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Improving FFT Frequency Measurement Resolution by Parabolic and Gaussian Spectrum Interpolation

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FFT FREQUENCY MEASUREMENT

Assume that a bandlimited compound signal $s(t)$ has been uniformly sampled with frequency f_s and contains a sinusoidal component $s_{in}(t)$, whose frequency f_{in} is to be measured. The Discrete Fourier Transform (DFT) magnitude spectrum of the signal sample sequence $s[n]=s(nT_s)$, usually computed using the Fast Fourier Transform (FFT) algorithm, is given by

$$S[k] = \left| \sum_{n=0}^{N-1} s[n] \exp\left(-\mathrm{j} \frac{2\pi n k}{N} \right) \right| \tag{1}$$

where $T_s = f_s^{-1}$ is the sampling period and N is the total number of samples. The discrete spectrum (1) is calculated at frequencies that are integer multiples of $\Delta_f = f_s/N$. If $S[k]$ has an observable local maximum corresponding to $s_{in}(t)$ at the spectrum bin k_m, f_{in} can be approximated as

$$f_{in} \cong k_m \, \Delta_f = k_m \frac{f_s}{N} = \frac{k_m}{N\, T_s} = \frac{k_m}{L} \tag{2}$$

where $L = N T_s$ is the sampling duration of the input signal. The largest approximation error $\varepsilon = \max(|k_m \Delta_f - f_{in}|)$ occurs for a frequency located exactly between two bins. This error can be considered as the resolution of the FFT frequency measurement and

$$\varepsilon = \frac{1}{2}\Delta_f = \frac{f_s}{2N} = \frac{1}{2NT_s} = \frac{1}{2L} \tag{3}$$

The resolution can be increased considerably by discrete spectrum interpolation, which has been used in tune measurement systems [1]. The method principle is sketched in Fig. 1. When the frequency component f_{in} is located exactly on a local maximum of the discrete magnitude spectrum at bin k_m, f_{in} can be calculated according to (2) with no error. This is the ideal case shown in Fig. 1a. When f_{in} increases, the amplitude of bin k_m–1 gets smaller and of bin k_m+1 bigger, as presented in Fig. 1b. The spectrum value $S[k_m]$ remains the biggest, until f_{in} is equidistant between bins k_m and k_m+1, see Fig. 1c. In this case, determining f_{in} with (2) leads to the largest error (3). When f_{in} is increased further, as shown in Fig. 1d, bin k_m+1 becomes the biggest. For f_{in} smaller than $k_m \Delta_f$ the analysis is similar resulting in symmetrical cases.

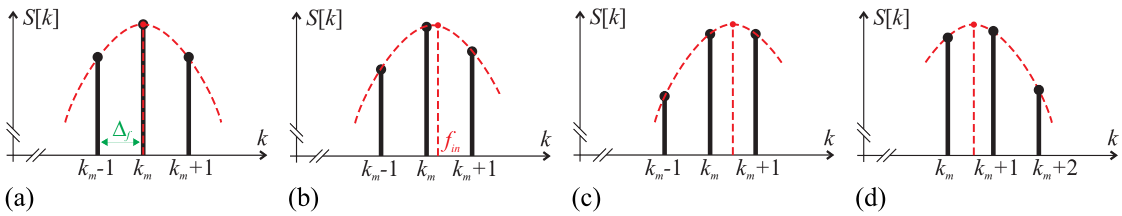

FIGURE 1. The principle of discrete spectrum interpolation.

This example illustrates the fact that continuous frequency f_{in} can be estimated, even if it is located between two bins, by calculating the maximum abscissa of an interpolation curve of the discrete spectrum peak. This maximum can be located between k_m–1/2 and k_m+1/2, where k_m is the index of the biggest bin within the range of interest. If one needs to resolve the cases presented in Fig. 1 and symmetrical ones, it is necessary for the interpolation to have at least three node points. As seen in Fig. 1c, for efficient interpolation the minimal width of the spectral peak is 3 bins.

If in the continuous-time Fourier Transform (FT)

$$S(f) = \int_{-\infty}^{\infty} s(t)\exp(-\mathrm{j}2\pi f t)dt \tag{4}$$

one assumes time domain sampling $t = nT_s$ and frequency domain sampling $f = k\Delta_f$, then the integral can be replaced by a sum with N elements and FT (4) becomes DFT as in (1) when taking T_s and Δ_f as units in the discrete time and frequency domains. For this reason discrete spectra in the paper are considered as continuous spectrum samples taken at multiples of Δ_f, allowing to examine discrete spectra between the bins by means of the FT (4). Replacing the ordinary frequency f in (4) by

$$\varphi = \frac{f}{\Delta_f} = f\frac{N}{f_s} = fNT_s = fL \tag{5}$$

referred to as the normalized frequency, adjusts continuous spectra to the scale of the discrete spectrum bin indexes. It allows expressing discrete and continuous spectra in one convenient scale (e.g. φ =10.5 for frequency lying exactly between bin 10 and 11).

Interpolated spectral peaks have shapes corresponding to the spectrum of the window function, since windowing can be considered as a modulation applied to each sinusoidal component of the analyzed signal. Consequently, the goal of the interpolation methods discussed hereafter is to find the maximum abscissa of such a shape while knowing only its three discrete spectrum samples.

Results of the methods are presented for weighted cosine windows with fast decaying small sidelobes, described by Nuttall [1]

$$w_c(t) = w_r(t) \sum_{i=0}^{3} c_i \cos\left(\frac{2\pi i}{L} t\right) \tag{6}$$

where $w_r(t)$ is the rectangular window with duration from $-L/2$ to $L/2$. Basic properties and coefficients of the weighted cosine windows are listed in Table 1. In the window names of the form aTb, a stands for the number of terms on which the main lobe width depends, and b for the order of the highest continuous derivative, determining the sidelobe asymptotic fall-off. As an example, 4T1 is a four-term window with the main lobe full width of 8 bins ($2a$) and continuous first derivative, resulting in the sidelobe decay of 18 dB/octave ($6b+12$).

Three Gaussian windows are also considered. They are defined as

$$w_g(t) = w_r(t) \exp\left(-r^2 \frac{t^2}{2L^2}\right) \tag{7}$$

where r is the ratio between the window length L and its standard deviation σ, i.e. the Gaussian shape is truncated at $\pm\sigma r/2$.

TABLE 1. Parameters of windows used in the paper.

Window	Main lobe -6 dB width [bin]	Highest sidelobe [dB]	Sidelobe fall-off [dB/oct]	Window coefficients c_0	c_1	c_2	c_3
Hanning	2.00	-31.5	18	1/2	1/2	0	0
Blackman	2.30	-58.1	18	0.42	0.50	0.08	0
3T1	2.36	-64.2	18	0.40897	0.5	0.09103	0
3T3	2.59	-46.7	30	3/8	1/2	1/8	0
4T1	2.69	-93.3	18	0.355768	0.487396	0.144232	0.012604
4T3	2.83	-82.6	30	0.338946	0.481973	0.161054	0.018027
4T5	3.07	-60.9	42	10/32	15/32	6/32	1/32
Gaussian $r=6$	2.26	-56.1	6	-	-	-	-
Gaussian $r=7$	2.62	-71.0	6	-	-	-	-
Gaussian $r=8$	3.00	-87.6	6	-	-	-	-

The window choice is a compromise between a few parameters, three of which are listed in Table 1. The main lobe width determines the minimal distance between two spectral peaks, which still can be resolved. The highest sidelobe and sidelobe fall-off characterize the spectral leakage to the component of interest from respectively close and far interference. As seen in Table 1 and Fig. 2, showing an example of magnitude spectra of four windows, spectral properties of Gaussian windows are inferior to those of the weighted cosine ones. This paper deals only with interpolation on idealized (i.e. not perturbed by noise nor interference) spectra and in this case only the shape of the main lobe is important, exclusively within the interpolation range of ± 1.5 φ units. This range of Fig. 2 is magnified in Fig. 3.

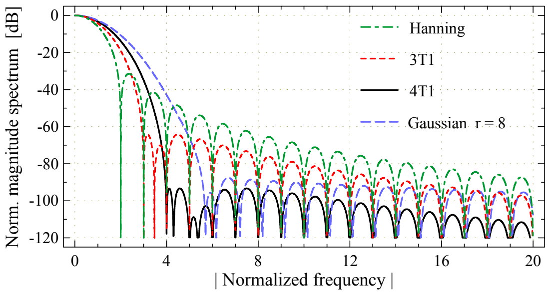

FIGURE 2. Window magnitude spectra.

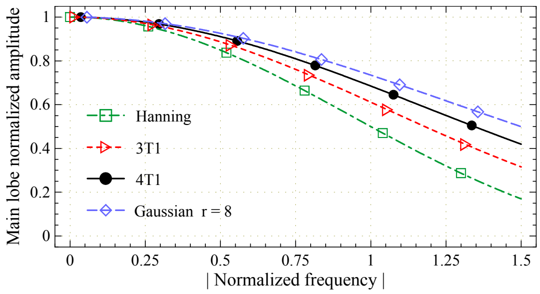

FIGURE 3. Magnified main lobes of Fig. 2.

PARABOLIC INTERPOLATION

As explained before, the goal of the interpolation process is to find the abscissa of the spectral peak maximum, knowing only three peak samples in the form of three bins of the discrete magnitude spectrum. The peak continuous shape, i.e. the window spectrum main lobe, does not need to be accurately reproduced as long as the maximum abscissa of the interpolation shape follows the measured frequency. This is why a simple parabolic interpolation (PI) can improve the discrete spectrum frequency resolution by an important factor even when, as shown in Fig. 3, the shape of the window spectrum main lobe is quite far from a parabola. To quantify this deviation, a window spectrum magnitude $W(\varphi)$ can be expanded into a (normalized to the coefficient upon φ^2) Maclaurin series of the form

$$W(\varphi) \cong a_0 + \varphi^2 + a_4\varphi^4 \tag{8}$$

Coefficient a_4, listed in Table 2, can be used as a measure of this deviation.

Let $S[k]$ be the discrete magnitude spectrum of N samples of a signal $s(t)$ containing a sinusoidal component of frequency $\varphi_{in}=f_{in}L$, and k_m be the index of the biggest bin of the corresponding discrete spectrum peak. Index k_m can be found if the bin constitutes a local maximum within a given range. Fitting a parabola

$$S_p(\varphi) = a(\varphi - \varphi_m)^2 + h \tag{9}$$

through interpolation nodes $S[k_m-1]$, $S[k_m]$, $S[k_m+1]$ and finding the abscissa of the interpolation maximum φ_m, gives

$$\varphi_{in} \cong \varphi_m = k_m + \Delta_m = k_m + \frac{S[k_m+1] - S[k_m-1]}{2\left(2S[k_m] - S[k_m+1] - S[k_m-1]\right)} \tag{10}$$

under condition $2S[k_m] > S[k_m+1] + S[k_m-1]$.

The quantity Δ_m in (10) is the abscissa correction of the discrete spectrum maximum. It is a real number linking both, the discrete and continuous spectra, ranging from $-{}^1/_2$ when $S[k_m-1] = S[k_m]$, to ${}^1/_2$ for $S[k_m+1] = S[k_m]$.

The shape of the interpolated magnitude spectrum peak corresponds to the spectrum of the windowing function applied to the signal samples. If window $w(t)$ with magnitude spectrum $W(\varphi)$ is used, then the interpolation error $E(\varphi_d) = \varphi_m - \varphi_{in}$ is [1]

$$E(\varphi_d) = -\frac{W(\varphi_d+1) - W(\varphi_d-1)}{2\big(2W(\varphi_d) - W(\varphi_d+1) - W(\varphi_d-1)\big)} - \varphi_d \tag{11}$$

where $\varphi_d = \varphi_{in} - k_m$. The interpolation error corresponding to four windows is shown in Fig. 4 and is given in units of Δ_f. For non-perturbed spectra the error is the same around each discrete spectrum bin, i.e. is periodic with period of Δ_f. The interpolation errors for other windows have similar shapes and can be characterized by the error maximum $E_{max} = \max(|E(\varphi_d)|)$ and its abscissa. They are listed in Table 2.

Performance of an interpolation method can be characterized by the interpolation gain, defined as the ratio of the FFT frequency resolution (3) and the method maximum error

$$G = \frac{\varepsilon}{E_{max}} = \frac{\Delta_f}{2E_{max}} \tag{12}$$

As listed in Table 2, the PI can increase the resolution of discrete spectra by more than one order of magnitude.

GAUSSIAN INTERPOLATION

The interpolation gain can be significantly improved by fitting a Gaussian shape to find the abscissa of the spectral peak maximum located between two discrete spectrum bins. Since a Gaussian curve

$$S_g(\varphi) = \exp\big(a'(\varphi - \varphi_m)^2 + h'\big) \tag{13}$$

is a parabola in the logarithmic scale, the Gaussian interpolation (GI) reduces to the PI on the natural logarithm of the magnitude spectrum.

Analogically to PI, the window spectrum main lobe deviation from a Gaussian shape can be quantified by coefficient b_4 of a normalized Maclaurin series expansion

$$\ln W(\varphi) \cong b_0 + \varphi^2 + b_4\varphi^4 \tag{14}$$

The coefficient is listed in Table 2. Its value is much smaller than the corresponding a_4 of (8), especially for Gaussian windows.

The GI can be derived from (10) using logarithmic spectrum values, i.e. the interpolation nodes $S[k_m-1]$, $S[k_m]$, $S[k_m+1]$ are replaced by natural logarithms $\ln(S[k_m-1])$, $\ln(S[k_m])$ and $\ln(S[k_m+1])$. Thus, after logarithm grouping, (10) becomes

$$\varphi_{in} \cong \varphi_m = k_m + \Delta_m = k_m + \frac{\ln\left(\dfrac{S[k_m+1]}{S[k_m-1]}\right)}{2\ln\left(\dfrac{S[k_m]^2}{S[k_m+1]\,S[k_m-1]}\right)} \tag{15}$$

Similarly, the GI error can be derived directly from (11)

$$E(\varphi_d) = -\frac{\ln W(\varphi_d+1) - \ln W(\varphi_d-1)}{2\big(2\ln W(\varphi_d) - \ln W(\varphi_d+1) - \ln W(\varphi_d-1)\big)} - \varphi_d \tag{16}$$

It is plotted in Fig. 5 for four windows. The errors for other windows have similar shapes and are characterized by the error maximum and its abscissa, as listed in Table 2. The interpolation gains are about two orders of magnitude for cosine weighted windows and well beyond three orders of magnitude for the Gaussian window of $r = 8$.

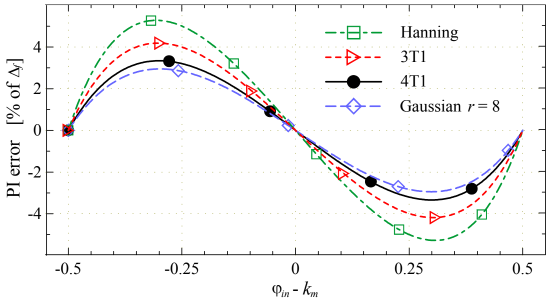

FIGURE 4. Parabolic interpolation errors.

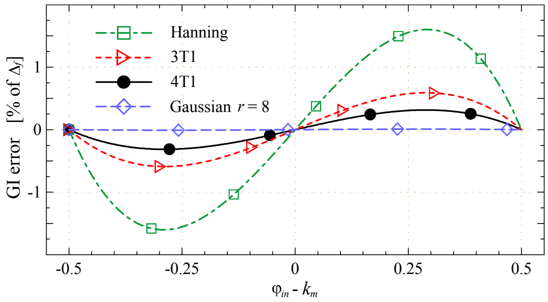

FIGURE 5. Gaussian interpolation errors.

TABLE 2. Parabolic and Gaussian interpolation performance.

Window	**Parabolic interpolation**				**Gaussian interpolation**			
	a_4 in (8)	E_{max} [% of Δ_f]	$\lvert\varphi_{in} - k_m\rvert$ @ E_{max}	Gain	b_4 in (14)	E_{max} [% of Δ_f]	$\lvert\varphi_{in} - k_m\rvert$ @ E_{max}	Gain
Hanning	0.259	5.28	0.307	9.5	0.064	1.60	0.291	31.2
Blackman	0.220	4.38	0.303	11.4	0.031	0.66	0.289	75.3
3T1	0.211	4.18	0.303	11.9	0.028	0.59	0.289	84.7
3T3	0.172	3.40	0.300	14.7	0.025	0.53	0.289	93.7
4T1	0.171	3.34	0.300	15.0	0.016	0.31	0.289	159
4T3	0.153	2.99	0.299	16.7	0.015	0.31	0.289	163
4T5	0.129	2.51	0.297	19.9	0.013	0.27	0.289	187
Gaussian $r=6$	0.252	4.95	0.305	10.1	0.015	0.24	0.281	208
Gaussian $r=7$	0.196	3.80	0.301	13.2	0.0038	0.052	0.279	970
Gaussian $r=8$	0.153	2.95	0.298	17.0	0.0007	0.0087	0.278	5756

MEASUREMENTS

The parabolic and Gaussian interpolation methods were examined with a laboratory setup shown schematically in Fig. 6, based on a tune measurement development system [1]. It consisted of a channel with a 14-bit analog to digital converter (ADC) preceded by an antialias low-pass filter (LPF), a memory for fast ADC sample storage, and a board with a floating-point digital signal processor (DSP). A PC was used to prepare processing software and to download it to the DSP board. To achieve input and clock frequencies of sufficient phase stability, the frequencies were generated by two Direct Digital Synthesizers (DDS), driven by the reference source of a frequency meter. During each acquisition, 2048 ADC samples of the sine wave input signal were stored in the memory. Then, the DSP successively performed windowing, the FFT and the power spectrum calculation. Next, the spectrum bin with the biggest amplitude was found and finally the input frequency f_{in} was calculated according to (10) or (15). The measurements were done around bin k_m=128, with the ADC clock frequency f_s = 1.25 MHz and f_{in} about 78 kHz (mid-range of the frequency span of the setup).

Measurement results are shown in Fig. 7. Crosses mark extreme values from 100 consecutive measurements with the same setup frequencies and dashed lines show theoretical errors as in Fig. 4 and 5. The measurement results were spread due to amplitude noise present in the analyzed spectra, which was converted during the interpolation process into a frequency jitter. This uncertainty was caused mostly by noise present in the input signal, originating in the DDS output 12-bit digital to analog converters. The quantization noise of the ADC might have contributed as well. The observed noise became more visible as the interpolation gain increased. Nevertheless, as seen in Fig. 7d, the total of the systematic and "noise" error was 107 ppm of Δ_f, that is some 65 mHz. The interpolation gain was close to 4700 and relative measurement error of about 0.8 ppm (for f_{in} close to only $f_s/16$).

The interpolation gain of 4700, obtained at the expense of performing the Gaussian interpolation (15) within some microseconds, is equivalent to the frequency resolution of an FFT measurement without interpolation with N samples and the sampling time L multiplied by this factor. For the presented measurement it corresponds to increasing N from 2048 to almost 10^7 and L from 1.6 ms to 7.5 s. Such an amount of data would increase the FFT calculation time by a factor of 10^4, from about 2 ms to 20 s.

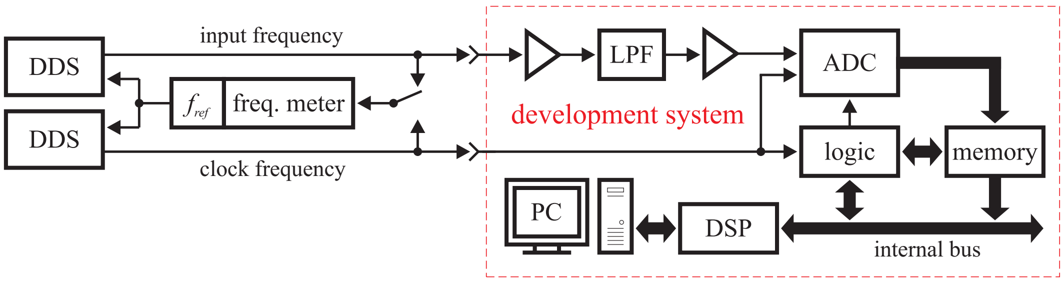

FIGURE 6. Measurement setup.

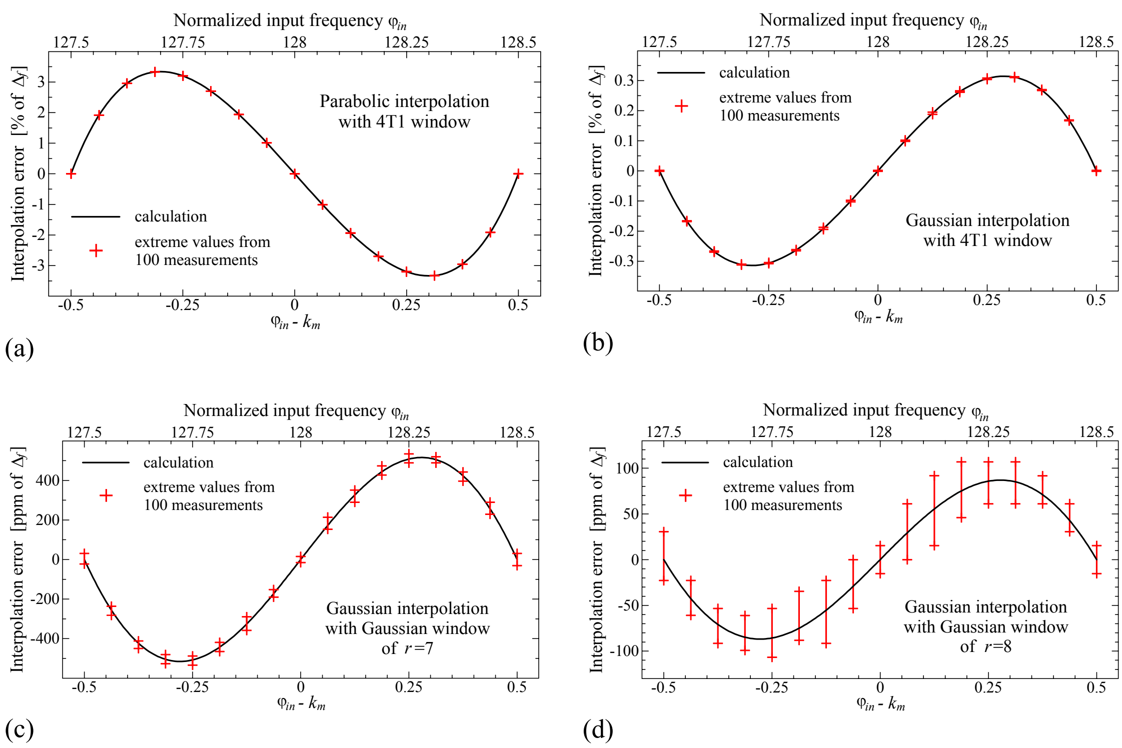

FIGURE 7. Measurement results of the parabolic and Gaussian interpolation methods.

CONCLUSIONS

Theoretical and experimental studies have been undertaken to enhance FFT frequency measurement resolution, using parabolic or Gaussian interpolations on the discrete magnitude spectrum to find abscissa of spectral peaks maxima located between discrete spectrum bins. The interpolation yield strongly depends upon the windowing function used and, to be significant, the window spectrum main lobe should be at least 3 bins wide.

This paper shows that the parabolic interpolation can improve the frequency resolution by more than one order of magnitude. The method computing cost is one division and one multiplication.

The frequency resolution improvement can still be better with Gaussian interpolation. A gain larger than two orders of magnitude is achievable with windows having very good spectral properties and well beyond three orders of magnitude when using steep Gaussian windows. The cost of the Gaussian interpolation is three divisions, three multiplications and calculation of two natural logarithms.

Both interpolation methods assume the measured frequency to be constant during acquisition and the spectral peak corresponding to the measured component to constitute a local maximum in a given band of the input signal spectrum. These methods do not help to resolve nearby spectral peaks and assume bin spacing to be small enough to avoid peak merging.

This paper describes systematic errors of the interpolation methods, assuming ideal discrete spectra. Soon, results will be published, concerning the behavior of the methods when spectra are perturbed by noise, interference from strong components and the exponential decay of the input signal.

A direct application of these methods are FFT-based tune measurement systems. The Gaussian method with 4T1 window is used in such a system for the CERN PS Booster accelerator. In the future, similar systems will be made for the PS and LEIR machines.

Further details concerning the methods and windowing can be found in [1], containing also a list of supplemental references.

ACKNOWLEDGMENTS

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1. M. Gasior, J.L. Gonzalez, "Improving FFT Frequency Measurement Resolution by Parabolic and Gaussian Interpolation", *AB-Note-2004-021 BDI*, CERN, February 2004, with all references therein.