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Improvement of approximation spaces using maximal left neighborhoods and ideals

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ABSTRACT Rough set theory was introduced by Pawlak, in 1982, as a methodology to discover structural relationships within imprecise and uncertain data. This theory has been generalized using the idea of neighborhood systems to be more efficient to get rid of uncertainty and deal with a wide scope of practical applications. Motivated by this idea, in this work, we initiate novel generalized rough set models using the concepts of "maximal left neighborhoods and ideals". Their basic features are studied and the relationships between them are revealed. The main merits of these models, as we prove, are first to preserve almost all major properties of approximation operators with respect to the Pawlak model. Second, they keep the monotonic property, which leads to an efficient evaluation of the uncertainty in the data, and third, these models enlarge the knowledge gotten from the information systems because they minimize the vagueness regions more than some previous models. We complete this manuscript by applying the proposed approach to analyze educational data and illustrate its role to improve the obtained classifications of objects and show the great performance of the present approach against other ones. Elucidative examples that support the obtained results are provided.

INDEX TERMS Approximation space, ideals, maximal left neighborhoods, rough set

I. INTRODUCTION

UNCERTAINTY is presented in several practical decision-making issues and real-life problems due to the incompleteness of knowledge. There are various approaches to handle uncertainty in these areas such as rough set theory, introduced by Pawlak [28], [29]. Rough set theory contributed to solve some issues such as characterization of a subset in terms of attribute values, finding dependency between the attributes, reduction of superfluous attributes, determining the most important attributes and decision rule generation.

This theory has rapidly progressed since it was advent in several directions; one of them is to reproduce the approximations operators and their related notions from neighborhoods generated by different binary relations instead of equivalence classes inspired by equivalence relations. This trend started by Yao [35], [36] in the nineties of the last century. He defined new approximation spaces induced from the right and left neighborhoods with respect to an arbitrary relation. These approximation spaces relax the strict condition of an equivalence relation and expand the scope of applications. On the other hand, these models lead to evaporate some properties of the original model given by Pawlak as well as the measures of accuracy exceed one in some cases, which requires some treatments [11]. Following the pioneering works of Yao, many researchers and scholars interested in the rough set theory introduced novel sorts of neighborhood systems and applied to establish new generalized rough set paradigms. Among these neighborhood systems, minimal left neighborhoods and minimal right neighborhoods [2], intersection of minimal left and right neighborhoods [25], and maximal neighborhoods [4], [9].

Mareay [27] familiarized four kinds of approximation spaces using new neighborhoods defined by the equality relation between Yao's neighborhoods. Further types of these sorts of neighborhoods were established in [8]. Recently, Al-shami [3] discussed new approximation spaces inspired by containment neighborhoods which are defined using the inclusion relation. Then, he and Ciucci [6] initiated novel rough set models induced from subset neighborhoods which are defined using the superset relation. These neighborhoods and their approximation spaces were exploited to rank suspected individuals of COVID-19. In fact, many characterizations of Pawlak's models are still valid by these rough paradigms [3], [6]; especially, those are related to the properties of Pawlak's approximation operators and the property of monotonicity. These rough paradigms were debated under arbitrary binary relations. In contrast, investigation of some rough paradigms was conducted under specific type of relations such as quasiorder [31] and similarity [32].

In 2013, Kandi et al. [23] proposed a new technique to study approximation spaces by using "ideal structure" to high the accuracy measure which refers to the completeness of knowledge. This development helps to deal with uncertainty and get rid of obstacles in decision-making problems as shown in [13], [22], [33]. Topological spaces are another significant approach to investigate approximation operators [5], [10], [24], [26], [37]. These approaches applied with ideal to construct new rough set models. In this trend, Hosny [14]–[16] has recently displayed topological approximation spaces via ideals. Afterwards, Güler et al. [12] studied rough approximations induced from containment neighborhoods via ideals. Another technique of studying rough set paradigms is presented by Abu-Donia [1]. He explored approximation spaces using a class of binary relations instead of one binary relation. This idea was exploited to build the previous approximation spaces in terms of finite number of binary relations and ideals; see, for example [7], [18], [20].

The aim of this study is to provide another interesting and novel version of approximation spaces induced from "maximal left neighborhoods and ideals". The main motivation for us to introduce and study this version is to improve the approximation operators and increase the accuracy values of subsets, and to preserve as many properties as possible of Pawlak's approximation spaces and the property of monotonicity.

This article has been structured in the following manner. Section 2 presents an overview of rough neighborhood systems and ideals, which is required for the understanding of this work. The aim of Section 3 is to establish four rough set models and discuss their essential characterizations. The currently proposed models are compared in Section 4 and shown their advantages in comparison with the previous models. After that, a numerical example is shown that the current approach can be effectively applied to some practical issues in Section 5. Finally, Section 6 concludes with a summary of this manuscript and a suggestion for further research.

II. PRELIMINARIES

This section is dedicated to mentioning the main notions and ideas that will be used in the coming sections.

Definition 2.1: [21] We call a non-empty family $\mathcal{P}$ of the power set of $U \neq \phi$ an "ideal" over U if it is closed under finite unions and subsets. That is, $V, W \in \mathcal{P} \Rightarrow V \cup W \in \mathcal{P}$, and if $V \in \mathcal{P}$ then every subset of V is a member of $\mathcal{P}$.

Definition 2.2: [22] Assume that $\mathcal{P}_1, \mathcal{P}_2$ are ideals on a set $U \neq \phi$. The smallest collection generated by $\mathcal{P}_1, \mathcal{P}_2$, denoted by $\mathcal{P}_1 \vee \mathcal{P}_2$, is defined as

$$\mathcal{P}_1 \vee \mathcal{P}_2 = \{G \cup F : G \in \mathcal{P}_1, F \in \mathcal{P}_2\}. \quad (1)$$

Proposition 2.1: [22] The collection $\mathcal{P}_1 \vee \mathcal{P}_2$ has the following properties.

(1) $\mathcal{P}_1 \vee \mathcal{P}_2 \neq \phi$,
(2) $V \in \mathcal{P}_1 \vee \mathcal{P}_2,\ W \subseteq V \Rightarrow W \in \mathcal{P}_1 \vee \mathcal{P}_2$,
(3) $V, W \in \mathcal{P}_1 \vee \mathcal{P}_2 \Rightarrow V \cup W \in \mathcal{P}_1 \vee \mathcal{P}_2$.

That is, the collection $\mathcal{P}_1 \vee \mathcal{P}_2$ is an ideal on U.

Definition 2.3: [28] Consider δ as an equivalence relation on a universe U and let $[\nu]_\delta$ be the equivalence class containing ν. It can be associated each subset V of U with two other sets called "lower approximation $\underline{apr}(V)$" and "upper approximation $\overline{apr}(V)$" given as follows.

$$\underline{apr}(V) = \{\nu \in U : [\nu]_\delta \subseteq V\}. \quad (2)$$
$$\overline{apr}(V) = \{\nu \in U : [\nu]_\delta \cap V \neq \phi\}. \quad (3)$$

The main characterizations of these approximation operators are listed in the following.

($\mathcal{L}_1$) $\underline{apr}(V^c) = [\overline{apr}(V)]^c$, where V^c is the complement of V.
($\mathcal{L}_2$) $\underline{apr}(U) = U$.
($\mathcal{L}_3$) $\underline{apr}(\phi) = \phi$.
($\mathcal{L}_4$) $\underline{apr}(V) \subseteq V$.
($\mathcal{L}_5$) $\underline{apr}(V \cap W) = \underline{apr}(V) \cap \underline{apr}(W)$
($\mathcal{L}_6$) $\underline{apr}(V \cup W) \supseteq \underline{apr}(V) \cup \underline{apr}(W)$
($\mathcal{L}_7$) $V \subseteq W \Rightarrow \underline{apr}(V) \subseteq \underline{apr}(W)$.
($\mathcal{L}_8$) $\underline{apr}(\underline{apr}(V)) = \underline{apr}(V)$.
($\mathcal{L}_9$) $\overline{apr}(V) \subseteq \underline{apr}(\overline{apr}(V))$.
($\mathcal{U}_1$) $\overline{apr}(V^c) = [\underline{apr}(V)]^c$.
($\mathcal{U}_2$) $\overline{apr}(U) = U$.
($\mathcal{U}_3$) $\overline{apr}(\phi) = \phi$.
($\mathcal{U}_4$) $V \subseteq \overline{apr}(V)$.
($\mathcal{U}_5$) $\overline{apr}(V \cup W) = \overline{apr}(V) \cup \overline{apr}(W)$.
($\mathcal{U}_6$) $\overline{apr}(V \cap W) \subseteq \overline{apr}(V) \cap \overline{apr}(W)$.
($\mathcal{U}_7$) $V \subseteq W \Rightarrow \overline{apr}(V) \subseteq \overline{apr}(W)$.
($\mathcal{U}_8$) $\overline{apr}(\overline{apr}(V)) = \overline{apr}(V)$.
($\mathcal{U}_9$) $\overline{apr}(\underline{apr}(V)) \subseteq \underline{apr}(V)$.

Definition 2.4: [28] Let δ be an equivalence relation on a universe U. Then accuracy measure $Acc_R(V)$ of any nonempty subset V is defined as follows: $Acc_R(V) = \frac{|\underline{apr}(V)|}{|\overline{apr}(V)|}$. If δ_1 and δ_2 are equivalence relations on a universe U such that $\delta_1 \subseteq \delta_2$. Then the approximations induced from these relations have the monotonic property if $Acc_{\delta_2}(V) \leq Acc_{\delta_1}(V)$.

Definition 2.5: [4], [9], [35], [36] Take δ as an arbitrary binary relation on a finite set $U \neq \phi$ and let $\nu \in U$. Then,

1) the right neighborhood of ν, denoted by $N_r(\nu)$ is given by $N_r(\nu) = \{\lambda \in U : (\nu, \lambda) \in \delta\}$.
2) the left neighborhood of ν, denoted by $N_l(\nu)$ is given by $N_r(\nu) = \{\lambda \in U : (\lambda, \nu) \in \delta\}$.
3) $\theta_r(\nu)$ is the union of all right neighborhoods containing ν.

4) $\theta_l(\nu)$ is the union of all left neighborhoods containing ν.
5) $\theta_u(\nu) = \theta_r(\nu) \cup \theta_l(\nu)$.

Theorem 2.1: [4] Let U be a universal set and δ_1, δ_2 be two binary relations on U. If $\delta_1 \subseteq \delta_2$, then $\theta_{1l}(\nu) \subseteq \theta_{2l}(\nu), \forall \nu \in U$.

Definition 2.6: [4] Let δ be a binary relation on a nonempty set U. For any subset $\phi \neq V \subseteq U$. The lower and upper approximations, boundary regions, accuracy and roughness of V induced from maximal left neighborhoods according to δ are defined respectively by:

$$L^{\delta}(V) = \{\nu \in U : \theta_l(\nu) \subseteq V\}. \quad (4)$$
$$U^{\delta}(V) = \{\nu \in U : \theta_l(\nu) \cap V \neq \phi\}. \quad (5)$$
$$Bnd^{\delta}_{\delta}(V) = U^{\delta}(V) - L^{\delta}(V). \quad (6)$$
$$Acc^{\delta}(V) = |\frac{L^{\delta}(V) \cap V}{U^{\delta}(V) \cup V}|. \quad (7)$$
$$Rough^{\delta}(V) = 1 - Acc^{\delta}(V). \quad (8)$$

Definition 2.7: [4] Let δ be a binary relation on a nonempty set U. For any subset $\phi \neq V \subseteq U$. The lower and upper approximations, boundary regions, accuracy and roughness of V induced from maximal union neighborhoods according to δ are defined respectively by:

$$Low^{\delta}(V) = \{\nu \in U : \theta_u(\nu) \subseteq V\}. \quad (9)$$
$$Upp^{\delta}(V) = \{\nu \in U : \theta_u(\nu) \cap V \neq \phi\}. \quad (10)$$
$$Boundary^{\delta}_{\delta}(V) = Upp^{\delta}(V) - Low^{\delta}(V). \quad (11)$$
$$Accuracy^{\delta}(V) = |\frac{Low^{\delta}(V) \cap V}{Upp^{\delta}(V) \cup V}|. \quad (12)$$
$$Roughness^{\delta}(V) = 1 - Accuracy^{\delta}(V). \quad (13)$$

Definition 2.8: [19] Let δ and $\mathcal{P}$ be binary relation and ideal on a set $U \neq \phi$. The first form of generalized approximations (lower and upper), boundary-regions, accuracy and rough values of a nonempty subset V of U produced by maximal union neighborhoods according to δ and $\mathcal{P}$ are respectively given by

$$Low_1{}^{\mathcal{P}\delta}(V) = \{\nu \in U : \theta_u(\nu) \cap V^c \in \mathcal{P}\}. \quad (14)$$
$$Upp_1{}^{\mathcal{P}\delta}(V) = \{\nu \in U : \theta_u(\nu) \cap V \notin \mathcal{P}\}. \quad (15)$$
$$Boundary_1{}^{\mathcal{P}\delta}(V) = Upp_1{}^{\mathcal{P}\delta}(V) - Low_1{}^{\mathcal{P}\delta}(V). \quad (16)$$
$$Accuracy_1{}^{\mathcal{P}\delta}(V) = \frac{|Low_1{}^{\mathcal{P}\delta}(V) \cap V|}{|Upp_1{}^{\mathcal{P}\delta}(V) \cup V}|. \quad (17)$$
$$Roughness_1{}^{\mathcal{P}\delta}(V) = 1 - Accuracy_1{}^{\mathcal{P}\delta}(V). \quad (18)$$

Definition 2.9: [19] Let δ and $\mathcal{P}$ be binary relation and ideal on a set $U \neq \phi$. The second form of generalized approximations (lower and upper), boundary-regions, accuracy and rough values of a nonempty subset V of U produced by maximal union neighborhoods according to δ and $\mathcal{P}$ are respectively given by

$$Low_2{}^{\mathcal{P}\delta}(V) = \{\nu \in A : \theta_u(\nu) \cap V^c \in \mathcal{P}\}. \quad (19)$$
$$Upp_2{}^{\mathcal{P}\delta}(V) = V \cup Upp_1{}^{\mathcal{P}\delta}(V). \quad (20)$$
$$Boundary_2{}^{\mathcal{P}\delta}(V) = Upp_2{}^{\mathcal{P}\delta}(V) - Low_2{}^{\mathcal{P}\delta}(V). \quad (21)$$
$$Accuracy_2{}^{\mathcal{P}\delta}(V) = \frac{|Low_2{}^{\mathcal{P}\delta}(V)|}{|Upp_2{}^{\mathcal{P}\delta}(V)|}, Upp_2{}^{\mathcal{P}\delta}(V) \neq \phi. \quad (22)$$
$$Roughness_2{}^{\mathcal{P}\delta}(V) = 1 - Accuracy_2{}^{\mathcal{P}\delta}(V). \quad (23)$$

Definition 2.10: [19] Let δ and $\mathcal{P}$ be binary relation and ideal on a set $U \neq \phi$. The third form of generalized approximations (lower and upper), boundary-regions, accuracy and rough values of a nonempty subset V of U produced by maximal union neighborhoods according to δ and $\mathcal{P}$ are respectively given by

$$Low_3{}^{\mathcal{P}\delta}(V) = \cup\{\theta_u(\nu) : \theta_u(\nu) \cap V^c \in \mathcal{P}\}. \quad (24)$$
$$Upp_3{}^{\mathcal{P}\delta}(V) = (Low_3{}^{\mathcal{P}\delta}(V^c))^c. \quad (25)$$
$$Boundary_3{}^{\mathcal{P}\delta}(V) = Upp_3{}^{\mathcal{P}\delta}(V) - Low_3{}^{\mathcal{P}\delta}(V). \quad (26)$$
$$Accuracy_3{}^{\mathcal{P}\delta}(V) = \frac{|Low_3{}^{\mathcal{P}\delta}(V) \cap V|}{|Upp_3{}^{\mathcal{P}\delta}(V) \cup V}|. \quad (27)$$
$$Roughness_3{}^{\mathcal{P}\delta}(V) = 1 - Accuracy_3{}^{\mathcal{P}\delta}(V). \quad (28)$$

Definition 2.11: [19] Let δ and $\mathcal{P}$ be binary relation and ideal on a set $U \neq \phi$. The fourth form of generalized approximations (lower and upper), boundary-regions, accuracy and rough values of a nonempty subset V of U produced by maximal union neighborhoods according to δ and $\mathcal{P}$ are respectively given by

$$Upp_4{}^{\mathcal{P}\delta}(V) = \cup\{\theta_u(\nu) : \theta_u(\nu) \cap V \notin \mathcal{P}\}. \quad (29)$$
$$Low_4{}^{\mathcal{P}\delta}(V) = (Upp_4{}^{\mathcal{P}\delta}(V^c))^c. \quad (30)$$
$$Boundary_4{}^{\mathcal{P}\delta}(V) = Upp_4{}^{\mathcal{P}\delta}(V) - Low_4{}^{\mathcal{P}\delta}(V). \quad (31)$$
$$Accuracy_4{}^{\mathcal{P}\delta}(V) = \frac{|Low_4{}^{\mathcal{P}\delta}(V) \cap V|}{|Upp_4{}^{\mathcal{P}\delta}(V) \cup V|}. \quad (32)$$
$$Roughness_4{}^{\mathcal{P}\delta}(V) = 1 - Accuracy_4{}^{\mathcal{P}\delta}(V). \quad (33)$$

III. SOME NEW ROUGH SET MODELS INDUCED FROM $\theta_L(\nu)$-NEIGHBORHOODS AND IDEALS

In this section, we display four types of rough set models defined by maximal left neighborhoods and ideals under any arbitrary relation. Their main features and characterizations are scrutinized and some counterexamples are provided to clarify the obtained facts and relationships.

A. FIRST TECHNIQUE TO GENERATE GENERALIZED ROUGH SETS VIA IDEALS

Definition 3.1: Let δ and $\mathcal{P}$ be binary relation and ideal on a set $U \neq \phi$. The first form of generalized approximations (lower and upper), boundary-regions, accuracy and rough values of a nonempty subset V of U produced by maximal left neighborhoods according to δ and $\mathcal{P}$ are respectively given by

$$L_1{}^{\mathcal{P}\delta}(V) = \{\nu \in U : \theta_l(\nu) \cap V^c \in \mathcal{P}\}. \quad (34)$$

$$U_1{}^{\mathcal{P}\delta}(V) = \{\nu \in U : \theta_l(\nu) \cap V \notin \mathcal{P}\}. \quad (35)$$

$$Bnd_1{}^{\mathcal{P}\delta}(V) = U_1{}^{\mathcal{P}\delta}(V) - L_1{}^{\mathcal{P}\delta}(V). \quad (36)$$

$$Acc_1{}^{\mathcal{P}\delta}(V) = \frac{|L_1{}^{\mathcal{P}\delta}(V) \cap V|}{|U_1{}^{\mathcal{P}\delta}(V) \cup V}|. \quad (37)$$

$$Rough_1{}^{\mathcal{P}\delta}(V) = 1 - Acc_1{}^{\mathcal{P}\delta}(V). \quad (38)$$

Proposition 3.1: Let $\mathcal{P}$ and $\mathcal{T}$ be ideals and δ be a binary relation on U such that $V, W \subseteq U$ Then,

(1) $U_1{}^{\mathcal{P}\delta}(\phi) = \phi$.

(2) $V \subseteq W \Rightarrow U_1{}^{\mathcal{P}\delta}(V) \subseteq U_1{}^{\mathcal{P}\delta}(W)$.

(3) $U_1{}^{\mathcal{P}\delta}(V \cap W) \subseteq U_1{}^{\mathcal{P}\delta}(V) \cap U_1{}^{\mathcal{P}\delta}(W)$.

(4) $U_1{}^{\mathcal{P}\delta}(V \cup W) = U_1{}^{\mathcal{P}\delta}(V) \cup U_1{}^{\mathcal{P}\delta}(W)$.

(5) $U_1{}^{\mathcal{P}\delta}(V) = (L_1{}^{\mathcal{P}\delta}(V^c))^c$.

(6) If $V \in \mathcal{P}$, then $U_1{}^{\mathcal{P}\delta}(V) = \phi$.

(7) If $\mathcal{P} \subseteq \mathcal{T}$, then $U_1{}^{\mathcal{T}\delta}(V) \subseteq U_1{}^{\mathcal{P}\delta}(V)$.

(8) If $\mathcal{P} = P(U)$, then $U_1{}^{\mathcal{P}\delta}(V) = \phi$.

(9) $U_1{}^{(\mathcal{P}\cap\mathcal{T})\delta}(V) = U_1{}^{\mathcal{P}\delta}(V) \cup U_1{}^{\mathcal{T}\delta}(V)$.

(10) $U_1{}^{(\mathcal{P}\vee\mathcal{T})\delta}(V) = U_1{}^{\mathcal{P}\delta}(V) \cap U_1{}^{\mathcal{T}\delta}(V)$.

Proof.

(1) $U_1{}^{\mathcal{P}\delta}(\phi) = \{\nu \in U : \theta_l(\nu) \cap \phi \notin \mathcal{P}\} = \phi$.

(2) Let $\nu \in U_1{}^{\mathcal{P}\delta}(V)$. Then, $\theta_l(\nu) \cap V \notin \mathcal{P}$. Since $V \subseteq W$ and $\mathcal{P}$ is an ideal. It follows that $\theta_l(\nu) \cap W \notin \mathcal{P}$. Therefore, $\nu \in U_1{}^{\mathcal{P}\delta}(W)$. Hence, $U_1{}^{\mathcal{P}\delta}(V) \subseteq U_1{}^{\mathcal{P}\delta}(W)$.

(3) It directly comes from (2).

(4) $U_1{}^{\mathcal{P}\delta}(V) \cup U_1{}^{\mathcal{P}\delta}(W) \subseteq U_1{}^{\mathcal{P}\delta}(V \cup W)$ according to (2). Let $\nu \in U_1{}^{\mathcal{P}\delta}(V \cup W)$. Then, $\theta_l(\nu) \cap (V \cup W) \notin \mathcal{P}$. It follows that $((\theta_l(\nu) \cap V) \cup (\theta_l(\nu) \cap W)) \notin \mathcal{P}$. Therefore, $\theta_l(\nu) \cap V \notin I$ or $\theta_l(\nu) \cap W \notin \mathcal{P}$, which gives $\nu \in U_1{}^{\mathcal{P}\delta}(V)$ or $\nu \in U_1{}^{\mathcal{P}\delta}(W)$. Then, $\nu \in U_1{}^{\mathcal{P}\delta}(V) \cup U_1{}^{\mathcal{P}\delta}(W)$. Thus, $U_1{}^{\mathcal{P}\delta}(V \cup W) \subseteq U_1{}^{\mathcal{P}\delta}(V) \cup U_1{}^{\mathcal{P}\delta}(W)$. Hence, $U_1{}^{\mathcal{P}\delta}(V \cup W) = U_1{}^{\mathcal{P}\delta}(V) \cup U_1{}^{\mathcal{P}\delta}(W)$.

(5) $(L_1{}^{\mathcal{P}\delta}(V^c))^c = (\{\nu \in U : \theta_l(\nu) \cap V \in \mathcal{P}\})^c = \{\nu \in U : \theta_l(\nu) \cap V \notin \mathcal{P}\} = U_1{}^{\mathcal{P}\delta}(V)$.

(6) The proof is straightforward by Definition 3.1.

(7) Let $\nu \in U_1{}^{\mathcal{T}\delta}(V)$. Then, $\theta_l(\nu) \cap V \notin \mathcal{T}$. Since $\mathcal{P} \subseteq \mathcal{T}$. So, $\theta_l(\nu) \cap V \notin \mathcal{P}$. Therefore, $\nu \in U_1{}^{\mathcal{P}\delta}(V)$. Hence, $U_1{}^{\mathcal{T}\delta}(V) \subseteq U_1{}^{\mathcal{P}\delta}(V)$.

(8) The proof is straightforward by Definition 3.1.

(9) $U_1{}^{(\mathcal{P}\cap\mathcal{T})\delta}(V) = \{\nu \in U : \theta_l(\nu) \cap V \notin \mathcal{P} \cap \mathcal{T}\}$
$= \{\nu \in U : \theta_l(\nu) \cap V \notin \mathcal{P}\}$ *or* $\{\nu \in U : \theta_l(\nu) \cap V \notin \mathcal{T}\}$
$= \{\nu \in U : \theta_l(\nu) \cap V \notin \mathcal{P}\} \cup \{\nu \in U : \theta_l(\nu) \cap V \notin \mathcal{T}\}$
$= U_1{}^{(\mathcal{P}\cup\mathcal{T})\delta}(V)$.

(10) $U_1{}^{(\mathcal{P}\vee\mathcal{T})\delta}(V) = \{\nu \in U : \theta_l(\nu) \cap V \notin \mathcal{P} \vee \mathcal{T}\}$
$= \{\nu \in U : \theta_l(\nu) \cap V \notin \mathcal{P} \cup \mathcal{T}\}$
$= \{\nu \in U : \theta_l(\nu) \cap V \notin \mathcal{P}\}$ *and* $\{\nu \in U : \theta_l(\nu) \cap V \notin \mathcal{T}\}$
$= \{\nu \in U : \theta_l(\nu) \cap V \notin \mathcal{P}\} \cap \{\nu \in U : \theta_l(\nu) \cap V \notin \mathcal{T}\}$
$= U_1{}^{(\mathcal{P}\cap\mathcal{T})\delta}(V)$.

Proposition 3.2: Let $\mathcal{P}$ and $\mathcal{T}$ be ideals and δ be a binary relation on U such that $V, W \subseteq U$ Then,

(1) $L_1{}^{\mathcal{P}\delta}(U) = U$.

(2) $V \subseteq W \Rightarrow L_1{}^{\mathcal{P}\delta}(V) \subseteq L_1{}^{\mathcal{P}\delta}(W)$.

(3) $L_1{}^{\mathcal{P}\delta}(V) \cup L_1{}^{\mathcal{P}\delta}(W) \subseteq L_1{}^{\mathcal{P}\delta}(V \cup W)$.

(4) $L_1{}^{\mathcal{P}\delta}(V \cap W) = L_1{}^{\mathcal{P}\delta}(V) \cap L_1{}^{\mathcal{P}\delta}(W)$.

(5) $L_1{}^{\mathcal{P}\delta}(V) = (U_1{}^{\mathcal{P}\delta}(V^c))^c$.

(6) If $V^c \in \mathcal{P}$, then $L_1{}^{\mathcal{P}\delta}(V) = U$.

(7) If $\mathcal{P} \subseteq \mathcal{T}$, then $L_1{}^{\mathcal{P}\delta}(V) \subseteq L_1{}^{\mathcal{T}\delta}(V)$.

(8) If $\mathcal{P} = P(U)$, then $L_1{}^{\mathcal{P}\delta}(V) = U$.

(9) $L_1{}^{(\mathcal{P}\cap\mathcal{T})\delta}(V) = L_1{}^{\mathcal{P}\delta}(V) \cap L_1{}^{\mathcal{T}\delta}(V)$.

Proof.

(1) $L_1{}^{\mathcal{P}\delta}(U) = \{\nu \in U : \theta_l(\nu) \cap \phi \in \mathcal{P}\} = U$.

(2) Let $\nu \in L_1{}^{\mathcal{P}\delta}(V)$. Then, $\theta_l(\nu) \cap V^c \in \mathcal{P}$. Since $W^c \subseteq V^c$ and $\mathcal{P}$ is an ideal. So, $\theta_l(\nu) \cap W^c \in \mathcal{P}$. Therefore, $\nu \in L_1{}^{\mathcal{P}\delta}(W)$. Hence, $L_1{}^{\mathcal{P}\delta}(V) \subseteq L_1{}^{\mathcal{P}\delta}(W)$.

(3) It directly comes from (2).

(4) $L_1{}^{\mathcal{P}\delta}(V) \cap L_1{}^{\mathcal{P}\delta}(W) \supseteq L_1{}^{\mathcal{P}\delta}(V \cap W)$ according to (2). Let $\nu \in L_1{}^{\mathcal{P}\delta}(V) \cap L_1{}^{\mathcal{P}\delta}(W)$. Then, $\theta_l(\nu) \cap V^c \in \mathcal{P}$ and $\theta_l(\nu) \cap W^c \in \mathcal{P}$. It follows that $(\theta_l(\nu) \cap (V^c \cup W^c)) \in \mathcal{P}$. So, $(\theta_l(\nu) \cap (V \cap W)^c) \in \mathcal{P}$. Therefore, $\nu \in L_1{}^{\mathcal{P}\delta}(V \cap W)$. Thus, $L_1{}^{\mathcal{P}\delta}(V) \cap L_1{}^{\mathcal{P}\delta}(W) \subseteq L_1{}^{\mathcal{P}\delta}(V \cap W)$. Hence, $L_1{}^{\mathcal{P}\delta}(V) \cap L_1{}^{\mathcal{P}\delta}(W) = L_1{}^{\mathcal{P}\delta}(V \cap W)$.

(5) $(U_1{}^{\mathcal{P}\delta}(V^c))^c = (\{\nu \in U : \theta_l(\nu) \cap V^c \notin \mathcal{P}\})^c = \{\nu \in U : \theta_l(\nu) \cap V^c \in \mathcal{P}\} = L_1{}^{\mathcal{P}\delta}(V)$.

(6) The proof is straightforward by Definition 3.1.

(7) Let $\nu \in L_1{}^{\mathcal{P}\delta}(V)$. Then, $\theta_l(\nu) \cap V^c \in \mathcal{P}$. Since $\mathcal{P} \subseteq \mathcal{T}$. It follows that $\theta_l(\nu) \cap V^c \in \mathcal{T}$. Therefore, $\nu \in L_1{}^{\mathcal{P}\delta}(V)$. Hence, $L_1{}^{\mathcal{P}\delta}(V) \subseteq L_1{}^{\mathcal{T}\delta}(V)$.

(8) The proof is straightforward by Definition 3.1.

(9) $L_1{}^{(\mathcal{P}\cap\mathcal{T})\delta}(V) = \{\nu \in U : \theta_l(\nu) \cap V^c \in \mathcal{P} \cap \mathcal{T}\}$
$= \{\nu \in U : \theta_l(\nu) \cap V^c \in \mathcal{P}\}$ *and* $\{\nu \in U : \theta_l(\nu) \cap V^c \in \mathcal{T}\}$
$= \{\nu \in U : \theta_l(\nu) \cap V^c \in \mathcal{P}\} \cap \{\nu \in U : \theta_l(\nu) \cap V^c \in \mathcal{T}\}$
$= L_1{}^{\mathcal{P}\delta}(V) \cap L_1{}^{\mathcal{T}\delta}(V)$.

With the help of the next counterexample, we elucidate that the converse of (2), (6), (7) and (8) of Proposition 3.1 and Proposition 3.2 is generally false. Also, we illustrate that the inclusion relations of (3) in Proposition 3.1 and Proposition 3.2 are proper, in general.

Example 3.1:

(i) Let
$$U = \{a, b, c, d\},$$
$$\mathcal{P} = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$
and
$$\delta = \{(a, c), (b, a), (c, a), (c, b), (d, b), (d, c)\}$$
be a binary relation defined on U thus $\theta_l(a) = \{a, d\}, \theta_l(b) = \{b, c\}, \theta_l(c) = \{b, c, d\}$ and $\theta_l(d) = \{a, c, d\}$. For (2), take
   (a) $V = \{a\}$ and $W = \{d\}$; then, ${U_1}^{\mathcal{P}\delta}(V) = \phi$ and ${U_1}^{\mathcal{P}\delta}(W) = \{a, c, d\}$. Therefore, ${U_1}^{\mathcal{P}\delta}(V) \subseteq {U_1}^{\mathcal{P}\delta}(W)$, but $V \nsubseteq W$.
   (b) $V = \{b\}$ and $W = \{a, c, d\}$; then, ${L_1}^{\mathcal{P}\delta}(V) = \{b\}$ and ${L_1}^{\mathcal{P}\delta}(W) = U$. Therefore, ${L_1}^{\mathcal{P}\delta}(V) \subseteq {L_1}^{\mathcal{P}\delta}(W)$, but $V \nsubseteq W$.

(ii) Let $U = \{a, b, c, d\}, \mathcal{T} = \{\phi, \{a\}\}, \mathcal{P} = \{\phi, \{d\}\}$ and $\delta = \{(a, a), (b, b), (c, c)\}$ be a binary relation defined on U; thus, $\theta_l(a) = \{a\}, \theta_l(b) = \{b\}, \theta_l(c) = \{c\}$ and $\theta_l(d) = \phi$.
   (1) For (6), take
      (a) $V = \{a, d\}$; then, ${U_1}^{\mathcal{T}\delta}(V) = \phi$. Therefore, ${U_1}^{\mathcal{T}\delta}(V) = \phi$, but $V \notin \mathcal{T}$.
      (b) $V = \{b, c\}$; then, ${L_1}^{\mathcal{T}\delta}(V) = U$. Therefore, ${L_1}^{\mathcal{T}\delta}(V) = U$, but $V^c \notin \mathcal{T}$.
   (2) For (7), take
      (a) $V = \{a, d\}$; then, ${U_1}^{\mathcal{P}\delta}(V) = \{a\}$ and ${U_1}^{\mathcal{T}\delta}(V) = \phi$. Therefore, ${U_1}^{\mathcal{T}\delta}(V) \subseteq {U_1}^{\mathcal{P}\delta}(V)$, but $\mathcal{P} \nsubseteq \mathcal{T}$.
      (b) $V = \{b, c\}$; then, ${L_1}^{\mathcal{P}\delta}(V) = \{b, c, d\}$ and ${L_1}^{\mathcal{T}\delta}(V) = U$. Therefore, ${L_1}^{\mathcal{P}\delta}(V) \subseteq {L_1}^{\mathcal{T}\delta}(V)$, but $\mathcal{P} \nsubseteq \mathcal{T}$.
   (3) For (8), take
      (a) $V = \{a, d\}$; then, ${U_1}^{\mathcal{T}\delta}(V) = \phi$, but $\mathcal{T} \neq P(U)$.
      (b) $V = \{b, c\}$; then, ${L_1}^{\mathcal{T}\delta}(V) = U$, but $\mathcal{T} \neq P(U)$.

(iii) Let $U = \{a, b, c, d\}, \mathcal{P} = \{\phi, \{d\}\}$ and $\delta = \Delta \cup \{(a, b), (a, c), (a, d)\}$ thus, $\theta_l(a) = U, \theta_l(b) = \{a, b\}, \theta_l(c) = \{a, c\}$ and $\theta_l(d) = \{a, d\}$. For (3), take $V = \{a, d\}, W = \{b, c\}$ and
   (a) $V \cap W = \phi$; then, ${U_1}^{\mathcal{P}\delta}(V) = U, {U_1}^{\mathcal{P}\delta}(W) = \{a, b, c\}$ and ${U_1}^{\mathcal{P}\delta}(V \cap W) = \phi$. Therefore, ${U_1}^{\mathcal{P}\delta}(V) \cap {U_1}^{\mathcal{P}\delta}(W) = \{a, b, c\} \neq \phi = {U_1}^{\mathcal{P}\delta}(V \cap W)$.
   (b) $V \cup W = U$; then, ${L_1}^{\mathcal{P}\delta}(V) = \{d\}, {L_1}^{\mathcal{P}\delta}(W) = \phi$ and ${U_1}^{\mathcal{P}\delta}(V \cup W) = U$. Therefore, ${L_1}^{\mathcal{P}\delta}(V) \cup {L_1}^{\mathcal{P}\delta}(W) = \{d\} \neq U = {L_1}^{\mathcal{P}\delta}(V \cup W)$.

Remark 3.1: Some properties of Pawlak are not satisfy by this type as we show in the following.

(i) Considering Example 3.1 (i), take
   (1) $V = \{a\}$; then, ${U_1}^{\mathcal{P}\delta}(V) = \phi$. Hence, $V \nsubseteq {U_1}^{\mathcal{P}\delta}(V)$.
   (2) $V = \{b, c, d\}$; then, ${L_1}^{\mathcal{P}\delta}(V) = U$. Hence, ${L_1}^{\mathcal{P}\delta}(V) \nsubseteq V$.
   (3) $V = U$; then, ${U_1}^{\mathcal{P}\delta}(U) = \{a, c, d\}$. Hence, ${U_1}^{\mathcal{P}\delta}(U) \neq U$.
   (4) $V = \phi$; then, ${L_1}^{\mathcal{P}\delta}(\phi) = \{b\}$. Hence, ${L_1}^{\mathcal{P}\delta}(\phi) \neq \phi$.

(ii) Considering Example 3.1 (iii), take
   (1) $V = \{b, c\}$; then, ${U_1}^{\mathcal{P}\delta}(V) = \{a, b, c\}$ and ${U_1}^{\mathcal{P}\delta}({U_1}^{\mathcal{P}\delta}(V)) = U$. Hence, ${U_1}^{\mathcal{P}\delta}(V) \neq {U_1}^{\mathcal{P}\delta}({U_1}^{\mathcal{P}\delta}(V))$.
   (2) $V = \{a, d\}$; then, ${L_1}^{\mathcal{P}\delta}(V) = \{d\}$ and ${L_1}^{\mathcal{P}\delta}({L_1}^{\mathcal{P}\delta}(V)) = \phi$. Hence, ${L_1}^{\mathcal{P}\delta}(V) \neq {L_1}^{\mathcal{P}\delta}({L_1}^{\mathcal{P}\delta}(V))$.

(iii) *Example 3.2:* Let $U = \{a, b, c, d\}, \mathcal{P} = \{\phi, \{a\}\}$ and $\delta = \Delta \cup \{(a, b), (a, c), (a, d), (b, a), (b, c), (b, d)\}$. Then $\theta_l(a) = \theta_l(b) = U, \theta_l(c) = \{a, b, c\}$ and $\theta_l(d) = \{a, b, d\}$. It is clear that, if
   (1) $V = \{c\}$; then, ${U_1}^{\mathcal{P}\delta}(V) = \{a, b, c\}$ and ${L_1}^{\mathcal{P}\delta}({U_1}^{\mathcal{P}\delta}(V)) = \{c\}$. Hence, ${U_1}^{\mathcal{P}\delta}(V) \nsubseteq {L_1}^{\mathcal{P}\delta}({U_1}^{\mathcal{P}\delta}(V))$.
   (2) $V = \{a, b, d\}$; then, ${L_1}^{\mathcal{P}\delta}(V) = \{d\}$ and ${U_1}^{\mathcal{P}\delta}({L_1}^{\mathcal{P}\delta}(V)) = \{a, b, d\}$. Hence,
$$ {U_1}^{\mathcal{P}\delta}({L_1}^{\mathcal{P}\delta}(V)) \nsubseteq {L_1}^{\mathcal{P}\delta}(V). $$

Proposition 3.3: Let $\mathcal{P}$ be an ideal and δ be a binary relation on U such that V is a nonempty subset of U. Then,
1) $0 \leq Acc_1^{\mathcal{P}\delta}(V) \leq 1$.
2) $Acc_1^{\mathcal{P}\delta}(U) = 1$.

Proof. We prove (1) only and (2) is straightforward. Since, $\phi \neq V \subseteq U$, then $U_1^{\mathcal{P}\delta}(V) \cup V \neq \phi$. Hence, $\phi \subseteq L_1^{\mathcal{P}\delta}(V) \cap V \subseteq U_1^{\mathcal{P}\delta}(V) \cup V$. Therefore, $0 \leq |L_1^{\mathcal{P}\delta}(V) \cap V| \leq |U_1^{\mathcal{P}\delta}(V) \cup V|$. So, $0 \leq \frac{|L_1^{\mathcal{P}\delta}(V) \cap V|}{|U_1^{\mathcal{P}\delta}(V) \cup V|} \leq 1$. It means that, $0 \leq Acc_1^{\mathcal{P}\delta}(V) \leq 1$.

Theorem 3.1: Let $\mathcal{P}$ and $\mathcal{T}$ be ideals and δ be a binary relation on U such that $\mathcal{P} \subseteq \mathcal{T}$. Then,
(1) ${Bnd_1}^{\mathcal{T}\delta}(V) \subseteq {Bnd_1}^{\mathcal{P}\delta}(V)$.
(2) ${Acc_1}^{\mathcal{P}\delta}(V) \leq {Acc_1}^{\mathcal{T}\delta}(V)$.
(3) ${Rough_1}^{\mathcal{T}\delta}(V) \leq {Rough_1}^{\mathcal{P}\delta}(V)$.

Proof.
(1) Let $\nu \in {Bnd_1}^{\mathcal{T}\delta}(V)$. Then, $\nu \in {U_1}^{\mathcal{T}\delta}(V) - {L_1}^{\mathcal{T}\delta}(V)$. So, $\nu \in {U_1}^{\mathcal{T}\delta}(V)$ and $\nu \in ({L_1}^{\mathcal{T}\delta}(V))^c$. Hence, $\nu \in {U_1}^{\mathcal{P}\delta}(V)$ and $\nu \in ({L_1}^{\mathcal{P}\delta}(V))^c$ according to (7) of Propositions 3.1 and 3.2. It follows that $\nu \in {Bnd_1}^{\mathcal{P}\delta}(V)$. Therefore, ${Bnd_1}^{\mathcal{T}\delta}(V) \subseteq {Bnd_1}^{\mathcal{P}\delta}(V)$.
(2) ${Acc_1}^{\mathcal{P}\delta}(V) = |\frac{{L_1}^{\mathcal{P}\delta}(V) \cap V}{{U_1}^{\mathcal{P}\delta}(V) \cup V}| \leq |\frac{{L_1}^{\mathcal{T}\delta}(V) \cap V}{{U_1}^{\mathcal{T}\delta}(V) \cup V}| = {Acc_1}^{\mathcal{T}\delta}(V)$.

(3) Straightforward by (2).

Remark 3.2: In Theorem 3.1 the converse of (1) and (2) is generally false. To validate this consider Example 3.1 (ii) and let $V = \{b, c\}$. Then,

(1) ${Bnd_1}^{\mathcal{T}\delta}(V) = \phi \subseteq \phi = {Bnd_1}^{\mathcal{P}\delta}(V)$, but $\mathcal{P} \nsubseteq \mathcal{T}$.
(2) ${Acc_1}^{\mathcal{P}\delta}(V) = 1 \leq 1 = {Acc_1}^{\mathcal{T}\delta}(V)$, but $\mathcal{P} \nsubseteq \mathcal{T}$.
(3) ${Rough_1}^{\mathcal{T}\delta}(V) = 0 \leq 0 = {Rough_1}^{\mathcal{P}\delta}(V)$, but $\mathcal{P} \nsubseteq \mathcal{T}$.

Theorem 3.2: Let $\phi \neq V \subseteq U$, $\mathcal{P}$ be an ideal on U and δ_1, δ_2 be two binary relations on U. If $\delta_1 \subseteq \delta_2$, then

(1) ${U_1}^{\mathcal{P}\delta_1}(V) \subseteq {U_1}^{\mathcal{P}\delta_2}(V)$.
(2) ${L_1}^{\mathcal{P}\delta_2}(V) \subseteq {L_1}^{\mathcal{P}\delta_1}(V)$.
(3) ${Bnd_1}^{\mathcal{P}\delta_1}(V) \subseteq {Bnd_1}^{\mathcal{P}\delta_2}(V)$.
(4) ${Acc_1}^{\mathcal{P}\delta_2}(V) \leq {Acc_1}^{\mathcal{P}\delta_1}(V)$.
(5) ${Rough_1}^{\mathcal{P}\delta_1}(V) \leq {Rough_1}^{\mathcal{P}\delta_2}(V)$.

Proof.

(1) Let $\nu \in {U_1}^{\mathcal{P}\delta_1}(V)$. Then, $\theta_{1l}(\nu) \cap V \notin \mathcal{P}$. Since $\theta_{1l}(\nu) \subseteq \theta_{2l}(\nu)$ (by Theorem 2.1 [4]). It follows that $\theta_{2l}(\nu) \cap V \notin \mathcal{P}$. Thus, $\nu \in {U_1}^{\mathcal{P}\delta_2}(V)$. Hence, ${U_1}^{\mathcal{P}\delta_1}(V) \subseteq {U_1}^{\mathcal{P}\delta_2}(V)$.
(2) Let $\nu \in {L_1}^{\mathcal{P}\delta_2}(V)$. Then, $\theta_{2l}(\nu) \cap V^c \in \mathcal{P}$. Since $\theta_{1l}(\nu) \subseteq \theta_{2l}(\nu)$ (by Theorem 2.1 [4]). It follows that $\theta_{1l}(\nu) \cap V^c \in \mathcal{P}$. Thus, $\nu \in {L_1}^{\mathcal{P}\delta_1}(V)$. Hence, ${L_1}^{\mathcal{P}\delta_2}(V) \subseteq {L_1}^{\mathcal{P}\delta_1}(V)$.
(3) Let $\nu \in {Bnd_1}^{\mathcal{P}\delta_1}(V)$. Then, $\nu \in {U_1}^{\mathcal{P}\delta_1}(V) - {L_1}^{\mathcal{P}\delta_1}(V)$. So, $\nu \in {U_1}^{\mathcal{P}\delta_1}(V)$ and $\nu \in ({L_1}^{\mathcal{P}\delta_1}(V))^c$. Thus, $\nu \in {U_1}^{\mathcal{P}\delta_2}(V)$ and $\nu \in ({L_1}^{\mathcal{P}\delta_2}(V))^c$ according to (1) and (2). Hence, $\nu \in {Bnd_1}^{\mathcal{P}\delta_2}(V)$. Therefore, ${Bnd_1}^{\mathcal{P}\delta_1}(V) \subseteq {Bnd_1}^{\mathcal{P}\delta_2}(V)$.
(4) ${Acc_1}^{\mathcal{P}\delta_2}(V) = |\frac{{L_1}^{\mathcal{P}\delta_2}(V) \cap V}{{U_1}^{\mathcal{P}\delta_2}(V) \cup V}| \leq |\frac{{L_1}^{\mathcal{P}\delta_1}(V) \cap V}{{U_1}^{\mathcal{P}\delta_1}(V) \cup V}| = {Acc_1}^{\mathcal{P}\delta_1}(V)$.
(5) Straightforward by (4).

To show that the inclusion and less than relation in Theorem 3.2 is proper, we provide the next example.

Example 3.3: Let

$$U = \{a, b, c, d\},$$

$$\mathcal{P} = \{\phi, \{b\}, \{c\}, \{d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{b, c, d\}\}$$

$$\delta_1 = \Delta \cup \{(a, b), (b, a)\} \text{ and } \delta_2 = \Delta \cup \{(a, b), (b, a), (c, a), (a, c)\}$$

be two relations defined on U; thus,

$$\theta_{1l}(a) = \theta_{1l}(b) = \{a, b\}, \theta_{1l}(c) = \{c\}, \theta_{1l}(d) = \{d\},$$

$$\theta_{2l}(a) = \theta_{2l}(b) = \theta_{2l}(c) = \{a, b, c\} \text{ and } \theta_{2l}(d) = \{d\}.$$

Take

(i) $V = \{a, d\}$; then,
  (1) ${U_1}^{\mathcal{P}\delta_1}(V) = \{a, b\} \neq \{a, b, c\} = {U_1}^{\mathcal{P}\delta_2}(V)$.
  (2) ${Acc_1}^{\mathcal{P}\delta_1}(V) = \frac{2}{3} \neq \frac{1}{2} = {Acc_1}^{\mathcal{P}\delta_2}(V)$.
  (3) ${Rough_1}^{\mathcal{P}\delta_1}(V) = \frac{1}{3} \neq \frac{1}{2} = {Rough_1}^{\mathcal{P}\delta_2}(V)$.
(ii) $V = \{b, c\}$; then, ${L_1}^{\mathcal{P}\delta_1}(V) = \{c, d\} \neq \{d\} = {L_1}^{\mathcal{P}\delta_2}(V)$.

B. SECOND TECHNIQUE TO GENERATE GENERALIZED ROUGH SETS VIA IDEALS

Definition 3.2: Let δ and $\mathcal{P}$ be binary relation and ideal on a set $U \neq \phi$. The second form of generalized approximations (lower and upper), boundary-regions, accuracy and rough values of a nonempty subset V of U produced by maximal union neighborhoods according to δ and $\mathcal{P}$ are respectively given by

$$ {L_2}^{\mathcal{P}\delta}(V) = \{\nu \in A : \theta_l(\nu) \cap V^c \in \mathcal{P}\}. \tag{39}$$

$$ {U_2}^{\mathcal{P}\delta}(V) = V \cup {U_1}^{\mathcal{P}\delta}(V). \tag{40}$$

$$ {Bnd_2}^{\mathcal{P}\delta}(V) = {U_2}^{\mathcal{P}\delta}(V) - {L_2}^{\mathcal{P}\delta}(V). \tag{41}$$

$$ {Acc_2}^{\mathcal{P}\delta}(V) = \frac{|{L_2}^{\mathcal{P}\delta}(V)|}{|{U_2}^{\mathcal{P}\delta}(V)|}, {U_2}^{\mathcal{P}\delta}(V) \neq \phi. \tag{42}$$

$$ {Rough_2}^{\mathcal{P}\delta}(V) = 1 - {Acc_2}^{\mathcal{P}\delta}(V). \tag{43}$$

Proposition 3.4: Let $\mathcal{P}$ and $\mathcal{T}$ be ideals and δ be a binary relation on U such that $V, W \subseteq U$ Then,

(1) $V \subseteq {U_2}^{\mathcal{P}\delta}(V)$ equality holds if $V = \phi$ or U.
(2) $V \subseteq W \Rightarrow {U_2}^{\mathcal{P}\delta}(V) \subseteq {U_2}^{\mathcal{P}\delta}(W)$.
(3) ${U_2}^{\mathcal{P}\delta}(V) \subseteq {U_2}^{\mathcal{P}\delta}({U_2}^{\mathcal{P}\delta}(V))$.
(4) ${U_2}^{\mathcal{P}\delta}(V \cap W) \subseteq {U_2}^{\mathcal{P}\delta}(V) \cap {U_2}^{\mathcal{P}\delta}(W)$.
(5) ${U_2}^{\mathcal{P}\delta}(V \cup W) = {U_2}^{\mathcal{P}\delta}(V) \cup {U_2}^{\mathcal{P}\delta}(W)$.
(6) ${U_2}^{\mathcal{P}\delta}(V) = ({L_2}^{\mathcal{P}\delta}(V^c))^c$.
(7) If $V \in \mathcal{P}$, then ${U_2}^{\mathcal{P}\delta}(V) = V$.
(8) If $\mathcal{P} \subseteq \mathcal{T}$, then ${U_2}^{\mathcal{T}\delta}(V) \subseteq {U_2}^{\mathcal{P}\delta}(V)$.
(9) If $\mathcal{P} = P(U)$, then ${U_2}^{\mathcal{P}\delta}(V) = V$.
(10) ${U_2}^{(\mathcal{P}\cap\mathcal{T})\delta}(V) = {U_2}^{\mathcal{P}\delta}(V) \cup {U_2}^{\mathcal{T}\delta}(V)$.
(11) ${U_2}^{(\mathcal{P}\vee\mathcal{T})\delta}(V) = {U_2}^{\mathcal{P}\delta}(V) \cap {U_2}^{\mathcal{T}\delta}(V)$.

Proof. Similar to Proposition 3.1.

Proposition 3.5: Let $\mathcal{P}$ and $\mathcal{T}$ be ideals and δ be a binary relation on U such that $V, W \subseteq U$ Then,

(1) ${L_2}^{\mathcal{P}\delta}(V) \subseteq V$ equality holds if $V = \phi$ or U.
(2) $V \subseteq W \Rightarrow {L_2}^{\mathcal{P}\delta}(V) \subseteq {L_2}^{\mathcal{P}\delta}(W)$.
(3) ${L_2}^{\mathcal{P}\delta}({L_2}^{\mathcal{P}\delta}(V)) \subseteq {L_2}^{\mathcal{P}\delta}(V)$.
(4) ${L_2}^{\mathcal{P}\delta}(V) \cup {L_2}^{\mathcal{P}\delta}(W) \subseteq {L_2}^{\mathcal{P}\delta}(V \cup W)$.
(5) ${L_2}^{\mathcal{P}\delta}(V \cap W) = {L_2}^{\mathcal{P}\delta}(V) \cap {L_2}^{\mathcal{P}\delta}(W)$.
(6) ${L_2}^{\mathcal{P}\delta}(V) = ({U_2}^{\mathcal{P}\delta}(V^c))^c$.
(7) If $V^c \in \mathcal{P}$, then ${L_2}^{\mathcal{P}\delta}(V) = V$.
(8) If $\mathcal{P} \subseteq \mathcal{T}$, then ${L_2}^{\mathcal{P}\delta}(V) \subseteq {L_2}^{\mathcal{T}\delta}(V)$.
(9) If $\mathcal{P} = P(U)$, then ${L_2}^{\mathcal{P}\delta}(V) = V$.
(10) ${L_2}^{(\mathcal{P}\cap\mathcal{T})\delta}(V) = {L_2}^{\mathcal{P}\delta}(V) \cap {L_2}^{\mathcal{T}\delta}(V)$.

Proof. Similar to Proposition 3.2.

Remark 3.3:

(i) It follows from Example 3.1 (i) that the converse of (2), (7) and (9) of Proposition 3.4 and Proposition 3.5 is generally false.

(a) For (2), take

(1) $V = \{a\}$ and $W = \{d\}$; then, ${U_2}^{\mathcal{P}\delta}(V) = \{a\} \subseteq \{a,c,d\} = {U_2}^{\mathcal{P}\delta}(W)$, but $V \nsubseteq W$.

(2) $V = \{a,b,c\}$ and $W = \{b,c,d\}$; then, ${L_2}^{\mathcal{P}\delta}(V) = \{b\} \subseteq \{b,c,d\} = {L_2}^{\mathcal{P}\delta}(W)$, but $V \nsubseteq W$.

(b) For (7), take

(1) $V = \{a,c,d\}$; then, ${U_2}^{\mathcal{P}\delta}(V) = V$, but $V \notin \mathcal{P}$.

(2) $V = \{b\}$; then, ${L_2}^{\mathcal{P}\delta}(V) = V$, but $V^c \notin \mathcal{P}$.

(c) For (9), take

(1) $V = \{a,c,d\}$; then, ${U_2}^{\mathcal{P}\delta}(V) = V$, but $\mathcal{P} \neq P(U)$.

(2) $V = \{b\}$; then, ${L_2}^{\mathcal{P}\delta}(V) = V$, but $\mathcal{P} \neq P(U)$.

(ii) It can be seen from Example 3.1 (ii) that the converse of (8) of Proposition 3.4 and Proposition 3.5 is generally false. To show that, let

(1) $V = \{a,d\}$. Then, ${U_2}^{\mathcal{T}\delta}(V) = V \subseteq V = {U_2}^{\mathcal{P}\delta}(V)$, but $\mathcal{P} \nsubseteq \mathcal{T}$.

(2) $V = \{b,c\}$; then, ${L_2}^{\mathcal{P}\delta}(V) = V \subseteq V = {L_2}^{\mathcal{T}\delta}(V)$, but $\mathcal{P} \nsubseteq \mathcal{T}$.

(iii) Example 3.1 (iii) illustrates that the inclusion relations of (3) and (4) of Proposition 3.4 and Proposition 3.5 are proper.

(a) For (3), take

(1) $V = \{b,c\}$; then, ${U_2}^{\mathcal{P}\delta}(V) = \{a,b,c\}$ and ${U_2}^{\mathcal{P}\delta}({U_2}^{\mathcal{P}\delta}(V)) = U$. Therefore, ${U_2}^{\mathcal{P}\delta}(V) = \{a,b,c\} \neq U = {U_2}^{\mathcal{P}\delta}({U_2}^{\mathcal{P}\delta}(V))$.

(2) $V = \{a,d\}$; then, ${L_2}^{\mathcal{P}\delta}(V) = \{d\}$ and ${L_2}^{\mathcal{P}\delta}({L_2}^{\mathcal{P}\delta}(V)) = \phi$. Therefore, ${L_2}^{\mathcal{P}\delta}(V) = \{d\} \neq \phi = {L_2}^{\mathcal{P}\delta}({L_2}^{\mathcal{P}\delta}(V))$.

(b) For (4), take $V = \{a,d\}, B = \{b,c\}$ and

(1) $V \cap W = \phi$. Hence, ${U_2}^{\mathcal{P}\delta}(V) = U$ and ${U_2}^{\mathcal{P}\delta}(W) = \{a,b,c\}$. Therefore, ${U_2}^{\mathcal{P}\delta}(V) \cap {U_2}^{\mathcal{P}\delta}(W) = \{a,b,c\} \neq \phi = {U_2}^{\mathcal{P}\delta}(V \cap W)$.

(2) $V \cup W = U$. Hence, ${L_2}^{\mathcal{P}\delta}(V) = \{d\}$ and ${L_2}^{\mathcal{P}\delta}(W) = \phi$. Therefore, ${L_2}^{\mathcal{P}\delta}(V) \cup {L_2}^{\mathcal{P}\delta}(W) = \{d\} \neq U = {L_2}^{\mathcal{P}\delta}(V \cup W)$.

Remark 3.4: Some properties given in the first type are not hold by this type as we show in the following.

(i) Considering Example 3.1 (i), take

(1) $V = \{a\} \in \mathcal{P}$; then, ${U_2}^{\mathcal{P}\delta}(V) = V$. Hence, if $V \in \mathcal{P} \nRightarrow {U_2}^{\mathcal{P}\delta}(V) = \phi$.

(2) $V^c = \{a\} \in \mathcal{P}$; then, ${L_2}^{\mathcal{P}\delta}(V) = V$. Hence, if $V^c \in \mathcal{P} \nRightarrow {L_2}^{\mathcal{P}\delta}(V) = U$.

(ii) Considering Example 3.1 (ii), take

(1) $\mathcal{T} = P(U)$ and $V = \{a,d\}$; then, ${U_2}^{\mathcal{T}\delta}(V) = V$. Hence, if $\mathcal{T} = P(U) \nRightarrow {U_2}^{\mathcal{T}\delta}(V) = \phi$.

(2) $\mathcal{T} = P(U)$ and $V = \{b,c\}$; then, ${L_2}^{\mathcal{T}\delta}(V) = V$. Hence, if $\mathcal{T} = P(U) \nRightarrow {L_2}^{\mathcal{T}\delta}(V) = U$.

Remark 3.5: Some properties of Pawlak are not satisfy by this type as we show in the following. In Example 3.2, take

(1) $V = \{c\}$; then, ${U_2}^{\mathcal{P}\delta}(V) = \{a,b,c\}$ and ${L_2}^{\mathcal{P}\delta}({U_2}^{\mathcal{P}\delta}(V)) = \{c\}$. Therefore, ${U_2}^{\mathcal{P}\delta}(V) = \{a,b,c\} \nsubseteq \{c\} = {L_2}^{\mathcal{P}\delta}({U_2}^{\mathcal{P}\delta}(V))$.

(2) $V = \{a,b,d\}$; then, ${L_2}^{\mathcal{P}\delta}(V) = \{d\}$ and ${U_2}^{\mathcal{P}\delta}({L_2}^{\mathcal{P}\delta}(V)) = \{a,b,d\}$. Therefore, ${U_2}^{\mathcal{P}\delta}({L_2}^{\mathcal{P}\delta}(V)) = \{a,b,d\} \nsubseteq \{d\} = {L_2}^{\mathcal{P}\delta}(V)$.

Proposition 3.6: Let $\mathcal{P}$ be an ideal and δ be a binary relation on U such that V is a nonempty subset of U. Then,

1) $0 \leq Acc_2^{\mathcal{P}\delta}(V) \leq 1$.
2) $Acc_2^{\mathcal{P}\delta}(U) = 1$.

Proof. It is similar to Proposition 3.3.

Theorem 3.3: Let $\mathcal{P}$ and $\mathcal{T}$ be ideals and δ be a binary relation on U such that $\mathcal{P} \subseteq \mathcal{T}$. Then,

(1) ${Bnd_2}^{\mathcal{T}\delta}(V) \subseteq {Bnd_2}^{\mathcal{P}\delta}(V)$.
(2) ${Acc_2}^{\mathcal{P}\delta}(V) \leq {Acc_2}^{\mathcal{T}\delta}(V)$.
(3) ${Rough_2}^{\mathcal{T}\delta}(V) \leq {Rough_2}^{\mathcal{P}\delta}(V)$.

Proof. Similar to the proof of Theorem 3.1.

Remark 3.6: In Theorem 3.3 the converse of (1) and (2) is generally false as illustrated in (ii) of Example 3.1. To show that let $V = \{b,c\}$. Then,

(1) ${Bnd_2}^{\mathcal{T}\delta}(V) = \phi \subseteq \phi = {Bnd_2}^{\mathcal{P}\delta}(V)$, but $\mathcal{P} \nsubseteq \mathcal{T}$.
(2) ${Acc_2}^{\mathcal{P}\delta}(V) = 1 \leq 1 = {Acc_2}^{\mathcal{T}\delta}(V)$, but $\mathcal{P} \nsubseteq \mathcal{T}$.
(3) ${Rough_2}^{\mathcal{T}\delta}(V) = 0 \leq 0 = {Rough_2}^{\mathcal{P}\delta}(V)$, but $\mathcal{P} \nsubseteq \mathcal{T}$.

Theorem 3.4: Let $\phi \neq V \subseteq U$, $\mathcal{P}$ be an ideal on U and δ_1, δ_2 be two binary relations on U. If $\delta_1 \subseteq \delta_2$, then

(1) ${U_2}^{\mathcal{P}\delta_1}(V) \subseteq {U_2}^{\mathcal{P}\delta_2}(V)$.

(2) ${L_2}^{\mathcal{P}\delta_2}(V) \subseteq {L_2}^{\mathcal{P}\delta_1}(V)$.

(3) ${Bnd_2}^{\mathcal{P}\delta_1}(V) \subseteq {Bnd_2}^{\mathcal{P}\delta_2}(V)$.

(4) ${Acc_2}^{\mathcal{P}\delta_2}(V) \leq {Acc_2}^{\mathcal{P}\delta_1}(V)$.

(5) ${Rough_2}^{\mathcal{P}\delta_1}(V) \leq {Rough_2}^{\mathcal{P}\delta_2}(V)$.

Proof. Similar to Theorem 3.2.

Remark 3.7: In Theorem 3.4 the inclusion and less than relation is proper as showed in Example 3.3. To validate that let $V = \{a,d\}$. Then,

(1) ${U_2}^{\mathcal{P}\delta_1}(V) = \{a,b,d\} \neq U = {U_2}^{\mathcal{P}\delta_2}(V)$.

(2) ${Bnd_2}^{\mathcal{P}\delta_1}(V) = \{b\} \neq \{b,c\} = {Bnd_2}^{\mathcal{P}\delta_2}(V)$.

(3) ${Acc_2}^{\mathcal{P}\delta_1}(V) = \frac{2}{3} \neq \frac{1}{2} = {Acc_2}^{\mathcal{P}\delta_2}(V)$.

(4) ${Rough_2}^{\mathcal{P}\delta_1}(V) = 0.3 \neq 0.5 = {Rough_2}^{\mathcal{P}\delta_2}(V)$.

C. THIRD TECHNIQUE TO GENERATE GENERALIZED ROUGH SETS VIA IDEALS

Definition 3.3: Let δ and $\mathcal{P}$ be binary relation and ideal on a set $U \neq \phi$. The third form of generalized approximations (lower and upper), boundary-regions, accuracy and rough values of a nonempty subset V of U produced by maximal union neighborhoods according to δ and $\mathcal{P}$ are respectively given by

$$ {L_3}^{\mathcal{P}\delta}(V) = \bigcup_{\nu\in U}\{\theta_l(\nu) : \theta_l(\nu)\cap V^c \in \mathcal{P}\}. \tag{44}$$

$$ {U_3}^{\mathcal{P}\delta}(V) = ({L_3}^{\mathcal{P}\delta}(V^c))^c. \tag{45}$$

$$ {Bnd_3}^{\mathcal{P}\delta}(V) = {U_3}^{\mathcal{P}\delta}(V) - {L_3}^{\mathcal{P}\delta}(V). \tag{46}$$

$$ {Acc_3}^{\mathcal{P}\delta}(V) = \frac{|{L_3}^{\mathcal{P}\delta}(V)\cap V|}{|{U_3}^{\mathcal{P}\delta}(V)\cup V}|. \tag{47}$$

$$ {Rough_3}^{\mathcal{P}\delta}(V) = 1 - {Acc_3}^{\mathcal{P}\delta}(V). \tag{48}$$

Proposition 3.7: Let $\mathcal{P}$ and $\mathcal{T}$ be ideals and δ be a binary relation on U such that $V, W \subseteq U$ Then,

(1) $V \subseteq W \Rightarrow {L_3}^{\mathcal{P}\delta}(V) \subseteq {L_3}^{\mathcal{P}\delta}(W)$.

(2) ${L_3}^{\mathcal{P}\delta}(V) \cup {L_3}^{\mathcal{P}\delta}(W) \subseteq {L_3}^{\mathcal{P}\delta}(V \cup W)$.

(3) ${L_3}^{\mathcal{P}\delta}(V \cap W) \subseteq {L_3}^{\mathcal{P}\delta}(V) \cap {L_3}^{\mathcal{P}\delta}(W)$.

(4) ${L_3}^{\mathcal{P}\delta}(V) = ({U_3}^{\mathcal{P}\delta}(V^c))^c$.

(5) If $\mathcal{P} \subseteq \mathcal{T}$, then ${L_3}^{\mathcal{P}\delta}(V) \subseteq {L_3}^{\mathcal{T}\delta}(V)$.

(6) ${L_3}^{(\mathcal{P}\cap\mathcal{T})\delta}(V) = {L_3}^{\mathcal{P}\delta}(V) \cap {L_3}^{\mathcal{T}\delta}(V)$.

Proof.

(1) Let $V \subseteq W$ and $\nu \in {L_3}^{\mathcal{P}\delta}(V)$. Then, $\exists\ y \in U$ such that $\nu \in \theta_l(y) \cap V^c \in \mathcal{P}$. Hence, $\nu \in \theta_l(y) \cap W^c \in \mathcal{P}$ (by $W^c \subseteq V^c$, and the properties of an ideal). Thus, $\nu \in {L_3}^{\mathcal{P}\delta}(W)$. Therefore, ${L_3}^{\mathcal{P}\delta}(V) \subseteq {L_3}^{\mathcal{P}\delta}(W)$.

(2) It is directly obtained by (1).

(3) It is directly obtained by (1).

(4) It immediately follows from Definition 3.3.

(5) Let $\mathcal{P} \subseteq \mathcal{T}$ and $\nu \in {L_3}^{\mathcal{P}\delta}(V)$. Then, $\exists\ y \in U$ such that $\nu \in \theta_l(y) \cap V^c \in \mathcal{P} \subseteq \mathcal{T}$. So, $\nu \in {L_3}^{\mathcal{T}\delta}(V)$, and hence ${L_3}^{\mathcal{P}\delta}(V) \subseteq {L_3}^{\mathcal{T}\delta}(V)$.

(6) ${L_3}^{(\mathcal{P}\cap\mathcal{T})\delta}(V) = \bigcup_{\nu\in U}\{\theta_l(\nu) : \theta_l(\nu)\cap V^c \in \mathcal{P}\cap\mathcal{T}\}$
$= (\bigcup_{\nu\in U}\{\theta_l(\nu) : \theta_l(\nu)\cap V^c \in \mathcal{P}\})$ and
$(\bigcup_{\nu\in U}\{\theta_l(\nu) : \theta_l(\nu)\cap V^c \in \mathcal{T}\})$
$= (\bigcup_{\nu\in U}\{\theta_l(\nu) : \theta_l(\nu)\cap V^c \in \mathcal{P}\}) \cap (\bigcup_{\nu\in U}\{\theta_l(\nu) : \theta_l(\nu)\cap V^c \in \mathcal{T}\})$
$= {L_3}^{\mathcal{P}\delta}(V) \cap {L_3}^{\mathcal{T}\delta}(V)$.

Proposition 3.8: Let $\mathcal{P}$ and $\mathcal{T}$ be ideals and δ be a binary relation on U such that $V, W \subseteq U$ Then,

(1) $V \subseteq W \Rightarrow {U_3}^{\mathcal{P}\delta}(V) \subseteq {U_3}^{\mathcal{P}\delta}(W)$.

(2) ${U_3}^{\mathcal{P}\delta}(V \cap W) \subseteq {U_3}^{\mathcal{P}\delta}(V) \cap {U_3}^{\mathcal{P}\delta}(W)$.

(3) ${U_3}^{\mathcal{P}\delta}(V) \cup {U_3}^{\mathcal{P}\delta}(W) \subseteq {U_3}^{\mathcal{P}\delta}(V \cup W)$.

(4) ${U_3}^{\mathcal{P}\delta}(V) = ({L_3}^{\mathcal{P}\delta}(V^c))^c$.

(5) If $\mathcal{P} \subseteq \mathcal{T}$, then ${U_3}^{\mathcal{T}\delta}(V) \subseteq {U_3}^{\mathcal{P}\delta}(V)$.

(6) ${U_3}^{(\mathcal{P}\cap\mathcal{T})\delta}(V) = {U_3}^{\mathcal{P}\delta}(V) \cup {U_3}^{\mathcal{T}\delta}(V)$.

Proof.

(1) Let $V \subseteq W$. Thus, $W^c \subseteq V^c$, and ${L_3}^{\mathcal{P}\delta}(W^c) \subseteq {L_3}^{\mathcal{P}\delta}(V^c)$ (by (1) in Proposition 3.7). So, $({L_3}^{\mathcal{P}\delta}(V^c))^c \subseteq ({L_3}^{\mathcal{P}\delta}(W^c))^c$. Consequently, ${U_3}^{\mathcal{P}\delta}(V) \subseteq {U_3}^{\mathcal{P}\delta}(W)$.

(2) The proof directly follows by (1).

(3) The proof directly follows by (1).

(4) The proof is straightforward by Definition 3.3.

(5) Let $\mathcal{P} \subseteq \mathcal{T}$ and $\nu \in {U_3}^{\mathcal{T}\delta}(V)$. Then, $\nu \in ({L_3}^{\mathcal{T}\delta}(V^c))^c \subseteq ({L_3}^{\mathcal{P}\delta}(V^c))^c$, (by (5) in Proposition 3.7). Thus, $\nu \in ({L_3}^{\mathcal{P}\delta}(V^c))^c = {U_3}^{\mathcal{P}\delta}(V)$. Therefore, ${U_3}^{\mathcal{T}\delta}(V) \subseteq {U_3}^{\mathcal{P}\delta}(V)$.

(6) ${U_3}^{(\mathcal{P}\cap\mathcal{T})\delta}(V) = ({L_3}^{(\mathcal{P}\cap\mathcal{T})\delta}(V^c))^c$
$= ({L_3}^{\mathcal{P}\delta}(V^c) \cap {L_3}^{\mathcal{T}\delta}(V^c))^c$ (by (6) in Proposition 3.7)
$= ({L_3}^{\mathcal{P}\delta}(V^c))^c \cup ({L_3}^{\mathcal{T}\delta}(V^c))^c$
$= {U_3}^{\mathcal{P}\delta}(V) \cup {U_3}^{\mathcal{T}\delta}(V)$.

Remark 3.8:

(1) In Proposition 3.7 and Proposition 3.8 the converse of (1) is generally false. To elucidate that consider Example 3.1 (i) and let
   (a) $V = \{a\}$ and $W = \{d\}$; then, ${U_3}^{\mathcal{P}\delta}(V) = \phi$ and ${U_3}^{\mathcal{P}\delta}(W) = \{a,d\}$. Therefore, ${U_3}^{\mathcal{P}\delta}(V) \subseteq {U_3}^{\mathcal{P}\delta}(W)$, but $V \nsubseteq W$.
   (b) $V = \{b\}$ and $W = \{a,c,d\}$; then, ${L_3}^{\mathcal{P}\delta}(V) = \{b,c\}$ and ${L_3}^{\mathcal{P}\delta}(W) = U$. Therefore, ${L_3}^{\mathcal{P}\delta}(V) \subseteq {L_3}^{\mathcal{P}\delta}(W)$, but $V \nsubseteq W$.

(2) In Proposition 3.7 and Proposition 3.8 the inclusion relation of (2) is proper as (iii) of Example 3.1 shows. For this, let $V = \{a,d\}$ and $W = \{b,c\}$. Then
   (a) ${U_3}^{\mathcal{P}\delta}(V) = U, {U_3}^{\mathcal{P}\delta}(W) = W$ and ${U_3}^{\mathcal{P}\delta}(V \cap W) = \phi$. Therefore, ${U_3}^{\mathcal{P}\delta}(V) \cap {U_3}^{\mathcal{P}\delta}(W) = W \neq \phi = {U_3}^{\mathcal{P}\delta}(V \cap W)$.
   (b) ${L_3}^{\mathcal{P}\delta}(V) = V, {L_3}^{\mathcal{P}\delta}(W) = \phi$ and ${L_3}^{\mathcal{P}\delta}(V \cup W) = U$. Therefore, ${L_3}^{\mathcal{P}\delta}(V) \cup {L_3}^{\mathcal{P}\delta}(W) = V \neq U = {L_3}^{\mathcal{P}\delta}(V \cup W)$.

(3) *Example 3.4:* Let $U = \{a,b,c,d\}, \mathcal{P} = \{\phi, \{a\}\}$ and $\delta = \{(a,a),(a,b),(a,d),(b,b),(b,d),(c,a),(c,b),(d,a)\}$. Then $\theta_l(a) = \theta_l(c) = U, \theta_l(b) = \{a,b,c\}$ and $\theta_l(d) = \{a,c,d\}$. To show that the inclusion relations of (3) of Proposition 3.7 and Proposition 3.8 are proper, take
   (a) $V = \{a,c,d\}, W = \{a,b,c\}$ and $V \cap W = \{a,c\}$; then, ${L_3}^{\mathcal{P}\delta}(V) = V, {L_3}^{\mathcal{P}\delta}(W) = W$ and ${L_3}^{\mathcal{P}\delta}(V \cap W) = \phi$. Therefore, ${L_3}^{\mathcal{P}\delta}(V) \cap {L_3}^{\mathcal{P}\delta}(W) = \{a,c\} \neq \phi = {L_3}^{\mathcal{P}\delta}(V \cap W)$.

(b) $V = \{b\}, W = \{d\}$ and $V \cup W = \{b,d\}$; then, ${U_3}^{\mathcal{P}\delta}(V) = V, {U_3}^{\mathcal{P}\delta}(W) = W$ and ${U_3}^{\mathcal{P}\delta}(V \cup W) = U$. Therefore, ${U_3}^{\mathcal{P}\delta}(V) \cup {U_3}^{\mathcal{P}\delta}(W) = \{b,d\} \neq U = {U_3}^{\mathcal{P}\delta}(V \cup W)$.

(4) The converse of (5) in Proposition 3.7 and Proposition 3.8 is generally false. To elucidate this consider (ii) of Example 3.1 and let

(a) $V = \{a,d\}$; then, ${U_3}^{\mathcal{P}\delta}(V) = \{a,d\}$ and ${U_3}^{\mathcal{T}\delta}(V) = \{d\}$. Therefore, ${U_3}^{\mathcal{T}\delta}(V) \subseteq {U_3}^{\mathcal{P}\delta}(V)$, but $\mathcal{P} \nsubseteq \mathcal{T}$.

(b) $V = \{b,c\}$; then, ${L_3}^{\mathcal{P}\delta}(V) = \{b,c\}$ and ${L_3}^{\mathcal{T}\delta}(V) = \{a,b,c\}$. Therefore, ${L_3}^{\mathcal{P}\delta}(V) \subseteq {L_3}^{\mathcal{T}\delta}(V)$, but $\mathcal{P} \nsubseteq \mathcal{T}$.

Remark 3.9: Some properties in the second type are not satisfy by this type as we show in the following.

(i) Considering Example 3.1 (i), take

(1) $V = \{a\}$; then, ${U_3}^{\mathcal{P}\delta}(V) = \phi$. Hence, $V \nsubseteq {U_3}^{\mathcal{P}\delta}(V)$.

(2) $V = \{b,c,d\}$; then, ${L_3}^{\mathcal{P}\delta}(V) = U$. Hence, ${L_3}^{\mathcal{P}\delta}(V) \nsubseteq V$.

(3) $V = U$; then, ${U_3}^{\mathcal{P}\delta}(U) = \{a,d\}$. Hence, ${U_3}^{\mathcal{P}\delta}(U) \neq U$.

(4) $V = \phi$; then, ${L_3}^{\mathcal{P}\delta}(\phi) = \{b,c\}$. Hence, ${L_3}^{\mathcal{P}\delta}(\phi) \neq \phi$.

(ii) *Example 3.5:* Let $U = \{a,b,c,d\}, \mathcal{P} = \{\phi,\{a\}\}$ and $\delta = \{(a,a)\}$ be a binary relation defined on U; thus, $\theta_l(a) = \{a\}$ and $\theta_l(b) = \theta_l(c) = \theta_l(d) = \phi$. Take

(1) $V = U$; then, ${L_3}^{\mathcal{P}\delta}(U) = \{a\}$. Hence, ${L_3}^{\mathcal{P}\delta}(U) \neq U$.

(2) $V = \phi$; then, ${U_3}^{\mathcal{P}\delta}(\phi) = \{b,c,d\}$. Hence, ${U_3}^{\mathcal{P}\delta}(\phi) \neq \phi$.

Remark 3.10: Some properties given in the firrst/second type are not satisfy by this type as we show in the following. In Example 3.5, take

(1) $V = \{b,c,d\}$; then, $V^c \in \mathcal{P}$ and ${L_3}^{\mathcal{P}\delta}(V) = \{a\}$. Hence, if $V^c \in \mathcal{P} \nRightarrow {L_3}^{\mathcal{P}\delta}(V) = U$ or V.

(2) $V = \{a\} \in \mathcal{P}$; then, ${U_3}^{\mathcal{P}\delta}(V) = \{b,c,d\}$. Hence, if $A \in \mathcal{P} \nRightarrow {U_3}^{\mathcal{P}\delta}(V) = \phi$ or V.

(3) $V = \{b,c,d\}$ and $\mathcal{P} = P(U)$; then, ${L_3}^{\mathcal{P}\delta}(V) = \{a\}$. Hence, if $\mathcal{P} = P(U) \nRightarrow {L_3}^{\mathcal{P}\delta}(V) = U$, or V.

(4) $V = \{a\}$ and $\mathcal{P} = P(U)$; then, ${U_3}^{\mathcal{P}\delta}(V) = \{b,c,d\}$. Hence, if $\mathcal{P} = P(U) \nRightarrow {U_3}^{\mathcal{P}\delta}(V) = \phi$, or V.

Proposition 3.9: Let $\mathcal{P}$ be an ideal and δ be a binary relation on U such that V is a nonempty subset of U. Then,

1) $0 \leq Acc_3^{\mathcal{P}\delta}(V) \leq 1$.
2) $Acc_3^{\mathcal{P}\delta}(U) = 1$.

Proof. It is similar to Proposition 3.3.

Theorem 3.5: Let $\mathcal{P}$ and $\mathcal{T}$ be ideals and δ be a binary relation on U such that $\mathcal{P} \subseteq \mathcal{T}$. Then,

(1) ${Bnd_3}^{\mathcal{T}\delta}(V) \subseteq {Bnd_3}^{\mathcal{P}\delta}(V)$.
(2) ${Acc_3}^{\mathcal{P}\delta}(V) \leq {Acc_3}^{\mathcal{T}\delta}(V)$.
(3) ${Rough_3}^{\mathcal{T}\delta}(V) \leq {Rough_3}^{\mathcal{P}\delta}(V)$.

Proof. Similar to Theorem 3.1.

Remark 3.11: It follows from (ii) of Example 3.1 that the converse of (1) and (2) of Theorem 3.5 is generally false. To demonstrate that let $V = \{b,c\}$. Then,

(1) ${Bnd_3}^{\mathcal{T}\delta}(V) = \{d\} \subseteq \{d\} = {Bnd_3}^{\mathcal{P}\delta}(V)$, but $\mathcal{P} \nsubseteq \mathcal{T}$.

(2) ${Acc_3}^{\mathcal{P}\delta}(V) = \frac{2}{3} \leq \frac{2}{3} = {Acc_3}^{\mathcal{T}\delta}(V)$, but $\mathcal{P} \nsubseteq \mathcal{T}$.

(3) ${Rough_3}^{\mathcal{T}\delta}(V) = \frac{1}{3} \leq \frac{1}{3} = {Rough_3}^{\mathcal{P}\delta}(V)$, but $\mathcal{P} \nsubseteq \mathcal{T}$.

The approximations operators, boundary-regions, measures of accuracy and roughness induced from the this type do not have monotonicity. The next example illustrates this fact.

Example 3.6: Let $U = \{a,b,c,d,e,f,g\}, \mathcal{P} = \{\phi,\{a\}\}$ and δ_1, δ_2 be two relations on U where $\delta_1 = \Delta \cup \{(a,c),(c,a),(c,g),(d,f),(e,g),(f,d),(g,c),(g,e)\}$. and $\delta_2 = \Delta \cup \{(a,c),(a,d),(a,e),(b,f),(c,a),(c,g),(d,a),(d,f),(e,a),(e,g),(f,b),(f,d),(g,c),(g,e)\}$. Thus, $\theta_{1l}(a) = \{a,c,g\}, \theta_{1l}(b) = \{b\}, \theta_{1l}(c) = \theta_{1l}(g) = \{a,c,e,g\}, \theta_{1l}(d) = \theta_{1l}(f) = \{d,f\}, \theta_{1l}(e) = \{c,e,g\}, \theta_{2l}(a) = \{a,c,d,e,f,g\}, \theta_{2l}(b) = \{b,d,f\}, \theta_{2l}(c) = \theta_{2l}(e) = \{a,c,d,e,g\}, \theta_{2l}(d) = \{a,b,c,d,e,f\}, \theta_{2l}(f) = \{a,b,d,f\}$ and $\theta_{2l}(g) = \{a,c,e,g\}$. Take

(1) $V = \{a,b,c,d,e,f\}$; then, ${L_3}^{\mathcal{P}\delta_1}(V) = \{b,d,f\}$ and ${L_3}^{\mathcal{P}\delta_2}(V) = V$. Therefore, ${L_3}^{\mathcal{P}\delta_1}(V) \nsupseteq {L_3}^{\mathcal{P}\delta_2}(V)$.
(2) $V = \{g\}$; then, ${U_3}^{\mathcal{P}\delta_1}(V) = \{a,c,e,g\}$ and ${U_3}^{\mathcal{P}\delta_2}(V) = \{g\}$. Therefore, ${U_3}^{\mathcal{P}\delta_1}(V) \nsubseteq {U_3}^{\mathcal{P}\delta_2}(V)$.
(3) $V = \{a,b,c,d,e,f\}$; then, ${L_3}^{\mathcal{P}\delta_1}(V) = \{b,d,f\}, {U_3}^{\mathcal{P}\delta_1}(V) = U, {L_3}^{\mathcal{P}\delta_2}(V) = V$ and ${U_3}^{\mathcal{P}\delta_2}(V) = U$. Therefore,

(a) ${Bnd_3}^{\mathcal{P}\delta_1}(V) = \{a,c,e,g\} \nsubseteq \{g\} = {Bnd_3}^{\mathcal{P}\delta_2}(V)$.
(b) ${Acc_3}^{\mathcal{P}\delta_1}(V) = \frac{3}{7} < \frac{6}{7} = {Acc_3}^{\mathcal{P}\delta_2}(V)$.
(c) ${Rough_3}^{\mathcal{P}\delta_1}(V) = \frac{4}{7} > \frac{1}{7} = {Rough_3}^{\mathcal{P}\delta_2}(V)$.

Although, $\delta_1 \subseteq \delta_2$.

D. FOURTH TECHNIQUE TO GENERATE GENERALIZED ROUGH SETS VIA IDEALS

Definition 3.4: Let δ and $\mathcal{P}$ be binary relation and ideal on a set $U \neq \phi$. The fourth form of generalized approximations (lower and upper), boundary-regions, accuracy and rough values of a nonempty subset V of U produced by maximal union neighborhoods according to δ and $\mathcal{P}$ are respectively given by

$$U_4{}^{\mathcal{P}\delta}(V) = \underset{\nu\in U}{\cup}\{\theta_l(\nu) : \theta_l(\nu)\cap V \notin \mathcal{P}\}. \quad (49)$$

$$L_4{}^{\mathcal{P}\delta}(V) = (U_4{}^{\mathcal{P}\delta}(V^c))^c. \quad (50)$$

$$Bnd_4{}^{\mathcal{P}\delta}(V) = U_4{}^{\mathcal{P}\delta}(V) - L_4{}^{\mathcal{P}\delta}(V). \quad (51)$$

$$Acc_4{}^{\mathcal{P}\delta}(V) = \frac{|L_4{}^{\mathcal{P}\delta}(V)\cap V|}{|U_4{}^{\mathcal{P}\delta}(V)\cup V|}. \quad (52)$$

$$Rough_4{}^{\mathcal{P}\delta}(V) = 1 - Acc_4{}^{\mathcal{P}\delta}(V). \quad (53)$$

Proposition 3.10: Let $\mathcal{P}$ and $\mathcal{T}$ be ideals and δ be a binary relation on U such that $V, W \subseteq U$ Then,

(1) $U_4{}^{\mathcal{P}\delta}(\phi) = \phi$.
(2) $V \subseteq W \Rightarrow U_4{}^{\mathcal{P}\delta}(V) \subseteq U_4{}^{\mathcal{P}\delta}(W)$.
(3) $U_4{}^{\mathcal{P}\delta}(V\cap W) \subseteq U_4{}^{\mathcal{P}\delta}(V)\cap U_4{}^{\mathcal{P}\delta}(W)$.
(4) $U_4{}^{\mathcal{P}\delta}(V\cup W) = U_4{}^{\mathcal{P}\delta}(V)\cup U_4{}^{\mathcal{P}\delta}(W)$.
(5) $U_4{}^{\mathcal{P}\delta}(V) = (L_4{}^{\mathcal{P}\delta}(V^c))^c$.
(6) If $V \in \mathcal{P}$, then $U_4{}^{\mathcal{P}\delta}(V) = \phi$.
(7) If $\mathcal{P} \subseteq \mathcal{T}$, then $U_4{}^{\mathcal{T}\delta}(V) \subseteq U_4{}^{\mathcal{P}\delta}(V)$.
(8) If $\mathcal{P} = P(U)$, then $U_4{}^{\mathcal{P}\delta}(V) = \phi$.
(9) $U_4{}^{(\mathcal{P}\cap\mathcal{T})\delta}(V) = U_4{}^{\mathcal{P}\delta}(V)\cup U_4{}^{\mathcal{T}\delta}(V)$.
(10) $U_4{}^{(\mathcal{P}\vee\mathcal{T})\delta}(V) = U_4{}^{\mathcal{P}\delta}(V)\cap U_4{}^{\mathcal{T}\delta}(V)$.

Proof.

(1) $U_4{}^{\mathcal{P}\delta}(\phi) = \cup\{\theta_l(\nu) : \theta_l(\nu)\cap\phi \notin \mathcal{P}\} = \phi$.
(2) Let $V \subseteq W$ and $\nu \in U_4{}^{\mathcal{P}\delta}(V)$. Then, $\exists\ y \in U$ such that $\nu \in \theta_l(y)$ and $\theta_l(y)\cap V \notin \mathcal{P}$. Thus, $\theta_l(y)\cap W \notin \mathcal{P}$. So, $\nu \in U_4{}^{\mathcal{P}\delta}(W)$. Consequently, $U_4{}^{\mathcal{P}\delta}(V) \subseteq U_4{}^{\mathcal{P}\delta}(W)$.
(3) It is immediately obtained by (2).
(4) $U_4{}^{\mathcal{P}\delta}(V\cup W) = \underset{\nu\in U}{\cup}\{\theta_l(\nu) : \theta_l(\nu)\cap(V\cup W) \notin \mathcal{P}\}$.
$= (\underset{\nu\in U}{\cup}\{\theta_l(\nu) : \theta_l(\nu)\cap V \notin \mathcal{P}\}) \cup (\underset{\nu\in U}{\cup}\{\theta_l(\nu) : \theta_l(\nu)\cap W \notin \mathcal{P}\})$.
$= (\underset{\nu\in U}{\cup}\{\theta_l(\nu) : \theta_l(\nu)\cap V \notin \mathcal{P}\})\ or\ (\underset{\nu\in U}{\cup}\{\theta_l(\nu) : \theta_l(\nu)\cap W \notin \mathcal{P}\})$.
$= U_4{}^{\mathcal{P}\delta}(V)\cup U_4{}^{\mathcal{P}\delta}(W)$.
(5)
$$(L_4{}^{\mathcal{P}\delta}(V^c))^c = ((U_4{}^{\mathcal{P}\delta}(V))^c)^c. \\ = U_4{}^{\mathcal{P}\delta}(V).$$
(6) The proof is straightforward by Definition 3.4.
(7) Let $\mathcal{P} \subseteq \mathcal{T}, \nu \in U_4{}^{\mathcal{T}\delta}(V)$. Then, $\exists\ y \in U$ such that $\nu \in \theta_l(y)$ and $\theta_l(y)\cap V \notin \mathcal{T}$. Thus, $\theta_l(y)\cap V \notin \mathcal{P}$ as $\mathcal{P} \subseteq \mathcal{T}$. So, $\nu \in U_4{}^{\mathcal{P}\delta}(V)$. Hence, $U_4{}^{\mathcal{T}\delta}(V) \subseteq U_4{}^{\mathcal{P}\delta}(V)$.
(8) The proof is straightforward by Definition 3.4.
(9) $U_4{}^{(\mathcal{P}\cap\mathcal{T})\delta}(V) = \underset{\nu\in U}{\cup}\{\theta_l(\nu) : \theta_l(\nu)\cap V \notin \mathcal{P}\cap\mathcal{T}\}$
$= (\underset{\nu\in U}{\cup}\{\theta_l(\nu) : \theta_l(\nu)\cap V \notin \mathcal{P}\})\ or\ (\underset{\nu\in U}{\cup}\{\theta_l(\nu) : \theta_l(\nu)\cap V \notin \mathcal{T}\})$
$= (\underset{\nu\in U}{\cup}\{\theta_l(\nu) : \theta_l(\nu)\cap V \notin \mathcal{P}\}) \cup (\underset{\nu\in U}{\cup}\{\theta_l(\nu) : \theta_l(\nu)\cap V \notin \mathcal{T}\})$
$= U_4{}^{\mathcal{P}\delta}(V)\cup U_4{}^{\mathcal{T}\delta}(V)$.
(10) $U_4{}^{(\mathcal{P}\vee\mathcal{T})\delta}(V) = \cup\{\theta_l(\nu) : \theta_l(\nu)\cap V \notin \mathcal{P}\vee\mathcal{T}\}$
$= \cup\{\theta_l(\nu) : \theta_l(\nu)\cap V \notin \mathcal{P}\cup\mathcal{T}\}$
$= (\cup\{\theta_l(\nu) : \theta_l(\nu)\cap V \notin \mathcal{P}\})\ and\ (\cup\{\theta_l(\nu) : \theta_l(\nu)\cap V \notin \mathcal{T}\})$
$= (\cup\{\theta_l(\nu) : \theta_l(\nu)\cap V \notin \mathcal{P}\})\cap(\cup\{\theta_l(\nu) : \theta_l(\nu)\cap V \notin \mathcal{T}\})$
$= U_4{}^{\mathcal{P}\delta}(V)\cap U_4{}^{\mathcal{T}\delta}(V)$.

Proposition 3.11: Let $\mathcal{P}$ and $\mathcal{T}$ be ideals and δ be a binary relation on U such that $V, W \subseteq U$ Then,

(1) $L_4{}^{\mathcal{P}\delta}(U) = U$.
(2) $V \subseteq W \Rightarrow L_4{}^{\mathcal{P}\delta}(V) \subseteq L_4{}^{\mathcal{P}\delta}(W)$.
(3) $L_4{}^{\mathcal{P}\delta}(V)\cup L_4{}^{\mathcal{P}\delta}(W) \subseteq L_4{}^{\mathcal{P}\delta}(V\cup W)$.
(4) $L_4{}^{\mathcal{P}\delta}(V\cap W) = L_4{}^{\mathcal{P}\delta}(V)\cap L_4{}^{\mathcal{P}\delta}(W)$.
(5) $L_4{}^{\mathcal{P}\delta}(V) = (U_4{}^{\mathcal{P}\delta}(V^c))^c$.
(6) If $V^c \in \mathcal{P}$, then $L_4{}^{\mathcal{P}\delta}(V) = U$.
(7) If $\mathcal{P} \subseteq \mathcal{T}$, then $L_4{}^{\mathcal{P}\delta}(V) \subseteq L_4{}^{\mathcal{T}\delta}(V)$.
(8) If $\mathcal{P} = P(U)$, then $L_4{}^{\mathcal{P}\delta}(V) = U$.
(9) $L_4{}^{(\mathcal{P}\cap\mathcal{T})\delta}(V) = L_4{}^{\mathcal{P}\delta}(V)\cap L_4{}^{\mathcal{T}\delta}(V)$.
(10) $L_4{}^{(\mathcal{P}\vee\mathcal{T})\delta}(V) = L_4{}^{\mathcal{P}\delta}(V)\cup L_4{}^{\mathcal{T}\delta}(V)$.

Proof.

(1) $L_4{}^{\mathcal{P}\delta}(U) = (U_4{}^{\mathcal{P}\delta}(\phi))^c = \phi^c = U$ by (1) in Proposition 3.10.
(2) Let $V \subseteq W$. Thus, $W^c \subseteq V^c$ and $U_4{}^{\mathcal{P}\delta}(W^c) \subseteq U_4{}^{\mathcal{P}\delta}(V^c)$ (by (2) in Proposition 3.10). Then, $(U_4{}^{\mathcal{P}\delta}(V^c))^c \subseteq (U_4{}^{\mathcal{P}\delta}(W^c))^c$. So, $L_4{}^{\mathcal{P}\delta}(V) \subseteq L_4{}^{\mathcal{P}\delta}(W)$.
(3) The proof is directly by (2).
(4) $L_4{}^{\mathcal{P}\delta}(V\cap W) = (U_4{}^{\mathcal{P}\delta}(V\cap W)^c)^c$
$= (U_4{}^{\mathcal{P}\delta}(V^c\cup W^c))^c$
$= (U_4{}^{\mathcal{P}\delta}(V^c)\cup U_4{}^{\mathcal{P}\delta}(W^c))^c$ (by (4) in Proposition 3.10)
$= (U_4{}^{\mathcal{P}\delta}(V^c))^c\cap(U_4{}^{\mathcal{P}\delta}(W^c))^c$
$= L_4{}^{\mathcal{P}\delta}(V)\cap L_4{}^{\mathcal{P}\delta}(W)$.
(5) The proof is straightforward by Definition 3.4.
(6) Let $V^c \in \mathcal{P}$; then, $L_4{}^{\mathcal{P}\delta}(V) = (U_4{}^{\mathcal{P}\delta}(V^c))^c = (\phi)^c = U$ according to Proposition 3.10 (6).
(7) Let $\mathcal{P} \subseteq \mathcal{T}$. Then, $U_4{}^{\mathcal{T}\delta}(V^c) \subseteq U_4{}^{\mathcal{P}\delta}(V^c)$ according to Proposition 3.10 (7). Thus, $(U_4{}^{\mathcal{P}\delta}(V^c))^c \subseteq (U_4{}^{\mathcal{T}\delta}(V^c))^c$. Hence, $L_4{}^{\mathcal{P}\delta}(V) \subseteq L_4{}^{\mathcal{T}\delta}(V)$.
(8) Let $\mathcal{P} = P(U)$; then, $L_4{}^{\mathcal{P}\delta}(V) = (U_4{}^{\mathcal{P}\delta}(V^c))^c = (\phi)^c = U$ according to Proposition 3.10 (8).
(9) $L_4{}^{(\mathcal{P}\cap\mathcal{T})\delta}(V) = (U_4{}^{(\mathcal{P}\cap\mathcal{T})\delta}(V^c))^c$
$= (U_4{}^{\mathcal{P}\delta}(V^c)\cup U_4{}^{\mathcal{T}\delta}(V^c))^c$ (by (9) in Proposition 3.10)
$= (U_4{}^{\mathcal{P}\delta}(V^c))^c\cap(U_4{}^{\mathcal{T}\delta}(V^c))^c$
$= L_4{}^{\mathcal{P}\delta}(V)\cap L_4{}^{\mathcal{T}\delta}(V)$.
(10) $L_4{}^{(\mathcal{P}\vee\mathcal{T})\delta}(V) = (U_4{}^{(\mathcal{P}\vee\mathcal{T})\delta}(V^c))^c$
$= (U_4{}^{\mathcal{P}\delta}(V^c)\cap U_4{}^{\mathcal{T}\delta}(V^c))^c$ by (10) in Proposition 3.10)
$= (U_4{}^{\mathcal{P}\delta}(V^c))^c\cup(U_4{}^{\mathcal{T}\delta}(V^c))^c$
$= L_4{}^{\mathcal{P}\delta}(V)\cup L_4{}^{\mathcal{T}\delta}(V)$.

Remark 3.12:

(1) To show that the converse of (2) of Proposition 3.10 and Proposition 3.11 is generally false take (i) of Example 3.1 and let
   (a) $V = \{a\}$ and $W = \{d\}$; then, $U_4{}^{\mathcal{P}\delta}(V) = \phi$ and $U_4{}^{\mathcal{P}\delta}(W) = U$. Therefore, $U_4{}^{\mathcal{P}\delta}(V) \subseteq U_4{}^{\mathcal{P}\delta}(W)$, but $V \nsubseteq W$.

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(b) $V = \{b\}$ and $W = \{a,c,d\}$; then, ${L_4}^{\mathcal{P}\delta}(V) = \phi$ and ${L_4}^{\mathcal{P}\delta}(W) = U$. Therefore, ${L_4}^{\mathcal{P}\delta}(V) \subseteq {L_4}^{\mathcal{P}\delta}(W)$, but $V \nsubseteq W$.

(2) In Proposition 3.10 and Proposition 3.11 the converse of (6), (7) and (8) is generally false as illustrated by (ii) of Example 3.1

(i) For (6), take

(a) $V = \{a,d\}$; then, ${U_4}^{\mathcal{T}\delta}(V) = \phi$. Therefore, ${U_4}^{\mathcal{T}\delta}(V) = \phi$, but $V \notin \mathcal{T}$.

(b) $V = \{b,c\}$; then, ${L_4}^{\mathcal{T}\delta}(V) = U$. Therefore, ${L_4}^{\mathcal{T}\delta}(V) = U$, but $V^c \notin \mathcal{T}$.

(ii) For (7), take

(a) $V = \{a,d\}$; then, ${U_4}^{\mathcal{P}\delta}(V) = \{a\}$ and ${U_4}^{\mathcal{T}\delta}(V) = \phi$. Therefore, ${U_4}^{\mathcal{T}\delta}(V) \subseteq {U_4}^{\mathcal{P}\delta}(V)$, but $\mathcal{P} \nsubseteq \mathcal{T}$.

(b) $V = \{b,c\}$; then, ${L_4}^{\mathcal{P}\delta}(V) = \{b,c,d\}$ and ${L_4}^{\mathcal{T}\delta}(V) = U$. Therefore, ${L_4}^{\mathcal{P}\delta}(V) \subseteq {L_4}^{\mathcal{T}\delta}(V)$, but $\mathcal{P} \nsubseteq \mathcal{T}$.

(iii) For (8), take

(a) $V = \{a,d\}$; then, ${U_4}^{\mathcal{T}\delta}(V) = \phi$, but $\mathcal{T} \neq P(U)$.

(b) $V = \{b,c\}$; then, ${L_4}^{\mathcal{T}\delta}(V) = U$, but $\mathcal{T} \neq P(U)$.

(3) Example 3.1 (iii) elaborates that the inclusion relations given in (3) of Proposition 3.10 and Proposition 3.11 are proper. To illustrate that, let $V = \{a,d\}$ and $W = \{b,c\}$. Then,

(a) ${U_4}^{\mathcal{P}\delta}(V) = {U_4}^{\mathcal{P}\delta}(W) = U$ and ${U_4}^{\mathcal{P}\delta}(V \cap W) = \phi$. Therefore, ${U_4}^{\mathcal{P}\delta}(V) \cap {U_4}^{\mathcal{P}\delta}(W) = U \neq \phi = {U_3}^{\mathcal{P}\delta}(V \cap W)$.

(b) ${L_4}^{\mathcal{P}\delta}(V) = {L_4}^{\mathcal{P}\delta}(W) = \phi$ and ${L_4}^{\mathcal{P}\delta}(V \cup W) = U$. Therefore, ${L_4}^{\mathcal{P}\delta}(V) \cup {L_4}^{\mathcal{P}\delta}(W) = \phi \neq U = {L_4}^{\mathcal{P}\delta}(V \cup W)$.

Remark 3.13: Some properties given in the second type are not satisfy by this type as we show in the following.

(i) Considering Example 3.1 (i), take

(1) $V = \{a\}$; then, ${U_4}^{\mathcal{P}\delta}(V) = \phi$. Hence, $V \nsubseteq {U_4}^{\mathcal{P}\delta}(V)$.

(2) $V = \{b,c,d\}$; then, ${L_4}^{\mathcal{P}\delta}(V) = U$. Hence, ${L_4}^{\mathcal{P}\delta}(V) \nsubseteq V$.

(ii) Considering Example 3.1 (ii), take

(1) $V = U$; then, ${U_4}^{\mathcal{P}\delta}(U) = \{a,b,c\}$. Hence, ${U_4}^{\mathcal{P}\delta}(U) \neq U$.

(2) $V = \phi$; then, ${L_4}^{\mathcal{P}\delta}(\phi) = \{d\}$. Hence, ${L_4}^{\mathcal{P}\delta}(\phi) \neq \phi$.

Proposition 3.12: Let $\mathcal{P}$ be an ideal and δ be a binary relation on U such that V is a nonempty subset of U. Then,

1) $0 \leq Acc_4^{\mathcal{P}\delta}(V) \leq 1$.
2) $Acc_4^{\mathcal{P}\delta}(U) = 1$.

Proof. It is similar to Proposition 3.3.

Theorem 3.6: Let $\mathcal{P}$ and $\mathcal{T}$ be ideals and δ be a binary relation on U such that $\mathcal{P} \subseteq \mathcal{T}$. Then,

(1) ${Bnd_4}^{\mathcal{T}\delta}(V) \subseteq {Bnd_4}^{\mathcal{P}\delta}(V)$.
(2) ${Acc_4}^{\mathcal{P}\delta}(V) \leq {Acc_4}^{\mathcal{T}\delta}(V)$.
(3) ${Rough_4}^{\mathcal{T}\delta}(V) \leq {Rough_4}^{\mathcal{P}\delta}(V)$.

Proof. Similar to Theorem 3.1.

Remark 3.14: According to (ii) of Example 3.1, the converse of (1) and (2) in Theorem 3.6 is generally false. To clarify that, let $V = \{b,c\}$. Then,

(1) ${Bnd_4}^{\mathcal{T}\delta}(V) = \phi \subseteq \phi = {Bnd_4}^{\mathcal{P}\delta}(V)$, but $\mathcal{P} \nsubseteq \mathcal{T}$.
(2) ${Acc_4}^{\mathcal{P}\delta}(V) = 1 \leq 1 = {Acc_4}^{\mathcal{T}\delta}(V)$, but $\mathcal{P} \nsubseteq \mathcal{T}$.
(3) ${Rough_4}^{\mathcal{T}\delta}(V) = 0 \leq 0 = {Rough_4}^{\mathcal{P}\delta}(V)$.

Theorem 3.7: Let $\phi \neq V \subseteq U$, $\mathcal{P}$ be an ideal on U and δ_1, δ_2 be two binary relations on U. If $\delta_1 \subseteq \delta_2$, then

(1) ${U_4}^{\mathcal{P}\delta_1}(V) \subseteq {U_4}^{\mathcal{P}\delta_2}(V)$.
(2) ${L_4}^{\mathcal{P}\delta_2}(V) \subseteq {L_4}^{\mathcal{P}\delta_1}(V)$.
(3) ${Bnd_4}^{\mathcal{P}\delta_1}(V) \subseteq {Bnd_4}^{\mathcal{P}\delta_2}(V)$.
(4) ${Acc_4}^{\mathcal{P}\delta_2}(V) \leq {Acc_4}^{\mathcal{P}\delta_1}(V)$.
(5) ${Rough_4}^{\mathcal{P}\delta_1}(V) \leq {Rough_4}^{\mathcal{P}\delta_2}(V)$.

Proof.

(1) Let $\nu \in {U_4}^{\mathcal{P}\delta_1}(V)$. Then, $\exists\ y \in U$ such that $\nu \in \theta_{1l}(y) \cap V \notin \mathcal{P}$. Since $\theta_{1l}(y) \subseteq \theta_{2l}(y)$ (by Theorem 2.1 [4]), it follows that $\nu \in \theta_{2l}(y) \cap V \notin \mathcal{P}$. Thus, $\nu \in {U_4}^{\mathcal{P}\delta_2}(V)$. Hence, ${U_4}^{\mathcal{P}\delta_1}(V) \subseteq {U_4}^{\mathcal{P}\delta_2}(V)$.

(2) $\nu \in {L_4}^{\mathcal{P}\delta_2}(V) = ({U_4}^{\mathcal{P}\delta_2}(V^c))^c \subseteq ({U_4}^{\mathcal{P}\delta_1}(V^c))^c$ (according to ${L_4}^{\mathcal{P}\delta_1}(V)$.

(3) Let $\nu \in {Bnd_4}^{\mathcal{P}\delta_1}(V)$. Then, $\nu \in {U_4}^{\mathcal{P}\delta_1}(V) - {L_4}^{\mathcal{P}\delta_1}(V)$. So, $\nu \in {U_4}^{\mathcal{P}\delta_1}(V)$ and $\nu \in ({L_4}^{\mathcal{P}\delta_1}(V))^c$. Thus, $\nu \in {U_4}^{\mathcal{P}\delta_2}(V)$ and $\nu \in ({L_4}^{\mathcal{P}\delta_2}(V))^c$ according to (1) and (2). Hence, $\nu \in {Bnd_4}^{\mathcal{P}\delta_2}(V)$. Therefore, ${Bnd_4}^{\mathcal{P}\delta_1}(V) \subseteq {Bnd_4}^{\mathcal{P}\delta_2}(V)$.

(4) ${Acc_4}^{\mathcal{P}\delta_2}(V) = \left|\frac{{L_4}^{\mathcal{P}\delta_2}(V) \cap V}{{U_4}^{\mathcal{P}\delta_2}(V) \cup V}\right| \leq \left|\frac{{L_4}^{\mathcal{P}\delta_1}(V) \cap V}{{U_4}^{\mathcal{P}\delta_1}(V) \cup V}\right| = {Acc_4}^{\mathcal{P}\delta_1}(V)$.

(5) Straightforward by (4).

Remark 3.15: According to Example 3.3, the inclusion and less than relation in Theorem 3.7 is proper. To clarify that, take

(i) $V = \{a,d\}$; then,

(1) ${U_4}^{\mathcal{P}\delta_1}(V) = \{a,b\} \neq \{a,b,c\} = {U_4}^{\mathcal{P}\delta_2}(V)$.

(2) ${Acc_4}^{\mathcal{P}\delta_1}(V) = 1 \neq \frac{2}{3} = {Acc_4}^{\mathcal{P}\delta_2}(V)$.

(3) ${Rough_4}^{\mathcal{P}\delta_1}(V) = 0 \neq \frac{1}{3} = {Rough_4}^{\mathcal{P}\delta_2}(V)$.

(ii) $V = \{b,c\}$; then, ${L_4}^{\mathcal{P}\delta_1}(V) = \{c,d\} \neq \{d\} = {L_4}^{\mathcal{P}\delta_2}(V)$.

IV. COMPARISON THE PROPOSED METHODS AND THEIR ADVANTAGES COMPARED TO THE PREVIOUS ONES

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A. COMPARISON THE PROPOSED METHODS IN TERMS APPROXIMATIONS AND ACCURACY MEASURES OF SUBSETS

Theorem 4.1: Let $\mathcal{P}$ be an ideal and δ be a binary relation on U such that $V \subseteq U$ Then,

(1) ${U_1}^{\mathcal{P}\delta}(V) \subseteq {U_2}^{\mathcal{P}\delta}(V)$.

(2) ${L_2}^{\mathcal{P}\delta}(V) \subseteq {L_1}^{\mathcal{P}\delta}(V)$.

(3) ${Bnd_1}^{\mathcal{P}\delta}(V) \subseteq {Bnd_2}^{\mathcal{P}\delta}(V)$.

(4) ${Acc_2}^{\mathcal{P}\delta}(V) = {Acc_1}^{\mathcal{P}\delta}(V)$.

(5) ${Rough_1}^{\mathcal{P}\delta}(V) = {Rough_2}^{\mathcal{P}\delta}(V)$.

Proof. directly follows from Definitions 3.1 and 3.2.

Remark 4.1: The inclusion and less than relations in the above theorem are proper. To elaborates that consider Example 3.3 and let $V = \{a, c, d\}$. Then,

(1) ${U_1}^{\mathcal{P}\delta_1}(V) = \{a, b\} \neq U = {U_2}^{\mathcal{P}\delta_1}(V)$.

(2) ${L_2}^{\mathcal{P}\delta_1}(V) = \{a, c, d\} \neq U = {L_1}^{\mathcal{P}\delta_1}(V)$.

(3) ${Bnd_1}^{\mathcal{P}\delta_1}(V) = \phi \neq \{b\} = {Bnd_2}^{\mathcal{P}\delta_1}(V)$.

Theorem 4.2: Let $\mathcal{P}$ be an ideal and δ be a reflexive relation on U such that $V \subseteq U$ Then,

1) ${L_2}^{\mathcal{P}\delta}(V) \subseteq {L_1}^{\mathcal{P}\delta}(V) \subseteq {L_3}^{\mathcal{P}\delta}(V)$.
2) ${U_3}^{\mathcal{P}\delta}(V) \subseteq {U_1}^{\mathcal{P}\delta}(V) \subseteq {U_2}^{\mathcal{P}\delta}(V)$.
3) ${Bnd^3}^{\mathcal{P}\delta}(V) \subseteq {Bnd^1}^{\mathcal{P}\delta}(V) \subseteq {Bnd^2}^{\mathcal{P}\delta}(V)$.
4) ${Acc^2}^{\mathcal{P}\delta}(V) \leq {Acc^1}^{\mathcal{P}\delta}(V) \leq {Acc^3}^{\mathcal{P}\delta}(V)$.
5) ${Rough^3}^{\mathcal{P}\delta}(V) \leq {Rough^1}^{\mathcal{P}\delta}(V) \leq {Rough^2}^{\mathcal{P}\delta}(V)$.

Proof.

(1) By Theorem 4.1, we have ${L_2}^{\mathcal{P}\delta}(V) \subseteq {L_1}^{\mathcal{P}\delta}(V)$. To prove, ${L_1}^{\mathcal{P}\delta}(V) \subseteq {L_3}^{\mathcal{P}\delta}(V)$. Let $\nu \in {L_1}^{\mathcal{P}\delta}(V)$, then $\theta_l(\nu) \cap V^c \in \mathcal{P}$. Hence, $\theta_l(\nu) \subseteq {L_3}^{\mathcal{P}\delta}(V)$. Since, δ is a reflexive relation, thus $\nu \in \theta_l(\nu) \subseteq {L_3}^{\mathcal{P}\delta}(V)$. Therefore, $\nu \in {L_3}^{\mathcal{P}\delta}(V)$.

(2) To prove, ${U_3}^{\mathcal{P}\delta}(V) \subseteq {U_1}^{\mathcal{P}\delta}(V)$. Let $\nu \in {U_3}^{\mathcal{P}\delta}(V) = ({L_3}^{\mathcal{P}\delta}(V^c))^c$, then $\nu \notin {L_3}^{\mathcal{P}\delta}(V^c)$. Hence, by Definition 3.3, we get $\theta_l(\nu) \cap V \notin \mathcal{P}$. It follows that $\nu \in {U_1}^{\mathcal{P}\delta}(V)$. By Theorem 4.1, we have ${U_1}^{\mathcal{P}\delta}(V) \subseteq {U_2}^{\mathcal{P}\delta}(V)$.

(3)-(5) Straightforward from (1) and (2).

Remark 4.2: In Theorem 4.2 the inclusion and less than relations are proper. To demonstrate that consider (iii) of Example 3.1 and let $V = \{b, c\}$. Then, ${U_3}^{\mathcal{P}\delta}(V) = \{b, c\} \subsetneq \{a, b, c\} = {U_1}^{\mathcal{P}\delta}(V)$. Moreover, take $V = \{a, d\}$, then

(1) ${L_1}^{\mathcal{P}\delta}(V) = \{d\} \subsetneq \{a, d\} = {L_3}^{\mathcal{P}\delta}(V)$.

(2) ${Bnd_3}^{\mathcal{P}\delta}(V) = \{b, c\} \subsetneq \{a, b, c\} = {Bnd_1}^{\mathcal{P}\delta}(V)$.

(3) ${Acc_1}^{\mathcal{P}\delta}(V) = \frac{1}{4} \lneq \frac{1}{2} = {Acc_3}^{\mathcal{P}\delta}(V)$.

(4) ${Rough^3}^{\mathcal{P}\delta}(V) = \frac{1}{2} \lneq \frac{3}{4} = {Rough^1}^{\mathcal{P}\delta}(V)$.

Theorem 4.3: Let $\mathcal{P}$ be an ideal and δ be a reflexive relation on U such that $V \subseteq U$ Then,

1) ${L_4}^{\mathcal{P}\delta}(V) \subseteq {L_1}^{\mathcal{P}\delta}(V) \subseteq {L_3}^{\mathcal{P}\delta}(V)$.
2) ${U_3}^{\mathcal{P}\delta}(V) \subseteq {U_1}^{\mathcal{P}\delta}(V) \subseteq {U_4}^{\mathcal{P}\delta}(V)$.
3) ${Bnd^3}^{\mathcal{P}\delta}(V) \subseteq {Bnd^1}^{\mathcal{P}\delta}(V) \subseteq {Bnd^4}^{\mathcal{P}\delta}(V)$.
4) ${Acc^4}^{\mathcal{P}\delta}(V) \leq {Acc^1}^{\mathcal{P}\delta}(V) \leq {Acc^3}^{\mathcal{P}\delta}(V)$.
5) ${Rough^3}^{\mathcal{P}\delta}(V) \leq {Rough^1}^{\mathcal{P}\delta}(V) \leq {Rough^4}^{\mathcal{P}\delta}(V)$.

Proof.

(1) By Theorem 4.2, we have ${L_1}^{\mathcal{P}\delta}(V) \subseteq {L_3}^{\mathcal{P}\delta}(V)$. To prove, ${L_4}^{\mathcal{P}\delta}(V) \subseteq {L_1}^{\mathcal{P}\delta}(V)$, let $\nu \in {L_4}^{\mathcal{P}\delta}(V) = {U_4}^{\mathcal{P}\delta}(V^c)^c$. Then, $\nu \notin {U_4}^{\mathcal{P}\delta}(V^c)$. Thus, by Definition 3.4, $\theta_l(\nu) \cap V^c \in \mathcal{P}$. It follows that $\theta_l(\nu) \subseteq {L_1}^{\mathcal{P}\delta}(V)$. Since, δ is a reflexive relation, then $\nu \in \theta_u(\nu) \subseteq {L_1}^{\mathcal{P}\delta}(V)$. Therefore, $\nu \in {L_1}^{\mathcal{P}\delta}(V)$.

(2) By Theorem 4.2, we have ${U_3}^{\mathcal{P}\delta}(V) \subseteq {U_1}^{\mathcal{P}\delta}(V)$. To prove ${U_1}^{\mathcal{P}\delta}(V) \subseteq {U_4}^{\mathcal{P}\delta}(V)$, let $\nu \in {U_1}^{\mathcal{P}\delta}(V)$. Then $\theta_l(\nu) \cap V \notin \mathcal{P}$. It follows that $\theta_u(\nu) \subseteq {U_4}^{\mathcal{P}\delta}(V)$. Since, δ is a reflexive relation, then $\nu \in \theta_l(\nu) \subseteq {U_4}^{\mathcal{P}\delta}(V)$. Therefore, $\nu \in {U_4}^{\mathcal{P}\delta}(V)$.

(3)-(5) Straightforward from (1) and (2).

Remark 4.3: To clarify that the inclusion and less than relations in Theorem 4.3 are proper, consider (iii) of Example 3.1 and take $V = \{b, c\}$. Then, ${U_1}^{\mathcal{P}\delta}(V) = \{a, b, c\} \subsetneq U = {U_4}^{\mathcal{P}\delta}(V)$. Moreover, take $V = \{a, d\}$; then,

(1) ${L_4}^{\mathcal{P}\delta}(V) = \phi \subsetneq \{d\} = {L_1}^{\mathcal{P}\delta}(V)$.

(2) ${Bnd_1}^{\mathcal{P}\delta}(V) = \{a, b, c\} \subsetneq U = {Bnd_4}^{\mathcal{P}\delta}(V)$.

(3) ${Acc_4}^{\mathcal{P}\delta}(V) = 0 \lneq \frac{1}{4} = {Acc_1}^{\mathcal{P}\delta}(V)$.

(4) ${Rough_1}^{\mathcal{P}\delta}(V) = \frac{3}{4} \lneq 1 = {Rough_4}^{\mathcal{P}\delta}(V)$.

Remark 4.4: It can be seen from the above findings that there are several techniques to study approximation operators and measures of the subsets. The third technique, displayed in Section III-C, is the best one to approximate the subsets because it reduces (or cancels) the boundary regions, and increases the measures of accuracy more than the other types displayed in the other sections.

In Tables 1 and 2, we compare the proposed four approaches in terms of Pawlak properties. In these tables $\bullet$ means that the property holds and $\circ$ means that the property does not hold.

TABLE 1. Comparison between the first and second methods in terms of the properties given in Definition 2.3.

	The first approach	The second approach
$\mathcal{L}_1$	$\bullet$	$\bullet$
$\mathcal{L}_2$	$\bullet$	$\bullet$
$\mathcal{L}_3$	$\circ$	$\bullet$
$\mathcal{L}_4$	$\circ$	$\bullet$
$\mathcal{L}_5$	$\bullet$	$\bullet$
$\mathcal{L}_6$	$\bullet$	$\bullet$
$\mathcal{L}_7$	$\bullet$	$\bullet$
$\mathcal{L}_8$	$\circ$	$\circ$
$\mathcal{L}_9$	$\circ$	$\circ$
$\mathcal{U}_1$	$\bullet$	$\bullet$
$\mathcal{U}_2$	$\circ$	$\bullet$
$\mathcal{U}_3$	$\bullet$	$\bullet$
$\mathcal{U}_4$	$\circ$	$\bullet$
$\mathcal{U}_5$	$\bullet$	$\bullet$
$\mathcal{U}_6$	$\bullet$	$\bullet$
$\mathcal{U}_7$	$\bullet$	$\bullet$
$\mathcal{U}_8$	$\circ$	$\circ$
$\mathcal{U}_9$	$\circ$	$\circ$

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TABLE 2. Comparison between the third and fourth approaches in terms of the properties in Definition 2.3.

	The third approach	The fourth approach
$\mathcal{L}_1$	•	•
$\mathcal{L}_2$	◦	•
$\mathcal{L}_3$	◦	◦
$\mathcal{L}_4$	◦	◦
$\mathcal{L}_5$	◦	•
$\mathcal{L}_6$	•	•
$\mathcal{L}_1$	•	•
$\mathcal{U}_1$	•	•
$\mathcal{U}_2$	◦	◦
$\mathcal{U}_3$	◦	•
$\mathcal{U}_4$	◦	◦
$\mathcal{U}_5$	◦	•
$\mathcal{U}_6$	•	•
$\mathcal{U}_7$	•	•

B. COMPARISON THE PROPOSED METHODS WITH THE PREVIOUS ONES

Theorem 4.4: Let $\mathcal{P}$ be an ideal and δ be a binary relation on U such that $V \subseteq U$ Then,

(1) ${U_1}^{\mathcal{P}\delta}(V) \subseteq U^{\delta}(V)$.
(2) $L^{\delta}(V) \subseteq {L_1}^{\mathcal{P}\delta}(V)$.
(3) ${Bnd_1}^{\mathcal{P}\delta}(V) \subseteq Bnd^{\delta}(V)$.
(4) $Acc^{\delta}(V) \leq {Acc_1}^{\mathcal{P}\delta}(V)$.
(5) ${Rough_1}^{\mathcal{P}\delta}(V) \leq Rough^{\delta}(V)$.

Proof.

(1) Let $\nu \in {U_1}^{\mathcal{P}\delta}(V)$. Then, $\theta_l(\nu) \cap V \notin \mathcal{P}$. Hence, $\theta_l(\nu) \cap V \neq \phi$. Therefore, $\nu \in U^{\delta}(V)$. So, ${U_1}^{\mathcal{P}\delta}(V) \subseteq U^{\delta}(V)$.
(2) Let $\nu \in L^{\delta}(V)$. Then, $\theta_l(\nu) \subseteq V$. Hence, $\theta_l(\nu) \cap V^c \in \mathcal{P}$. Therefore, $\nu \in {L_1}^{\mathcal{P}\delta}(V)$. So,$L^{\delta}(V) \subseteq {L_1}^{\mathcal{P}\delta}(V)$.
(3)-(5) It is immediately obtained by (1) and (2).

Remark 4.5:

In Theorem 4.4 the inclusion and less than relations are proper as illustrated by Example 3.3. To this end, Take $V = \{a, d\}$. Then,

(1) ${U_1}^{\mathcal{P}\delta_1}(V) = \{a, b\} \neq \{a, b, d\} = U_{\delta_1}(V)$.
(2) ${L_1}^{\mathcal{P}\delta_1}(V) = U \neq \{d\} = L_{\delta_1}(V)$.
(3) ${Bnd_1}^{\mathcal{P}\delta_1}(V) = \phi \neq \{a, b\} = Bnd_{\delta_1}(V)$.
(4) ${Acc_1}^{\mathcal{P}\delta_1}(V) = \frac{2}{3} \neq \frac{1}{3} = Acc_{\delta_1}(V)$.
(5) ${Rough_1}^{\mathcal{P}\delta_1}(V) = \frac{1}{3} \neq \frac{2}{3} = Rough_{\delta_1}(V)$.

Theorem 4.5: Let $\mathcal{P}$ be an ideal and δ be a binary relation on U such that $V \subseteq U$ Then,

(1) ${U_1}^{\mathcal{P}\delta}(V) \subseteq Upp^{\delta}(V)$.
(2) $Low^{\delta}(V) \subseteq {L_1}^{\mathcal{P}\delta}(V)$.
(3) ${Bnd_1}^{\mathcal{P}\delta}(V) \subseteq Boundary^{\delta}(V)$.
(4) $Accuracy^{\delta}(V) \leq {Acc_1}^{\mathcal{P}\delta}(V)$.
(5) ${Rough_1}^{\mathcal{P}\delta}(V) \leq Roughness^{\delta}(V)$.

Proof.

(1) Let $\nu \in {U_1}^{\mathcal{P}\delta}(V)$. Then, $\theta_l(\nu) \cap V \notin \mathcal{P}$. Hence, $\theta_u(\nu) \cap V \notin \mathcal{P}$. So, $\theta_u(\nu) \cap V \neq \phi$. Therefore, $\nu \in Upp^{\delta}(V)$. So, ${U_1}^{\mathcal{P}\delta}(V) \subseteq Upp^{\delta}(V)$.
(2) Let $\nu \in Low^{\delta}(V)$. Then, $\theta_u(\nu) \subseteq V$. Hence, $\theta_u(\nu) \cap V^c \in \mathcal{P}$. Since, $\theta_l(\nu) \subseteq \theta_u(\nu)$. So, $\theta_l(\nu) \cap V^c \in \mathcal{P}$. Therefore, $\nu \in {L_1}^{\mathcal{P}\delta}(V)$. So,$Low^{\delta}(V) \subseteq {L_1}^{\mathcal{P}\delta}(V)$.
(3)-(5) The proof is immediately by (1) and (2).

Remark 4.6: In Theorem 4.5 the inclusion and less than relations are proper. To clarify that consider Example 3.3 and let $V = \{a, d\}$. Then,

(1) ${U_1}^{\mathcal{P}\delta_1}(V) = \{a, b\} \neq \{a, b, d\} = Upp_{\delta_1}(V)$.
(2) ${L_1}^{\mathcal{P}\delta_1}(V) = U \neq \{d\} = Low_{\delta_1}(V)$.
(3) ${Bnd_1}^{\mathcal{P}\delta_1}(V) = \phi \neq \{a, b\} = Boundary_{\delta_1}(V)$.
(4) ${Accu_1}^{\mathcal{P}\delta_1}(V) = \frac{2}{3} \neq \frac{1}{3} = Accuracy_{\delta_1}(V)$.
(5) ${Rough_1}^{\mathcal{P}\delta_1}(V) = \frac{1}{3} \neq \frac{2}{3} = Roughness_{\delta_1}(V)$.

According to Theorems 4.4 and 4.5 it can be seen that the present methods reduce the boundary region with the comparison of Al-shami's methods [4]. This means that the current approximation spaces are proper generalizations of Al-shami's approximations [4].

One can easily prove the next result which shows that Al-shami's approximations [4] are special cases of the current approximations.

Proposition 4.1:

(1) If the ideal $\mathcal{P}$ is the empty set, then the approximation spaces given herein and the approximation spaces given in Definition 2.6 [4] are identical.
(2) If the ideal $\mathcal{P}$ is the empty set and binary relation is a similarity relation, then the approximation spaces given herein and the approximation spaces given in Definition 2.7 [4] are identical.

Theorem 4.6: Let $\mathcal{P}$ be an ideal and δ be a binary relation on U such that $V \subseteq U$ Then,

(1) ${U_1}^{\mathcal{P}\delta}(V) \subseteq {Upp_1}^{\mathcal{P}\delta}(V)$.
(2) ${Low_1}^{\mathcal{P}\delta}(V) \subseteq {L_1}^{\mathcal{P}\delta}(V)$.
(3) ${Bnd_1}^{\mathcal{P}\delta}(V) \subseteq {Boundary_1}^{\mathcal{P}\delta}(V)$.
(4) ${Accuracy_1}^{\mathcal{P}\delta}(V) \leq {Acc_1}^{\mathcal{P}\delta}(V)$.
(5) ${Rough_1}^{\mathcal{P}\delta}(V) \leq {Roughness_1}^{\mathcal{P}\delta}(V)$.

Proof.

(1) Let $\nu \in {U_1}^{\mathcal{P}\delta}(V)$. Then, $\theta_l(\nu) \cap V \notin \mathcal{P}$. Since, $\theta_l(\nu) \subseteq \theta_u(\nu)$. Hence, $\theta_u(\nu) \cap V \notin \mathcal{P}$. Therefore, $\nu \in {Upp_1}^{\mathcal{P}\delta}(V)$. So, ${U_1}^{\mathcal{P}\delta}(V) \subseteq U^{\delta}(V)$.
(2) Let $\nu \in {Low_1}^{\mathcal{P}\delta}(V)$. Then, $\theta_u(\nu) \cap V^c \in \mathcal{P}$. Hence, $\theta_l(\nu) \cap V^c \in \mathcal{P}$. Therefore, $\nu \in {L_1}^{\mathcal{P}\delta}(V)$. So, ${Low_1}^{\mathcal{P}\delta}(V) \subseteq {L_1}^{\mathcal{P}\delta}(V)$.
(3)-(5) It is immediately obtained by (1) and (2).

Remark 4.7: The inclusion and the less than in Theorem 4.6 can not be replaced by equality relation in general. In Example 3.1 (i), take $\mathcal{P} = \{\phi, \{d\}\}$ and $V = \{b\}$, then ${U_1}^{\mathcal{P}\delta}(V) = \{b, c\} \neq \{a, b, c\} = {Upp_1}^{\mathcal{P}\delta}(V)$. Additionally, in Example 3.1 (i), take $\mathcal{P} = \{\phi, \{a\}\}$ and $V = \{a, c, d\}$, then

(1) ${Low_1}^{\mathcal{P}\delta}(V) = \{d\} \neq \{a, d\} = {L_1}^{\mathcal{P}\delta}(V)$.
(2) ${Bnd_1}^{\mathcal{P}\delta}(V) = \{b, c\} \neq \{a, b, c\} = {Boundary_1}^{\mathcal{P}\delta}(V)$.
(3) ${Accuracy_1}^{\mathcal{P}\delta}(V) = \frac{1}{4} \neq \frac{1}{2} = {Acc_1}^{\mathcal{P}\delta}(V)$.

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(4) $Rough_1{}^{\mathcal{P}\delta}(V) = \frac{1}{2} \neq \frac{3}{4} = Roughness_1{}^{\mathcal{P}\delta}(V)$.

Theorem 4.7: Let $\mathcal{P}$ be an ideal and δ be a binary relation on U such that $V \subseteq U$ Then,

(1) $U_2{}^{\mathcal{P}\delta}(V) \subseteq Upp_2{}^{\mathcal{P}\delta}(V)$.
(2) $Low_2{}^{\mathcal{P}\delta}(V) \subseteq L_2{}^{\mathcal{P}\delta}(V)$.
(3) $Bnd_2{}^{\mathcal{P}\delta}(V) \subseteq Boundary_2{}^{\mathcal{P}\delta}(V)$.
(4) $Accuracy_2{}^{\mathcal{P}\delta}(V) \leq Acc_2{}^{\mathcal{P}\delta}(V)$.
(5) $Rough_2{}^{\mathcal{P}\delta}(V) \leq Roughness_2{}^{\mathcal{P}\delta}(V)$.

Proof. Similar to the proof of Theorem 4.6.

Remark 4.8: In Theorem 4.7 the inclusion and the less than relations are proper as (i) of Example 3.1 shows. To this end, let $\mathcal{P} = \{\phi, \{d\}\}$ and $V = \{b\}$. Then $U_2{}^{\mathcal{P}\delta}(V) = \{b, c\} \neq \{a, b, c\} = Upp_2{}^{\mathcal{P}\delta}(V)$. Additionally, in Example 3.1 (i), take $\mathcal{P} = \{\phi, \{a\}\}$ and $V = \{a, c, d\}$, then

(1) $Low_2{}^{\mathcal{P}\delta}(V) = \{d\} \neq \{a, d\} = L_2{}^{\mathcal{P}\delta}(V)$.
(2) $Bnd_2{}^{\mathcal{P}\delta}(V) = \{b, c\} \neq \{a, b, c\} = Boundary_2{}^{\mathcal{P}\delta}(V)$.
(3) $Accuracy_2{}^{\mathcal{P}\delta}(V) = \frac{1}{4} \neq \frac{1}{2} = Acc_2{}^{\mathcal{P}\delta}(V)$.
(4) $Rough_2{}^{\mathcal{P}\delta}(V) = \frac{1}{2} \neq \frac{3}{4} = Roughness_2{}^{\mathcal{P}\delta}(V)$.

Theorem 4.8: Let $\mathcal{P}$ be an ideal and δ be a binary relation on U such that $V \subseteq U$ Then,

(1) $U_3{}^{\mathcal{P}\delta}(V) \subseteq Upp_3{}^{\mathcal{P}\delta}(V)$.
(2) $Low_3{}^{\mathcal{P}\delta}(V) \subseteq L_3{}^{\mathcal{P}\delta}(V)$.
(3) $Bnd_3{}^{\mathcal{P}\delta}(V) \subseteq Boundary_3{}^{\mathcal{P}\delta}(V)$.
(4) $Accuracy_3{}^{\mathcal{P}\delta}(V) \leq Acc_3{}^{\mathcal{P}\delta}(V)$.
(5) $Rough_3{}^{\mathcal{P}\delta}(V) \leq Roughness_3{}^{\mathcal{P}\delta}(V)$.

Proof. Similar to the proof of Theorem 4.6.

Remark 4.9: In Theorem 4.8 the inclusion and the less than relations are proper as illustrated by (iii) of Example 3.1. To this end, let $V = \{a, d\}$. Then,

(1) $U_3{}^{\mathcal{P}\delta}(V) = \{a, b, c\} \neq U = Upp_3{}^{\mathcal{P}\delta}(V)$.
(2) $Low_3{}^{\mathcal{P}\delta}(V) = \phi \neq \{d\} = L_3{}^{\mathcal{P}\delta}(V)$.
(3) $Bnd_3{}^{\mathcal{P}\delta}(V) = \{a, b, c\} \neq U = Boundary_3{}^{\mathcal{P}\delta}(V)$.
(4) $Accuracy_3{}^{\mathcal{P}\delta}(V) = 0 \neq \frac{1}{4} = Acc_3{}^{\mathcal{P}\delta}(V)$.
(5) $Rough_3{}^{\mathcal{P}\delta}(V) = \frac{3}{4} \neq 1 = Roughness_3{}^{\mathcal{P}\delta}(V)$.

Theorem 4.9: Let $\phi \neq V \subseteq U,\ \mathcal{P}$ be an ideal on U and δ be a binary relation on a non-empty set U. Then,

(1) $U_4{}^{\mathcal{P}\delta}(V) \subseteq Upp_4{}^{\mathcal{P}\delta}(V)$.
(2) $Low_4{}^{\mathcal{P}\delta}(V) \subseteq L_4{}^{\mathcal{P}\delta}(V)$.
(3) $Bnd_4{}^{\mathcal{P}\delta}(V) \subseteq Boundary_4{}^{\mathcal{P}\delta}(V)$.
(4) $Accuracy_4{}^{\mathcal{P}\delta}(V) \leq Acc_4{}^{\mathcal{P}\delta}(V)$.
(5) $Rough_4{}^{\mathcal{P}\delta}(V) \leq Roughness_4{}^{\mathcal{P}\delta}(V)$.

Proof. The proof is similar to that of Theorem 4.6.

Remark 4.10: In Theorem 4.9 the inclusion and the less than relations are proper. To illustrate that consider (i) of Example 3.1. Then,

(i) Take $\mathcal{P} = \{\phi, \{a\}\}$ and $V = \{b\}$, then
  (1) $U_4{}^{\mathcal{P}\delta}(V) = \{b, c, d\} \neq U = Upp_4{}^{\mathcal{P}\delta}(V)$.
  (2) $Bnd_4{}^{\mathcal{P}\delta}(V) = \{b, c, d\} \neq U = Boundary_4{}^{\mathcal{P}\delta}(V)$.

(ii) Take $\mathcal{P} = \{\phi, \{a\}\}$ and $V = \{a, c, d\}$, then
  (1) $Low_4{}^{\mathcal{P}\delta}(V) = \phi \neq \{a\} = L_4{}^{\mathcal{P}\delta}(V)$.
  (2) $Accuracy_4{}^{\mathcal{P}\delta}(V) = 0 \neq \frac{1}{4} = Acc_4{}^{\mathcal{P}\delta}(V)$.
  (3) $Rough_4{}^{\mathcal{P}\delta}(V) = \frac{3}{4} \neq 1 = Roughness_4{}^{\mathcal{P}\delta}(V)$.

One can easily prove the next result which shows that Hosny's and Al-shami's approximations [19] are special cases of the current approximations.

Proposition 4.2: If the binary relation is a similarity relation, then the approximation spaces given herein and the approximation spaces given in [19] are identical.

Finally, we draw attention to that the maximal left neighborhoods and maximal right neighborhoods are independent of each other. In Example 3.4, the maximal left neighborhood of b is $\theta_l(b) = \{a, b, c\} \neq \{a, b, d\} = \theta_r(b)$ the maximal right neighborhood of b. Consequently, the approximation spaces generated by the maximal left neighborhoods and maximal right neighborhoods via ideals are independent of each other. To validate that, in Example 3.4 take $V = \{b\}$, then the lower, upper approximations and boundary by the previous third method in [17] are ϕ, ϕ and ϕ, respectively. Meantime, the lower, upper approximations and boundary by the present third method are $\phi, \{c\}$ and $\{c\}$, respectively. Therefore, the previous boundary $= \phi \nsubseteq \{c\} =$ the present boundary. If we take another set say $W = \{a, c, d\}$, then the lower, upper approximations and boundary by the previous third method in [17] are $\phi, \{c\}$ and $\{c\}$, respectively. While, the lower, upper approximations and boundary by the present third method are $\{a, c, d\}, \phi$ and ϕ, respectively. So, the present boundary $= \phi \nsubseteq \{c\} =$ the previous boundary. That is, the proposed approaches and their counterparts introduced in [17] are different in general.

V. NUMERICAL EXAMPLE

In this section, we analysis the obtained computations from six students exams in four subjects in terms of the current first approach and those were introduced in two recent manuscripts [4], [19]. We illustrate this matter by considering six students $S = \{s_i : i = 1, 2, ..., 6\}$ have been examined in four subjects; say, biology, chemistry, mathematics, and physics. The evaluation of students' performance is given by five ranks or levels as follows.

$$Rank1:\ excellent$$
$$Rank2:\ very\ good$$
$$Rank3:\ good$$
$$Rank4:\ fair$$
$$Rank5:\ failed$$

These ranks are ordered as follows: excellent $\succ$ very good $\succ$ good $\succ$ fair $\succ$ failed, where $\succ$ means "greater than".

U	biology	chemistry	mathematics	physics
s_1	good	fair	excellent	excellent
s_2	very good	good	excellent	fair
s_3	very good	good	failed	good
s_4	excellent	fair	very good	excellent
s_5	very good	fair	very good	excellent
s_6	good	good	excellent	fair

TABLE 3. Information system of students' rank for each subject

The relation that associates between students is defined by: xRy iff student x has at least two subjects with a rank greater than the rank of the corresponding subjects of student y. For instance, $s_4 \delta s_3$ because the student's ranks s_4 in biology, mathematics, and physics are greater than the student's ranks s_3 in these subjects. But, $(s_3, s_4) \notin \delta$ because the student s_3 has only one subject's rank greater than student s_4.

The approximation space of the students' information system is constructed firstly by converting Table 3 to the following binary relation: $\delta = \{(s_1, s_3), (s_2, s_1), (s_2, s_4), (s_2, s_5), (s_4, s_2), (s_4, s_3), (s_4, s_6), (s_5, s_3), (s_5, s_6), (s_6, s_4)\}$. It should be noted that this relation is irreflexive because $(s, s) \notin \delta$ for each $s \in S$, not symmetry because $(s_3, s_1) \notin \delta$ in spite of $(s_1, s_3) \in \delta$, also, it is not transitive because $(s_5, s_4) \notin \delta$ in spite of $(s_5, s_6) \in \delta$ and $(s_6, s_4) \in \delta$.

Secondly, the left neighborhood N_l and maximal left neighborhood M_l are calculated for each $s \in S$

$$\begin{array}{ll} N_l(s_1) = N_l(s_5) = \{s_2\} & N_l(s_4) = \{s_2, s_6\} \\ N_l(s_2) = \{s_4\} & N_l(s_6) = \{s_4, s_5\} \\ N_l(s_3) = \{s_1, s_4, s_5\} & \end{array}$$

$$\begin{array}{l} M_l(s_1) = M_l(s_4) = M_l(s_5) = \{s_1, s_4, s_5\} \\ M_l(s_2) = M_l(s_6) = \{s_2, s_6\} \\ M_l(s_3) = \phi \end{array}$$

Let $\mathcal{T} = \{\phi, \{s_4\}\}$ be an ideal on S. Then, we examine the performance of the present approach in Definition 3.1 and their counterparts approaches studied in [4], [19]. To this end, let $F = \{s_3, s_4, s_6\}$. In what follows, we calculate its lower and upper approximations, boundary regions and accuracy values utilizing a method of maximal left neighborhoods given in [4] and the first method given herein.

- It follows from Al-shami's approach [4](see, Definition 2.6) that:

$$\begin{cases} L^\delta(F) = & \{s_3\}; \\ U^\delta(F) = & \{s_1, s_2, s_4, s_5, s_6\}; \\ Bnd^\delta(F) = & \{s_1, s_2, s_4, s_5, s_6\}; \\ Acc^\delta(F) = & \frac{1}{6}; \\ Rough^\delta(F) = & \frac{5}{6}. \end{cases} \tag{54}$$

- It follows from our approach given in Definition 3.1 that

$$\begin{cases} L_1{}^{\mathcal{P}\delta}(F) = & \{s_3\}; \\ U_1{}^{\mathcal{P}\delta}(F) = & \{s_2, s_6\}; \\ Bnd_1{}^{\mathcal{P}\delta}(F) = & \{s_2, s_6\}; \\ Acc_1{}^{\mathcal{P}\delta}(F) = & \frac{1}{4}; \\ Rough_1{}^{\mathcal{P}\delta}(F) = & \frac{3}{4}. \end{cases} \tag{55}$$

The proposed approach is compared with its counterpart given in [19] by calculating the right neighborhood N_r, then the maximal right neighborhood M_r and finally the maximal union neighborhood M_u for each $s \in S$.

$$\begin{array}{ll} N_r(s_1) = \{s_3\} & N_r(s_4) = \{s_2, s_3, s_6\} \\ N_r(s_2) = \{s_1, s_4, s_5\} & N_r(s_5) = \{s_3, s_6\} \\ N_r(s_3) = \phi & N_r(s_6) = \{s_4\} \end{array}$$

$$\begin{array}{l} M_r(s_1) = M_r(s_4) = M_r(s_5) = \{s_1, s_4, s_5\} \\ M_r(s_2) = M_r(s_3) = M_r(s_6) = \{s_2, s_3, s_6\} \end{array}$$

$$\begin{array}{l} M_u(s_1) = M_u(s_4) = M_u(s_5) = \{s_1, s_4, s_5\} \\ M_u(s_2) = M_u(s_3) = M_u(s_6) = \{s_2, s_3, s_6\} \end{array}$$

According to the approach given in [19] (see, Definition 2.8) we find that

$$\begin{cases} Low_1{}^{\mathcal{P}\delta}(F) = & \phi; \\ Upp_1{}^{\mathcal{P}\delta}(F) = & \{s_2, s_3, s_6\}; \\ Boundary_1{}^{\mathcal{P}\delta}(F) = & \{s_2, s_3, s_6\}; \\ Accuracy_1{}^{\mathcal{P}\delta}(F) = & 0; \\ Roughness_1{}^{\mathcal{P}\delta}(F) = & 1. \end{cases} \tag{56}$$

It follows from the above calculations that the boundary regions of the set F generated by approaches given in [4] and [19] are $\{s_1, s_2, s_4, s_5, s_6\}$ and $\{s_2, s_3, s_6\}$, respectively. Whereas, the boundary region of the set F generated by the suggested approach introduced in Definition 3.1 is $\{s_2, s_6\}$, which implies that the uncertainty/vagueness area is minimized by the proposed approach more than approaches displayed in [4], [19]. Hence, a decision made according to the calculations of the present approach is more accurate.

According to the above discussion, it can be seen that there are various methods or approaches used to approximate the subsets. The current technique " maximal left neighborhoods and ideals" is a vital tool to eliminate the ambiguity of the data in the real-life issues and produces more accurate decisions since it decreases the boundary region by enlarging the lower approximations and dwindling the upper approximations, and hence, increases the value of accuracy compared to the other types such those were discussed in [4], [19].

VI. CONCLUSIONS

One of the recent successfully tool to handel uncertainty problems is rough set. It was proposed with the goal of the induction of approximations of concepts; it offers mathematical tools to discover patterns hidden in data. This manuscript had been written to contribute to this field by introducing some novel kinds of approximation spaces generated by "maximal left neighborhoods and ideals" which generalize the old concepts and get preferable results by reducing the boundary regions.

First, we scrutinized their main properties and provided some illustrative counterexample to elucidate the obtained results. By the way, it was proved that the current approach preserved main characterizations of Pawlak's model and kept the property of monotonicity. Then, we compared between the proposed methods and discussed their advantages compared to the previous methods in terms of improvement the approximation operators and accuracy measures. Finally, a numerical example was given and demonstrated how the current methods expanded the knowledge obtained from the information systems.

As it is well-known that the interior and closure topological operators behave similarly to the lower and upper approximations; so, in forthcoming works, we plan to study the counterparts of these models via topological structures. In addition, we will benefit from the hybridization of rough set theory with some approaches such as soft sets and fuzzy sets [30], [34] to introduce these approximation spaces via these hybridized frames and show their role in efficiently dealing with uncertain knowledge.

Conflicts of interest: The authors declare that they have no competing interests.

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