagger \qquad (38)$$

where v is an $M \times 1$ column vector. We therefore arrive at the form (31) for $H(z)$. It remains to be shown that v has unit norm. For this, note that the quantity inside the square brackets in (31), which is $H(z)R^\dagger$, is lossless (in particular, unitary for $z = -1$) so that $I - 2vv^\dagger$ is unitary. From Appendix B (property 3) it follows that v should have unit norm. This completes the proof of Lemma 3.1.

Proof of Theorem 3.1: The "if" part follows trivially because (32) is a product of lossless systems and is hence lossless. Consider next the "only if" part. Suppose $H_k(z)$ is an $M \times M$ causal FIR lossless system with degree k. Let the impulse response coefficients be $h_k(n)$ so that $H_k(z) = \sum_{n=0}^{k} h_k(n)z^{-n}$. (Notice that $h_k(k)$ could be null even though $H_k(z)$ has degree k.) We will show that it can be realized as

$$H_k(z) = [I - v_k v_k^\dagger + v_k v_k^\dagger z^{-1}]H_{k-1}(z), \qquad (39)$$

i.e., as in Fig. 5(a), where

1) The quantity v_k is an appropriate $M \times 1$ column vector with unit norm, and
2) $H_{k-1}(z)$ is an $M \times M$ causal FIR lossless system of degree $k - 1$.

This step is equivalent to *extracting* a lossless system

$$V_k(z) = [I - v_k v_k^\dagger + v_k v_k^\dagger z^{-1}] \qquad (40)$$

of degree one from $H_k(z)$ to obtain the reduced degree lossless system $H_{k-1}(z)$. The system $H_{k-1}(z)$ will be called the remainder of the *extraction process*. If we begin with $k = N - 1$ and repeat this *extraction process* a finite number of times, we obtain the representation of (32) [i.e., the structure of Fig. 5(b)], for $H_{N-1}(z)$. The final remainder $H_0(z)$ is a zero-degree lossless system (i.e., constant unitary matrix), and is denoted by H_0.

Given a causal FIR $M \times M$ lossless $H_k(z)$ with degree k, it only remains to show how to construct v_k of unit norm, such that $H_{k-1}(z)$ is a causal FIR $M \times M$ lossless matrix of degree $k - 1$. With v_k restricted to have unit

norm, $V_k(z)$ in (40) is clearly lossless (by Lemma 3.1), so that $V_k^{-1}(z) = \tilde{V}(z)$. As a result, (39) is equivalent to

$$[I - v_k v_k^\dagger + v_k v_k^\dagger z] H_k(z) = H_{k-1}(z). \qquad (41)$$

From (41) we see that $H_{k-1}(z)$ is FIR for FIR $H_k(z)$. Since $H_k(z)$ is also causal, we see that $H_{k-1}(z)$ is causal if and only if v_k is such that

$$v_k^\dagger h_k(0) = 0 \qquad (42)$$

where $h_k(0)$ is the 0th impulse response coefficient of $H_k(z)$. The losslessness of $H_k(z)$ ensures that $h_k(0)$ is singular (Property 3.2), so that there exists a unit-norm column vector v_k satisfying (42). Assume that v_k has been so chosen. Since $V_k(z)$ and $H_k(z)$ are paraunitary, it is clear from (41) that $H_{k-1}(z)$ is paraunitary. In summary, $H_{k-1}(z)$ is a causal lossless FIR system. It only remains to show that its degree is $k - 1$. For this, we invoke Property 3.1, which says that for $M \times M$ causal FIR lossless systems, the determinant has the form (28) where $N - 1$ is the degree. Accordingly, we have det $H_k(z) = c_1 z^{-k}$, and det $V_k(z) = c_2 z^{-1}$. Taking determinants on both sides of (41), we immediately see that det $H_{k-1}(z) = cz^{-(k-1)}$, so that the degree of $H_{k-1}(z)$ is $k - 1$ indeed. This completes the proof of Theorem 3.1.

Theorem 3.1 gives us the most general form of $M \times M$ causal FIR lossless system. This form can therefore be used in place of $E(z)$ in Fig. 3(a). The elements of v_k and H_0 are then optimized to minimize the objective function (2). In this way, we are guaranteed that the search will be conducted over the set of all FIR lossless systems of a given degree. Even though this property was shared by the method in [17], we find that the method just described is more convenient and leads to faster convergence.

As mentioned in Section II, it is of great interest to be able to initialize one of the analysis filters, say $H_0(z)$. This is equivalent to initializing one column of $E^T(z)$ and seeking to find the remaining columns such that $E^T(z)$ is lossless and (2) is minimized. In the next section, details of this will be presented. In preparation for this, we now derive the most general form of $M \times 1$ causal FIR lossless vectors.

Consider an $M \times 1$ column vector $P_{N-1}(z)$ of the form

$$P_{N-1}(z) = \sum_{n=0}^{N-1} p_{N-1}(n) z^{-n} \qquad (43)$$

where $p_{N-1}(n)$ are $M \times 1$ constants, with $p_{N-1}(N-1) \neq 0$. Clearly, $P_{N-1}(z)$ can be implemented with $N - 1$ scalar delays so that its degree is $N - 1$. We say that $P_{N-1}(z)$ is lossless if $\tilde{P}_{N-1}(z) P_{N-1}(z) = 1$ for all z. This is equivalent to the power-complementary property of the $M - 1$ components of $P_{N-1}(z)$. The following result will be used in the next section.

Theorem 3.2: Let $P_{N-1}(z)$ be an $M \times 1$ causal FIR system. Then it is lossless of degree $N - 1$ if and only if it can be written in the form

$$P_{N-1}(z) = U_{N-1}(z) U_{N-2}(z) \cdots U_1(z) P_0 \qquad (44)$$

where P_0 is a constant $M \times 1$ vector of unit norm, and $U_k(z)$ are $M \times M$ degree-one FIR lossless matrices of the form

$$U_k(z) = [I - u_k u_k^\dagger + u_k u_k^\dagger z^{-1}], \qquad (45)$$

where u_k are $M \times 1$ unit-norm vectors.

The proof of the above Theorem is again based on a repeated application of the degree-reduction step. There is, however, a fundamental difference because determinants are not meaningful anymore. Given a causal FIR $M \times 1$ lossless system $P_k(z)$ of degree $k \neq 0$, the degree-reduction step seeks to generate a remainder $P_{k-1}(z)$ such that it is a lower degree causal FIR $M \times 1$ lossless system. This is done by attempting to express $P_k(z)$ as

$$P_k(z) = [I - u_k u_k^\dagger + z^{-1} u_k u_k^\dagger] P_{k-1}(z). \qquad (46)$$

The choice of u_k is crucial in the degree reduction process. Since u_k has unit norm, (45) is lossless so that (46) can be rewritten as

$$P_{k-1}(z) = [I - u_k u_k^\dagger + z u_k u_k^\dagger] P_k(z). \qquad (47)$$

The remainder function $P_{k-1}(z)$ is causal if and only if $u_k^\dagger p_k(0) = 0$. Since $p_k(0)$ is a column vector, there exist many unit-norm vectors u_k satisfying this condition. In particular, the following choice:

$$u_k = \frac{p_k(k)}{\| p_k(k) \|} \qquad (48)$$

works because, due to losslessness of $P_k(z)$, the condition $p_k^\dagger(k) p_k(0) = 0$ holds (Property 3.2). With u_k so chosen, the coefficient of z^{-k} in (47) becomes

$$[I - u_k u_k^\dagger] p_k(k) = \| p_k \| [I - u_k u_k^\dagger] u_k$$
$$= \| p_k \| [u_k - u_k] = 0$$

which proves that $P_{k-1}(z)$ has degree $k - 1$. In summary, the choice of u_k according to (48) ensures that $P_{k-1}(z)$ is a causal $M \times 1$ lossless system of degree $k - 1$. Repeated application of this step results in the form (44), proving the theorem.

One consequence of Theorem 3.1 is that a causal $M \times M$ FIR lossless system $H_{N-1}(z)$ of degree $N - 1$ can be realized as in Fig. 6(a) where H_0 is $M \times M$ unitary and $V(z)$ is a special form of $M \times M$ lossless system of degree $N - 1$, viz.,

$$V(z) = V_{N-1}(z) V_{N-2}(z) \cdots V_1(z), \qquad (49)$$

with $V_k(z)$ as in (40). Similarly, Theorem 3.2 says that a causal $M \times 1$ FIR lossless system $P_{N-1}(z)$ of degree $N - 1$ can be realized as in Fig. 6(b) where P_0 is a unit-norm vector and where $U(z)$ is an $M \times M$ lossless system of degree $N - 1$, with exactly the same form as $V(z)$, i.e.,

$$U(z) = U_{N-1}(z) U_{N-2}(z) \cdots U_1(z). \qquad (50)$$

Both realizations of Fig. 6 are clearly minimal in delays.

Uniqueness of the Factorization of $P_{N-1}(z)$: Except for a scale factor of unit magnitude, (48) is the *only choice*
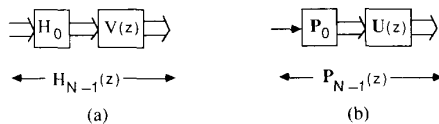
Fig. 6. Overall representation of the factorization of $H_{N-1}(z)$ and $P_{N-1}(z)$.
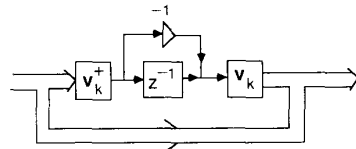
Fig. 7. Implementation of $V_k(z)$ using one scalar delay and $2M$ scalar multipliers.

that results in a reduced order causal lossless system P_{k-1}. This is because $[I - u_k u_k^\dagger]$ has rank $M - 1$, so that there is precisely one vector u (except for a scale factor) such that $[I - u_k u_k^\dagger] u = 0$. Since the choice $u = c u_k$ works, it is the only possibility. The scale factor c of unit magnitude disappears in the expression $u_k u_k^\dagger$ so that $U_k(z)$, and hence $U(z)$, are unique for a given $P_{N-1}(z)$. From (44) we see that $P_{N-1}(1) = P_0$, so that the vector P_0 is unique as well.

Nonuniqueness of the Factorization of $H_{N-1}(z)$: Notice that the choice of v_k according to (42) is necessary and sufficient to obtain a lower degree causal lossless system $H_{k-1}(z)$ from $H_k(z)$. Unless the rank of $h_k(0)$ is precisely $M - 1$, the choice of v_k is not unique, so that $V_k(z)$ in (33) are not unique. So the realization of $H_{N-1}(z)$ as in Fig. 5(b) is, in general, nonunique. However, from (32) we see that $H_{N-1}(1) = H_0$, so that H_0 is unique. This implies that $V(z)$, which is equal to $H_{N-1}(z) H_0^{-1}$, is unique. Summarizing, an $M \times M$ causal FIR lossless system of degree $N - 1$ can be factorized as $H_{N-1}(z) = V(z)H_0$, where H_0 and $V(z)$ are unique, even though the building blocks $V_k(z)$ may be nonunique.

Implementation of (33) with Minimum Number of Delays: It is possible to implement each building block of the form (33) with only one delay because the degree of $V_k(z)$ is unity. Fig. 7 shows an implementation of $V_k(z)$ with one scalar delay and $2M$ multipliers. A cascade of building blocks of this form, terminated on the left with H_0, completes the minimal implementation of (32).

IV. The Degree of Freedom After Initialization

Assume that we have designed a causal FIR filter $H_0(z)$ as in Section II, so that it is a spectral factor of an Mth band filter $G_0(z)$. This completely determines the 0th row

$$e_0^T(z) = [E_{00}(z) \quad E_{01}(z) \quad \cdots \quad E_{0,M-1}(z)] \quad (51)$$

of $E(z)$. Because of the Mth band property of $G_0(z)$, the $M \times 1$ column-vector $e_0(z)$ turns out to be lossless (Appendix A). Let $N - 1$ denote the degree of $e_0(z)$. We know from Theorem 3.2 that $e_0(z)$ can be represented as

$$e_0(z) = U(z)P_0 \quad (52)$$

where $U(z)$ is as in (50), and P_0 is an $M \times 1$ unit-norm vector. If we construct an $M \times M$ unitary matrix

$$H_0 = [P_0 \quad P_1 \quad \cdots \quad P_{M-1}] \quad (53)$$

with 0th column equal to P_0 (say, by using Gram-Schmidt procedure), then the $M \times M$ FIR system

$$S(z) = U(z)H_0 \quad (54)$$

is clearly lossless with degree $N - 1$. We can now define the lossless polyphase matrix as $E(z) \triangleq S^T(z)$ and obtain the FIR analysis filters $H_k(z)$ according to Fig. 3(a). This shows that once a spectral factor $H_0(z)$ of an Mth band filter $G_0(z)$ of arbitrary order is obtained, it is easy to find a set of analysis filters $H_k(z)$, $1 \le k \le M - 1$ of the perfect-reconstruction system. The number of freedoms that can be exercised in the above construction of $H_k(z)$, $1 \le k \le M - 1$, is equal to the number of freedoms available in constructing an $M \times M$ unitary matrix, whose 0th column is fixed. This number is given by

$$N_f' = M^2 - 2M + 1 \quad (55)$$

if the filter coefficients (and hence H_0) are complex, whereas if the coefficients are real, this number is precisely equal to N_f given in (8) (see Appendix D for details).

Given the FIR filter $H_0(z)$, does this technique cover *every possible* set of causal FIR filters $H_k(z)$, $1 \le k \le M - 1$, such that $E(z)$ is lossless? In general, the answer is *no*, as demonstrated by the following counterexample: consider the trivial transfer function $H_0(z) = 1$; then

$$e_0(z) = [1 \quad 0 \quad \cdots \quad 0]^T \quad (56)$$

so that we have $U(z) = I$ in (52). This means that $H_k(z)$, $1 \le k \le M - 1$ have degrees equal to 0. Evidently, this does not cover the PR system [13] given by

$$H_k(z) = z^{-k},$$

$$F_k(z) = z^{-(M-1-k)}, \quad 0 \le k \le M - 1, \quad (57)$$

for which $E(z) = I$ (which is lossless).

According to Theorem 4.1, which we shall see next, the answer to the above question is *yes*, provided the degree of $H_0(z)$ is equal to the maximum possible degree it can have, based on the implementation of Fig. 3 (i.e., degree of $H_0(z) = M(N - 1) + M - 1$). Since this is a reasonable assumption, in practice, we shall assume so for the rest of the paper. This is equivalent to the statement that the degree of $e_0(z)$ is equal to that of $E(z)$. The main result to be proved is the following.

Theorem 4.1: Let $H_{N-1}(z)$ be any $M \times M$ causal FIR lossless system with degree $N - 1$, whose 0th column $P_{N-1}(z)$ has degree $N - 1$. Let (44) be the unique factorization of $P_{N-1}(z)$ where $U_k(z)$ are as in (45) with unit-norm u_k's. Let (32) be an *arbitrary* factorization of $H_{N-1}(z)$ with $V_k(z)$ as in (33) with unit-norm v_k's. Then

$$V_k(z) = U_k(z), \quad 1 \le k \le N - 1, \quad (58)$$

and P_0 is the 0th column of H_0.

Proof: Since the degrees of $P_{N-1}(z)$ and $H_{N-1}(z)$ are both equal to $N - 1$, they have realizations of the forms (44) and (32), respectively, with $v_k \neq 0$ and $u_k \neq 0$ for any k. This implies, in particular, that $P_{N-1}(z)$ can be written in two forms, viz. (44), *and* the following:

$$P_{N-1}(z) = V_{N-1}(z) \, V_{N-2}(z) \cdots V_1(z) H_{00} \quad (59)$$

where H_{00} is the 0th column of H_0. By uniqueness of the factorization for $P_{N-1}(z)$ (see the end of Section III), the relation (58) immediately follows. Moreover, by uniqueness of P_0, it is clear that we have $P_0 = H_{00}$. This completes the proof.

The theorem obviously holds if the phrase "0th column" is replaced with "any column." The consequence is that if *any* column $P_{N-1}(z)$ of an $M \times M$ causal FIR lossless system $H_{N-1}(z)$ of degree $N - 1$ has degree $N - 1$, then we have to synthesize (or factorize) only this column, in order to obtain the factorization of the entire $H_{N-1}(z)$. The factorization of this column $P_{N-1}(z)$ places in evidence the degree-one building blocks $V_k(z)$, $1 \leq k \leq N - 1$. The terminating matrix H_0 is evaluated as $H_0 = H_{N-1}(1)$, completing the factorization of $H_{N-1}(z)$. It is clear that if the column $P_{N-1}(z)$ of degree $N - 1$ is fixed *a priori*, the number of degrees of freedom available in the choice of the remaining columns of $H_{N-1}(z)$ is equal to the number of freedoms available in choosing the unitary matrix H_0, one of whose columns is fixed.

Recall from Section III that the factorization of $H_{N-1}(z)$ is in general not unique, because the choice of v_k satisfying (42) is not unique unless the rank of $h_k(0)$ is precisely $M - 1$. The above theorem, on the other hand, implies that the factorization of $H_{N-1}(z)$ is indeed unique if one of the columns has the full degree $N - 1$. The corollary is that whenever there exists a column of $H_{N-1}(z)$ which has degree $N - 1$, the 0th coefficient $h_{N-1}(0)$ of $H_{N-1}(z)$ *must* have rank $M - 1$. A second proof of this fact, which removes the mystery associated with such a coincidence, is given in Appendix C.

V. IMPOSING PAIRWISE SYMMETRY ON THE ANALYSIS FILTERS

The final phase of improvement in our design of perfect-reconstruction FIR QMF banks is the recognition that, in many applications, the responses of the analysis filters $H_k(z)$ and $H_{M-1-k}(z)$ are not arbitrary. To be specific, it is often acceptable to let $|H_{M-1-k}(e^{j\omega})|$ be an image of $|H_k(e^{j\omega})|$ with respect to $\omega = \pi/2$. We can exploit this to our advantage for the reason that if $H_k(z)$ and $H_{M-1-k}(z)$ are related in this manner, then the number of independent elements in the parameterization of the polyphase matrix $E(z)$ is reduced almost by a factor of two. This feature, combined with the initialization scheme described in Section II, results in an improved design procedure for $H_k(z)$, $0 \leq k \leq M - 1$.

The appropriate step now would be to identify a structural form for $E(z)$ so that it will reflect the symmetry of pairs of analysis filters, while at the same time being lossless. Two such structures are presented in considerable
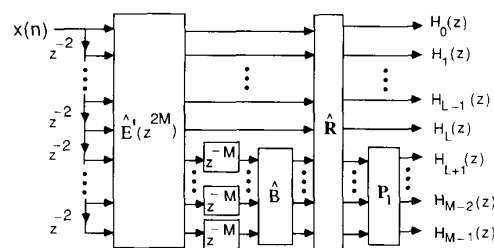

Fig. 8. An FIR analysis bank structure which imposes the relation (62) while retaining $E(z)$ to be lossless.

details in [15]—one for odd M and the other for even M. As in [15], the filter coefficients are assumed to be real. The design example to be presented in the next section pertains to odd M, so we shall restrict our discussions to odd M here.

Fig. 8 is a reproduction of Fig. 9 in [15]. Here $L = (M - 1)/2$ and $\hat{E}'(z)$ is an FIR lossless matrix with real coefficients. The matrix $\hat{B}$ is $L \times L$ real orthogonal and P_1 is the permutation matrix

$$P_1 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}. \quad (60)$$

Finally, $\hat{R}$ is given by

$$\hat{R} = \begin{bmatrix} \dfrac{1}{\sqrt{2}} I_L & 0 & \dfrac{1}{\sqrt{2}} I_L \\ 0 & 1 & 0 \\ \dfrac{1}{\sqrt{2}} I_L & 0 & -\dfrac{1}{\sqrt{2}} I_L \end{bmatrix}. \quad (61)$$

It is shown in [15] that under this condition, the conventional polyphase matrix $E(z)$ defined according to Fig. 3(a) is guaranteed to be lossless, and that the analysis filters are automatically related by

$$H_k(z) = H_{M-1-k}(-z). \quad (62)$$

The FIR synthesis bank which results in perfect reconstruction can be obtained in the usual way [i.e., by using (1)].

The FIR lossless matrix $\hat{E}'(z)$ in Fig. 8 contains the majority of parameters to be optimized during the design of analysis filters. (The matrix $\hat{B}$ has only $\binom{L}{2}$ parameters.) Comparing the structure of Fig. 8 to the conventional structure of Fig. 3(a), we see that for a given order of the analysis filters, the number of parameters in $\hat{E}'(z)$ is about half the number in $E(z)$ because of the appearance of z^{2M} rather than z^M in Fig. 8. This is precisely due to the fact that the symmetry of (62) is *structurally* imposed by Fig. 8. Consequently, in minimizing (2), fewer parameters need to be optimized than if Fig. 3(a) were directly used. Moreover, due to the relation (62), the ob-

jective function (2) can be simplified to have only $L + 1$ terms.

Initialization: It is important to adapt the initialization scheme, described in Section II, to the more efficient structure of Fig. 8. Notice that, in view of (62), the transfer function $H_L(z)$ is its own image, i.e., $H_L(z) = H_L(-z)$. In other words, $H_L(z)$ has the form $H'_L(z^2)$. It turns out to be more convenient to initialize this transfer function rather than $H_0(z)$. The first step would therefore be to design $H_L(z)$ as a bandpass filter centered around $\pi/2$. This is equivalent to designing a low-pass filter of half the order and replacing z with $-z^2$. This can be done by using the eigenfilters approach of Section II. Once $H_L(z)$ is known, the Lth row $\hat{e}_L^T(z)$ of the matrix $\hat{E}'(z)$ is completely known, because of the form (61) of $\hat{R}$ in Fig. 8. By using the technique of Section III, we can synthesize $\hat{e}_L(z)$ as

$$\hat{e}_L(z) = U(z)P_0 \tag{63}$$

where $U(z)$ is the special lossless matrix (50) (but with real coefficients). The complete lossless matrix $\hat{E}'(z)$ is then given by

$$\hat{E}'(z) = U(z)H_0. \tag{64}$$

Here H_0 is a real constant orthogonal matrix whose Lth column is P_0, and whose remaining columns are free. This gives us N_f degrees of freedom [(8)] in the structure. These parameters, and the $\binom{L}{2}$ parameters in the choice of $\hat{B}$, are available as free parameters in the design of the remaining $M - 1$ analysis filters. In practice, these parameters can be randomly initialized and optimized. In fact, the parameters which were initialized because of the knowledge of $H_L(z)$ can also be reoptimized so that all the M filters come out to be good. Further details will be presented along with the design example in the next section.

It should be noted that in the structure of Fig. 8, the lengths of the filters $h_k(n)$, $k \neq L$ are typically higher than $h_L(n)$ by a few samples, because of the extra delay elements to the right of $\hat{E}'(z^{2M})$. However, the transfer functions at the output of $\hat{E}'(z^{2M})$ have typically the same order. It is therefore not a serious loss of generality to assume that the degree of $\hat{E}'(z)$ is equal to that of its Lth row.

VI. A DESIGN EXAMPLE

Consider a three-channel QMF bank (i.e., $M = 3$). We shall design a perfect-reconstruction system with real coefficient FIR analysis filters $H_0(z)$, $H_1(z)$, and $H_2(z)$ of orders 55, 52, and 55, respectively. The design will be based on the structure of Fig. 8. The first step is to design a 3rd band zero-phase filter $G_1(z)$ of order 104, with $G_1(e^{j\omega}) \geq 0$ for all ω, and obtain a spectral factor $H_1(z)$. This step can be simplified by noting that $G_1(z)$ has the form $G'_1(-z^2)$ where $G'_1(z)$ is a 3rd band zero-phase *low-pass* filter of order 52 with $G'_1(e^{j\omega}) \geq 0$. With $H'_1(z)$ denoting a spectral factor of $G'_1(z)$ (so that $H'_1(z)$ has order 26), we have $H_1(z) = H'_1(-z^2)$.

In terms of the notations in Section II, let $H'_1(z)$ be of the form $H'_1(z) = H'_{10}(z) H'_{11}(z)$ where $H'_{10}(z)$ has order l_0 and $H'_{11}(z)$ has order l_1. Clearly, we have $l_0 + l_1 = 26$ so that from (27a), $p_1 = 2$. Thus, from (27b) we obtain $2l_0 - l_1 = -2$. These relations give the unique values $l_0 = 8$, and $l_1 = 18$. We can now proceed to design the low-pass filter $H'_1(z)$ of order 26 in the manner described in Section II, and obtain $H_1(z) = H'_1(-z^2)$. The objective function analogous to (16) to be minimized is

$$\int_{\pi/3+\epsilon}^{\pi} \left| H'_{10}(e^{j\omega}) \right|^2 \left| H'_{11}(e^{j\omega}) \right|^2 d\omega.$$

In the example, $\epsilon = 0.2\pi$ was used. Notice that the order of $G'_1(z)$ is 52, whereas that of $G'_{10}(z)$ is only 16. The spectral factorization of $G'_{10}(z)$, which is required in the procedure of Section II, is therefore much simpler than a direct factorization of $G'_1(z)$. Fig. 9 shows the response of the bandpass filter $G_1(z)$ and its spectral factor $H_1(z)$. Note that $G_1(z)$ has several zeros on the unit circle, and has about 140 dB stopband attenuation.

With the above $H_1(z)$ used as an "initial value" for the transfer function $H_1(z)$ in Fig. 8, the middle row $[\hat{e}_1(z)]^T$ of the 3×3 matrix $\hat{E}'(z)$ is completely determined. We then synthesize $\hat{e}_1(z)$ as $\hat{e}_1(z) = U(z)P_1$, thereby initializing the vectors u_k and P_1.

We can express $[\hat{E}'(z)]^T$ as $[\hat{E}'(z)]^T = U(z)H_0$ where the 3×3 matrix H_0 is $H_0 = [H_{00} \ P_1 \ H_{02}]$. Thus, H_0 is a real orthogonal matrix with middle column determined. It has only one extra degree of freedom (by Appendix D). Since $\hat{B}$ is 1×1, it has no free parameters. (The matrices $\hat{B}$ and P_1 are not necessary for $M = 3$.) In the example, the extra freedom in H_0 was randomly initialized, and all the parameters [including the u_k vectors in $\hat{E}'(z)$] were reoptimized to minimize (2). Fig. 10 shows the complete lattice structure, where $V_k(z)$ are 3×3 degree-one lossless sections as in (33). The resulting analysis filter responses are shown in Fig. 11, whereas the impulse response coefficients are given in Table I. The convergence of the optimization was found to be much faster than our earlier techniques (in fact comparable to the design of a single-rate linear-phase lowpass filter of same order using [24]).

VII. CONCLUDING REMARKS

The main purpose of this paper has been to provide an improved approach for designing M-band perfect-reconstruction FIR QMF banks based on lossless polyphase matrices. The improvements arise from judicious initialization of the analysis filters in the optimization process, more efficient characterization of FIR lossless systems by avoiding rotation operators, and finally the incorporation of symmetry between pairs of analysis filters. The improvements obtained are in terms of faster convergence of the optimization procedure, and in terms of better stopband attenuation of analysis filters for a given filter length and transition bandwidth. The techniques developed in this paper work for arbitrary M (i.e., arbitrary number of channels).
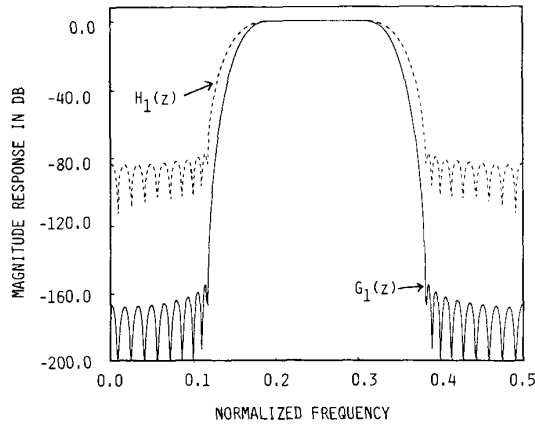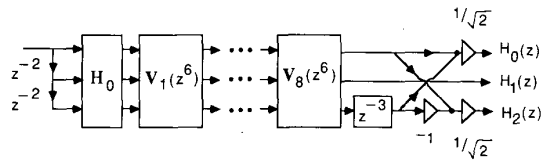
Fig. 9. The 3rd band filter $G_1(z)$ and its spectral factor $H_1(z)$.

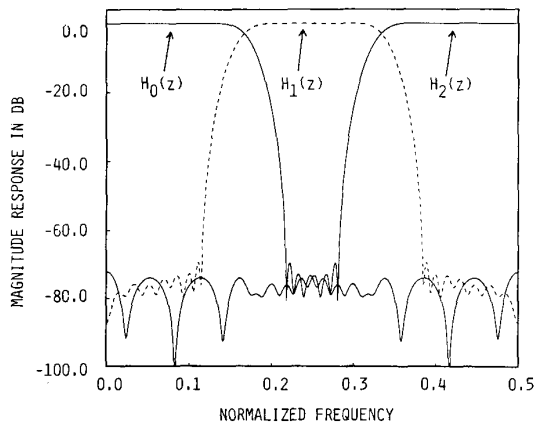Fig. 10. The cascaded lossless structure for the analysis bank of the design example.

Fig. 11. Responses of the reoptimized analysis filters.

APPENDIX A
LOSSLESSNESS OF $E(z)$ AND MTH BAND PROPERTY OF $\tilde{H}_k(z) H_k(z)$

The analysis filter $H_0(z)$ can be written as

$$H_0(z) = [1 \quad z^{-1} \quad \cdots \quad z^{-(M-1)}]e_0(z^M) \quad (A.1)$$

where $e_0^T(z)$ is the 0th row of $E(z)$. By replacing z with zW^{-k} for $0 \leq k \leq M - 1$, we can arrive at M equations like (A.1). Stacking these together, we obtain

$$\begin{bmatrix} H_0(z) \\ H_0(zW^{-1}) \\ \vdots \\ H_0(zW^{-(M-1)}) \end{bmatrix} = W\Lambda(z) e_0(z^M) \quad (A.2)$$

where W is the $M \times M$ DFT matrix, and $\Lambda(z)$ is a diagonal matrix of delays:

$$\Lambda(z) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & z^{-1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & z^{-(M-1)} \end{bmatrix} \quad (A.3)$$

Since $E(z)$ is lossless, the vector $e_0(z)$ is lossless. The matrices W and $\Lambda(z)$ are clearly lossless. The product in (A.2) is therefore lossless (i.e., power complementary), so that

$$\sum_{k=0}^{M-1} \tilde{H}_0(zW^{-k}) H_0(zW^{-k}) = c \quad (A.4)$$

where c is a nonzero constant. This proves that $G_0(z) \triangleq \tilde{H}_0(z) H_0(z)$ is an Mth band filter. In other words, $H_0(z)$ is a spectral factor of an Mth band filter. The same is evidently true of $H_k(z)$, $1 \leq k \leq M - 1$.

Conversely, we see that if $G_0(z)$ is an Mth band filter, then $H_0(z)$ satisfies (A.4), so that the left-hand side of (A.2) is lossless. It can then be verified that $e_0(z)$ is lossless as well.

APPENDIX B
PROPERTIES OF DIADICS

Let v be an $M \times 1$ column vector, $v \neq 0$. Then $vv^\dagger$ is said to be a (Hermitian-)diadic. Some important properties of diadics are listed below.

1) The rank of such a matrix is unity (since all the columns are scalar multiples of v) so that $M - 1$ of the M eigenvalues are equal to zero. The only nonzero eigenvalue is equal to $v^\dagger v$, with corresponding eigenvector equal to v (proof: $vv^\dagger v = v^\dagger vv$ by associativity).

2) Let v be of unit norm so that $v^\dagger v = 1$. Then the matrix $I - vv^\dagger$ has rank $M - 1$. The *only* nonnull w such that $[I - vv^\dagger]w = 0$ has the form $w = cv$, where $c \neq 0$ is a scalar.

3) Consider a matrix of the form $A = I - 2vv^\dagger$, with $v \neq 0$. Assume that v is such that A is unitary. Thus, A is unitary and Hermitian so that its eigenvalues are ± 1. Since $1 - 2v^\dagger v$ is an eigenvalue, it should equal ± 1. With $v \neq 0$, this eigenvalue *must be* -1, so that $v^\dagger v = 1$. In summary, if a matrix of the form $I - 2vv^\dagger$ is unitary where v is a column vector, then v has unit norm.

4) Let u and v be unit-norm vectors such that

$$u^\dagger v \neq 0. \quad (A.5)$$

Consider the matrix

$$A = [I - vv^\dagger][I - uu^\dagger]. \quad (A.6)$$

Clearly, $Au = 0$ by property 2. It turns out that any nonnull vector w for which

$$Aw = 0 \quad (A.7)$$

is such that

$$w = cu, \quad c \neq 0. \quad (A.8)$$

TABLE I
IMPULSE RESPONSE COEFFICIENTS OF THE REOPTIMIZED ANALYSIS FILTERS
IN THE DESIGN EXAMPLE

m	Coefficients $h_0(n)$	Coefficients $h_1(n)$	Coefficients $h_2(n)$
0	$-1.4128229067239 \times 10^{-3}$	$1.5056378297974 \times 10^{-5}$	$-1.4128229067239 \times 10^{-3}$
1	0.	0.	0.
2	$2.1414058710181 \times 10^{-2}$	$6.6399724053250 \times 10^{-5}$	$2.1414058710181 \times 10-2$
3	$9.1308218484911 \times 10^{-2}$	0.	$-9.1308218484911 \times 10^{-2}$
4	$2.2641530391958 \times 10^{-1}$	$8.8793526437529 \times 10^{-5}$	$2.2641530391958 \times 10^{-1}$
5	$3.9757826027754 \times 10^{-1}$	0.	$-3.9757826027754 \times 10^{-1}$
6	$5.1966245198362 \times 10^{-1}$	$-1.6061927086436 \times 10^{-4}$	$5.1966245198362 \times 10^{-1}$
7	$4.9519441407873 \times 10^{-1}$	0.	$-4.9519441407873 \times 10^{-1}$
8	$2.9453172543002 \times 10^{-1}$	$-5.9188010394438 \times 10^{-4}$	$2.9453172543002 \times 10^{-1}$
9	$5.2368950426968 \times 10^{-3}$	0.	$-5.2368950426968 \times 10^{-3}$
10	$-2.0994092811453 \times 10^{-1}$	$8.2758834028361 \times 10^{-4}$	$-2.0994092811453 \times 10^{-1}$
11	$-2.3111384039953 \times 10^{-1}$	0.	$2.3111384039953 \times 10^{-1}$
12	$-7.8847322029914 \times 10^{-2}$	$1.9924611673375 \times 10^{-3}$	$-7.8847322029914 \times 10^{-2}$
13	$1.0104466433718 \times 10^{-1}$	0.	$-1.0104466433718 \times 10^{-1}$
14	$1.6112218318323 \times 10^{-1}$	$-4.6270092409176 \times 10^{-3}$	$1.6112218318323 \times 10^{-1}$
15	$7.4057850344872 \times 10^{-2}$	0.	$-7.4057850344872 \times 10^{-2}$
16	$-5.8686064368913 \times 10^{-2}$	$-1.7834882160760 \times 10^{-3}$	$-5.8686064368913 \times 10^{-2}$
17	$-1.1225434647491 \times 10^{-1}$	0.	$1.1225434647491 \times 10^{-1}$
18	$-5.4550575653918 \times 10^{-2}$	$1.4722039145031 \times 10^{-2}$	$-5.4550575653918 \times 10^{-2}$
19	$4.0281521826881 \times 10^{-2}$	0.	$-4.0281521826881 \times 10^{-2}$
20	$7.7614334886641 \times 10^{-2}$	$-1.1233905811727 \times 10^{-2}$	$7.7614334886641 \times 10^{-2}$
21	$3.4995958094679 \times 10^{-2}$	0.	$-3.4995958094679 \times 10^{-2}$
22	$-3.0376976569541 \times 10^{-2}$	$-2.1645331175599 \times 10^{-2}$	$-3.0376976569541 \times 10^{-2}$
23	$-5.1824955200610 \times 10^{-2}$	0.	$5.1824955200610 \times 10^{-2}$
24	$-1.9388392668025 \times 10^{-2}$	$4.7813912371864 \times 10^{-2}$	$-1.9388392668025 \times 10^{-2}$
25	$2.3116098694287 \times 10^{-2}$	0.	$-2.3116098694287 \times 10^{-2}$
26	$3.2571783564021 \times 10^{-2}$	$-1.0294204447447 \times 10^{-2}$	$3.2571783564021 \times 10^{-2}$
27	$8.5801203815075 \times 10^{-3}$	0.	$-8.5801203815075 \times 10^{-3}$
28	$-1.6753271999023 \times 10^{-2}$	$-7.7918425976633 \times 10^{-2}$	$-1.6753271999023 \times 10^{-2}$
29	$-1.8667164769370 \times 10^{-2}$	0.	$1.8667164769370 \times 10^{-2}$
30	$-2.2382174791812 \times 10^{-3}$	$1.0751032541568 \times 10^{-1}$	$-2.2382174791812 \times 10^{-3}$
31	$1.1096519113841 \times 10^{-2}$	0.	$-1.1096519113841 \times 10^{-2}$
32	$9.4068169151632 \times 10^{-3}$	$6.7499047357221 \times 10^{-3}$	$9.4068169151632 \times 10^{-3}$
33	$-6.9083772961599 \times 10^{-4}$	0.	$6.9083772961599 \times 10^{-4}$
34	$-6.5014160081924 \times 10^{-3}$	$-1.8443792714387 \times 10^{-1}$	$-6.5014160081924 \times 10^{-3}$
35	$-3.8902922683508 \times 10^{-3}$	0.	$3.8902922683508 \times 10^{-3}$
36	$1.4380900534430 \times 10^{-3}$	$2.2127681299343 \times 10^{-1}$	$1.4380900534430 \times 10^{-3}$
37	$3.2229563292928 \times 10^{-3}$	0.	$-3.2229563292928 \times 10^{-3}$
38	$1.1229696945844 \times 10^{-3}$	$-6.1951636758746 \times 10^{-3}$	$1.1229696945844 \times 10^{-3}$
39	$-1.1471284909200 \times 10^{-3}$	0.	$1.1471284909200 \times 10^{-3}$
40	$-1.2482566010628 \times 10^{-3}$	$-3.3826132198080 \times 10^{-1}$	$-1.2482566010628 \times 10^{-3}$
41	$-5.8375452585888 \times 10^{-5}$	0.	$5.8375452585888 \times 10^{-5}$
42	$5.9489610310688 \times 10^{-4}$	$5.6761980190258 \times 10^{-1}$	$5.9489610310688 \times 10^{-4}$
43	$3.0406128784832 \times 10^{-4}$	0.	$-3.0406128784832 \times 10^{-4}$
44	$-1.5404701840277 \times 10^{-4}$	$-5.5034960832577 \times 10^{-1}$	$-1.5404701840277 \times 10^{-4}$
45	$-1.9182526916816 \times 10^{-4}$	0.	$1.9182526916816 \times 10^{-4}$
46	$1.3642678226151 \times 10^{-5}$	$3.5985541240956 \times 10^{-1}$	$1.3642678226151 \times 10^{-5}$
47	$8.4501036966324 \times 10^{-5}$	0.	$-8.4501036966324 \times 10^{-5}$
48	$2.4216033174065 \times 10^{-6}$	$-1.5882736075304 \times 10^{-1}$	$2.4216033174065 \times 10^{-6}$
49	$-3.4346594598878 \times 10^{-5}$	0.	$3.4346594598878 \times 10^{-5}$
50	$-6.5174428259106 \times 10^{-7}$	$4.2746400100218 \times 10^{-2}$	$-6.5174428259106 \times 10^{-7}$
51	$1.3132483913474 \times 10^{-5}$	0.	$-1.3132483913474 \times 10^{-5}$
52	$7.6751821402677 \times 10^{-8}$	$-5.0339744493869 \times 10^{-3}$	$7.6751821402677 \times 10^{-8}$
53	$-3.5344439963839 \times 10^{-6}$	0.	$3.5344439963839 \times 10^{-6}$
54	0.	0.	0.
55	$4.1622921997806 \times 10^{-7}$	0.	$-4.1622921997806 \times 10^{-7}$

Property 4 implies that A has rank $M - 1$. To prove property 4, note that (A.7) implies

$$[I - uu^\dagger]w = vv^\dagger[I - uu^\dagger]w \qquad (A.9)$$

which shows that

$$[I - uu^\dagger]w \qquad (A.10)$$

is either an eigenvector of $vv^\dagger$ (with unit eigenvalue) or the null vector. If the former is true then

$$[I - uu^\dagger]w = cv \qquad (A.11)$$

by Property 1. However, (A.11) implies $u^\dagger v = 0$ violating (A.5) unless $c = 0$. The vector (A.10) is thus null, proving, by Property 2, that w has the form (A.8).

APPENDIX C
ON THE RANK OF $h_{N-1}(0)$

Let $H_{N-1}(z)$ be an $M \times M$ lossless system of degree $N - 1$ given by $H_{N-1}(z) = \sum_{n=0}^{K} h_{N-1}(n)z^{-n}$. We then have $K \leq N - 1$. We know from Property 3.2 that $h(0)$ is singular. If there exists a column of $H_{N-1}(z)$, denoted $P_{N-1}(z)$, such that its degree is $N - 1$, then at least one column of $h_{N-1}(N - 1)$ is nonzero, so that $K = N - 1$. This means that in the factorization (32) with $V_k(z)$ as in (33), the product

$$v_{N-1}v_{N-1}^{\dagger}v_{N-2}v_{N-2}^{\dagger} \cdots v_1 v_1^{\dagger} \qquad (A.12)$$

is a nonnull vector. This in turn implies the following:

$$v_k^{\dagger}v_{k-1} \neq 0, \qquad 2 \leq k \leq N - 1. \qquad (A.13)$$

Now consider the expression for $h_{N-1}(0)$ in terms of the representation (32)

$$h_{N-1}(0) = [I - v_{N-1}v_{N-1}^{\dagger}][I - v_{N-2}v_{N-2}^{\dagger}]$$
$$\cdots [I - v_1 v_1^{\dagger}]H_0. \qquad (A.14)$$

Since H_0 is unitary, the rank of $h_{N-1}(0)$ is equal to that of $h_{N-1}(0)H_0^{\dagger}$. It turns out that under the condition (A.13), this rank is precisely equal to $M - 1$. This can be proved by repeated application of the argument given in Property 4 of Appendix B.

APPENDIX D
EXPRESSIONS FOR N_f AND N_f'

Let $U = [u_0\ u_1\ \cdots\ u_{M-1}]$ be an $M \times M$ unitary matrix. The unitary property can be expressed in terms of the $\binom{M}{2}$ constraints

$$u_k^{\dagger}u_l = 0, \qquad 0 \leq k \leq M - 1,$$
$$k + 1 \leq l \leq M - 1 \qquad (A.15)$$

and the M constraints

$$u_k^{\dagger}u_k = 1, \qquad 0 \leq k \leq M - 1. \qquad (A.16)$$

The left-hand sides of each of the $\binom{M}{2}$ constraints in (A.15) are, in general, complex, whereas those in (A.16) are real. Altogether, we therefore have a total of $2\binom{M}{2} + M$ real constraints because of (A.15) and (A.16). An arbitrary $M \times M$ matrix with complex entries has $2M^2$ real-valued degrees of freedom. Subtracting the $2\binom{M}{2} + M$ constraints due to unitariness, we arrive at the number of degrees of freedom [21] $N_{unitary} = M^2$ for an $M \times M$ unitary matrix. A similar calculation with real matrices shows that the number of real-valued degrees of freedom for an $M \times M$ real orthogonal matrix is equal to $\binom{M}{2}$. These freedoms are commonly expressed as planar rotation angles [21], [18].

Now assume that one particular column, say u_0, of the $M \times M$ unitary matrix U is fixed. In the complex case, this takes away $2M - 1$ real-valued degrees of freedom (because the column has M complex entries constrained by $u_0^{\dagger}u_0 = 1$). This leaves behind $N_f' = M^2 - 2M + 1$ freedoms which is (55). For the case of real matrices, a similar calculation shows that the number of degrees of

freedom in a real orthogonal matrix with a fixed column is equal to $N_f = \binom{M}{2} - (M - 1)$ which is (8).

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