Letters to the Editor

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Improvement of the Landau Theory and the Critical Indices

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The inconsistency inherent in the Landau theory[1] of the second order phase transition is removed in such a way that most of the results obtained by the static scaling theory[2] may be reproduced. Consider a spherical region of volume $v=(4\pi/3)R^3=(4\pi/3)K^{-3}$ in a system and define the local order parameter $\eta(\boldsymbol{r})$ as

$$\eta(\boldsymbol{r})=(1/v)\int_v m(\boldsymbol{r}+\boldsymbol{r}')\,d\boldsymbol{r}', \qquad (1)$$

where $m(\boldsymbol{r})$ denotes the local order parameter associated with the atom at the position $\boldsymbol{r}$, and the integral extends over the above mentioned spherical region centered at $\boldsymbol{r}$. From the definition (1) it follows that $\eta_{\boldsymbol{k}}\simeq 0$ for $|\boldsymbol{k}|>K$. In the Landau theory R is kept fixed but in our treatment R is changed with temperature in such a way that at all temperatures the fluctuation of $\eta(\boldsymbol{r})$ remains small enough to ensure the consistency of Landau's idea, i.e. the linearity of the fluctuation. In the paramagnetic region, $\varepsilon=(T-T_c)/T_c>0$, this means that in the expression of the free energy density,

$$g(\boldsymbol{r})=g_0-a\eta(\boldsymbol{r})^2+b\eta(\boldsymbol{r})^4+c[\nabla\eta(\boldsymbol{r})]^2, \qquad (2)$$

the term $b\eta(\boldsymbol{r})^4$ may be omitted, or

$$\lambda a\sim b\langle\eta(\boldsymbol{r})^2\rangle \quad (\lambda\ll 1). \qquad (3)$$

Let R change in such a way that λ does not depend on ε. It is assumed at the same time that the term $c[\nabla\eta(\boldsymbol{r})]^2$ contributes always with the same importance as the term, $a\eta(\boldsymbol{r})^2$. This implies that

$$a\sim cK^2. \qquad (4)$$

From (2)$\sim$(4), the fluctuation of the k-th mode $\phi_k=\langle\eta_k\eta_{-k}\rangle$ is obtained in the form $\phi_k\sim K^x f(k,K)$, where $f(k,K)$ has the Ornstein-Zernike form and x is determined by the K-dependence of a. The above form of ϕ_k implies that $K(\varepsilon)$ behaves like the inverse correlation range κ, or $K\sim\varepsilon^{\nu}$.*) Similarly for $T<T_c$, we have $K\sim|\varepsilon|^{\nu'}$. From (3) and (4), and noting that a is proportional to the inverse susceptibility, we obtain

$$a\sim\varepsilon^{\gamma}, \qquad (5a)$$

$$b\sim\lambda\varepsilon^{-d\nu+2\gamma} \qquad (5b)$$

and

$$c\sim\varepsilon^{\gamma-2\nu}. \quad (\varepsilon>0) \qquad (5c)$$

These are our basic equations from which the relations among the critical indices are derived.

(*a*) *Spatial correlation function* $\phi(R)$. Assuming the form of $\phi(R)$ as $\phi(R)\sim\exp(-\kappa R)/R^{d-2+\eta}$ $(R<R_c)$, we obtain $(1/v)\int_v\phi(R)\,d\boldsymbol{R}\sim\kappa^{d-2+\eta}\sim\varepsilon^{\nu(d-2+\eta)}$. Since $\eta(\boldsymbol{r})$ is approximately constant in the region of volume v, we obtain another equation $(1/v)\int_v\phi(R)\,d\boldsymbol{R}\sim\langle\eta(\boldsymbol{r})^2\rangle\sim\varepsilon^{d\nu-\gamma}$. Compar-

*) Throughout the present note, the standard notations are used for the critical indices.[3]

ing these two equations, the relation $\gamma/(2-\eta)=\nu$ is obtained.

(*b*) *Specific heat C.* Expressing $g(\boldsymbol{r})$ in terms of η_K's, the singular part of the free energy Ψ is given by

$$\Psi=-\beta^{-1}\ln\int e^{-\int\beta(g(\boldsymbol{r})-g_0)d\boldsymbol{r}}d\{\eta_K\}$$

$$\sim\kappa^d\ln\varepsilon^{-\gamma}\sim\gamma\varepsilon^{d\nu}\ln\varepsilon. \quad (\varepsilon>0) \qquad (6)$$

Noting that $C\propto-(\partial^2\Psi/\partial\varepsilon^2)$, we obtain $d\nu=2-\alpha$. Similarly for $\varepsilon>0$, the relation $d\nu'=2-\alpha'$ is obtained. In particular for $\alpha=0$, C diverges logarithmically, and if $\gamma=\gamma'$ the same coefficient multiplies the logarithmically, when $\varepsilon>0$ and $\varepsilon<0$. Our argument on the specific heat is quite analogous to that of Douglass.[4)]

(*c*) *Magnetization* $\langle\eta\rangle$. Making a slight conversion of our procedure, let us keep R at a fixed value $\varepsilon_0^{-\nu}$ and change the temperature from ε_0 down to $-\varepsilon_0$. In this case $a(T,\kappa(\varepsilon_0))$ may be considered to be an analytic function of temperature, while $a(T,\kappa(\varepsilon))$ may not. b and c are assumed, following Landau, to be approximately constant near T_c. If we ignore the difference of two temperatures at which $a(T,\kappa(\varepsilon_0))$ and $a(T,\kappa(\varepsilon))$ vanish respectively, we have at $\varepsilon=-\varepsilon_0$ the equations

$$a\sim-\varepsilon_0^{\gamma}, \qquad (7a)$$

$$b\sim\lambda\varepsilon_0^{-d\nu+2\gamma} \qquad (7b)$$

and

$$c\sim\varepsilon_0^{\gamma-2\nu}. \qquad (7c)$$

The above temperature difference may properly be taken into account and leads to results which are identical with Eqs. (7a)$\sim$(7c). Substituting (7a)$\sim$(7c) into (2) and expressing (2) in the form

$$g(\boldsymbol{r})=g_0-a'(\eta(\boldsymbol{r})-\eta_0)^2+b'(\eta(\boldsymbol{r})-\eta_0)^3$$
$$+d'(\eta(\boldsymbol{r})-\eta_0)^4+c'[\nabla(\eta(\boldsymbol{r})-\eta_0)]^2,$$

where $\eta_0=(-a/2b)^{1/2}$, we find that the nonlinear fluctuation terms are negligible at $\varepsilon=-\varepsilon_0$. More accurately, $\lambda a'\sim b'\langle(\eta(\boldsymbol{r})-\eta_0)^2\rangle^{1/2}$ and $\lambda a'\sim d'\langle(\eta(\boldsymbol{r})-\eta_0)^2\rangle$. Therefore $R(\sim\varepsilon_0^{-\nu})$ is just the correlation range at $\varepsilon=-\varepsilon_0$, or $\nu=\nu'$. Since a' is proportional to a, we have a relation $\gamma=\gamma'$. Further we note that the Landau theory should give a correct result at $\varepsilon=-\varepsilon_0$. Then $\eta_0(\sim\varepsilon_0^{(d\nu-\gamma)/2})$ may be replaced by $\langle\eta\rangle$, giving the relation $2\beta=d\nu-\gamma$. Finally we investigate the effect of the external field. In order to produce at $\varepsilon=-\varepsilon_0$ an additional magnetization comparable with the spontaneous magnetization, an external field B of the order of $\varepsilon_0^{\gamma}\langle\eta\rangle(\sim\langle\eta\rangle^{1+(\gamma/\beta)})$ is necessary. This is easily seen when we note that, in the existence of the external field, the term $-B\eta(\boldsymbol{r})$ comes into the expression of $g(\boldsymbol{r})$. The induced magnetization will not be reduced seriously by changing the temperature towords T_c. Therefore the equation $B\sim\langle\eta\rangle^{1+(\gamma/\beta)}$ may hold at T_c, leading to the relation $\delta=1+(\gamma/\beta)$. Thus we have succeeded in deriving all the independent relations among the critical indices known to us. The details will be discussed in a separate paper. The author wishes to express his sincere gratitude to Professor K. Tomita for his valuable advice.

1) L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Pergamon Press, London, 1958), Chap. 14.
2) L. P. Kadanoff, Physics **2** (1966), 263.
3) M. E. Fisher, *Procccdings of the Conference on Critical Phenomena* (N. B. S. Publication, Washington, D. C., 1966), 21.
4) D. H. Douglass, unpublished.