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Improving Networks Reliability

Jamal H. Nouh
Western Michigan University

IMPROVING NETWORKS RELIABILITY

by

Jamal H. Nouh

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IMPROVING NETWORKS RELIABILITY

Jamal H. Nouh, Ph.D.

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One of the basic concepts associated with a network is reliability. In this dissertation some new techniques to improve network reliability are introduced. Several new network structures are defined by adding multiple edges. Chapter I gives a brief overview of the history of network reliability and different reliability measures. It also provides a background for the chapters that follow.

In Chapter II a new sequence associated to the edges of a graph G is defined. The traffic vector of an edge of G of order n is defined as

$$TV(e) = (\pi_1(e), \pi_2(e), \ldots, \pi_{n-1}(e))$$

where $\pi_i(e)$ is the number of paths of length i that contain e and is studied in the case in which the graph is a tree.

A probabilistic graph is a graph $G = (V, E)$ together with a probability assignment to the edges and vertices of G. The vertices and the edges G are subject to failure with probability q, where $0 \leq q \leq 1$. In this dissertation we assume that the vertices of G are absolutely reliable (never fail), but the edges of E are down (i.e., in the fail state) independently with probability q.

In Chapter III we introduce pair-connected reliability of a graph G. It is the expected number of vertices that are connected in a probabilistic graph G. In order to maximize the pair-connected reliability, we use the concept of a traffic vector to characterize those edges in G which are the best choice to be improved, in order to maximize the pair-connected reliability .

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Nouh, Jamal Hussain, Ph.D.

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U·M·I
300 N. Zeeb Rd.
Ann Arbor, MI 48106

In Chapter IV the global reliability of a graph G is defined as the probability that a graph G is connected. Two methods are described for improving the global reliability of a network G.

The first method is multiple edge enhancement and the second is edge improvement or replacement. The first method consists of adding to a given network G, multiple edges between vertices that are already joined by an edge in G. The second method consists of replacing or improving existing edges in G by more reliable ones.

In Chapter V the K-terminal reliability is defined as the probability that, in a given probabilistic graph, the vertices in the set $K \subseteq E(G)$ are connected. The effect of enhancement or replacement edges on the K-terminal reliability for several classes of graphs are stated. Chapter VI is devoted to possible other extensions of this research.

To the memories of my father Hussein and my mother Nowara.

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CHAPTER I

INTRODUCTION

1.1 Definitions, Notation, and Historical Review

Reliability is concerned with the ability of networks to carry out certain network operations. An important step is the identification of necessary network operations. A widely used model for communication networks in which elements (vertices and edges) are subject to failure is that of a probabilistic graph G. Given a graph G with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$, G is called a probabilistic graph, if each element in $V(G)$ and $E(G)$ is assigned a certain probability, say $p(v_i)$ and $p(e_j)$, for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. The number $p(v_i)$ denotes the probability that v_i exists in $V(G)$ and $p(e_j)$ denotes the probability that e_i exists in $E(G)$. If the vertex v_i exists in $V(G)$, then it is considered to be in the up state; otherwise, it is in the failed state. Similarly, if e_i exists in $E(G)$, then it is in the up state; otherwise, it is in the failed state. We call a graph G with n vertices and m edges, an (n,m)-graph. Here it is assumed that vertices are fail safe (i.e., never been in the failed state) but that each edge $e \in E(G)$ is down (that is, in a failed state) independently with probability q, where $0 \leq q \leq 1$, and $p = 1 - q$ will denote the probability that each edge is in the up state. For $S \subseteq E(G)$, the graph G is said to be in the up state S, if the edges that are in the up state are precisely those in S. The spanning subgraph of G induced by a set of edges S is denoted by $< S >$.

Perhaps the most common operation is communication from a source node s to a target node t.

For a probabilistic graph G, and specified nodes s, t, we define the two-terminal reliability to be the probability that there exists at least one st-path in G and the probability will be denoted by $R_{s,t}(G, q)$. In the directed case, the problem is called st-connectedness (see Colbourn [15]). The K-terminal reliability $R_K(G, q)$ is one which measures the probability of having all pairs of vertices in K connected in G, where $K \subseteq V(G)$.

Another common operation in networks is broadcasting. In order to model such an operation, we define the all-terminal reliability to be the probability that for any pair v_1, v_2 of vertices in G, there exists a path from v_1 to v_2 (equivalently, G has at least a set S in the up state and $< S >$ is spanning tree) and probability will be denoted by $R(G, q)$. If G is a directed graph (digraph), then $R_u(G, q)$ is the probability that the digraph G contains a directed path from a vertex u to every other vertex in G. Recently, a new reliability measure called pair-connected reliability has been introduced for graphs (see Siegrist and Slater [6] and Boesch [10]). The pair-connected reliability of a given graph G is the expected number of pairs of connected vertices.

These reliability measures have the following practical application: Assume the graph G models a system in which a subset K of vertices in G represents sufficient processing capability and/or data storage capacity for a processor to execute efficiently. It is important in this case to have the vertices in K connected. The K-terminal reliability measures the probability of having all vertices in K connected. On the other hand, if our graph G represents a communication system, in which it is important for each node to communicate with others, in this case, what is important is the expected number of vertices in the probabilistic graph G which stay connected. Pair-connected reliability measures the expected number of pairs of vertices in G which are connected.

Herein, we survey some of the known results, and graph theory notions which are relevant as models to the analysis and synthesis of the network problem.

For the graph-theoretic ideas and reliability notation, we follow the books by Chartrand and Lesniak [13] and Colbourn [14], respectively.

In addition to the above reliability measures, other reasonable measures can be defined. A general mechanism for defining a reliability problem is therefore in order. Given a probabilistic graph $G = (V, E)$, we define

$$R(G, q, f) = \sum_{S \in \Omega} f(S) \cdot R(S) \tag{1.1}$$

to be a general reliability measure of G, where Ω is the power set of E (that is, the set of all possible states for the system). If $S \in \Omega$, then $R(S)$ is the probability that G is in the up state S, which under the assumption of independent and equal probability of failure q, means $R(S) = p^k q^{m-k}$ where $|S| = k$ and $m = |E(G)|$. Note that (Ω, R) is a probability space, f is a random variable[1] defined in this space, and $R(G, q, f)$ is the expected value of f.

Different choices for the function f provide a variety of reliability measures. For the global reliability (all terminal reliability), the formula in (1.1) becomes

$$R(G, q) = \sum_{S \in \Omega} f(S) \cdot R(S) \tag{1.2}$$

where

$$f(S) = \begin{cases} 1 & \text{if } <S> \text{ is connected} \\ 0 & \text{otherwise.} \end{cases}$$

The function f is dropped from $R(G, q, f)$ for simplicity. Since $f(S)$ takes the value 0 or 1 and (Ω, R) is a probability space, the formula in (1.2) measures the expected value that G is connected, namely it is the probability that G contains a spanning tree, in the up state. Observe that $0 \leq R(G, q) \leq 1$.

For the k-terminal reliability, the function f in (1.1) is defined as follows:

$$f(S) = \begin{cases} 1 & \text{if } <K> \text{ is connected in } G - S \\ 0 & \text{otherwise.} \end{cases}$$

In this case, the function $R(G, q, f)$ is $R_K(G, q)$. If $|K| = 2$, it is called a two-terminal reliability. If $K = \{s, t\}$, then formula (1.1) is written as $R_{s,t}(G, q) =$

[1] a real function defined on Ω.

$R(G,q,f)$. Thus $0 \leq R_{s,t}(G,q) \leq 1$ represents the probability that there exists an st-path in G.

For the pair-connected reliability of G, the function $f(S)$ in (1.1) is denoted by $PC(S)$, and it is equal to the number of pairs of vertices that are connected in $< S >$. In the case of the pair-connected reliability, formula (1.1) can be written in the form

$$R(G,q) = \sum_{S \in \Omega} PC(S) \cdot R(S).$$

This is the expected value of the number of pairs of vertices that are connected in G.

Ball and Provan [9] showed that computing $R_{s,t}(G,q)$ is NP-hard, even if G is a planar graph of maximum degree 3. Moreover it can be shown that computing $PC(G,q)$ is NP-hard in the case where G is planar of maximum degree 4.

All the reliability measures we have introduced are number P-complete problems (see Colbourn [15]). Moreover, Gilbert [20], and Frank and Gaul [19] established formulas for the all-terminal and two-terminal reliability of the complete graph K_n of n vertices.

Theorem 1.1 ([19]) *If $G = K_n$ is the complete graph of n vertices, then*

$$A_n = R(K_n,q) = 1 - \sum_{j=1}^{n-1} C_{j-1}^{n-1} A_j q^{j(n-j)}$$

where $A_j = R(K_j,q)$.

Note that A_n has a recurrence relation in terms of A_j, for $j < n$.

In the case of the two-terminal reliability, the following formula is obtained for the complete graph.

Theorem 1.2 ([19]) *For the complete graph K_n*

$$R_{s,t}(K_n,q) = 1 - \sum_{j=1}^{n-1} C_{j-1}^{n-2} A_j q^{j(n-j)}$$

where $A_j = R(K_j,q)$, and $C_m^n = \frac{n!}{m!(n-m)!}$.

Similar formulas can obtained for complete bipartite graphs.

Two graphs $G_1$and G_2 are in series connection, if G_1 and G_2 have only one vertex in common. Such a connection is denoted by $G_1 \cdot G_2$. In general, a set of graphs $G_1, G_2, \ldots, G_r$ is in series connection if no two graphs have more than one vertex in common, and the only cycles are those in $G_i, i = 1, 2, ..., r$.

The following results can be found in Amin, Siegrist and Slater [2]. Let A be a set of graphs $G_1, G_2, ..., G_r$. If G is a graph obtained from A by series connection, then

$$R(G,q) = \prod_{i=1}^{r} R(G_i, q) \; .$$

For pair connected reliability, suppose $G = G_1 \cdot G_2$ is a graph obtained from G_1 and G_2 by series connection, and let u be the common vertex between G_1 and G_2, define $PC(G_1(u), q)$ to be the expected number of vertices that are connected to u in G_1. This function can be written in the form

$$PC(G_1(u), q) = \sum_{v \in V - \{u\}} E_p(I_{G_1}(u, v))$$

where

$$I_{G_1}(u,v) = \begin{cases} 1 & \text{if } u \text{ is connected to } v \\ 0 & \text{otherwise} \end{cases}$$

and $E_p(I_{G_1}(u, v))$ is the probability that u and v are connected in the probabilistic graph G.

The following result is due to Amin, Siegrist and Slater [3].

Theorem 1.3 ([3]) *If* $G = G_1 \cdot G_2$, *then*

$$PC(G,q) = PC(G_1, q) + PC(G_2, q) + PC(G_1(u), q) \cdot PC(G_2(u), q)$$

For a given tree T, the calculation of $R(T, q)$ can be computed easily. Given a tree T of order n, then $R(T, q) = p^n$, where p is the probability of having the tree T in state $\{e\}$ for all $e \in E(T)$.

In general the calculation of global and K-terminal reliability are not trivial. Moore and Shannon [26] used the following reduction formula in finding K-reliability.

Let G/e be the graph obtained from G by contraction of the edge e, and let $G-e$ be the graph obtained from G after deleting the edge e.

Theorem 1.4 (Reduction Formula) *If G is a graph, then*

$$R_k(G,q) = pR_K(G/e\ ,q) + (1-p)R(G-e\ ,q).$$

For K-terminal reliability, the above reduction formula uses the following transformation:

If e_1 and e_2 are two parallel edges (edges connecting the same vertices), assume e_1 has probability of failure $q_1 = 1-p_1$, and e_2 has probability failure $q_2 = 1-p_2$, then e_1 and e_2 can be replaced by one edge e with reliability (probability of being functional) p, where $p = 1 - q_1 \cdot q_2$.

On the other hand, if $e_1 = u_1 v$, and $e_2 = vu_2$ are incident edges and v is the common vertex with $v \notin K$,then we can replace e_1 and e_2 by e, with $p(e) = p_1 \cdot p_2$; for $v \in K$, e_1 and e_2 can be replaced by e with probability $p = \frac{p_1 p_2}{1-q_1 q_2}$.

For global connectivity, the function $R(G,q)$ in (1.1) can be expressed in terms of the number of induced connected subgraphs of G,

$$R(G,q) = \sum_{r=0}^{|E|} m_r p^r (1-p)^{|E|-r}$$

where m_r is the number of induced connected subgraph in G of size r and $|E|$ is the size of G.

An (n,m)-graph G is said to be uniformly optimally reliable if

$$R(G,q) \geq R(H,q)$$

for all (n,m)-graphs, and all q, $0 < q < 1$. Boesch [11] conjectured that uniformly optimally reliable graphs always exist. In fact, he showed that such graphs exist for classes of graphs with order at most 6.

Opposite to uniformly optimally reliable graphs is the uniformly least reliable graph, it is the one in which every other graph, with the same order and size, is

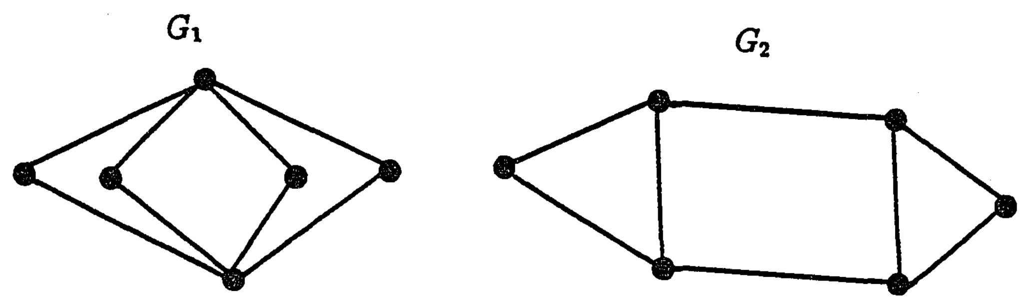

Figure 1.1

more reliable. Boesch [10] observed that all trees with n vertices have the same reliability.

Several authors studying the area of network reliability (see Evans and Smith [17]), have concluded that the synthesis of reliable networks is not a pure graph theory problem, because the decision of which of two graphs is better, is dependent on the probability q. Given two (n,m)-graphs, G_1 and G_2, let $R(G_1, q)$, and $R(G_2, q)$, be the global reliability polynomials of G_1 and G_2, respectively. For different values of q the two functions $R(G_1, q)$, and $R(G_2, q)$ may cross. This can be seen by the following example Kelmans [25] which is the smallest example where two $R(G, q)$ functions cross.

Example : The graphs G_1 and G_2 shown in Figure 1.1 has the global reliability

$$R(G_1, q) = 4q^2(1-q)^6 + 24q^3(1-q)^5 + \sum_{k=4}^{8} C_k^8 p^k (1-p)^{8-k}$$

$$R(G_2, q) = 3q^2(1-q)^6 + 26q^3(1-q)^5 + \sum_{k=4}^{8} C_k^8 q^k (1-q)^{8-k}.$$

Comparing the two functions $R(G_1, q)$ and $R(G_2, q)$, the following can be concluded:

$$R(G_1, q) \quad > \quad R(G_2, q) \quad for \quad 0 < q < \frac{1}{3},$$

$$R(G_1, q) = R(G_2, q) \quad for \quad q = \frac{1}{3},$$
$$R(G_1, q) < R(G_2, q) \quad for \quad \frac{1}{3} < q < 1.$$

The definition of optimally reliable graphs can be extended to pair-connected reliability. For the pair-connected reliability, Amin, Siegrist, and Slater [6] showed, that the star $K_{1,n-1}$ is the uniformly optimally reliable tree with n vertices. In the same paper, it was shown that the path P_n is the uniformly least reliable tree on n vertices. In [6] the same authors have shown that there do not exist uniformly optimal (n,m)-graphs, except in the extreme cases when $m \leq n-1$ or $m \geq C_2^n - 1$, where $C_2^n = \frac{n!}{(n-2)!2!}$. When $m \leq n-1$, it follows easily from the fact that all graphs with $m < n-1$ have the same reliability; namely 0, and the star $K_{1,n-1}$ is the uniformly optimal graph for $m = n-1$. On the other hand, when $m \geq C_2^n - 1$, all (n, m)-graphs are isomorphic.

For trees, the computation of $PC(T, q)$ is straightforward (see Siegrist [29]). For an arbitrary graph G, the distance distribution of G is defined as $D(G) = (d_1(G), d_2(G), ..., d_{n-1}(G))$, where $d_i(G)$ denotes the number of pairs of vertices at distance i. The following result shows how to compute the pair-connected reliability of a tree T from its distance distribution.

Theorem 1.5 *The distance distribution $D(T)$ of a tree T completely determines $PC(T,q)$, namely $PC(T,q) = \sum_{i=1}^{n-1} d_i(T)p^i$.*

For any graph G,

$$R_{s,t}(G, q) \geq p^{dist(s,t)}.$$

where $R_{s,t}(G, q)$ is the two-terminal reliability of G and, dist(s, t) is the distance between s and t. The next result follows.

Theorem 1.6 *For any graph G,*

$$PC(G, q) \geq \sum_{i=1}^{n-1} d_i(G)p^i$$

For trees and series-parallel graphs, efficient algorithms for computing $PC(G,q)$ are described in Amin, Siegrist, and Slater [3]. For certain classes of graphs formulas for $PC(G,q)$ have been determined (see [5], [6]). Here, we present the formula for the n-cycle C_n and the wheel $W_{n+1} = K_1 + C_n$ on $n+1$ vertices.

$$PC(C_n,q) = n\frac{p-p^n}{q} - \frac{n(n-1)}{2}p^n \ ;$$

$$PC(W_{n+1},q) = n[1 - \frac{q^3}{(1-pq)^2}][1-(pq)^n] - n^2\frac{(pq)^{n+1}}{(1-pq)^2} +$$
$$\frac{n(n-1)}{2}[1-\frac{q^3}{(1-pq)^2}]^2 + [\frac{q^4(1+3p^2)+2p^4q^3}{(1-pq)^4}][n\frac{(pq)-(pq)^n}{1-pq} - \frac{n(n-1)}{2}(pq)^n] +$$
$$\frac{pq^4}{(1-pq)^3}[\frac{-n^2(pq)^n + n^2(pq)^{n+1} + npq - n(pq)^{n+1}}{(1-pq)^2} - \frac{n^2(1-n)}{2}(pq)^n]$$
$$+(\frac{pq}{1-pq})^2(\frac{n^2(n-1)(n+1)}{12})(pq)^n \ .$$

A lot of work has been done on network synthesis. The result of such work is important for the design of reliable networks.

If a network is described by its underlying graph G, then the cost of building a network could be measured by the number of edges. We assume the cost of each edge is constant. Such an assumption is sometimes valid in practical problems which allow for a simplified model. Various optimization problems are suggested by this model; for example, one might try to find the maximum value of the edge connectivity λ over the class of all graphs with prescribed values of n and m . This is an example of an extremal graph problem. If κ and λ represent the vertex and edge connectivity of G, respectively, then one may ask for the maximum value of κ or λ in the class of (n,m)-graphs. The first publication related to this topic can be found in Harary [21]. His result is stated in the theorem below. Let $(n, \lambda \geq k)$ denote the size of the graph of order n and edge connectivity at least k. Similarly, let $(n, \kappa \geq k)$ denote the size of the graph of a graph of order n, size m and vertex connectivity at least k.

Theorem 1.7 ([21])

$$min(n, \lambda \geq k) = min(n, \kappa \geq k) = \begin{cases} \lceil nk/2 \rceil & if\, \lambda,\ k \geq 2 \\ n-1 & if\, \lambda = \kappa = 1 \end{cases}$$

Note that $\lceil x \rceil$ denotes the smallest integer not smaller than x; $\lfloor x \rfloor$ is the largest integer not larger than x. The graph used by Harary [21], to prove the theorem is called the *elementary Harary* graph and is denoted by $H(n, k)$. If n vertices are labeled $0, 1, 2, ..., n-1$, then $H(n, k)$ can be constructed by joining each node $i, 0 \leq i \leq n-1$, to the node $i \pm 1, i \pm 2, ..., i \pm \lfloor \frac{k}{2} \rfloor$. The graph $H(n, k)$ has $m = \lceil \frac{nk}{2} \rceil, \kappa = \lambda = k$ and is regular when both n, k are not odd. For further discussion about optimization on graphical parameters, see Harary [21].

In the following section, we study network synthesis from a different perspective.

1.2 Back-up Links in Networks

As mentioned in Section 1.1, most studies in the area of network synthesis concentrate on finding reliable (n,m)-graphs for specified values of n and m. In this dissertation, we describe how to optimize the reliability of a given graph G by means of adding multiple edges to G or by replacing some edges in G, by more reliable edges.

There are many cases where a network already exists, or the logical design of an (n,m)-graph does not follow an optimal reliable graph. For a given network, one may ask the following question: If the reliability of a set of r edges in a given (n,m)-graph G is to be improved, what is the best choice among all subsets of r edges in G, that should be considered in order to optimize a given a reliability measure of G?

In this dissertation we introduce two methods to enhance network reliability: (1) multiple edge enhancement, and (2) edge improvement or replacement.

The first method consists of adding in a given network multiple edges to a network with the restriction that edges may only be added between vertices which are already joined by an edge. The latter method consists of replacing or improving existing edges.

Let G be an (n,m)-graph. If some edges in G are replaced by more reliable edges or enhanced by multiple edges, then the new graph has a new probability assignment to its edges. Such an assignment will be denoted by Γ. If the edges in G are labeled $e_1, e_2, ..., e_m$, then

$$\Gamma = \{(e_1, \Delta p_1), (e_2, \Delta p_2), ..., (e_m, \Delta p_m)\}$$

where the ordered pair $(e_i, \Delta p_i)$ indicates that there is an increase in the reliability of e_i by an amount Δp_i. Under the assumption that edges in G have the same reliability, namely p, the new reliability of the edge e_i will be $p + \Delta p_i$ after enhancement. The restriction of only improving or replacing existing edges or adding multiple edges is to preserve the functionality of the network.

Throughout the discussion, we always assume that edges in the graph G (which represents the network) have the same reliability. Hereafter, the additional edges used to improve reliability of the network are assumed to have the same reliability as the edges in G, unless otherwise stated.

In this dissertation we investigate the improvement of three different network reliability measures: the global reliability, K-terminal reliability, and pair-connected reliability. We present some examples to illustrate the above stated measures, and possible ways to improve those measures.

Example 1: In this example we consider the question: Is there an optimal method, with respect to the global reliability, to add two edges to a path of length 4, P_5 and is the method independent of p?

Let $P_5'[2]$ be any graph obtained from P_5 by adding two multiple edges. Let e_1, e_2, e_3, e_4 be a labeling of the edges of P_5 taken according to their order from one of its end vertices. The answer to the above stated question can be resolved by the following cases:

Case 1: No more than one multiple edge is added between a pair of vertices of P_5. Let G_1 be a graph obtained from P_5 by adding one multiple edge to e_1 and one to e_2. Then $R(G_1, \Gamma_1) = p_1^2 \cdot p^2$, where $p_1 = 2p - p^2$, and

$$\Gamma_1 = \{(e_1, p - p^2), (e_2, p - p^2), (e_3, 0), (e_4, 0)\}.$$

In this case, the choice of the edges in $E(T)$ to be improved has no effect on the global reliability.

Case 2: Allow more than one multiple edge between two vertices of P_5. Let G_2 be the graph created from P_5 by adding two multiple edges to the edge e_1, then $R(G_2, \Gamma_2) = p_2.p^3$, where $p_2 = 3p - 3p^2 + p^3$.

$$\Gamma_2 = \{(e_1, 2p - 3p^2 + p^3), (e_2, 0), (e_3, 0), (e_4, 0)\}$$

Again, as in Case 1, the choice of the edge e_i has no effect on the global reliability. To observe which one of the above cases increases the global reliability the most, consider the following difference:

$$\begin{aligned} \Delta R(p) &= R(G_2, \Gamma_2) - R(G_1, \Gamma_1) \\ &= p_1^2.p^2 - p_2 p^3 = p^2(p_1^2 - p_2 p) \\ &= p^2[p^2(4 - 4p_p^2) - p^2(3 - 3p + p^2)] \\ &= p^4[1 - p]. \end{aligned}$$

For $p \in (0, 1)$, the difference function $\Delta R(p)$ is always positive. Therefore the choice in Case 1 is always better for all values of p.

Example 2: Is there an optimal way with respect to K-terminal reliability to add two multiple edges to the cycle C_6, with vertex set V and edge set E , such that the K-reliability for $K = \{s, t\}$ where s and t are two vertices in V with $d(s, t) = 3$, is maximum? (see Figure 1.2). Suppose the vertices of C_6 are labeled $v_0, v_1, \ldots, v_5$ such that $e_i = v_i v_{(i+1) mod 6}$ are the edges. Without loss of generality we assume that $s = v_0$ and $t = v_3$. Let the path from s to t containing v_1 be P_1 and the path from s to t containing v_5 be P_2. We proceed by considering two cases:

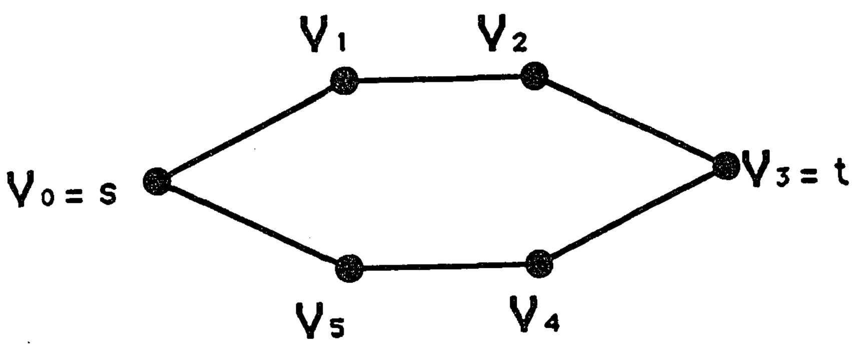

Figure 1.2

Case 1: Both additional edges are added to P_1. Let G_1 be the graph obtained from C_6 by adding one multiple edge to both e_1, e_2 on P_1.

$$R_{s,t}(G_1, \Gamma_1) = 1 - [1 - p_1^2 p][1 - p^3]$$

where

$$p_1 = 2p - p^2, and \quad \Gamma_1 = \{(e_1, p - p^2), (e_2, p - p^2)\} \cup \{(e_i, 0)\}$$

for $i = 1, 2, ..., 6$

Case 2: One edge is added to P_1 and the other is added to P_2.

Let G_2 be the graph obtained from C_6 by adding one multiple edge to both e_1 and e_4 on P_1 and P_2, respectively

$$\begin{aligned} R_{s,t}(G_2, \Gamma_2) &= 1 - [1 - p_1 p^2][1 - p_1 p^2] \\ &= [1 - p_1 p^2]^2 \end{aligned}$$

where $p_1 = 2p - p^2$ and

$$\Gamma_2 = \{(e_1, p - p^2), (e_4, p - p^2)\} \cup \{(e_i, 0)\}$$

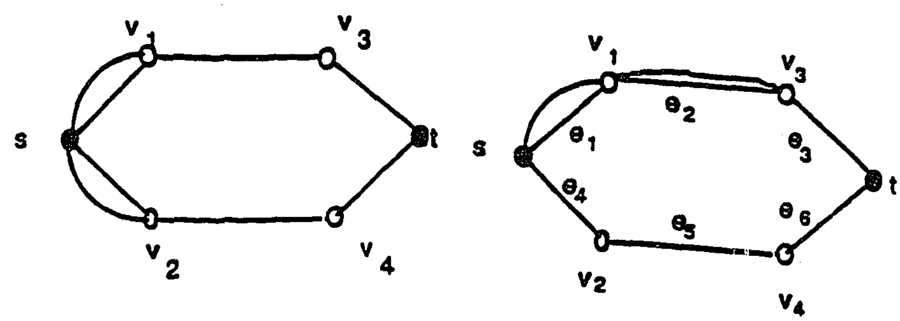

Figure 1.3

for all $i = 1, 2, ..., 6$. The graphs in Figure 1.3 show the above cases.

To see which case is better, consider the following difference:

$$\begin{aligned}
\Delta R(p) &= R_{s,t}(G_1, \Gamma_1) - R_{s,t}(G_2, \Gamma_2) \\
&= [1 - p_1 p^2]^2 - [1 - p^3][1 - P_1^2 p] \\
&= [p^8 - 4p^7 + 4p^6 + 2p^4 - 4p^3 + 1] - [1 - 2p^2 + 2p^5 - p^6] \\
&= p^8 - 4p^7 + 5p^6 - 2p^5 + 2p^4 - 4P^3 + 2p^2.
\end{aligned}$$

The graph of the function $\Delta R(p)$ is shown in Figure 1.4. Observe that $\Delta R(1) = \Delta R(0) = 0$, and $\Delta R(p) > 0$, for $0 < p < 1$. We conclude choice 1 is the better choice, namely improving one path rather than two paths.

If a probabilistic graph G is given, and m is a positive integer $m \leq |E(G)|$, a natural question is: Which is the best set of edges in $E(G)$ of size m to improve, so we obtain the most reliable graph from G with respect to pair-connected reliability? We consider the following example:

Example 3: Let P_5 be the path in Example 1. Let m be the number of extra edges needed to be used in the enhancement. What is the best choice of one edge among $E(P_5)$, so we can increase $PC(P_5, q)$ the most? By Theorem 1.3, the

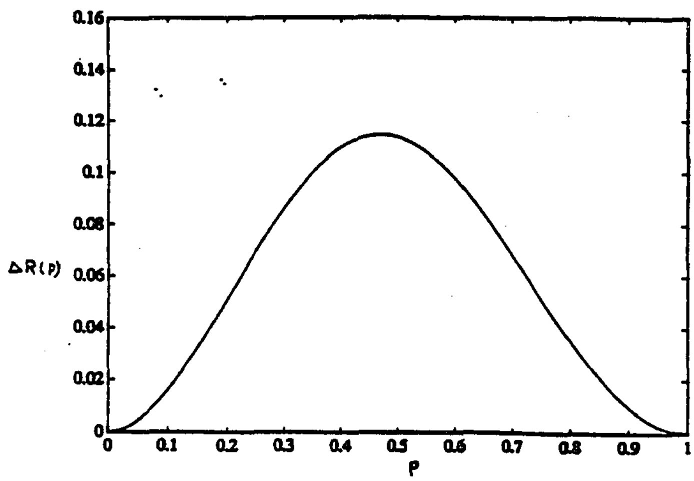

Figure 1.4

pair-connected reliability of the tree T is

$$PC(T,q) = \sum_{i=1}^{n-1} D_i p_i$$

where, D_i is the distance distribution of the vertices in T. The above formula can be modified as follows

$$(T,\Gamma) = \sum_{i=1}^{n-1} R(P_i).$$

Note that the sum is taken over all paths P_i of length i, $R(P_i)$ is the probability that P_i is connected, and Γ is the probability distribution of E. This modification allows us to find the pair-connected reliability of the tree when the edges have different probability assignment. Now label the edges in P_5 as e_1, e_2, e_3, e_4 according to their location from one of the end vertices of T. By symmetry of the edges in P_5, it is necessary to consider the following two cases:

Case 1: Enhancing the edge e_1.

Let G_1 be the graph obtained from the path P_5, by adding new multiple edge on e_1. Then

$$PC(G_1,\Gamma_1) = (p_1 + 3p) + (p_1 p + 2p^2) + (p_1 p^2 + p^3) = (p_1 p^3)$$

where

$$\Gamma_1 = \{(e_1, p - p^2), (e_2, 0), (e_3, 0), (e_4, 0)\}$$

and $p_1 = 2p - p^2$.

Case 2: Enhancing the edge e_2.

Let G_2 be the graph obtained from P_5 by adding one multiple edge to e_2, then

$$PC(G_2, \Gamma_2) = (p_1 + 3p) + (2p_1pp + p^2) + (2p_1p^2) + (p_1p^3)$$

where,

$$\Gamma_2 = \{(e_2, p - p^2), (e_1, 0), (e_3, 0), (e_4, 0)\}.$$

To see the difference between pair-connected reliabilities, consider the following :

$$\begin{aligned} PC(G_2, \Gamma_2) - PC(G_1, \Gamma_1) &= (p_1p + p^2) + (p_1p^2 + p^3) \\ &= p(p_1 - p) + p^2(p_1 - p) \\ &= (p_1 - p)(p + p^2) \\ &\geq 0 \end{aligned}$$

for all $p \in (0,1)$. From the above analysis, we conclude that improving the edge e_2 is the best choice, for all values of p.

In Chapter II we define the traffic vector of an edge for a given graph, and we study the traffic vector distributions of the edges of trees. The set S of traffic vectors is said to be graphic, if there exists a tree T of order $|S| + 1$, such that the set of edges in T has the set S, as its traffic vector distribution. We prove that the problem of whether a set of traffic vectors is graphic or not is an NP-complete problem.

In Chapter III, we use the traffic vectors in improving the pair-connected reliability. In particular, if k edges in T are to be improved, then we use the traffic vector analysis to find a subset S, with $|S| = k$, and $S \subseteq E(T)$, such that improving S, increases the pair-connected reliability of T the most.

In Chapter IV, we study how to improve global reliability of tree networks, unicyclic networks and multi-ring connection networks.

In Chapter V, we present the analysis of improving the two-terminal reliability for parallel and series connection graphs.

In Chapter VI, new reliability measures are presented with some suggestions to improve these reliability measures by using the two methods mentioned in this chapter. In addition, open questions and possible research problems are given.

CHAPTER II

TRAFFIC VECTORS

2.1 Traffic Vectors

A sequence for a graph is simply an invariant which consists of a list of numbers rather than a single number. In this chapter, we would like to introduce the concept of the traffic vector sequence. A number of graph sequences are discussed in literature (see Buckley and Harary [12]). Given a graph G and a set S of edges in G, the induced subgraph on S is denoted by Ind(S). The traffic vector of S will be a sequence of numbers which describes the number of paths of different lengths containing S. In the following chapters, we will use the the traffic vector sequence in investigating of improving network reliability . We shall adopt the notations of Harary and Buckley [12].

Definition 1 *Let $G = (V, E)$ be a graph with order n and let $E \subset S$. The traffic vector distribution of S is defined as:*

$$TV_G(S) = (\pi_1(S), \pi_2(S), \ldots, \pi_{n-1}(S)),$$

where $\pi_i(S)$ denotes, the number of paths of length i in G which contain all the edges of S.

We restrict the study of traffic vectors to acyclic graphs.

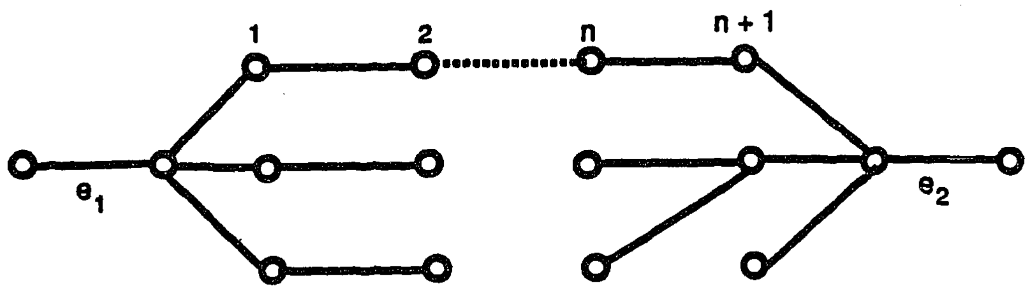

Figure 2.1

Remark 1 *Given a forest F and a set S of edges in F, if k is the length of a minimal path in F which contains S, then $\pi_i(S) = 0$ for all $0 \leq i < k$.*

We will write $\pi_i(e)$ for $\pi_i(S)$ when $S = \{e\}$. If F is known, we can drop the subscript F in $TV_F(S)$ and simply write $TV(S)$. In a forest $F = (V, E)$, if $E = \{e_1, e_2, \ldots, e_{n-1}\}$ then the set of traffic vectors of the edges in E is called the *Traffic Vector Distribution* of F.

An edge $e = uv$ is called an *end edge* if one of the vertices u or v has degree one. In a tree T, the traffic vector of an edge e in T doesn't identify e uniquely. In fact, the following example contains two non-isomorphic edges which have the same traffic vector in T. (see Figure 2.1). We proceed to construct two non-isomorphic trees T_1 and T_2, such that the end edges in both have the same traffic vector (see Figure 2.2).

$$\begin{aligned} TV_T(e_i) &= TV_{T'}(e_i') = (1,1,4,7,4)\ , \ i = 1,2,3,4 \\ TV(e_i) &= TV_{T'}(e_i') = (1,2,4,6,4)\ , \ i = 5,6,7,8 \\ TV(e_9) &= TV(e_{10}) = (1,4,8,4) \end{aligned}$$

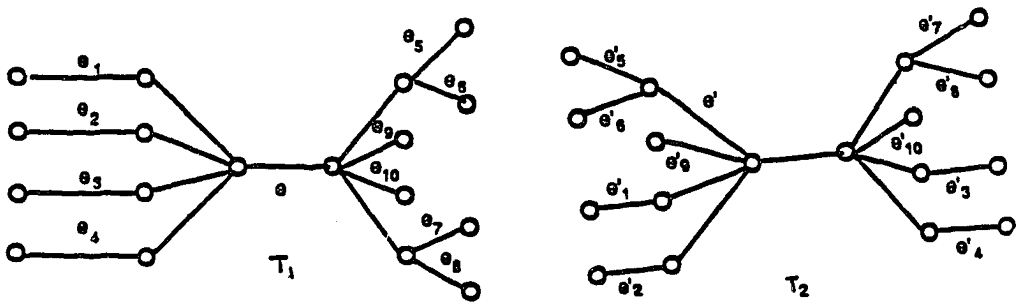

Figure 2.2

The traffic vector distribution of the set of end edges in T_1, and T_2 are

$$\{(1,1,4,7,4)^4,(1,2,4,6,4)^4,(1,4,8,4)^2\},$$

where the notation $[TV]^i$ indicates that there are i edges with the same traffic vector TV. The two trees T_1 and T_2 have different traffic vectors distributions. For example, in the tree T_1, $TV(e) = (1,5,11,11,4)$ but there are no edge in T_2 has this traffic vector. The question of whether the traffic vector distribution of $E(T)$ uniquely determines T remains open.

Let $G = (V, E)$ be a graph of order n, for any $v \in V$, the *Distant Degree Sequence* of v in G is $DDS_G(v) = (d_0(v), d_1(v), \ldots, d_{n-1}(v))$, where $d_i(v)$ is the number of vertices of distant i from v. Note that $d_0(v) = 1$ for all v. Given a tree T and an edge $e = uv$, the subtrees in $T - e$ which contain u and v will be denoted by T_u and T_v respectively. The *Edge Degree Distribution* of an edge e $= uv$ in T is the sequence $EDD(uv) = (\overline{S_1}, S_2)$, where $S_1 = DD_{T_u}(u)$ *and* $S_2 = DD_{T_v}(v), \overline{S_1}$

is sequence S_1 written in reverse order. Note that the sequence $EDD(uv)$ is dependent on the order uv.

Example: Consider a tree T, let $e = uv$ be an edge in T with

$$DD_{T_u}(u) = (1, 2, 3) \ \ and$$

$$DD_{T_v}(v) = (1, 4, 6).$$

The Edge Degree Distribution of $e = uv$ is $EDD(uv) = (3, 2, 1, 1, 4, 6)$

Lemma 1 *Suppose $e = uv$ is an edge in a tree T. If*

$$EDD(uv) = (n_l, n_{l-1}, \ldots, n_0, m_0, m_1, \ldots, m_k), \ \ where \ \ n_0 = m_0 = 1$$

with $k \geq l$, then

$$\pi_i(e) = \sum_{k=0}^{i-1} n_k m_{i-k-1} \quad \forall\, i, \ i = 1, 2, \ldots, (l + k - 1).$$

Proof: This follows directly from the fact that any path of length i containing e consists of a path of length k in T_u and a path of length $i - k - 1$ in T_v together with the edge e. □

Theorem 2.8 *Let T be a tree of order n and $n \geq 2$, let S be a non-empty set in $E(T)$. If k is the smallest integer such that $\pi_k(S) \neq 0$ then*

$$1 \leq \sum_{i=1}^{n-1} \pi_i(S) \leq \lceil \frac{n-k+1}{2} \rceil \lfloor \frac{n-k+1}{2} \rfloor.$$

Proof: Since $\pi_k(S) \neq 0$ this implies that there exists a uv-path P of length k which contains all edges of S. If $k > |S|$, then P contains edges not in S and hence $|E(P)| \geq |S|$..

Let T_u and T_v be the subtrees of T in $T - E(P)$ which contain u and v respectively. The total number of paths containing S is

$$\sum_{i=1}^{n-1} \pi_i(S) = |V(T_u)||V(T_v)| = |V(T_u)|[n - M - |V(T_u)|] \tag{2.1}$$

where

$$M = |V(T)| - |V(T_v)| - |V(T_u)|$$

The fact that $M \geq |E(P)|$, together with $|V(T_u)| \geq 1$ and $|V(T_v)| \geq 1$ implies that $n-1 \geq M \geq k-1$. Observe that $|V(T)| = |V(T_v)| + |V(T_u)| + M$. Therefore $|V(T_v)| = |V(T)| - |V(T_u)| - M$. Letting $|V(T_u)| = x$, equation (2.1) can be written as

$$f(x) = \sum_{i=1}^{n-1} \pi_i(s) = x(n - M - x).$$

For $x \geq 1$, the function $f(x)$ has a minimum value at $x = 1$ a maximum value at $x = (\frac{n-M}{2})$. Since x assumes only integer values, $f(x)$ has a minimum values at $x = 1$ and maximum value at $x = \lfloor\frac{n-M}{2}\rfloor$ or $\lceil\frac{n-M}{2}\rceil$. The fact that $M \geq k-1$ implies that

$$\lceil\frac{n-M}{2}\rceil \leq \lceil\frac{n-k+1}{2}\rceil$$

and

$$\lfloor\frac{n-M}{2}\rfloor \leq \lfloor\frac{n-k+1}{2}\rfloor.$$

Therefore

$$\sum_{i=1}^{n-1} \pi_i(s) \leq \lceil\frac{n-M}{2}\rceil(n - \lfloor\frac{n-M}{2}\rfloor) \leq \lceil\frac{n-k+1}{2}\rceil\lfloor\frac{n-k+1}{2}\rfloor$$

and

$$\sum_{i=1}^{n-1} \pi_i(s) \geq 1(n - m - 1).$$

□

The graphs G_1 and G_2, shown in Figure 2.3 illustrate that the upper bounds given in the above result are sharp.

Corollary 1 *If T is a tree of order n and $S = \{e\} \subset E(T)$ then*

$$n - 1 \leq \sum_{i=1}^{n-1} \pi_i(e) \leq \lceil\frac{n}{2}\rceil\lfloor\frac{n}{2}\rfloor.$$

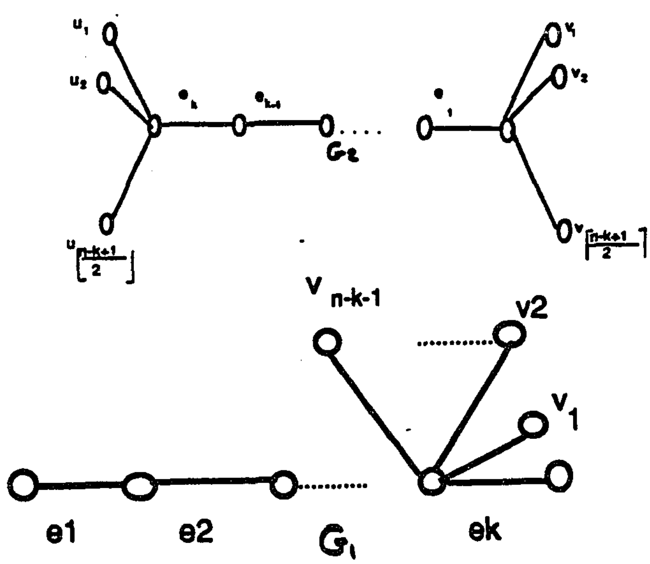

Figure 2.3

Proof: The upper bound follows immediately from Theorem 2.8 with $k = 1$ and with the fact that the size of the minimal path which contains e is one. To show the lower bound, let $e = uv$ with T_u and T_v be the components in $T - e$ which contains u and v respectively. Observe that $\sum_{i=1}^{n-1} \pi_i(e) = |V(T_u)| \cdot |V(T_v)|$. If $|V(T_u)| = x$, then $\sum_{i=1}^{n-1} \pi_i(e) = x(n-x)$. This is a function of x which has maximum value at $x = 1$. Therefore $\sum_{i=1}^{n-1} \pi_i(e) \geq n - 1$. □

Lemma 2 *Let T be a tree of order n. If $S \subset E(T)$, then $\pi_{|S|+1}(S) \leq n - (|S| + 1)$*

Proof: Let P be a minimal uv-path which contains S. Since any path containing S must contain $E(P)$, it follows that $\pi_k(S) = \pi_k(E(P))$ for all k. Therefore,

$$\pi_{|S|+1}(S) \leq \pi_{|E(P)|+1}(E(P)) \leq deg(u) + deg(v) \leq n - (|S| + 1)$$

□

Theorem 2.9 *An edge $e = uv$ in a tree of order n is an end edge, if and only if*

$$\sum_{i=1}^{n-1} \pi_i(e) = (n-1).$$

Proof: Let T be a tree of order 1, then T has no edges and $\sum_{i=1}^{n-1} \pi_i(e) = 0 = 1(1-1)$. Assume $n \geq 2$. For any edge $e = uv \in E(T)$,

$$\sum_{i=1}^{n-1} \pi_i(e) = |V(T_u)| \cdot |V(T_v)|$$

where, T_u and T_v are the component of $T - e$ which contain u, and v respectively. If e is an end edge, then one of the subtrees T_u or T_v is isomorphic to K_1 and

$$\sum_{i=1}^{n-1} \pi_i(e) = 1(n-1).$$

Conversely, let e be an edge in T with $\sum_{i=1}^{n-1} \pi_i(e) = (n-1)$. Thus, $|V(T_u)| \cdot |V(T_v)| = (n-1)$
and $|V(T_u)| + |V(T_v)| = n$. Therefore e is an end edge. □

Definition 2 *Let T be a tree of order n and let S_1 and S_2 be two subsets of $E(T)$ having the same cardinality. The traffic vector $TV(S_1)$* dominates *$TV(S_2)$, if $\forall j$, where $j = 1, 2, \ldots, n-1$*

$$\sum_{i=1}^{j} \pi_i(S_1) \geq \sum_{i=1}^{j} \pi_i(S_2)$$

$TV(S_1)$ strictly dominates $TV(S_2)$, if $TV(S_1)$ dominates $TV(S_2)$ and there exists j such that

$$\sum_{i=1}^{j} \pi_i(S_1) > \sum_{i=1}^{j} \pi_i(S_2).$$

We will simply denote dominates and strictly dominates by $TV(S_1) \geq TV(S_2)$ and $TV(S_1) > TV(S_2)$, respectively.

An edge e_0 in T is called a dominant edge if

$$TV(e_0) > TV(e)$$

for all $e \in E(T)$ An edge set S is called a dominant set, if $\forall j, j = 1, 2, \ldots, n-1$, and $I \in E(G)$ with $|S| = |I|$.

$$\sum_{i=1}^{j} \pi_i(S) \geq \sum_{i=1}^{j} \pi_i(I)$$

2.2 Analysis of Dominant Edges

Theorem 2.10 *If an end edge in a tree T is a dominant edge, then T is isomorphic to $K_{1,n}$.*

Proof: Suppose T is a tree which is not isomorphic to $K_{1,n}$, let e be an end edge of T. The fact that $T \not\cong K_{1,n}$ implies that there exists an edge e_0 in $E(T)$ such that e_0 is incident to e and e_0 is not an end edge.

Claim: $TV(e_0) > TV(e)$. In order to see this, let $A = \{P_1, P_2, \ldots, P_{\pi_i(e)}\}$ be the set of paths of length i which contain e. The set A can be partitioned into two sets A_1 and A_2. The first set, A_1, consists of paths which contain both e and e_0 and the second set, A_2, consists of paths which contain e but not e_0. If $P_i \in A_2$ then P_i can be modified by replacing e by e_0 to become a path containing e_0. Hence, $\pi_i(e_0) \geq \pi_i(e)$. To show $TV(e_0) > TV(e)$, note that if $e = uv$ and $e_0 = vw$, then

$$\pi_2(e) = deg(v) - 1$$

and

$$\pi_2(e_0) = deg(v) + deg(w) - 2 > deg(v) - 1.$$

Hence, $\pi_2(e_0) > \pi_2(e)$, and this implies the result. □

Remark 2 *If P is a minimal path containing a set S in a tree T, then $TV(S) = TV(E(P))$.*

Remark 3 *If T is a tree of order n and S is a set of end edges in T, then*

$$\sum_{i=1}^{n-1} \pi_i(S) = \begin{cases} n-1 & \text{for } |S| = 1 \\ 1 & \text{for } |S| = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Observe that, if e_0 is a dominant edge in a tree T then the number of paths containing e_0 is maximum, namely

$$\sum_{i=1}^{n-1} \pi_i(e_0) \geq \sum_{i=1}^{n-1} \pi_i(e) \ \ \forall e \in E(T).$$

Remark 4 *If* e_1 *and* e_2 *are dominant edges in* T*, then*

$$\pi_i(e_1) = \pi_i(e_2),\ \forall i = 1, 2, \ldots, n-1.$$

Proof: The fact that e_1 and e_2 are dominant edges implies

$$\sum_{i=1}^{j}(\pi_i(e_1) - \pi_i(e_2)) = 0,\ \forall j, j = 1, 2, \ldots, n-1$$

which implies $\pi_i(e_1) = \pi_i(e_2),\ \forall i.$ □

Theorem 2.11 *A complete binary tree* T *has a dominant edge if and only if the height of* T *is at most* 2.

Proof: Let T be a binary tree of order n (note that n must be odd), and height K, $K \geq 3$. Assume to the contrary that e_0 is a dominant edge, then

$$\sum_{i=1}^{n-1} \pi_i(e_0) \geq \sum_{i=1}^{n-1} \pi_i(e),\ \forall e \in E(T)$$

Let v be the root of T and let e_1 and e_2 be the two edges in T which are incident to v. Necessarily, $e_0 = e_1$ or e_2 see figure 2.4. This follows from the fact that

$$\sum_{i=1}^{n-1} \pi_i(e_1) = \sum_{i=1}^{n-1} \pi_i(e_2) = \lceil n/2 \rceil \lfloor n/2 \rfloor = \frac{n^2-1}{4}$$

which is an upper bound on the number of paths containing edge of T when n is odd. Since e_1 and e_2 are symmetric in T, let us assume that $e_0 = e_1$. Let e_3 be an edge incident to e_1 in T , such that $e_3 \neq e_2$. Observe the following:

$$\sum_{i=1}^{2} \pi_i(e_3) = 4 > \sum_{i=1}^{2} \pi_i(e_1) = 3.$$

But this contradicts the fact that e_1 is a dominant edge, therefore T has no dominant edge. □

A tree F_v is called a *fan*, if F_v has exactly one vertex v with degree more than two and the vertex v is called the root.

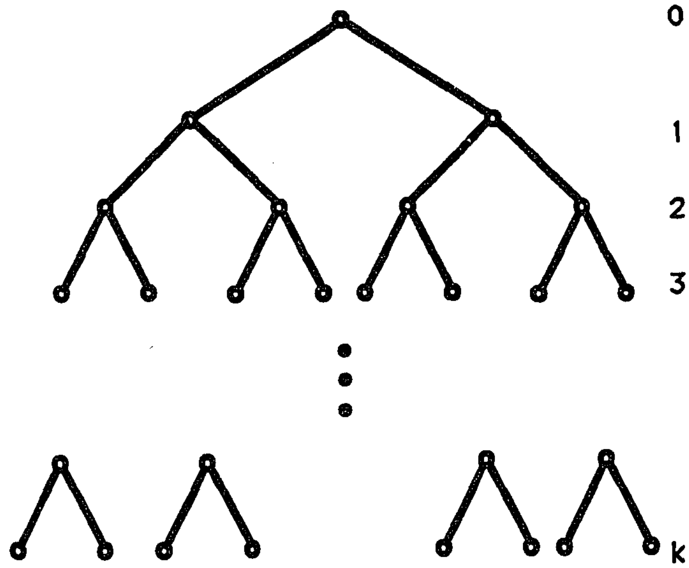

Figure 2.4

Theorem 2.12 *Given a tree T with edge $e = uv$. Let T_u and T_v be the two components in $T - e$ which contain u and v respectively. If T_u and T_v are two isomorphic fans, then e is a strictly dominant edge.*

Proof: Let T be a tree which has the property that $T - e_0$ consists of two identical fans, F_v and F_u and let $e \in E(T) - e_0$. Without loss of generality, we assume $e \in E(F_u)$.

Claim: e_0 strictly dominates e. Let $A = \{P_1, P_2, ..., P_{\pi_i(e)}\}$ be the set of paths with length i which contain e. The set A can be partitioned into two sets, A_1 and A_2, where A_1 consists of paths which contain e and e_0 and A_2 consists of paths containing e but not e_0. By definition, $\pi_i(e) = |A_1| + |A_2|$. Next, we will show that for each path in A_2, there exists a path containing e_0 but not e. If $P \in A_2$, then P contains only edges from $E(F_u)$. By using the second copy F_v, one can construct a path P' corresponding to P which contains e_0 but not e. Therefore, $\pi_i(e_0) \geq \pi_i(e)$.

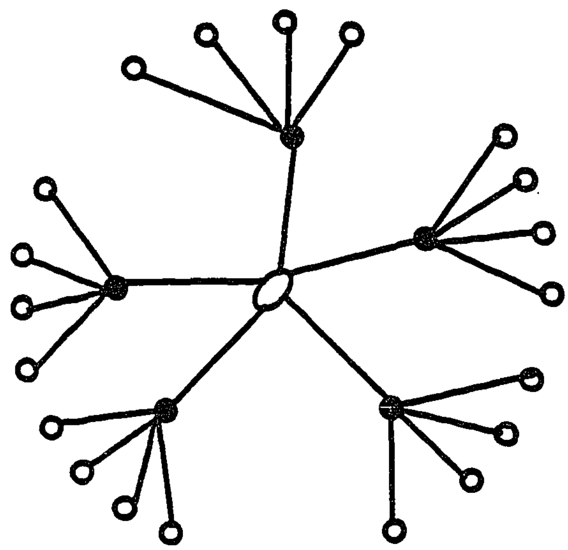

Figure 2.5

In order to show that $TV(e_0) > TV(e)$, let $e = xy$ define T_x and T_y to be the two components in $T - e$ containing x and y , respectively. Since $e \neq e_0$ either $|V(T_x)| < k$ or $|V(T_y)| < k$, where the order of F_v is k. Without loss of generality, let $|V(T_x)| = l < k$.

$$\sum_{i=1}^{n-1} \pi_i(e) = l(n-l) < k^2 = \sum_{i=1}^{n-1} \pi_i(e_0)$$

Therefore $TV(e_0) > TV(e)$. □

The next result shows that there exists a tree T with arbitrary number of dominant edges.

The *Power Star* K_n^m with a tree constructed by identifying every end vertex in $K_{1,n}$ to the center of the star $K_{1,m}$.

Example: The star K_5^4 is shown in Figure 2.5.

Lemma 3 *If n and m are two positive integer such that $n \geq 2m+1$ then there exists a tree T of order n, such that T has exactly m strictly dominant edges.*

Proof: The proof is by construction. Let $l = \lfloor\frac{n-m-1}{m}\rfloor$. Construct a tree T, from the power star K_m^l by attaching to the center of K_m^l an extra r edges, where $r = (n-m-1) - m\lfloor\frac{n-m-1}{m}\rfloor$. It is not difficult to see that the tree T has exactly m strictly dominant edges. □

Next, exhibit the traffic vector distribution for paths. Let P_{n+1} be a path of length n. Label the edges in P_{n+1} according to their location from one of the two end vertices, say $e_1, e_2, \ldots, e_n$. By simple analysis, it is not difficult to see that, for all $i = 1, 2, \ldots, \lceil n/2\rceil$

$$TV(e_i) = (1, 2, \ldots, i-1, i, i, \ldots, i, i-1, \ldots, 2, 1),\ \forall i \leq \lceil n/2\rceil$$

and by the symmetry of the edges on the path

$$TV(e_i) = TV(e_{n-i}),\ \forall i \geq \lceil n/2\rceil.$$

Now, if n is odd then the path P_{n+1} has an edge $e_{\lceil n/2\rceil}$ with traffic vector

$$TV(e_{\lceil n/2\rceil}) = (1, 2, \ldots, \lceil n/2\rceil, \ldots, 2, 1)$$

which dominates $TV(e_i)$, $\forall i, i = 1, 2, \ldots, n$. For even n, the path P_{n+1} has two dominant edges, namely $e_{\lceil n/2\rceil}$ and $e_{\lfloor n/2\rfloor}$ which have the traffic vector

$$TV(e_{\lceil n/2\rceil}) = TV(e_{\lfloor n/2\rfloor}) = (1, 2, \ldots, \lceil n/2\rceil, \lceil n/2\rceil, \ldots, 2, 1).$$

Let $T = (V, E)$ be a tree and let u and $v \in V$. A contraction of T on u, v is a tree T' which is the tree constructed from T in the following way: If P represents the path between u, and v in T, then define T_u and T_v to be the two components in $T - E(P)$ which contain u, and v respectively. Now construct T' by joining the vertices u and v in the two trees T_u, and T_v by an edge. If uv is an edge, then the contraction of T on u and v, is T. Figure 2.6 shows a contraction of a tree on two vertices u and v. The contraction defined above exists and is unique for any two vertices of a tree.

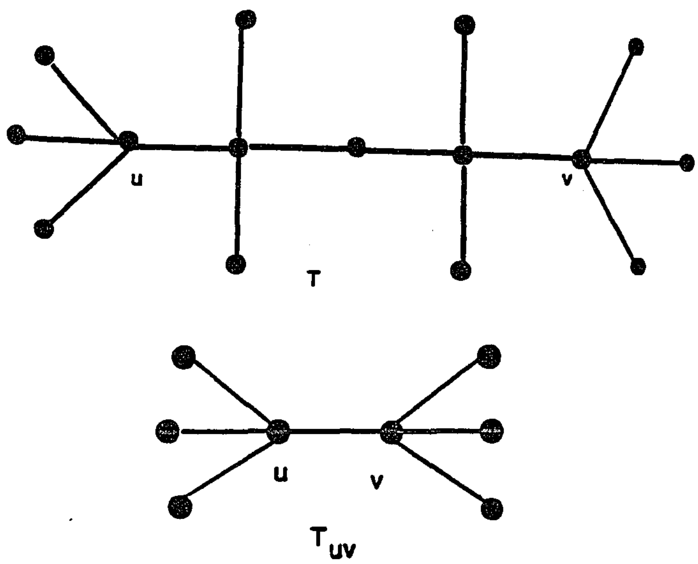

Figure 2.6

Remark 5 *Given a tree T, let P_{k+1} be uv-path in T, then*

$$\pi_{k+j}(E(P_{k+1})) \text{ in } T \text{ is equal to } \pi_{j+1}(uv) \text{ in } T_{uv}$$

where T_{uv} is the tree obtained from T, by a contraction on u and v.

Remark 6 *If P_{k+1} is the minimal path which contains the edge set S in a tree T, then $TV(E(P_{k+1})) = TV(S)$.*

Theorem 2.13 *Let S_1 and S_2 be two subsets of $E(T)$ with $|S_1| = |S_2|$ and let P_1 and P_2 be the minimal paths which contain S_1 and S_2 respectively. If P_1 is a subpath of P_2, then $TV(S_1) \geq TV(S_2)$.*

Proof: Using Remark 6 it is enough to show that $TV(E(P_1)) \geq TV(E(P_2))$. Let P_1 be a subpath of P_2, then every path of length i which contains $E(P_2)$ must contain $E(P_1)$. Hence, $\pi_i(E(P_1)) \geq \pi_i(E(P_2))$. In fact if P_1 is a proper subgraph of P_2 then $TV(E(P_1)) > TV(E(P_2))$. To show this, note that $\pi_i(E(P_2)) = 0$ for

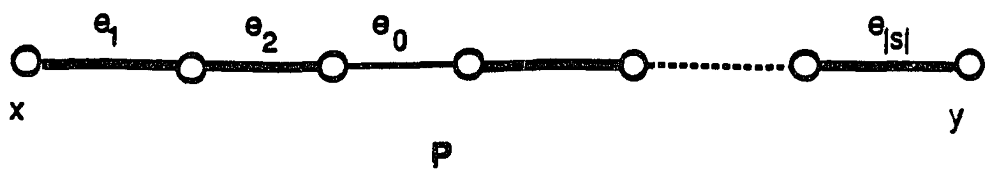

Figure 2.7

all i, $0 \leq i < k$ where $k = |E(P_2)|$. Since P_1 is a proper subpath of P_2 there exists an edge e in $E(P_2) - E(P_1)$. Therefore $|E(P_1)| \leq k - 1$ which implies

$$\pi_{k-1}(E(P_1)) \geq 1 > \pi_{k-1}(E(P_2)) = 0$$

□

Theorem 2.14 *Let T be a tree, with diameter d and let $S \subseteq E(T)$ with $|S| \leq d$. If Ind(S) is disconnected, then S is not a dominant edge set.*

Proof: Let T be a tree with $S \subseteq E(T)$. The fact that T has diameter d implies that there exist a path of length L in T for all $L, L \leq d$.

Case 1: There is no minimal path in T which contains S. In this case $TV(S)$ is the zero vector, since the traffic vector of a path P of length $|S|$ in T has a nonzero traffic vector, therefore, $TV(P)$ strictly dominates $TV(S)$, and S is not a dominant edge set.

Case 2: There is a minimal xy-path P in T which contains S. Since Ind(S) is not connected, there exists an edge e in $E(P)$ such that $e \notin S$ (see Figure 2.7). Let $e_1 = xv$ be the edge in $E(P)$ which is incident to x. The set $(S - \{e_1\}) \cup \{e_0\}) = S'$ has cardinality equal to $|S|$. If the minimal path which contain S' is P', then P' is a subpath of P. By theorem 2.14, $TV(E(P')) > TV(E(P))$ and $TV(S') > TV(S)$. Again, this implies that S is not a dominant edge set. □.

Remark 7 *If S is a dominant set, then Ind(S) contains a path of length $|S|$.*

Caveat lector: a dominant edge set has different meaning than a set of dominant edges. A set S_1 in a graph G is a dominant set if $TV(S_1) \geq TV(I)$,
for all $I \subseteq E(G)$, with $|S_1| = |I|$. On the other hand, the set S_2 is called a set of dominant edge, if $S_2 \subseteq E(G)$ and for all $e \in S_2$, e is dominant edge.

2.3 Characterizing the Set of Dominant Edges

Let S be the set of dominant edges. A natural question to ask is: what is the structure of the induced subgraph on S?

Theorem 2.15 *Let S be a subset of the edges of a tree T with the property that $e \in S$ implies that the total number of paths containing e, is maximum, then Ind(S) is a connected subtree.*

Proof: If $|S| = 1$, then Ind(S) is connected. Let $|S| > 1$ and assume to the contrary that Ind(S) is disconnected. There exists two edges $e_1 = xy$ and $e_2 = uv$ such that e_1 and $e_2 \in S$ and $Ind(e_1, e_2)$ is disconnected graph. Without loss of generality let x and u be the two vertices in $\{x, y, u, v\}$ with maximum distance between them. Let e_0 be an edge on xu-path, incident to e_1 such that $e_0 \notin S$. (see Figure 2.8).

Define the fallowing:

T_x be the component in $T - e_1$ which contain x.
T_y be the component in $T - \{e_1, e_0\}$ which contain y.
T_c be the component in $T - \{e_2, e_0\}$ which contain u.
T_v be the component in $T - e_2$ which contain v.

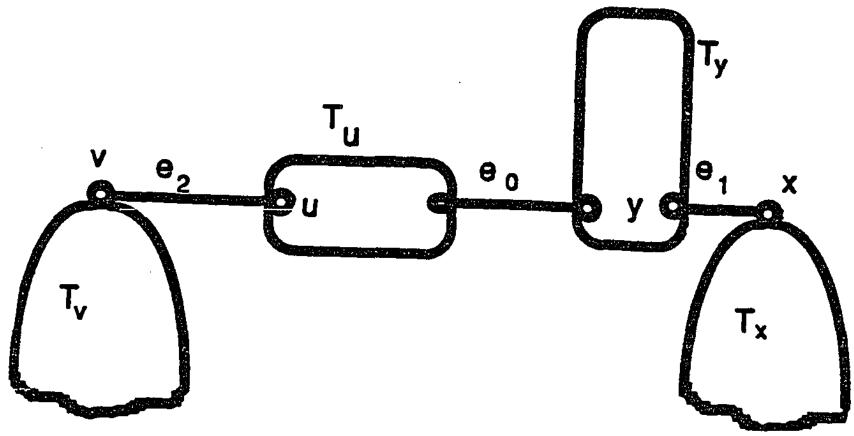

Figure 2.8

Let $a = |V(T_x)|$, $b = |V(T_y)|$, $c = |V(T_u)|$ and $d = |V(T_v)|$. Since $e_1, e_2 \in S$,

$$\sum_{i=1}^{n-1} \pi(e_1) = \sum_{i=1}^{n-1} \pi(e_2) \tag{2.2}$$

Claim 1: $a = d$.

In order to see this, observe that there are $a \cdot d$ different paths containing both e_1, e_2. By using and equation (2.2), we have the following:

$$a(b+c) = d(b+c)$$

This implies $a = d$.

Claim 2: $b \cdot c = 0$.

Assume to the contrary that $b \geq 1$, and,$c \geq 1$, then

$$\sum_{i=1}^{n-1} \pi_i(e_0) = (d+c)(a+b)$$

$$\sum_{i=1}^{n-1} \pi_i(e_1) = a(b+c+d).$$

By claim (1), $a = d$. Therefore, together with the fact that e_1 is a dominant edge we have:

$$\sum_{i=1}^{n-1} \pi_i(e_1) - \sum_{i=1}^{n-1} \pi_i(e_0) = (a+c)(a+b)+$$
$$a(b+c+a) = -bc \leq 0.$$

Therefore, $bc = 0$. Hence, e_0 does not exist. Therefore, $Ind(S)$ is a connected subtree. □.

Theorem 2.16 *If T is a tree, and S is a set of edges in T with the property that $e \in S$, then e is contained in a maximum number of paths, then $Ind(S) \cong K_{1,|S|}$.*

Proof: For $|S| = 1$ or 2, the result follows from Theorem 2.15. Let $|S| \geq 3$, by the previous theorem $< S >$ is connected. Assume to the contrary that Ind(S) $\ncong K_{1,|S|}$. In this case there exists a path P of length 3 in Ind(S) , say $V(P) = \{u_1, u_2, u_3, u_4\}$, and $E(P) = \{e_1, e_0, e_2\}$. If e_0 is the edge which is incident to e_1 and e_2, then by using exactly the same argument as in Theorem 2.15, we can show that $\sum_{i=1}^{n-1} \pi_i(e_0) > \sum_{i=1}^{n-1} \pi_i(e_1)$, which contradicts the fact that e_0 and e_1 belong to the same total number of paths. Therefore, $< S > \cong K_{1,|S|}$. □

Corollary 2 *If T is a tree and S is a set of dominant edges in T, then Ind(S) $\cong K_{1,|S|}$.*

We will denote the set of dominant edges in a tree T by SDE(T).

Corollary 3 *The only tree T which has $SDE(T) = E(T)$ is the tree $K_{1,n}$.*

Theorem 2.17 *Let v be a vertex with degree m, in a tree of order n where $m \geq 1$, and let $N(v) = \{v_1, v_2, \ldots, v_m\}$ be the neighbor set of v. If $DD_T(v_i) = DD_T(v_j)$ for all $1 \leq i, j \leq m$, then for evry $1 \leq i \leq n$*

$$\sum_{i=1}^{n-1} \pi_i(vv_i) = max\{\sum_{i=1}^{n-1} \pi_i(e_i)|e \in T\}$$

$1 \leq i \leq m\}$ *has the property that $\sum_{i=1}^{n-1} \pi_i(e_i)$ is maximum in T.*

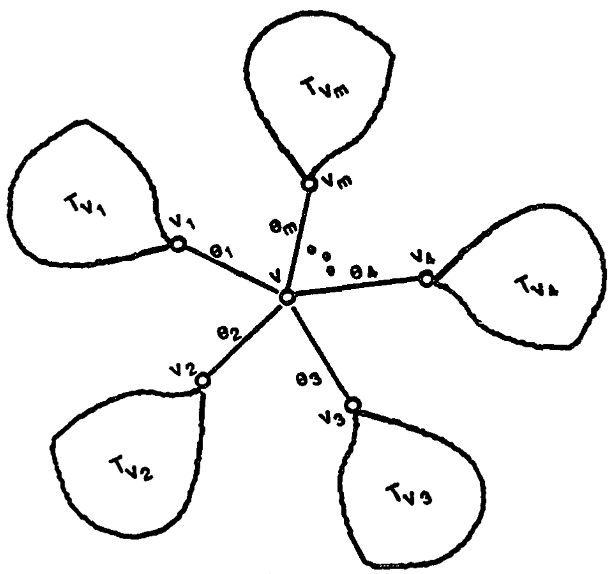

Figure 2.9

Proof:

Let T_{v_1}, T_{v_2}, and T_v be the components in $T - \{vv_1, vv_2\}$ which contain v_1, v_2, and v, respectively. Let

$$DD_{T_{v_1}}(v_1) = (m_0, m_1, \ldots, m_l)$$

and

$$DD_{T_{v_2}}(v_2) = (n_0, n_1, \ldots n_k).$$

First: We will show that $DD_{T_{v_1}}(v_1) = DD_{T_{v_2}}(v_2)$. Consider the subtree T_0 in T which consists of the subtrees T_{v_1} and T_{v_2}, together with the edges vv_1 and vv_2. (see Figure 2.9).

Claim: $DD_{T_0}(v_1) = DD_{T_0}(v_2)$.

To show the claim, observe that, every vertex is the same distance from v_1 as it is from v_2. Using this observation together with the fact that $DD_T(v_1) = DD_T(v_2)$, we see that the number of vertices at distance i from v_1 in T_{v_1} is the same as the number of vertices at distance i from v_2 in T_{v_2}, namely $DD_{T_0}(v_1) = DD_{T_0}(v_2)$.

Therefore $(m_0, m_1, \ldots, m_l) = (n_0, n_1, \ldots n_k)$, which implies $l = k$. Since vv_1, vv_2 are arbitrarily taken from $N(v)$, the Distance Distribution of v_i in T_{v_i} is the same for all $i = 1, 2, \ldots, m$.

Second: Let e_0 be an edge of T which is not incident to v.

Claim:

$$\sum_{i=1}^{n-1} \pi_i(vv_1) > \sum_{i=1}^{n-1} \pi_i(e_0).$$

Let

$$x = |V(T_{v_1})|$$

since $DD_{T_{v_1}}(v_1) = DD_{T_{v_2}}(v_2)$

$$|V(T_{v_1})| = |V(T_{v_2})|.$$

Since v_1 and v_2 are taken arbitrarily from $N(v)$, it follows that,

$$|V(T_{v_i})| = |V(T_{v_j})|$$

for all $1 \leq i, j \leq m$. Therefore

$$\sum_{i=1}^{n-1} \pi_i(vv_1) = x[(m-1)x + 1] = x(n-x) = x(mx - x + 1).$$

The fact $e_0 \in E(T) - S$ implies that $e_0 \in T_{v_j}$ for some $1 \leq j \leq m$. Let $e_0 = uw$, let T_u, and T_w be the two components in $T - e_0$ which contain u, w respectively, and $|V(T_u)| = x_1$ and $|V(T_w)| = x_2$. Assume $x_1 \leq x_2$ then $x_1 \leq x$. Therefore

$$|V(T_w)| = x_2 > (m-1)x + 1.$$

The following is true:

$$\sum_{i=1}^{n-1} \pi_i(e_0) = x_1 x_2 = x_1(n - x_1) = x_1(mx + 1 - x_1).$$

Suppose $H(x) = x(n-x)$, then $H(x)$ is an increasing function of x in $[0, n/2]$. Therefore, if $x_1 < x$, then $H(x) > H(x_1)$. Thus

$$\sum_{i=1}^{n-1} \pi_i(vv_1) > \sum_{i=1}^{n-1} \pi_i(e_0).$$

Hence, e_i has maximum number of paths containing e_i in T, for all $i = 1, 2, \ldots, m$. □

In the above theorem, the fact that $DD_T(v_i) = DD_T(v_j)$ for all neighbors of a vertex v in T does not imply that the edge $e_i = vv_i$ is the dominant edge in T. This can be shown with the example of a complete binary tree.

Theorem 2.18 *Let T be a tree of order n and let $K_{1,m}$ be a subtree in T where $m > 1$. Suppose $E(K_{1,m}) = \{e_i, e_i = vu_i, 1 \leq i \leq m\}$ be a set of dominant edges, then*

$$DD_T(u_i) = DD_T(u_j)$$

$$\forall\, i\ and\ j; 1 \leq i\ \ and\ \ j \leq n-1.$$

Proof:

Let T_{u_1} be the component in $T - e_1$ which contains u_1 *and* T_{u_2} be the component in $T - e_2$ which contains u_2. By an argument similar to the one in the previous theorem, we see that

$$TV_T(e_1) = TV_T(e_2) \quad implies \;\; TV_{T'}(e_1) = TV_{T'}(e_2) \tag{2.3}$$

where T' is the tree consisting of the two subtrees T_{u_1} and T_{u_2} together with the two edges u_1v and vu_2. The fact that e_1 and e_2 are dominant edges, implies that

$$\sum_{i=1}^{n-1} \pi_i(e_1) = \sum_{i=1}^{n-1} \pi_i(e_2).$$

Therefore, $|V(T_{u_1})| = |V(T_{u_2})|$. Let $DD_{T_{u_1}}(u_1) = (n_0, n_1, \ldots, n_k)$ and $DD_{T_{u_2}}(u_2) = (m_0, m_1, \ldots, m_l)$. By using (2.3) we see that $\pi_2(e_1) = \pi_2(e_2)$, and $n_1 + 1 = m_1 + 1$. We use induction on j to show that $n_j = m_j$ for all $0 \leq j \leq min(k, l)$. The result is true for $N = 1, 2$. Assume the result is true for all $j; j < N$. In the tree T'

$$\pi_{N+1}(e_1) = \sum_{i=0}^{i=N} n_{N-i} m_i \;\; and \;\; \pi_{N+1}(e_2) = \sum_{n=0}^{N} m_{N-i} n_i.$$

Since $\pi_{N+1}(e_1) = \pi_{N+1}(e_2)$ we have

$$m_N + \sum_{i=0}^{N-1} n_{N-i} m_i = n_N + \sum_{i=0}^{N-1} n_i m_{N-i}.$$

By the induction hypothesis

$$\sum_{i=0}^{N-1} n_{N-i} m_i = \sum_{i=0}^{N-1} n_i m_{N-i}.$$

Therefore $n_N = m_N$.

Claim: $k = l$

Assume to the contrary that $k > l$, then

$$\sum_{i=0}^{k-1} n_{k-i-1} m_i = \sum_{i=0}^{l-1} n_{l-i-1} m_i$$

that is $n_j = 0$ for all $K < j < L$ which results in a contradiction. Let T_v be the component in $T - \{e_1, e_2\}$ which contains v. By adding the distance degree sequence of v in the tree T_v, properly, one can see that $DD_T(u_i) = DD_T(u_j)$ $\forall i, j$. □

Corollary 4 *If T is the tree in the previous theorem $DD_{T_{u_i}}(u_i) = DD_{T_{u_j}}(u_j)$ for all $1 \leq i, j \leq m$.*

Corollary 5 *Let $S = \{e_1, e_2, ..., e_m\}$ be a set of a dominant edges. If $e_i = (v, u_i)$ $i = 1, 2, ..., m$, then $EDS(e_i) = EDS(e_j)$, for all i, j, where $1 \leq i, j \leq m$.*

Proof: This follows from Theorem 2.18 together with the definition of an Edge Degree Sequence EDS(e). □

Corollary 6 *Let T be a tree of order n, let*

$$S = \{vu_i, i = 1, 2, 3, ..., m\}$$

be a maximal dominant edge set. If $e_0 = wv$ is an edge in T then $\forall i = 1, 2, ..., m$

$$|V(T_w)| < |V(T_{u_i})|$$

where T_w, and T_{u_i} are the components in $T - (S \cup \{e_0\})$ which contain w and u_i, respectively.

Proof: By Corallary 4, $DD_{T_{u_i}}(u_i) = DD_{T_{u_j}}(u_j)$, $\forall i,j : 0 \le i,j \le m$. Therefore, $V(T_{u_i})| = |V(T_{u_j})|$ for all $0 \le i,j \le m$. If

$$|V(T_w)| = L \ \ and \ \ |V(T_{u_i})| = N$$

then

$$\sum_{i=1}^{n-1} \pi_i(vu_k) = N((m-1)N + L + 1)$$

for all $1 \le k \le m$ and

$$\sum_{i=1}^{n-1} \pi_i(vw) = L(mN + 1).$$

Since vu_i is dominant edge, then

$$N((m-1)N + L + 1) > L(mN + 1)$$

$$mN^2 - N^2 + N > mNL - NL + L$$

$$N(mN - N + 1) > L(mN - N + 1).$$

Since N and m are greater than 1, and $mN - N + 1 \neq 0$, it follows that $N > L$. □

By Theorem 2.14 and Remark 7 we know that a dominant set is always a path thus, throughout the following discussion we will refer to a dominant set S as a path $P_{|S|+1}$.

2.4 Dominant Edges and the Center of a Tree

The center of a tree is the set of all vertices in T with minimum eccentricity; (see Chartrand and Lesniak [13]). If Ind(S(T)) and Ind(C(T)) denote the induced subtrees of the dominant edge set and the center of T, respectively, then we can show that Ind(S(T)) and Ind(C(T)) may can arbitrarily far apart in T.

Lemma 4 *Let T be a tree constructed by identifying the center of $K_{1,n}$ with an end vertex of the path P_{m+1}. If $n \ge m-3$ then T has a dominant edge.*

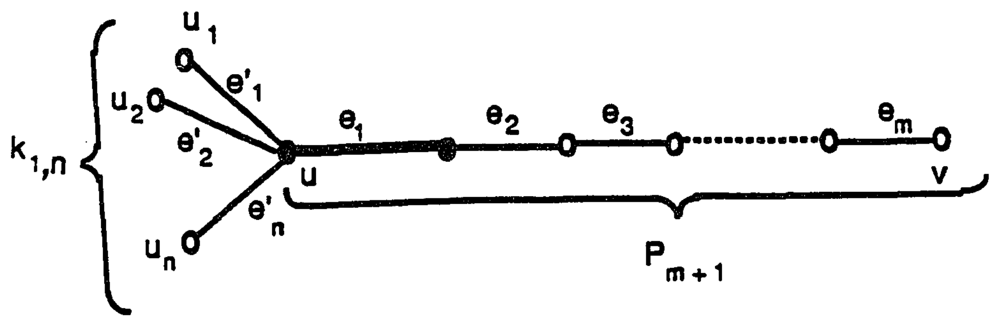

Figure 2.10

Proof: Label the vertices in $E(P_{m+1})$ according to their distance from the center of the star $K_{1,n}$, i.e. $E(P_{m+1}) = \{e_1, e_2, ..., e_m\}$. For any e_i , $i = 1, 2, 3, ..., m$

$$TV(e_i) = (1, 2, \ldots, i, n+i, \ldots, n+i, n+i-1, \ldots, n+1, n)$$

$$TV(e_1) = (1, n+1, \ldots, n+1, n)$$

where $TV(e_i)$ and $TV(e_1)$ are vectors of dimension $m+1$. It is not difficult to see that the following is true, for all k, $0 \leq k \leq m+1$

$$\sum_{i=1}^{k} \pi_i(e_1) - \pi_i(e_i) \geq 0$$

if and only if, $n \geq m - 2i + 1$ (see Figure 2.10). Therefore, e_1 is a dominant edge in T whenever $n \geq m - 2i + 1$. □

Let G_1 and G_2 be two subgraphs of a graph G. Then the distance for G_1 to G_2 denoted by $d(G_1, G_2)$, is defined to be the minimum. $\{d(u_i, v_i) | u_i \in V(G_1), v_i \in V(G_2)\}$.

Theorem 2.19 *Given a positive integer $n \in Z^+$, there exists a tree T such that*

$$d(Ind(S(T)), Ind(C(T))) \geq n$$

Proof Consider the tree constructed in Lemma 4, the center of T can be located with arbitrary distance from the center v of $K_{1,n}$. This can be done simply by choosing m to be large enough, so that the center of T is at distance n from v. Now choose n so that $n \geq m-3$ and Lemma 4 will guarantee that e_1 is a dominant edge. Observe that the location of the center at the tree T is independent of n for $n \geq 1$. □

Given a tree T, let v be a vertex in $V(T)$. The neighbor vertices of v denoted by $N(v)$ is the set of all vertices adjacent to v.

Theorem 2.20 *Let v be a vertex of a tree T of order n and let*

$$K_{1,m} = \quad Ind(\{v\} \cup N(v)).$$

If $m \geq 2$ and $E(K_{1,m})$ is the dominant edge set in T, then v is the only center of T.

Proof Let T be a tree of order n and let $v \in V(T)$, with $N(v) = \{u_1, u_2, ..., u_m\}$. Define T_{u_i} to be the tree in $T - \{vu_i\}$ which contains u_i. By Theorem 2.18,

$$DD_{T_{u_i}}(u_i) = DD_{T_j}(u_j)$$

for all $1 \leq i, j \leq m$. Let $e(x)$ be the eccentricity of the vertex x. $e(u_i) = e(u_j), \forall i, j, 1 \leq i, j \leq m$. By the symmetric structure of $K_{1,m}$, $e(v) = e(u_1) + 1$. We will show that if $x \in V(T) - \{v\}$, then $e(x) > e(v)$. Since $x \neq v$, , it follows that $x \in V(T_{u_i})$ for some $i, 1 \leq i \leq m$. Let y be a vertex in $V(T_{u_j})$, $j \neq i$, such that $d(y, v) = e(v)$. Such a vertex exists, since $m \geq 2$. Now

$$d(x, y) = d(x, v) + d(v, y) = d(x, v) + 1 + e(v) > e(v) + 1.$$

Therefore $e(x) \geq e(v)$; hence x is not a center vertex. □

2.5 Traffic Sequences

A sequence $(\pi_1, \pi_2, \ldots, \pi_k)$ is called a traffic sequence, if $\pi_1 = 1$ and π_i is a positive integer, for all $i = 1, 2, ..., k$. Such a sequence is denoted by TV. Recall

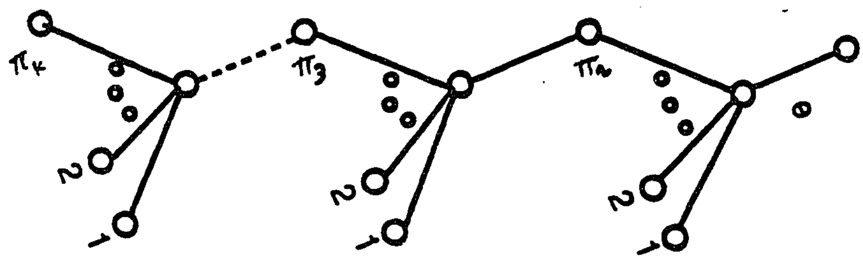

Figure 2.11

that for a set S of traffic sequence, S is realizable (graphical), if there exists a forest F which has S as its traffic vector distribution.

Lemma 5 *For any traffic sequence TV, there exists a tree T such that T has an edge e_0 with $TV(e_0) = TV$.*

Proof Let $TV = (1, \pi_2, \pi_3, \ldots, \pi_k)$. The required tree can be constructed as in Figure 2.11.□

For any set S of traffic vectors, the above lemma suggests a method for constructing a forest F with a subset A in $E(F)$, such that A has the same traffic vector distribution as S.

Remark 8 *Given a set of traffic vectors $S = \{TV_1, TV_2, \ldots, TV_n\}$. Where $TV_i = (1, \pi_{i2}, \pi_{i3}, ..., \pi_{in})$, $i = 1, 2, ..., m$. The following is true:*

(1) S is not graphic if there exists $TV_i \in S$, such that, $\pi_j > 0$ for some $j \geq m+1$.

(2) If S contains TV_i, with $\pi_{ij} > (m/2)^2$, then S is not graphic.

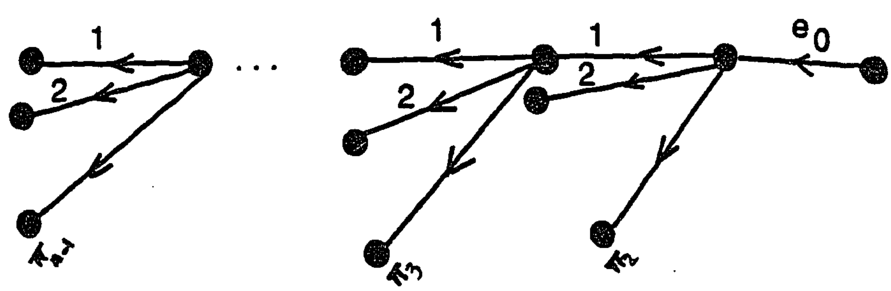

Figure 2.12

Definition 3 *Let e be an edge in a directed tree of order n. The directed traffic vector is defined to be* $\vec{TV}(e) = (\pi_1(e), \pi_2(e), \ldots, \pi_{n-1}(e))$*, where* $\pi_i(e)$ *is the number of directed path in T which contains e.*

Remark 9 *Let TV be a given traffic sequence. There exists a directed tree T which has an edge e, having TV as a traffic vector.*

Proof: Let $TV = (\pi_1, \pi_2, \ldots, \pi_{n-1})$. We will construct the directed tree T as shown in Figure 2.12. It is not difficult to see that the edge e_0 in $E(T)$ has TV as its traffic vector. □

Theorem 2.21 *Given a set of directed traffic sequence S, there exists a directed tree T, such that E(T) has a subset* E_1*, with* $|E_1| = |S|$*, and the directed traffic vector distribution of the edges in* E_1 *is the same as S.*

Proof: We will show that the required set S is graphic by construction. Let $S = \{TV_1, TV_2, \ldots, TV_{|S|}\}$ where $TV_i = (\pi_{i1}, \pi_{i2}, \ldots, \pi_{i(n-1)})$. The graph in Figure 2.13

illustrates a tree with a subset S' of edges with $|S'| = |S|$ and the set $S' = \{e_1, e_2, \ldots, e_{|S|}\}$. This can be seen in Figure 2.13. It is not difficult to see that the set of edges $\{e_1, e_2, \ldots, e_{|S|}\}$ has traffic vector distributions exactly as in S. □

Our next result shows that the problem of determining whether a set of traffic sequence is graphic or not is NP-complete. This is accomplished by showing that the known NP-complete partition problem (see [24]) is reducible to a particularization of the decision problem for tree realizability when given a set of traffic sequence.

Even Partition Problem:

Instant: A finite set A with $|A| = 2k$, and a 'size' $S(a) \in Z^+$ for each $a \in A$.

Question: Is there a subset A' of A such that $|A'| = k$ and $\sum_{a \in A'} S(a) = \sum_{a \in (A-A')} S(a)$?

The Realization Problem1G:

Input: A set $S = \{\pi_{i1}, \pi_{i2}, \ldots, \pi_{in} | \pi_{ij}$ is a positive integer for $i \leq i \leq m \, and \, 1 \leq j \leq n\}$.

Question: Does there exist a tree with S as its traffic vector sequence to its edges?.

Theorem 2.22 *The Realization Problem is a NP-complete problem.*

Proof: It is enough to show that the realization for trees of diameter 5 is a NP-complete problem. We reduce the partition problem to the realization problem.

Instant of Partition Problem:

A set $A = \{n_i | 1 \leq i \leq 2K\}$ and n_i is a positiv integer, and $S(n_i) = n_i$. Define $2N = \sum_{i=1}^{2K} n_i$ and $n = 2K + 2N + 2$. We define the following set of traffic sequences:

$$C = \{(1, 2K, 2N + K^2, 2NK, N^2)$$

$$(1, n_i + K, n_iK + K + N - n_i, n_i(K + N - n_i) + N, n_iN), \; 1 \leq i \leq 2k$$

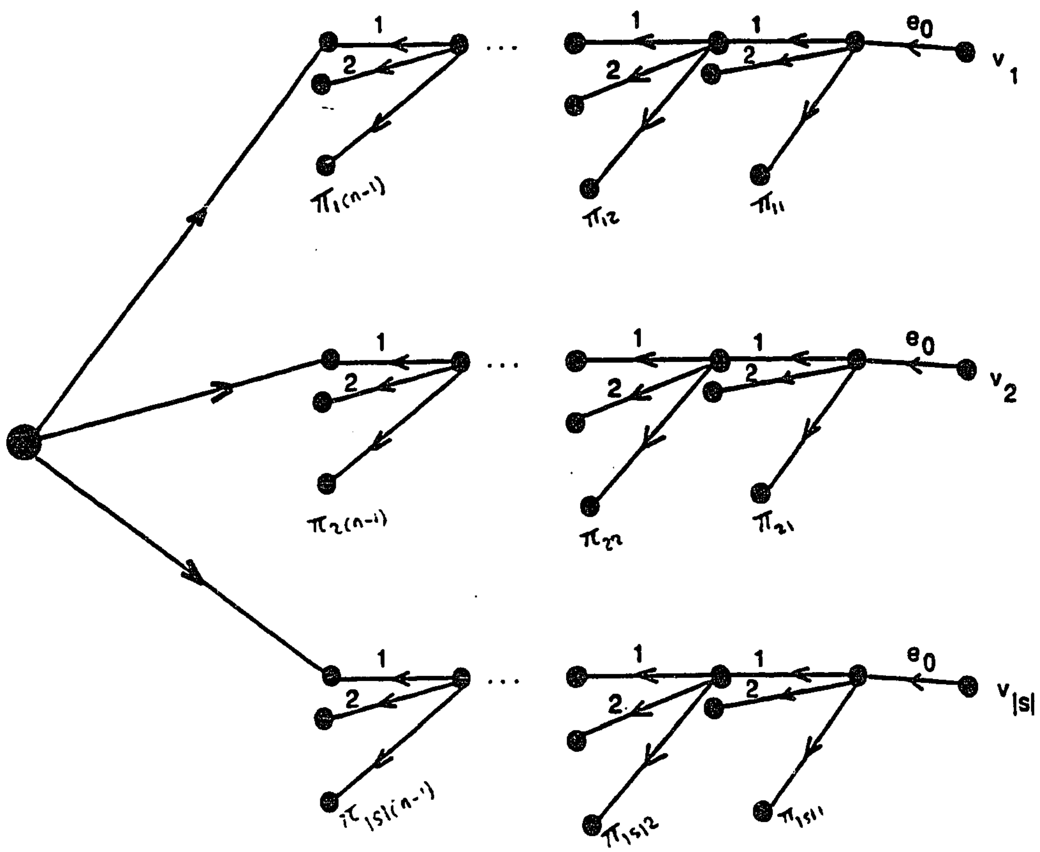

Figure 2.13

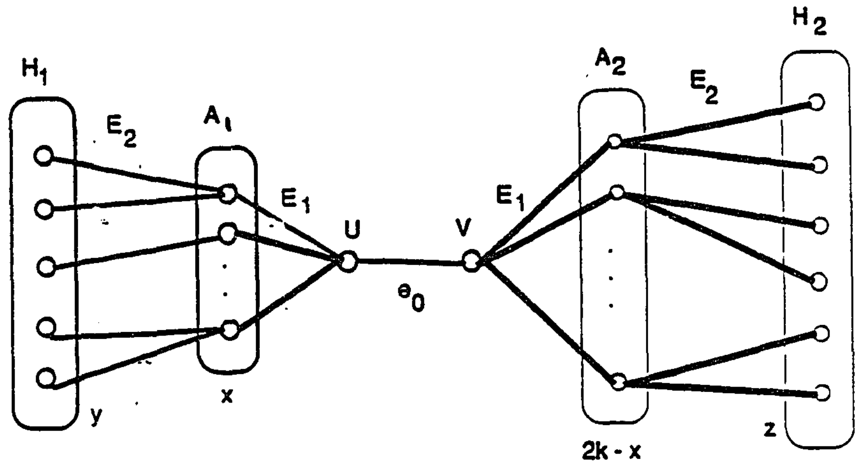

Figure 2.14

$$(1, n_i, K, K + N - n_i, N)^{n_i},\ 1 \leq i \leq 2K\}$$

where $\sum_{i=1}^{2K} n_i = 2N$ and $n = 2N + 2K + 2$ and $(TV)^i$ means there are i occurrence of the traffic vector TV.

We show that the set C is graphic if and only if the answer to the partition problem is yes. Let A be the instant of the partition problem with answer no. We show that C is not graphic.

Assume C is graphic, and let T be a tree of order n and C is the traffic vectors of its edges. Let $e_0 = uv$ be the edge in T with $TV(e_0) = (1, 2K, 2N + K^2, 2NK, N^2)$ since $TV(e_0)$ has only five components, there are three possible structures for T.

First structure:

Let T_u and T_v be the two components $T - \{e_0\}$ which contain u and v respectively. Let A_1 and H_1 be the set of vertices of distance 1 and 2 from u respectively and let A_2 and H_2 be the set of vertices with distance 1 and 2 from v respectively (see Figure 2.14). If $|A_1| = x$, then $|A_2| = 2K - x$. Let $|H_1| = y$, $|H_2| = z$. By

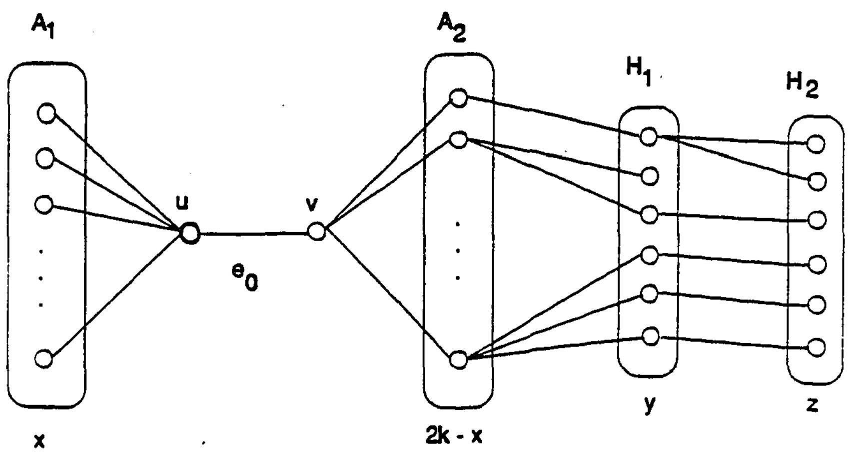

Figure 2.15

considering $TV(e_0)$ it follows

$y + x(2K - x) + z = 2N + K$

$yz = N^2$

$y(2K - x) + x(z) = 2NK.$

Solving the above system implies the following: $x = K$ and $y = z = N$.

Second structure:

Define T_u and T_v as in the first structure and let A_2, H_1 and H_2 be the set of vertices in T_v with distance 1, 2 and 3 from v. Let A_1 be the set of vertices of distance 1 from u in the the tree T_u (see Figure 2.15). Let$|A_1| = x$, $|A_2| = 2K - x$ and let $|H_1| = y$, $|H_2| = z$. Since

$$\sum_{i=1}^{5} \pi_i(e_0) = (N + K + 1)^2 = (n/2)^2.$$

This implies that $(x + 1)(2K - x + y + z + 1) = (N + K + 1)^2$. Moreover, observe that

$$\pi_3(e_0) = x(2K - x) + y = 2N + K^2$$

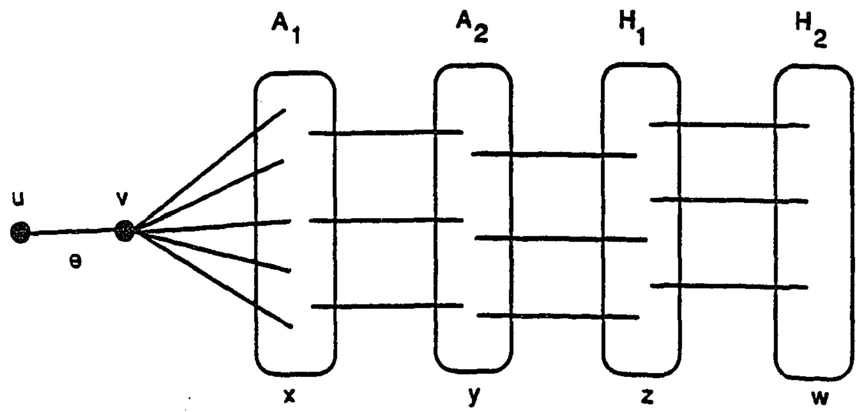

Figure 2.16

and $z + xy = 2NK$. It can be shown that the above system of equations implies that $y \leq 0$, which is a contradiction.

Third structure:

Define the trees T_u and T_v as above. This structure has $|V(T_u) = 1$. Let A_1, A_2, H_1 and H_2 be the set of vertices with distance 1,2,3 and 4 respectively from v (see Figure 2.16).

$$\sum \pi_i(e_0) = (K + N + 1)^2 \neq 1(2K + 2N + 1)$$

for all $K, N \in z^+$. Hence, this is impossible.

We consider the first structure: Since $|H_1| = N_1$ and $|A_1| = K$, there are exactly K edges from H_1 to A_1. Let $E_1 = \{e_1, e_2, \ldots, e_K\}$ be the set of edges from u to A_1 and $E_2 = \{e'_1, e'_2, \ldots, e'_N\}$ be the set of edges from A_1 to H_1. Since edges in E_1 are sharing the same vertex u, this implies that $\pi_2(e_i) = K + r_i$, where r_i is the number of edges in E_2 which are incident to e_i. By the structure of the tree $\sum r_i = N$.

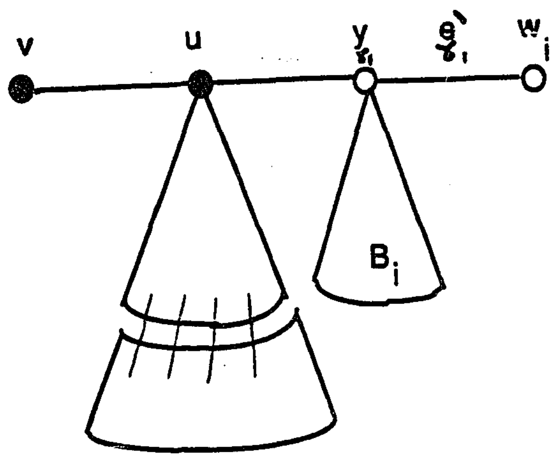

Figure 2.17

Next we claim that the edges in E_2 are those with a traffic vector equal to $(1, n_i, K, K+N-n_i, N)$. This follows from the fact that

$$1 + n_i + k + K + N - n_i + N = 2N + 2N + 1 = 1(n-1).$$

By the result of Chapter II the edge corresponding to this traffic vector must be an end edge (see Figure 2.17). There are exactly $2N$ end edges in T, and exactly

$$\sum_{i=1}^{n-1} n_i = 2N$$

as traffic vectors. Therefore, the end edges in T must have traffic vectors of the form $(1, n_i, K, K+N-n_i, N)$. Clearly, if such a tree exists, then the $2K$ traffic vectors which are of the form

$$(1, n_i + K, N_i K + K + N - n_i, n_i(K + N - n_i) + N, n_i N)$$

must be assigned to the edges in E_1. Since there are $2K$ edges in E_1, this assignment is one to one.

Let $e'_{\gamma_1} \in E_2$ be an edge connecting a vertex w_i in H_1 to a vertex y_{γ_1} in A_1. Let B_{γ_1} be the set of vertices in E_2 which are adjacent to y_{γ_1}, for e'_{γ_1}), $\pi_2(e'_{\gamma_1}) = n_{\gamma_1}$ where $n_{\gamma_1} = n_i$ for some i, $1 \leq i \leq 2K$. Therefore, $|B_{\gamma_1}| = n_{\gamma_1} - 1$; namely, y_{γ_1} is adjacent to n_{γ_1} vertices in H_1. By a similar argument, we can show that each vertex y_i in A_1 is adjacent to exactly n_i.

Observe that if y_{n_i} is a vertex in A_1 which is incident to the edges $e_1, e_2, \ldots, e_{n_i}$ in H_1, then

$$TV(e_s) = TV(e_t), \forall s,t; 1 \leq s,t \leq n_i$$

and $\pi_2(e_s) = n_i \; \forall 1 \leq s \leq n_i$. This follows from the fact that we have exactly n_i traffic vectors of those. But this implies that the set of traffic vectors $n_i -$ $(1, n_i, k, k+N-n_i, N)$ form a star with a center: say $y_i \in A_1$. There are k sets of those, hence we have k stars, with k distinct centers $\{y_1, y_2, \ldots, y_k\}$ in A_1.

By exactly similar analysis, the structure of the edges from A_2 to H_2 is similar to the one from A_1 to H_1. It consists of K distinct vertices, each representing a center of a star K_{1,n_i} for some $1 \leq n_i \leq 2K$.

Next, we study the structure of the edges connecting the vertex u to vertices in A_1. Let e_{γ_1} be the edge uy. Then we have degree $u = k+1$, degree $y_{\gamma_1} = n_{\gamma_1} + 1$. Therefore, $\pi_2(e_{\gamma_1}) = n_{\gamma_1} + k$ (see Figure 2.18). Thus

$$TV(e_{\gamma_1}) = (1, n_{\gamma_1} + K, n_iK + K + N - n_i, n_i(K + N - n_i) + N, n_iN). \quad (2.4)$$

Next we show that, if there is a partition to A, then C is graphic. There are exactly K edges from u to A_1, and K edges from v to A_2, where each edge has a traffic vector exactly as in 2.4. The fact that $|H_1| = |H_2| = N$ implies that there exist $n_{\gamma_1}, ..., n_{\gamma_K}$ such that $\sum_{i=1}^{K} n_{\gamma_i} = N$. Therefore the set $\{n_{\gamma_1}, ..., n_{\gamma_K}\}$ is partition to the set A, which is a contradiction.

Suppose $A = \{n_1, n_2, \ldots, n_{2k}\}$ has a partition. Let

$$A_1 = \{n_1, n_2, \ldots, n_k\}$$

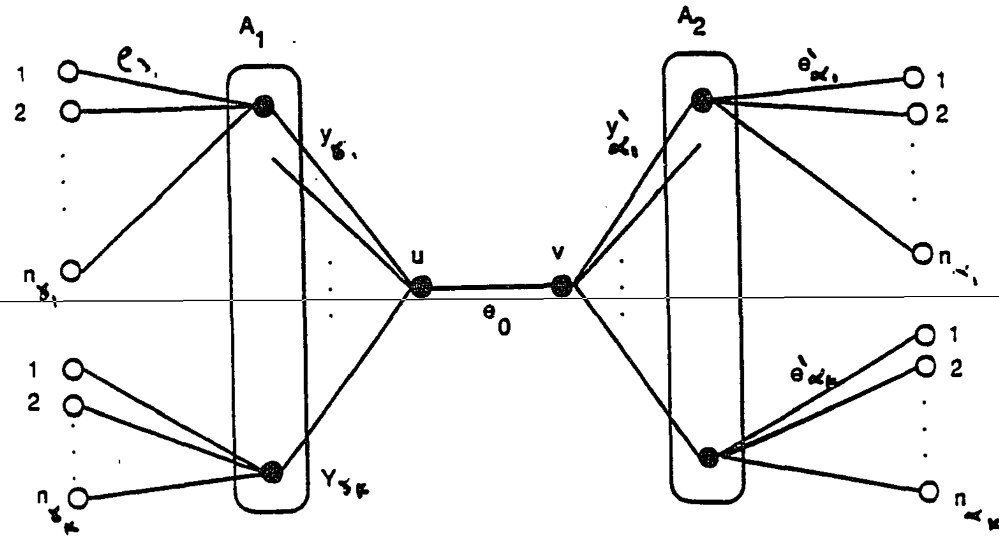

Figure 2.18

and

$$A_2 = \{m_1, m_2, \ldots, m_k\}$$

be a partition on the set A with the following property $\sum_i n_i = \sum_i m_i = N$. The tree shown in Figure 2.19 has three sets of edges: the set E_0 which consists of one edge, namely uv and has

$$E_0 = \{(1, 2k, 2N + k^2, 2Nk, N^2)\}$$

as its traffic vector. The second set E_1 consists of edges incident to uv and has the following traffic vectors:

$$E_1 = \{(1, n_i + k, n_ik + k + N - n_i, n_i(k + N - n_i) + N, n_iN); i = 1, 2, \ldots, 2k\}.$$

The third set is $E_2 = E(T) - (E_0 \cup E_1)$ and has the following traffic vectors:

$$E_3 = \{n_i - (1, n_i, k, k + N - n_i, N); i = 1, 2, \ldots, 2k\}.$$

The union of the above three sets of traffic vector is equal to C.

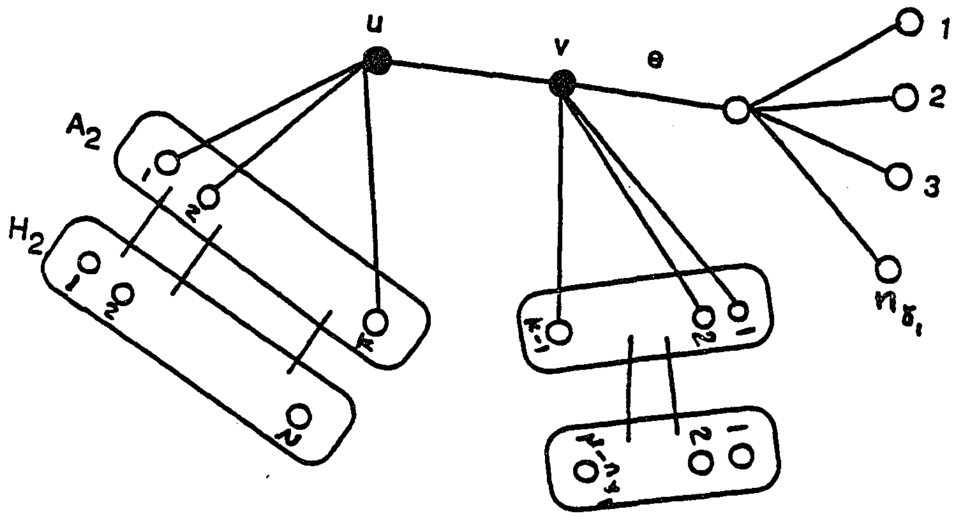

Figure 2.19

Assume A has no partition. We show that such a tree does not exist. Assume to the contrary that such a tree exists, then it should be of the form as shown in Figure 2.20. Moreover, the degree of the vertices in A_1 or A_2 are of the form $n_i + 1; 1 \leq i \leq 2k$; namely. Each vertex in A_1 or A_2 is adjacent to n_i vertices in H_1 or H_2, respectively. By the same analysis in this theorem, $|H_1| = |H_2| = N$. Therefore, there exists a set of elements $n_{\alpha_1}, n_{\alpha_2}, \ldots, n_{\alpha_k}$, such that $\sum_{i=1}^{k} n_{\alpha_i} = N$, which contradicts the fact that A has no partition, and this completes the proof.□

Our next result gives an algorithm which allows us to calculate the traffic vector distribution of a tree from the set of degree sequences.

Theorem 2.23 *Let $e = uv$ be an edge in a tree T, if $DDS(u)$ and $DDS(v)$ are given, then $TV(e)$ can be calculated.*

Proof Let T be a tree of order n. Let $e = uv$ be an edge with

$$DDS_T(u) = (d_0(u), d_1(u), \ldots, d_{n-1}(u))$$

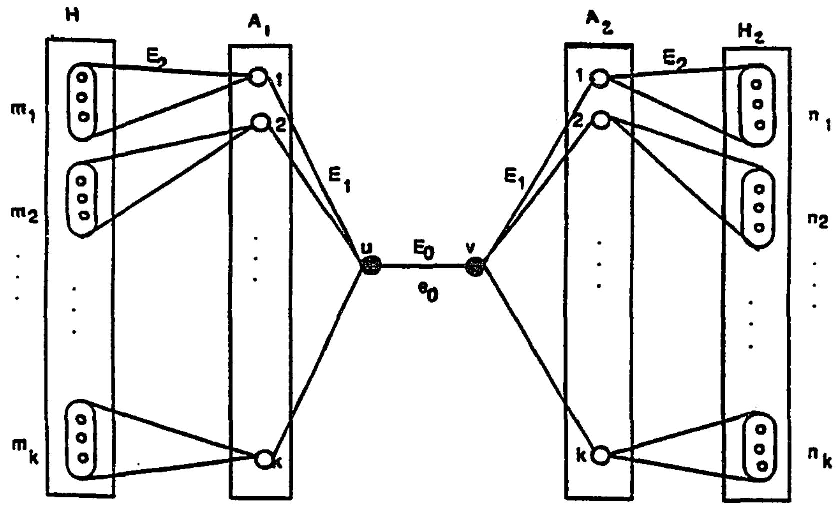

Figure 2.20

$$DDS_T(v) = (d_0(v), d_1(v), \ldots, d_{n-1}(v)).$$

Let T_u and T_v be the components in $T - e$ which contain the vertices u and v, respectively. First, we evaluate $DDS_{T_u}(u)$ and $DDS_{T_v}(v)$ from $DDS_T(u), DDS_T(v)$. If

$$DD_{T_u}(u) = (S_0(u), S_1(u), \ldots, S_{n-1}(u)$$

and

$$DDS_{T_v}(v) = (S_0(v), S_1(v), \ldots, S_{n-1}(v))$$

then the following is true:

$S_K(u) = d_k(v) - S_{k-1}(u)$ and $S_k(v) = d_k(u) - S_{k-1}(v)$, where $d_0(v) = 0, d_0(u) = 0$. By using the above recursive formula we can now evaluate the traffic vector of e. If $TV(e) = (\pi_1(e), \pi_2(e), \ldots, \pi_{n-1}(e))$ then

$$\pi_i(e) = \sum_{k=1}^{n-1} S_k(u) S_{n-k-1}(v), \quad i = 1, 2, \ldots, n-1$$

□

This algorithm can use the known algorithm for finding the distance degree sequence to evaluate the traffic vector.

CHAPTER III

IMPROVING PAIR-CONNECTED RELIABILITY

3.1 Introduction

In this chapter the improvement of pair-connected reliability of trees is discussed. We characterize those edges of a tree whose improvement maximizes the pair-connected reliability.

Let $G = (V, E)$ be a graph and let $S \in E$. Define $PC(S)$ to be the number of connected pairs of vertices in $< S >$. Recall that the function $PC(G, q)$ gives the expected number of pairs of vertices that are connected in the random graph (probabilistic graph) G. The function PC(G,q) seems particularly appropriate for communication networks, in which the goal is to maintain communication between as many pairs of sites as possible. Basic results on algorithms and computational complexity for pair-connected reliability can be found in [3], and [6].

In Chapter I, we defined the pair-connected reliability of a graph $G = (V, E)$ to be

$$PC(G,q) = \sum_{S \in \Omega} PC(S)R(S) \tag{3.1}$$

where $R(S)$ is the probability that G is in the state of S. In general determining the pair-connected reliability for a graph is difficult. In fact, there is no known polynomial time algorithm for finding the pair-connected reliability for a given graph. In order to illustrate the concept of pair-connected reliability, we consider the following example:

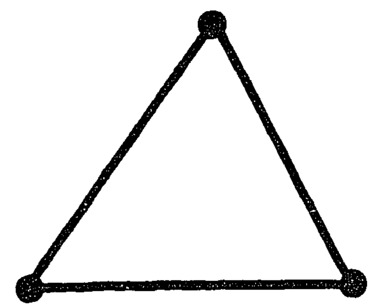

Figure 3.1

Let $G = K_3$ be the complete graph of order 3, and label the vertices in G by a, b, c (see Figure 3.1). The three states corresponding to one edge in a failed condition are: $S_1 = \{ab, bc\}$, $S_2 = \{bc, ca\}$, $S_3 = \{ca, ab\}$. Because $PC(S_1) = PC(S_2) = PC(S_3) = 3$, the three terms in the summation (3.1) are $3p^2q^1, 3p^2q^1$ and $3p^2q^1$.

In general, the pair-connected reliability polynomial can be written as

$$PC(G,q) = B_1pq^{m-1} + B_2p^2q^{m-2} + \ldots + B_{m-1}p^{m-1}q + B_mp^m \qquad (3.2)$$

where each B_i represents the total number of pairs of connected vertices taken over all subgraphs with exactly i edges. For the graph G in Figure 3.1, we have

$$PC(G;q) = 3p^3 + 9p^2q^1 + 3pq^2$$

or recalling that $q = (1-p)$

$$PC(G;q) = 3p + 3p^2 - 3p^3.$$

Recall from Chapter I, that for trees, the computation of $PC(T,q)$ is straightforward. Let $D(G) = (d_1(G), d_2(G), \ldots, d_{n-1}(G))$ be the distance distribution of G, where $d_i(G)$ denotes the number of pairs of vertices in G with distance i between them. The distance distribution $D(T)$ of a tree T completely determines $PC(T, q)$, namely

$$PC(T,q) = \sum_{i=1}^{n} d_i(T)p^i. \tag{3.3}$$

Slater [29] shows that the star $K_{1,n-1}$ is the optimal tree on n vertices, with respect to pair-connected reliability and the path P_n is the least reliable tree on n vertices.

As in Chapter I, we consider a probabilistic graph $G = (V, E)$ in which each edge $e \in E$ fails independently with probability q. In this chapter, we will improve the pair-connected reliability of trees by using the edge replacement or improvement and by multiple edge enhancement. For the following discussion, our basic probabilistic graph is tree T. If $S = \{e_1, e_2, \ldots, e_k\}$ is the set of edges in $E(T)$ which will be improved, then $S* = \{(e_1, \Delta_1), (e_2, \Delta_2), \ldots, (e_k, \Delta_k)\}$ denotes the new probability assignment of the edges in S, where (e_i, Δ_i) indicates that the edge e_i has been changed to have new reliability $p + \Delta_i$. Note that the reliability of the edges in $E(T) - S$ remains p . Next we find the pair-connected reliability in a tree T when the edges on T have different probability. The pair connected reliability of a graph G may be formulated in terms of two terminal reliability $R_{u,v}(G, q)$ (The probability that u and v are connected in G (see [6])). If G is a graph, then

$$PC(G,q) = \sum_{u,v \in V(G)} R_{u,v}(G,q) \tag{3.4}$$

where the sum is taken over all undirected pairs of distinct vertices in $V(G)$, and $R_{u,v}(G, q)$ is the probability of having u and v connected in G. For a tree T there exists a unique path between every pair of vertices in $V(T)$, therefore if u and v are two vertices in T, then the probability that u and v are connected is the same as the probability that no edge in the path from u to v has failed. If P is a uv-path in T with $E(P) = \{e_1, e_2, ..., e_i\}$, then

$$R_{u,v}(T,q) = \prod_{j=1}^{i} p_j(e_j)$$

when $p_j(e_j)$ is the probability that the edge e_j is functioning. For the trees we simply denote $R_{u,v}(T,q)$ by $R(I_{i+1})$, where I_{i+1} is the uv-path. For a tree T, if Γ denotes the probability assignment of $E(T)$, then formula (3.4) becomes

$$PC(T,\Gamma) = \sum_{I_{i+1}<T} R(I_{i+1}) \tag{3.5}$$

where the sum is taken over all paths I_{i+1} in T, of length i and $R(I_{i+1})$ is the probability of having I_{i+1} connected. If $E(I_{i+1}) = \{e_1, e_2, ..., e_i\}$, and $p(e_j) = p_j$, for $j = 1, 2, ..., i$, then $R(I_{i+1}) = \prod_{j=1}^{i} p_j$. By convention, if S^* (the new probability assignment of the edges in S) is known, then $R(T,\Gamma) = R(T, S^*, q)$.

Lemma 6 *Let e be an edge in a tree T of order n . If $\{e_1\}^* = \{(e_1, \Delta p)\}$ then*

$$PC(T, \{e_1\}^*, q) = \sum_{i=1}^{n-1} [D_i - \pi_i(e_1)]p^i + \pi_i(e_1)p^{i-1}(p + \Delta p)$$

where D_i is the number of the paths of length i in T, and $\pi_i(e_1)$ is the number of paths of length i in T which contain the edge e_1.

Proof: If A_i is the set of paths of length i in T, then A_i can be partitioned into two sets, A_{1i} and A_{2i}, where A_{1i} consists of paths of length i in T which do not contain e_1, and A_{2i} consists of paths of length i in T which contain e_1. Necessarily, $|A_i| = |A_{1i}| + |A_{2i}| = D_i$. Let

$$TV(e_1) = (\pi_1(e_1), \pi_2(e_1), ..., \pi_{n-1}(e_1))$$

be the traffic vector of e_1, then by definition of $\pi_i(e_1)$, it follows that: $|A_{1i}| = D_i - \pi_i(e_1)$ and $|A_{2i}| = \pi_i(e_1)$. By using the equation in (3.5)

$$\begin{aligned} &R(T, \{e_1\}^*, q) = \\ &\sum_{i=1}^{n-1} |A_{1i}|p^i + |A_{2i}|p^{i-1}(p + \Delta p) = \\ &\sum_{i=0}^{n-1} [D_i - \pi_i(e_1)]p^i + \pi_i(e_1)p^{i-1}(p + \Delta p). \end{aligned}$$

□ Next we use the concept of traffic vectors, for measuring the effect on pair-connected reliability by edge improvement.

Theorem 3.24 *Suppose* $TV(e_1) = (\pi(e_1), \ldots, \pi_{n-1}(e_1))$ *and* $TV(e_1) = (\pi_1(e_1), \pi_2(e_2) \ldots, \pi_{n-1}(e_2))$ *are traffic vectors of* e_1 *and* e_2 *in tree* T *respectively. Let* e_1 *and* e_2 *be edges of tree* T, *of order* $n+1$. *If* $\{e_1\}^* = \{(e_1, \Delta p)\}$, *and* $\{e_2\}^* = \{(e_2, \Delta p)\}$ *then,*

$$PC(T, \{e_1\}^*, q) - PC(T, \{e_2\}^*, q) = \Delta p \sum_{i=1}^{n} [\pi_i(e_1) - \pi_i(e_2)] p^{i-1}$$

Proof: Let $D(T) = (D_1, D_2, \ldots, D_n)$ be the distance distribution of T. By using Lemma 6 the following is true:

$$PC(T, \{e_1\}^*, q) = \sum_{i=1}^{n-1} [D_i - \pi_i(e_1)] p^i + \pi_i(e_1).p^{i-1}(p + \Delta p)$$

and

$$PC(T, \{e_2\}^*, q) = \sum_{i=1}^{n-1} [D_i - \pi_i(e_2)] p^i + \pi_i(e_1).p^{i-1}(p + \Delta p).$$

By taking the difference in the above equations,

$$\Delta PC = PC(T, \{e_1\}^*, q) - PC(T, \{e_2\}^*, q) =$$
$$\sum_{i=0}^{n-1} ([\pi_i(e_2) - \pi_i(e_1)] p^i + [\pi_i(e_2) - \pi_i(e_1)] p^{i-1}(p + \Delta p)) =$$
$$\sum_{i=0}^{n-1} ([\pi_i(e_2) - \pi_i(e_1)] p^i +$$
$$[\pi_i(e_1) - \pi_i(e_2)] p^i) + \sum_{i=0}^{n-1} [\pi_i(e_1) - \pi_i(e_2)] p^{i-1} \cdot \Delta p =$$
$$\Delta p \sum_{i=0}^{n-1} [\pi_i(e_1) - \pi_i(e_2)] p^{i-1}.$$

□

The above result indicates that the choice between edges is independent of the amount of improvement Δp but depends on the traffic vectors of the edges e_1 and e_2 or the probability p.

Corollary 7 *Let* e_1 *and* e_2 *be the edges of a tree* T *of order* $(n + 1)$ *with*

$$TV(e_1) \geq TV(e_2).$$

If $\{e_1\}^* = \{(e_1, \Delta p)\}$,*and* $\{e_2\}^* = \{(e_2, \Delta p)\}$, *then*

$$PC(T, \{e_1\}^*, q) \geq PC(T, \{e_1\}^*, q)$$

Proof: Let $\Delta PC = PC(T, \{e_1\}^*, q) - PC(T, \{e_2\}^*, q)$. By Theorem 3.24

$$\Delta PC = \Delta p \sum_{i=1}^{n-1} [\pi_i(e_1) - \pi_i(e_2)] p^{i-1}.$$

Since $TV(e_1) \geq TV(e_2)$, this implies that

$$\sum_{i=1}^{j} [\pi_i(e_1) - \pi_i(e_2)] \geq 0$$

for all $j = 1, 2, \ldots, n$. Hence for each term in ΔPC with negative coefficient, we can associate one or more terms of lower exponents for which the sum of its coefficients equals the absolute value of the negative coefficient. By noting that for $j > i$ and $p^i > p^j$, we conclude that $\Delta PC \geq 0$, for $0 < p < 1$. □

Definition 4 *An edge* e_0 *of a tree* T *is called a uniform edge if*

$$PC(T, \{(e_0, \Delta p)\}, q) \geq PC(T, \{(e, \Delta p)\}, q)$$

for all $e \in E(T)$ *and for all* $0 \leq \Delta p < 1 - p$.

We now address the relationship between the uniform edge and the dominant edge.

Corollary 8 *If* e *is a dominant edge in* T, *then* e *is a uniform edge.*

Proof: The result follows immediately by using Corollary 7. □

The above result shows that a tree T with a dominant edge always has a uniform edge.

Theorem 3.25 *The only tree in which an end edge is a uniform edge is the tree* $K_{1,n}$.

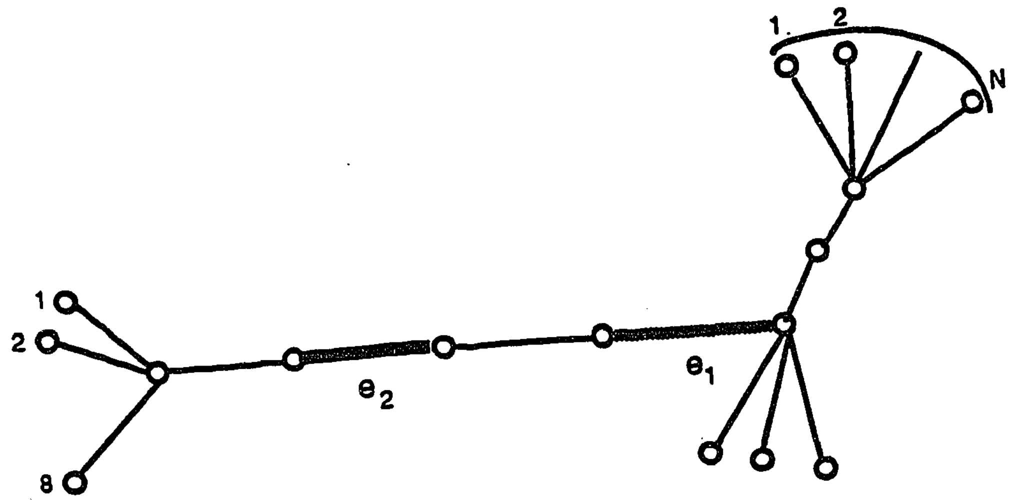

Figure 3.2

Proof: Suppose $e_0 \in E(T)$ is both a uniform and an end edge. Assume that $T \not\cong K_{1,n}$. By Theorem 2.10, there exists an edge $e \in E(T)$ such that $TV(e) > TV(e_0)$. By Theorem 3.24 and Corollary 7, we have the following:

$$PC(T, \{(e, \Delta p)\}) - PC(T, \{(e_0, \Delta p)\}, q) > 0.$$

Hence, e_0 is not a uniform edge, which is a contradiction. □

If an edge e_1 dominates an edge e_2, then it follows from Corollary 7, that $PC(T, \{e_1\}^*, q) \geq PC(T, \{e_2\}^*, q)$. The converse to this statement is not true. This can be seen in the following example. Let T be the tree shown in Figure 3.2. The tree T has two edges e_1 and e_2 with the following traffic vectors: $TV(e_1) = (1, 5, 6, 6 + N, h_1, h_2, \ldots)$ and $TV(e_2) = (1, 2, 10, 10, l_1, l_2, \ldots)$ respectively. Since $\sum_{i=1}^{3} \pi_i(e_1) - \pi_i(e_2) = -1$, the edge e_1 does not dominate e_2. On the other hand, let

$$H(p) = PC(T, \{(e_1, \Delta p)\}^*, q) - PC(T, \{(e_2, \Delta p)\}^*, q).$$

By using Theorem 3.24,

$$H(p) = \Delta p[3p^1 - 4p^2 + (N - 4)p^3 + \sum_{i=5}[\pi_i(e_1) - \pi_i(e_2)]p^{i-1}].$$

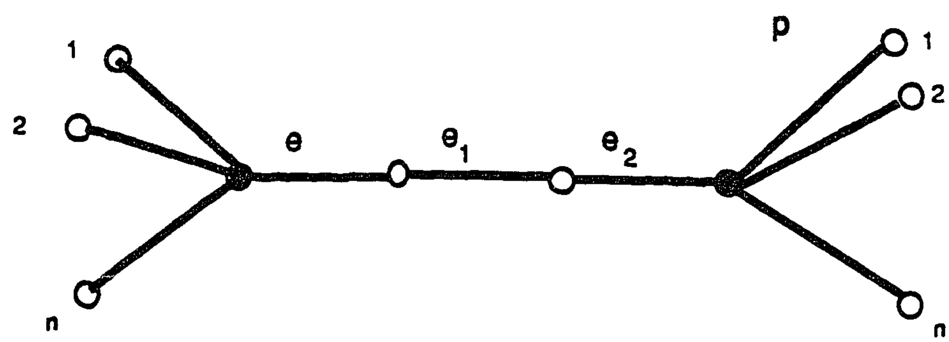

Figure 3.3

It is not difficult to see that $\sum_{i=5}^{8}[\pi_i(e_1) - \pi_i(e_2)]p^{i-1} \geq 0$ for all $i \geq 5$. Therefore

$$H(p) \geq \Delta p[p(3 - 4p + (N - 4)p^2].$$

For $p \in (0, 1)$, the sign of $H(p)$ depends upon the sign of the quadratic function $(N - 4)p^2 - 4p + 3$ which has no real root when N satisfies $16 - 12(N - 4) < 0$. Therefore, for $N \geq 6$,

$$(N - 5)p^2 - 4p + 3 > 0.$$

For Δp, $H(p) > 0$ for all $N \geq 6$.

The above example, with the fact that any dominant edge is a uniform edge, suggests the following question: Is there a uniform edge which is not a dominant edge? This question remains open.

In the process of improving network reliability the choice between edges does not depend only on their traffic vectors. The value of the probability p for the edges in T may change the decision. The following example shows an edge e contained in a number of paths which is not a uniform edge.

Example: Consider the tree T (see Figure 3.3). In T, $TV(e_1) = (1, 2, 2n +$

$1, 2n, n^2)$ and $TV(e_2) = (1, n+1, n+1, 2n, n^2)$.

$$\begin{aligned}
\Delta PC = & \\
& PC(T, \{(e_1, \Delta p)\}^*, q) - PC(T, \{(e_2, \Delta p)\}^*, q) = \\
& \Delta p \sum_{i=1}^{2n+3} [\pi_i(e_1) - \pi_i(e_2)] p^{i-1} = \\
& p \cdot \Delta p [np + (1-n)].
\end{aligned}$$

For positive values of Δp and p, we have the following: $np + (1-n) \geq 0$ if and only if $p \geq \frac{n-1}{n}$. Therefore,

$$\Delta PC = \begin{cases} \geq 0 & \text{for } p \geq \frac{n-1}{n} \\ \leq 0 & \text{otherwise.} \end{cases}$$

Thus, the edge e_1 is a better choice for $p \geq \frac{n-1}{n}$, and the edge e_2 is a better choice for $p \leq \frac{n-1}{n}$, with respect to improving pair-connected reliability.

Theorem 3.26 *Let T be a tree of order $n+1$. If e_0 is a uniform edge in T, then*

$$\sum_{i=1}^{n} \pi_i(e_0) \geq \sum_{i=1}^{n} \pi_i(e)$$

for all $e \in V(T)$.

Proof: Let $TV(e_0) = (\pi_1(e_0), \pi_2(e_0), \ldots, \pi_n(e_0))$ and $TV(e) = (\pi_1(e), \pi_2(e), \ldots, \pi_n(e))$ Thus

$$\begin{aligned}
\Delta PC = & \\
& PC(T, \{(e_0, \Delta p)\}, q) - PC(T, \{(e, \Delta p)\}, q) \Delta PC = \\
& \sum_{i=1}^{n} [\pi_i(e_0) - \pi_i(e)] p^{i-1} \Delta PC = \\
& a_1 p^0 + a_2 p^2 + \ldots + a_n p^{n-1} = H(p)
\end{aligned}$$

where $a_i = \pi_i(e_0) - \pi_i(e)$. Assume to the contrary that e_0 is a uniform edge and $\sum_{i=1}^{n} \pi_i(e_0) < \sum_{i=1}^{n} \pi_i(e)$. This implies that $\sum_{i=1}^{n-1} a_i < 0$, therefore $\lim_{p \to 1} H(p) < 0$. Thus, there exists $p \in (0,1)$ such that $H(p) < 0$, which contradicts the fact that $H(p) \geq 0$ for all p. □

Corollary 9 *Let T be a tree of order n, and let S be the set of edges in $E(T)$ with the property, $\sum_{i=1}^{n-1}\pi_i(e)$ is maximum. Given $e_0 \in S$ and $e \in E(T) - S$, there exists p_0, $0 < p_0 < 1$ such that $\forall p \in (p_0, 1); 0 \leq \Delta pleq < 1 - p$*

$$PC(T, \{e_0, \Delta p\}, q) > PC(T, \{e, \Delta p\}, q).$$

Proof: Let e_0 be an edge in S, then

$$\sum_{i=1}^{n-1}\pi_i(e_0) > \sum_{i=1}^{n-1}\pi_i(e) \;\; \forall e \in E(T). \tag{3.6}$$

$$H(p) = PC(T, \{e_0\}^*, q) - PC(T, \{e\}^*, q) =$$

$$\Delta p \sum_{i=1}^{n-1}(\pi_i(e_0) - \pi_i(e))p^{i-1}$$

The above expression is a polynomial function of p. If $H(p) = a_2p^1 + a_3p^2 + \ldots + a_{n-1}p^{n-2}$, then by using (3.6) we have $\sum_{i=2}^{n-1} a_i > 0$. Therefore, $\lim_{p\to 1} H(p) > 0$. Thus there exists p_0 with $0 < p_0 < 1$ such that $H(p) > 0$ for all $p > p_0$, which implies the desired result. □

Corollary 10 *If e is a uniform edge in a tree T, then the number of paths containing e is maximum.*

Corollary 11 *If $S = \{e_1, e_2, \ldots, e_m\}$ is a set of uniform edges then the $Ind(S)$ is isomorphic to $K_{1,|S|}$.*

Proof: This follows from Theorem 3.26 together with Theorem 2.15 for the result.

3.2 Improving More Than One Edge

Let T be a tree of order $n + 1$. Given $S = \{e_1, e_2, \ldots, e_k\} \subseteq E(T)$ and a positive number r, the set $A = \{t_1, t_2, \ldots, t_k\}$ is called k-partition to r if t_i is non-negative for all $i = 1, 2, \ldots, k$ and $\sum_{i=1}^{k} t_i = r$. Define the set

$$S^* = \{(e_1, \Delta p_1), (e_2, \Delta p_2), \ldots, (e_k, \Delta p_k)\}$$

to be a new probability assignment to the edges in S, where $\Delta p = \sum_{i=1}^{k} \Delta p_i$, and $\Delta p \leq k(1-p)$. We wish to find a partition of Δp, which maximizes the increase in the pair-connected reliability of T. The following result shows the best partition of Δp in the case of $|S| = 2$. Throughout the following sections, S will denote a set of edges in the graph which are to be improved and S^* will be the new probability assignment for the edges of S.

Theorem 3.27 *Let e_1 and e_2 be edges in a tree T of order $n+1$. Suppose $S = \{e_1, e_2\}$ and $S^*(t) = \{(e_1, t\Delta p), (e_2, (1-t)\Delta p)\}$, where $0 < t < 1$, and $\Delta p \leq 1-p$ then the function*

$$H(t) = PC(T, S^*(t), q) - PC(T, \{(e_1, \Delta p)\}, q)$$

has a maximum value when

$$t = \frac{1}{2} + \frac{\sum[\pi_i(e_1) - \pi_i(e_2)]p^{i-1}}{\sum[\pi_i(S)]p^{i-2}}.$$

Proof: Paths of length i in T can be partitioned into three sets:

Set 1: paths containing e_1 but not e_2.

Set 2: paths containing e_2 but not e_1.

Set 3: paths containing both e_1 and e_2. Therefore,

$$PC(T, S^*(t), q) = \sum_{i=1}^{n}([D_i - (\pi_i(e_1) + \pi_i(e_2) - \pi_i(S))]p^i +$$

$$[\pi_i(e_2) - \pi_i(S)][p + (1-t)\Delta p]p^{i-1} + [\pi_i(e_1) - \pi_i(S)][p + t\Delta p]p^{i-1} +$$

$$[\pi_i(S)][p + t\Delta p][p + (1-t)\Delta p]p^{i-2}) \qquad (3.7)$$

where, D_i is the number of ordered pairs of vertices in T with distance i between them, $\pi_i(e_1)$ and $\pi_i(e_2)$ are the number of paths of length i which contain e_1 and e_2 respectively, and $\pi_i(S)$ is the number of paths of length i which contain both edges e_1 and e_2. Simplifying the equation in (3.7) gives the following:

$$PC(T, S^*(t), q) = \sum_{i=1}^{n}[D_i - \pi_i(S)]p^i +$$

$$[(\pi_i(e_1))\Delta pt + (1-t)\Delta p\pi_i(e_2) - \Delta p\pi_i(S)]p^{i-1} + (\pi_i(S)(\Delta p)^2(t-t^2)p^{i-2}.$$

On the other hand,

$$PC(T, \{(e_1, \Delta p)\}) = \sum_{i=1}^{n}[D_i - \pi_i(e_1)]p^i + \pi_i(e_1)p^{i-1}(p+\Delta p).$$

Now the function

$$H(t) = PC(T, S^*(t), q) - PC(T, \{(e_1, \Delta p)\}, q) =$$

$$\sum_{i=1}^{n} -\pi_i(S)p^i + [\pi_i(e_1)\Delta p\, t - \Delta p\, \pi_i(e_1) + (1-t)\Delta p\, \pi_i(e_2) - \Delta p\, \pi_i(S)]p^{i-1} +$$

$$t(\Delta p)^2(1-t)\pi_i(e_1, e_2)p^{i-2}.$$

The function $H(t)$ has a maximum value in $[0,1]$ at the critical points. Thus, if $H'(t)$ is the derivative of $H(t)$, then

$$H'(t) = \sum \Delta p[\pi_i(e_1) - \pi_i(e_2)]p^{i-1} + \pi_i(S)(\Delta p)^2(1-2t)p^{i-2}.$$

Letting $H'(t) = 0$, and solving for t implies

$$t = t_0 = \frac{1}{2} + \frac{\sum_{i=1}^{n-1}[\pi_i(e_1) - \pi_i(e_2)]p^{i-1}}{\sum_{i=2}^{n-1}[\pi_i(S)]p^{i-2}}.$$

Since $H''(t) < 0$ for all $t \in (0,1)$, this implies that $H(t_0)$ has no minimum value in (0, 1) at $t = t_0$.□

Corollary 12 *In the previous theorem suppose $TV(e_1) = TV(e_2)$, then the maximum value of $H(t)$ occurs when $t = \frac{1}{2}$.*

Given $S = \{e_1, e_2, \ldots, e_k\} \subseteq E(T)$ and

$$S^*(A) = \{(e_1, t_1\Delta p), (e_2, t_2\Delta p), \ldots, (e_k, t_k\Delta p)\}$$

where $A = \{t_1, t_2, \ldots, t_k\}$ is k-partition to 1, what is the best k-partion of 1 which maximizes $PC(T, S^*(A), q)$ the most ? We conjecture the following: If A contains zero element, then $PC(T, S^*(A), q)$ is not maximum.

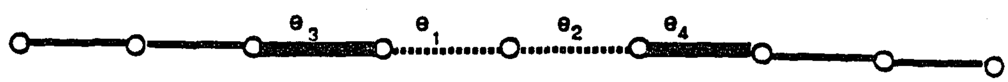

Figure 3.4

The increase in reliability depends not only on the way we distribute Δp among the set of edges which are to be improved, but also upon the choice of the edges to be improved. To see this, consider the example:

Let T be the path P_9. Label the edges in $E(P_9)$ according to their location from one of the ends, say $E(P_9) = \{e_1, e_2, ..., e_8\}$. Assume we wish to improve two edges in this path, each edge by the amount of $\frac{\Delta p}{2}$ where, $p < \frac{\Delta p}{2} < 1$ (see Figure 3.4). If $S = \{e_1, e_2\}$ then $S^*(A) = \{(e_1, \cdot\frac{1}{2}\Delta p), (e_2, \cdot\frac{1}{2}\Delta p)\}$ where $A = \{\cdot\frac{1}{2}, \cdot\frac{1}{2}\}$. Note that we are taking a fixed partition to 1, namely $\{\cdot 5, \cdot 5\}$.

Consider the following choices of improving two edges in $E(R_9)$:

(1) e_4, e_5 are the two edges needed to be improved.

Let $p_0 = p + \Delta p$. $S_1^* = \{(e_4, \Delta p), (e_5, \Delta p)\}$ then

$$PC(T, S_1^*, q) = (6p + 2p_0) + (4p^2 + 2pp_0 + p_0^2) +$$

$$(2p^3 + 2p^2p_0 + 2pp_0^2) + (2p^3p_0 + 3p^2p_0^2) + (4p^3p_0^2) + 3p^4p_0^2 + 2p^5p_0^2 + 1p^6p_0^2.$$

(2) e_3 and e_6 are the two edges which need to be improved.

Let $S_2^* = \{(e_3, \Delta p), (e_6, \Delta p)\}$, then

$$PC(T, S_2^*, q) = (6p + 2p_0) + (3p^2 + 4pp_0) + (6p^2p_0) +$$

$$(4p^3p_0 + p^2p_0^2) + (2p^4p_0 + 2p^3p_0^2 + 3p^4p_0^2 + 2p^5p_0^2 + p^6p_0^2).$$

Consider the difference

$$\Delta PC = PC(T, S_1^*, q) - PC(T, S_2^*, q) =$$

$$(p_0 - p)^2 + 2(p_0 - p)[2pp_0 - 2p^2] + 2p^2p_0(p_0 - p) + 2p^3p_0(p_0 - p).$$

For $p_0 > p$ we get $\Delta PC > 0$. Therefore e_4 and e_5 is a better choice than e_6 and e_3.

Next we improve network reliability by adding multiple edges. Let e be an edge in a tree T. If n new multiple edges are added to e then the new reliability assignment of e is $\{(e, \Delta_n p)\}$, where $\Delta_n p = (1-p)(1-(1-p)^n)$. Note that $0 \leq \Delta p \leq 1 - p$. If $S^* = \{(e_1, \Delta_n p)(e_2, \Delta_n p), \ldots, (e_k, \Delta_n p)\}$, then we will denote S^* by simply $S_n^*(e_1, e_2, \ldots, e_k)$.

Definition 5 *An edge set $S = \{e_1, e_2, \ldots, e_m\}$ is called an m-uniform set in T, if a given $S^* = \{(e_i, \Delta p); i = 1, 2, \ldots, m\}$ then*

$$PC(T, S^*, q) \geq PC(T, I^*, q)$$

for all $I = \{e_1', e_2', \ldots, e_m'\} \subseteq E(T)$ with $|I| = m$ and $I^* = \{(e_i', \Delta p); i = 1, 2, \ldots, m\}$.

Theorem 3.28 *Let e_1 and e_2 be two dominant edges in a tree T. If $\{e_1, e_2\}$ is a dominant set, then $\{e_1, e_2\}$ is a 2-uniform set.*

Proof: Let $S^* = \{(e_1, \Delta p), (e_2, \Delta p)\}$, and let e_1' and e_2' be two edges in $E(T)$ and

$$I^* = \{(e_1', \Delta p), (e_2', \Delta p)\}.$$

Let x_i be the number of paths of length i which contain e_1 but not e_2, y_i be the number of paths of length i which contain e_2 but not e_1, and z_i be the number of paths which contain both e_1 and e_2. Similarly, let x_i' be the number of paths containing e_1' but not e_2', y_i' be the number of paths of length i which contain e_2'

but not e'_1, and z'_i be the number of paths of length i which contain both e'_1 and e'_2. If D_i denotes the number of paths of length i in T, then the following is true:

$$PC(T, S^*, q) = \sum_{i=1}^{n-1}[D_i - (x_i + y_i + z_i)]p^i +$$

$$(x_i + y_i)p^{i-1}(p + \Delta p) + z_i p^{i-2}(p + \Delta P)^2$$

and

$$PC(T, I^*, q) = \sum[D_i - (x'_i + y'_i + z'_i)]p^i + (x_i + y_i)p^{i-1}(p + \Delta p) + z_i p^{i-2}(p + \Delta P)^2.$$

Consider the difference:

$$\Delta PC = PC(T, S^* q) - PC(T, I^*, q) =$$

$$\sum(x'_i - x_i + y'_i - y_i + z'_i - z_i)p^i + [(x_i - x'_i) + (y_i - y'_i)]p^{i-1}(p + \Delta p) + (z_i - z'_i)p^{i-2}(p + \Delta P)^2.$$

Observe that

$$x_i + z_i = \pi_i(e_1)$$

$$y_i + z_i = \pi_i(e_2)$$

$$x'_i + z'_i = \pi_i(e'_1)$$

$$y'_i + z'_i = \pi_i(e'_2).$$

Therefore,

$$\Delta PC = \sum[\pi_i(e_1) - \pi_i(e'_1) + \pi_i(e_2) - \pi_i(e'_2)]p^{i-1}\Delta p + (z_i - z'_i)p^{i-2}(\Delta p)^2.$$

The fact that e_1 and e_2 are dominant edges and $\{e_1, e_2\}$ is a dominant set implies that $\Delta PC \geq 0$ for all Δp and for all $p \in (0, 1)$. □

Theorem 3.29 *If e_1 and e_2 are the only dominant edges in a tree T and the common vertex has degree 2, then $\{e_1, e_2\}$ is a 2-dominant set.*

Proof To the contrary, assume there exist two edges e_1' and e_2' in T such that

$$\sum_{i=1}^{j} \pi_i(e_1', e_2') > \sum_{i=1}^{j} \pi_i(e_1, e_2).$$

For some $0 \leq j \leq diam(T)$ we show that e_1 or e_2 is not a dominant edge. Without loss of generality, let $Ind(e_1', e_2')$ be the path $u_1'v'u_2'$. Let A_1 be the set of paths of length $L \leq j$ which contain $\{e_1, e_2\}$. Suppose that T_{u_1}, and T_{u_2} are the components in $T - \{e_1, e_2\}$, which contain u_1 and u_2, respectively. Let $T_{u_i}(A_1)$ $; i = 1, 2$, be the subtree induced from the two parts of those paths in A_1 which are contained in T_{u_1} and T_{u_2} respectively. Let $y_1 = |V(T_{u_1}(A_1))|$, and $y_2 = |V(T_{u_2}(A_1))|$, then

$$\sum_{i=1}^{j} \pi_i(e_1, e_2) = y_1 y_2$$

(see Figure 3.5). Similarly, let A_2 be the set of all paths containing e_1', e_2' with length $L \leq j$.

Let $T_{u_1'}$ and $T_{u_2'}$ be the components of $T - \{e_1', e_2'\}$ which contain u_1' and u_2', respectively and let $x_1 = |V(T_{u_1'}(A_2))|$, and $x_2 = |V(T_{u_2'}(A_2))|$. Then

$$\sum_{i=1}^{j} \pi_i(e_1', e_2') = x_1 \cdot x_2.$$

By assumption $x_1 \cdot x_2 > y_1 \cdot y_2$. Without loss of generality, assume $x_1 \leq x_2$, and $y_1 \leq y_2$. The number of paths of length L, where $L \leq j$, which contain e_1 is $y_1(1 + y_2)$. On the other hand, the number of paths of length $L \leq j$ which contain e_2' is $(x_1 + 1)x_2$. If $x_1 x_2 > y_1 y_2$, $x_1 \leq x_2$ and $y_1 \leq y_2$, then $x_2 > y_1$ which implies $(x_1 + 1)x_2 > y_1(1 + y_2)$. This contradicts the fact that e_1 is a dominant edge. □

Corollary 13 *A tree T with exactly two dominant edges e_1 and e_2 has $\{e_1, e_2\}$ as a 2-uniform set, if the common vertex has degree two.*

Proof Using Theorem 3.28 together with Theorem 3.29 will imply the result. □

The improvement of network reliability by changing the probability of more than two edges is not easy to analyze. Our goal is always to come up with the

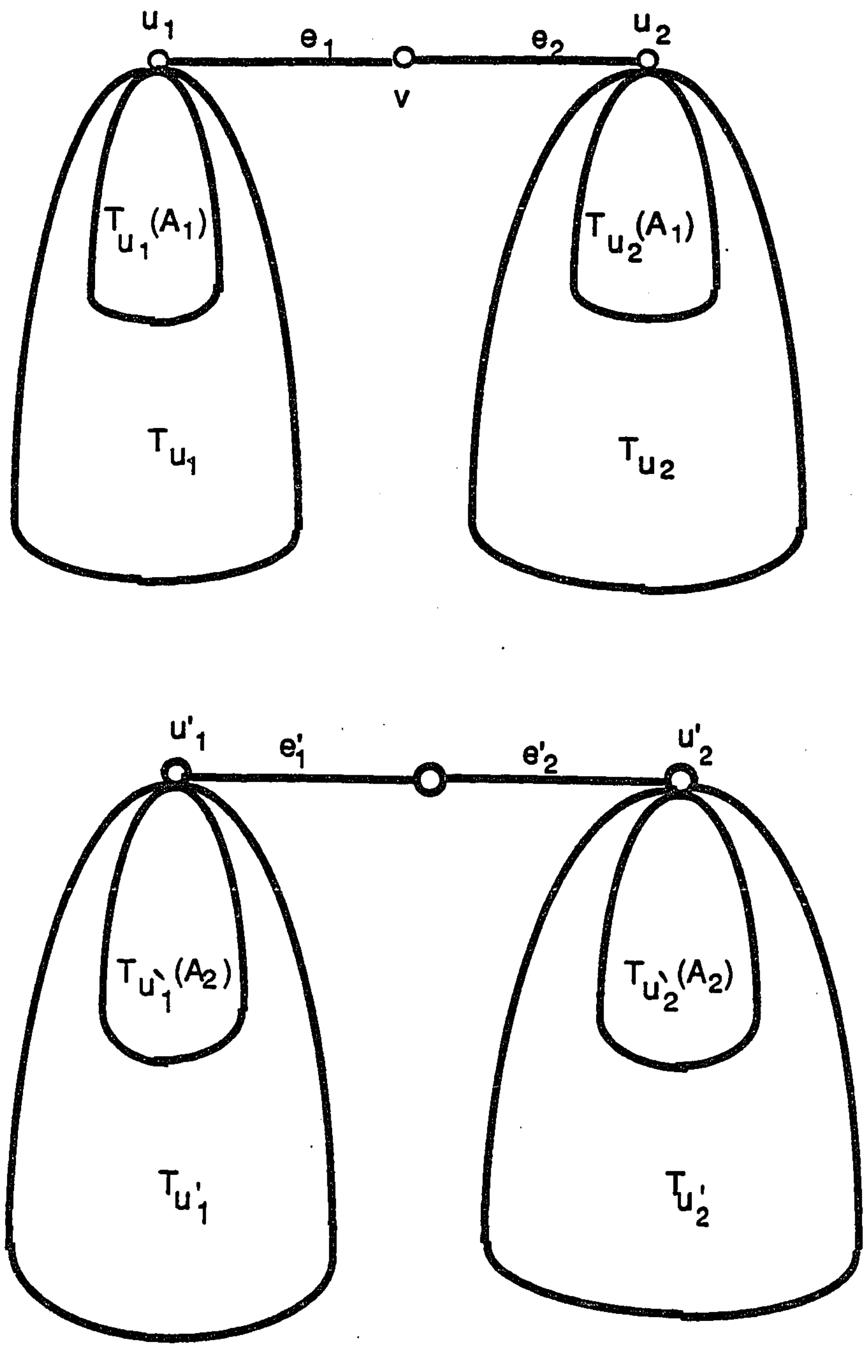

Figure 3.5

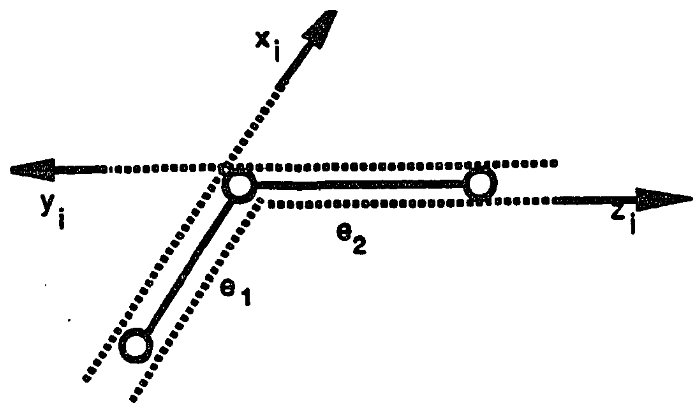

Figure 3.6

best choice of a set of edges, and having it remain the best for all values of p with $0 < p < 1$. Unfortunately, for most networks no such choices exist, unless we restrict the value of p.

The next result shows that the set of dominant edges contains a K-uniform set when the value of p is restricted.

Theorem 3.30 *Let e_1 and e_2 be two dominant edges in a tree T of order n. If $p > \frac{\Delta p}{2}$, then the set $\{e_1, e_2\}$ is a uniform set.*

Proof: Let e_1' and e_2' be any two edges in $E(T)$. Let x_i' and y_i' represent the number of paths of length i, which pass through e_1' but not e_2', and e_2' but not e_1', respectively. Moreover, let x_i and y_i be the number of paths which pass through e_1 but not e_2 and e_2 but not e_1, respectively (see Figure 3.6 and 3.7).

Let $\pi_i(e_1, e_2) = z_i$ and $\pi_i(e_1', e_2') = z_i'$. By an analysis similar to the one given in the proof of Theorem 3.28, if

$$S^* = \{(e_1, \Delta p), (e_2, \Delta p)\}$$

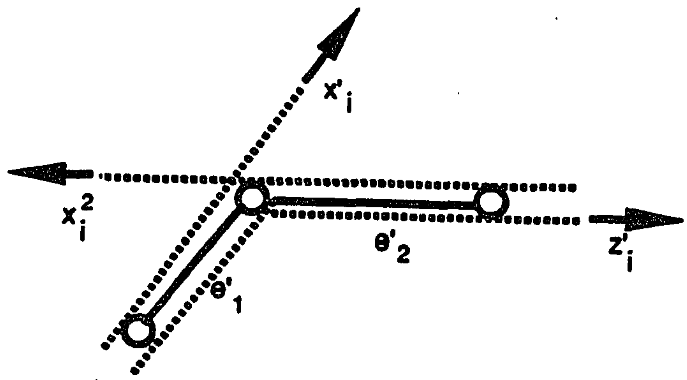

Figure 3.7

$$I^* = \{(e'_1, \Delta p), (e'_2, \Delta p)\}$$

then

$$\Delta R = PC(T, S^*, q) - PC(T, I^*, q)$$

$$= \sum_{i=1}^{n-1} [\pi_i(e_1) - \pi_i(e'_1) + \pi_i(e_2) - \pi_i(e'_2)] p^{i-1} \Delta p + (z_i - z'_i) p^{i-2} (\Delta p)^2$$

$$= \sum_{i=1}^{n-1} [\pi_i(e_1) - \pi_i(e'_1) + \pi_i(e_2) - \pi_i(e'_2) p + (z_i - z'_i) \Delta p] p^{i-2} \Delta p.$$

The fact that $x_i - x'_i > z'_i - z_i$ and $y_i - y'_i > z'_i - z_i$ implies the following:

$$h' = \pi_i(e_1) - \pi_i(e'_1) + \pi_i(e_2) - \pi_i(e'_2) \geq 2(z'_i - z_i) = 2h$$

Therefore, $\Delta R \geq 0$ if and only if $h' \cdot p - h\Delta p \geq 0$ or

$$p > \frac{h}{h'} \Delta p \geq \frac{1}{2} \Delta p$$

□

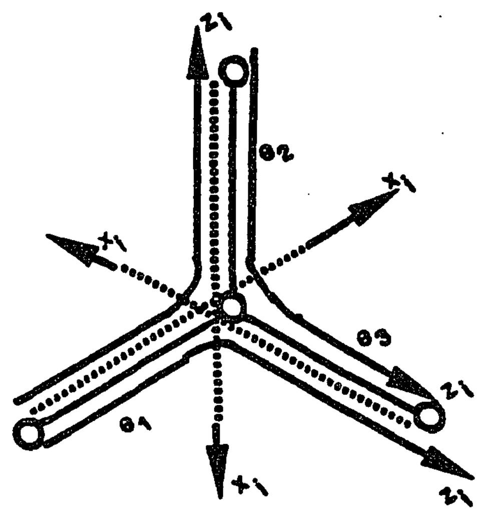

Figure 3.8

Theorem 3.31 *Let e_1, e_2, e_3 be three dominant edges in T of order $n+1$. If $p > \frac{\Delta p}{2}$, then $\{e_1, e_2, e_3\}$ is a 3-uniform set in T.*

proof: Observe that if $\{e_1, e_2, e_3\} = S$ is a set of three dominant edges, then $< S > \cong K_{1,3}$. Since $S = \{e_1, e_2, e_3\}$ is a set of dominant edges, then the number of paths which contain exactly one edge in S is independent of the edge. Moreover the number of paths of length i which contain exactly two edges in S is independent of the pair of edges. Let x_i be the number of paths of length i which contain exactly one edge in S and let z_i be the number of paths of length i which contain exactly two edges in S. (See Figure 3.8 and 3.9) Now, let $I = \{e'_1, e'_2, e'_3\} \subseteq E(T)$. For $j = 1, 2, 3$, let x_i^j denote the number of paths of length i which contain e'_j but do not intersect $I - \{e'_j\}$. If $\pi_i(e'_s, e'_t) = z_i^{st}$; $1 \leq s, t \leq 3$ then

$$3x_i - (x_i^1 + x_i^2 + x_i^3) > z_i - z_i^{12} + z_i - z_i^{23} + z_i - z_i^{13}$$

Let

$$S^* = \{(e_1, \Delta p), (e_2, \Delta p), (e_3, \Delta p)\}.$$

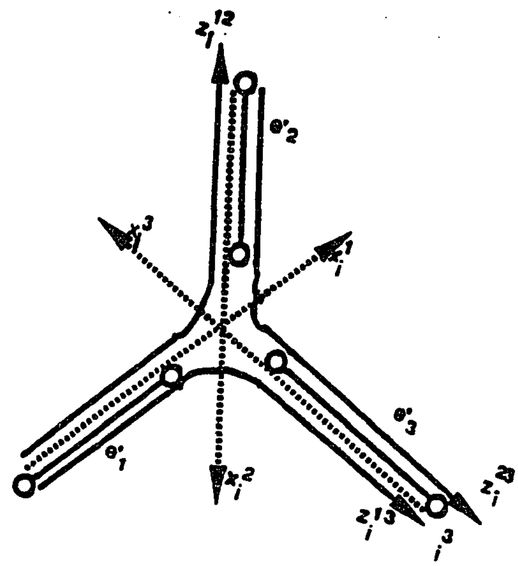

Figure 3.9

and

$$I^* = \{(e_1', \Delta p), (e_2', \Delta p), (e_3', \Delta p)\}$$

$$PC(T, S^*, q) = \sum_{i=1}^{n} \{[D_i - (3x_i + 3z_i)]p^i + 3x_i p^{i-1}(p + \Delta p) + 3z_i p^{i-2}(p + \Delta p)\}$$

and

$$PC(T, I^*, q) = \sum_{i=1}^{n} \{[D_i - (x_i^1 + x_i^2 + x_i^3 + z_i^1 + z_i^2 + z_i^3)]p^i +$$

$$(z_i^1 + x_i^2 + x_i^3)p^{i-1}(p + \Delta p) + (z_i^{12} + z_i^{23} + z_i^{13})p^{i-2}(p + \Delta p)\}$$

$If \quad \Delta PC = PC(T, S^*, q) - PC(T, I^*, q) \quad , then$

$$\Delta PC =$$

$$\sum_{i=1}^{n} \Delta p p^{i-1} \{[3x_i + 6z_i - x_i^1 - x_i^2 - x_i^3 - 2z_i^{12} - 2z_i^{23} - 2z_i^{13}] +$$

$$(\Delta p)^2 p^{i-2}[3z_i - z_i^{12} - z_i^{23} - z_i^{13}]\} =$$

$$\sum_{i=1}^{n} \{\Delta p p^{i-2}[(3x_i - x_i^1 - x_i^2 - x_i^3 + 6z_i - 2z_i^{12} - 2z_i^{23} - 2z_i^{13})p -$$

$$\Delta p(z_i^{12} + z_i^{23} + z_i^{13} - 3z_i))]\}$$

Observe that, for all $i = 1, 2, \ldots, n$

$$\begin{aligned} x_i + 2z_i &= \pi_i(e_i); \\ x_i^1 + z_i^{12} + z_i^{13} &= \pi_i(e_1'); \\ x_i^2 + z_i^{23} + z_i^{12} &= \pi_i(e_2'); \quad \textit{and} \\ x_i^3 + z_i^{13} + z_i^{23} &= \pi_i(e_3'). \end{aligned}$$

Equation(3.10) can be written as

$$\Delta PC = \sum \Delta p p^{i-2}[(\pi_i(e_1) - \pi_i(e_1') + \pi_i(e_2) - \pi_i(e_2') +$$

$$\pi_i(e_3) - \pi_i(e_3'))p - \Delta p(z_i^{12} + z_i^{23} + z_i^{13} + 3z_i)].$$

Using the fact that e_1, e_2, e_3 are dominant edges in T, we have

$$\begin{aligned} h' &= 3\pi_i(e_1) - \pi_i(e_1') - \pi_i(e_2') - \pi_i(e_3') \\ &> 2[z_i^{12} + z_i^{23} + z_i^{13} - 3z_i] \\ &= 2h. \end{aligned}$$

Therefore, $\Delta PC \geq 0$ if and only if $ph' - h\Delta p \geq 0$, or

$$p \geq \frac{h}{h'}\Delta p \geq \frac{\Delta p}{2}$$

□

The above argument can be extended to obtain the following result.

Theorem 3.32 *Let $S = \{e_1, e_2, \ldots, e_k\}$ be a set of dominant edges in a tree T of order n, such that $k < n$. If $p > \frac{\Delta p}{2}$, then S is a k-uniform set.*

For some k, $k \leq |S|$, the following example shows that a set of dominant edges S does not always contain a k-uniform set. Consider the tree in Figure 3.10. The set

$$S = \{e_1, e_2, e_3, e_4, e_5\}$$

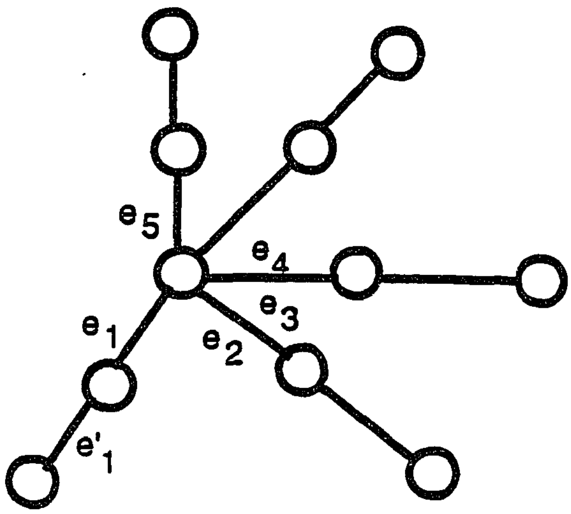

Figure 3.10

is the maximal set of dominant edges. We show that there are no 2-uniform sets in S. Without loss of generality, assume that $\{e_1, e_2\}$ be a 2-uniform set. $TV(e_1) = TV(e_2) = (1,5,8,4)$, and $TV(\{e_1, e_2\}) = (0,1,2,1)$. Let e_1' be an end edge incident to e_1, then $TV(e_1') = (1,1,4,4)$, and $TV(\{e_1', e_1\}) = (0,1,4,4)$. Let

$$S^* = \{(e_1, \Delta p), (e_2, \Delta p)\}$$

$$I^* = \{(e_1, \Delta p), (e_1', \Delta p)\}.$$

Then

$$\Delta PC = PC(T, S^*, q) - PC(T, I^*, q) =$$

$$\sum \Delta p \cdot p^{i-2}[(\pi_i(e_1) + \pi_i(e_2) - \pi_i(e_1) - \pi_i(e_1'))p +$$

$$\Delta p(\pi_i(e_1, e_2) - \pi_i(e_1, e_1')] =$$

$$\Delta p[(4p) + 0\Delta p] + p\Delta p[(8-4)p - 2\Delta p]p\Delta p[1p - 3\Delta p] =$$

$$4p\Delta p + 4p^2\Delta p - 2p(\Delta p)^2 - 3p(\Delta p)^2 + p^2\Delta p =$$

$$2p\Delta p[2 + 2p - \Delta p - 1.5\Delta p + p\Delta p] =$$

$$2\Delta pp[2 + 2p - 2.5\Delta p + p\Delta p].$$

Letting $p \rightarrow 0$ and $\Delta p \rightarrow 1$ implies there exist p and Δp for which ΔPC is negative.

Observe that for $p \geq \frac{1}{3}$ and $0 \leq \Delta p \leq 1 - p$ we have $p \geq \frac{\Delta p}{2}$. In most network applications, the reliability of the layout of the network is at least $\frac{1}{3}$, therefore one can apply the above result in improving pair-connected reliability. In general, if $p > \frac{1}{3}$ then $p > \frac{\Delta p}{2}$ for all $0 \leq \Delta p \leq 1 - p$.

3.3 k-Reliable Trees

Suppose that we wish to improve the pair-connected reliability of one of two trees of network T_1 or T_2 which have the same order. Let k be the number of edges whose probability we decide to improve. One can ask whether we should choose the network T_1 or T_2. A tree T_0 of order n is called a k-reliable tree, if for any other tree with the same order, there exist $S = \{e_1, e_2, \ldots, e_k\} \subseteq E(T_0)$ and

$$PC(T_0, \{(e_i, \Delta p) | i = 1, 2, \ldots, k\}, q) \geq PC(T, S^*, q)$$

where $S^* = \{(e'_i, \Delta p) | i = 1, 2, \ldots, k; e'_i \in S\}$. In this section we investigate the question of finding the k-reliable tree of given order.

Theorem 3.33 *There exists a 1-reliable tree T_0 of order n for any $n \geq 1$.*

Proof: The proof is by construction. Let T_0 be the tree obtained by identifying the centers of the stars $K_{1,\lfloor\frac{n-2}{2}\rfloor}$ and $K_{1,\lceil\frac{n-2}{2}\rceil}$ with the end vertices of K_2 (see Figure 3.11). To show that T_0 is a 1-uniform tree, let e_0 be the edge in T_0 which is not an end edge. Note that

$$TV(e_0) = (1, \lfloor\frac{n-2}{2}\rfloor, \lceil\frac{n-2}{2}\rceil).$$

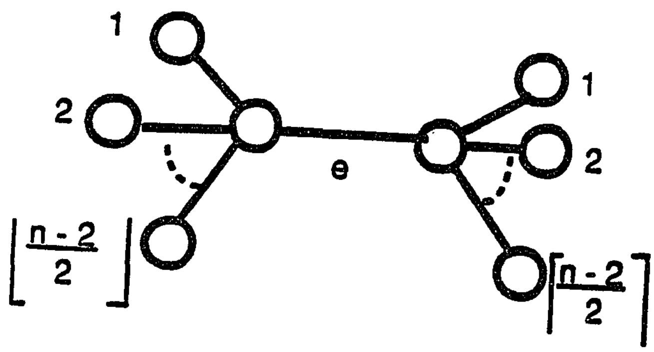

Figure 3.11

Let T be a tree of order n, let e be an edge in T. Claim: $TV(e_0) \geq TV(e)$. Observe that $\pi_2(e_0) = n - 2$ which is an upper bound, and

$$\sum_{i=1}^{n-1} \pi_i(e_0) = (\lfloor \frac{n}{2} \rfloor)(\lceil \frac{n}{2} \rceil)$$

which is an upper bound. Therefore $TV(e_0) \geq TV(e)$. The fact that

$$PC(T_0, \{(e_0, \Delta p)\}, q) - PC(T, \{(e, \Delta p)\} = \Delta p \sum_{i=1}^{n-1} [\pi_i(e_0) - \pi_i(e)] p^{i-1} \geq 0$$

implies the result. □

CHAPTER IV

GLOBAL RELIABILITY

4.1 Tree Networks

In this chapter a study of network enhancement is introduced using two methods:

(1) multiple edge replacement, and

(2) reliability improvement of edges.

The first method consists of adding additional edges to the network, with the restriction that edges are added only between vertices which are already joined by an edge. The latter method consists of replacing edges of the network by more reliable edges. Throughout this chapter, we will only consider global reliability measure.

Recall from Chapter I that the global connectivity of G, under the assumption of independent edge failure is given by:

$$R(G,q) = \sum_{S \in \Omega} f(S) \cdot R(S) \tag{4.1}$$

where q is the probability of failure of any edge in G, Ω is the power set of E, and $f(S)$ is defined as

$$f(S) = \begin{cases} 1 & \text{if } < S > \text{ is a connected spanning subgraph of } G \\ 0 & \text{otherwise.} \end{cases}$$

If G is a λ-edge connected graph, we need at least λ edges to disconnect G. Therefore $f(S) = 1$ for all $|S| \geq m - \lambda + 1$.

For a global reliability, Formula 4.1 can be rewritten as follows

$$R(G,q) = \sum_{S \in \Omega} m_i R(S) \tag{4.2}$$

where m_i is the number of induced connected subgraphs of size i in G. Observe that for (n,m)-graph G, $m_i = 0$ for all $i < n-1$.

In general, given a graph G of order n, if $\Gamma = \{p_1, p_2, \ldots, p_m\}$ is the probability assignment of the edges in $E(G)$, then Formula 4.2 can be rewritten as

$$R(G,\Gamma) = \sum_{S \in \Omega} f(S) \cdot R(S).$$

where $f(S)$ is defined as above and $R(S)$ is the probability that G is in state S. If $S = \{e_1, e_2 \ldots, e_k\}$, then

$$R(S) = \prod_{i=1}^{k} p(e_i) \cdot [1 - p(e_i)]$$

where $p(e_i)$ is the reliability of the edge e_i (the probability of having G in state $\{e_i\}$).

If the p_i's in Γ, are equal to p, then $R(G,\Gamma) = R(G,q)$, where $q = 1-p$. The function $R(G,\Gamma)$ gives the probability that for every pair v_1 and v_2 of nodes in G, there is a path from v_1 to v_2; equivalently, that is the probability that the graph G contains at least one spanning tree. If G is a directed graph and s is a source node, then in 4.1

$$f(S) = \begin{cases} 1 & \text{if } < S > \text{ contains a directed path from } s \text{ to every other vertex in } G \\ 0 & \text{otherwise.} \end{cases}$$

This chapter is devoted to studying the analysis of improving global reliability in tree networks, using the stated methods.

If T is a tree of order n, then $T_n[k]$ will denote the class of all graphs obtained from T by adding k multiple edges. The graphs in $T_n[k]$ may have different reliabilities, depending on how the k extra edges are added. In this section, we investigate the problem of finding an optimally reliable graph with respect to global reliability in $T_n[k]$.

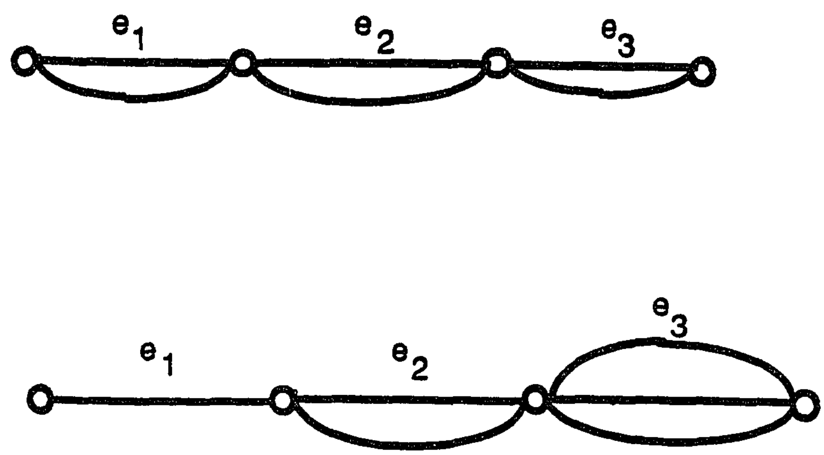

Figure 4.1

As in Chapter III, if the only edges in a graph G which are improved by multiple edges or by edge replacement are S, then the global reliability of G is denoted by $R(G, S^*, q)$ where S^* is the probability assignment to S. If there is no ambiguity, then we simply write $R(G, S^*)$.

Example: Consider the following graphs, G_1 and G_2 from the class $T_4[3]$ which are shown in Figure 4.1. Let $\{e_1, e_2, e_3\}$ be any labeling of the set $E(T_4)$ and $A = \{e_1, e_2\}$ with $A^* = \{(e_1, 2p-p), (e_2, 2p-p), (e_3, 2p-p)\}$, then

$$R(G_1, A^*) = (2p-p)^3.$$

Let $B = \{e_2, e_3\}$ with $B^* = \{(e_2, 2p-p^2), (e_3, 3p-3p^2+p^3)\}$ then

$$R(G_2, B^*) = p \cdot (2p-p)(3p-3p^2+p^3).$$

We can easily show that $R(G_1, A) > R(G_2, B)$, for all $0 < p < 1$. Therefore, the graph G_1 is more reliable than the graph G_2.

In general, if $\Gamma = \{p_1, p_2, \ldots, p_{n-1}\}$ is the probability assignment of the edges in a tree T then

$$R(T, \Gamma) = \prod_{i=1}^{n-1} p_i.$$

Definition 6 *If T is a tree of order n and the edges in $E(T)$ are labeled $\{e_1, e_2, \ldots, e_{n-1}\}$, then $T_n(t_1, t_2, \ldots, t_{n-1})$, will denote the graph obtained from T by enhancing the edge e_i by t_i extra edges.*

Example: In graphs G_1 and G_2 of the previous example the new graphs are represented by $T_4(1,1,1)$ and $T_4(0,1,2)$ respectively.

Recall from Chapter II that the probability of having two vertices u and v connected, if there are k edges between them, is $1-(1-p)^k$. The increase in reliability between u and v is

$$1-(1-p)^k - p = (1-p) - (1-p)^k = \Delta p.$$

Let $A = \{t_1, t_2, \ldots, t_{n-1}\}$ be a partition of a positive integer k and let F be the set of all one-to-one functions from $E(T)$ to A. For f_1 and f_2 in F define

$$G_1 = T_n(f_1(e_1), f_1(e_2), \ldots, f_1(e_{n-1}))$$

$$G_2 = T_n(f_2(e_1), f_2(e_2), \ldots, f_2(e_{n-1})).$$

G_1 and G_2 have the same global reliability. To see this, consider

$$R(G_1, q) = \prod_{i=1}^{n-1} [1-(1-p)^{f_1(e_i)+1}]$$

and

$$R(G_2, q) = \prod_{i-1}^{n-1} [1-(1-p)^{f_2(e_i)+1}].$$

The terms in the first product are exactly the same as in the second, except the order may be different, therefore $R(G_1, q) = R(G_2, q)$.

The graphs in Figure 4.2 and Figure 4.3, represent $G_1 = T_4(2,0,0)$, and $G_2 = T_4(0,2,0)$ respectively. The two graphs are not isomorphic, even though G_1 and G_2 have the same global reliability.

The following combinatorial result will be useful for the next theorem.

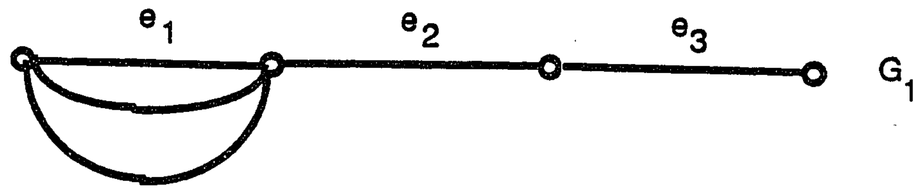

Figure 4.2

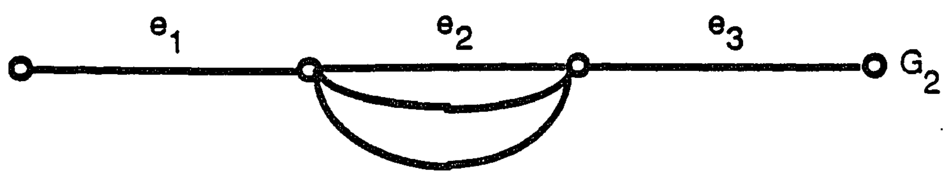

Figure 4.3

Lemma 7 *Let A and B be two sets with $|A| = k - s$, and $|B| = k + s$, where $0 \le s \le k$. Let $H(s,r)$ be the number of ways of choosing r element from $A \cup B$, such that at most $|A|-1$ and $|B|-1$ elements are taken from A and B respectively. For fixed r, $0 \le r \le 2k-2$, the function $H(s,r)$ has a maximum value when $s = 0$.*

Proof: The number of ways of choosing r elements from $A \cup B$ without restriction is

$$\sum_{i=0}^{r} C_i^{k+s} \cdot C_{r-i}^{k-s} = C_r^{2k}.$$

If the function $H(s,r)$ denotes the number of ways of choosing r elements from $A \cup B$ when at least one must be left, then $H(s,r)$ can be described as

$$H(0,r) = \begin{cases} C_r^{2k} & 0 \le r < k \\ C_r^{2k} - 2C_{r-k}^{k} & k \le r \le 2k-2 \end{cases}$$

and for $s \ge 1$

$$H(s,r) = \begin{cases} C_r^{2k} & 0 \le r < k-s \\ C_r^{2k} - C_{r-k+s}^{k+s} & k-s \le r \le k+s \\ C_r^{2k} - [C_{r-k+s}^{k+s} + C_{r-k-s}^{k-s}] & k+s \le r \le 2k-2. \end{cases}$$

Claim : $H(0,r) \ge H(s,r)$ for all r.

Case 1: $0 \le r \le k-s$ then $H(0,r) = H(s,r)$.

Case 2: $k-s \le r < k$

$$H(s,r) = C_r^{2k} - C_{r-k+s}^{k+s}, \ \ and \ \ H(0,r) = C_r^{2k}$$

hence $H(0,r) \ge H(s,r)$ for all $s = 0, 1, \ldots, k$

Case 3: $k \le r < 2k-2$

$$H(0,r) = C_r^{2k} - 2C_{r-k}^{k}, H(s,r) = C_r^{2k} - C_{r-k+s}^{k+s}$$

It is enough to show that $C_{r-k+s}^{k+s} \ge 2C_{r-k}^{k}$. To see this, observe the following:

$$C_{r-k+s}^{k+s} = \frac{(k+s)!}{(r-k+s)!(2k-r)!}$$

$$2C^k_{r-k} = \frac{2k!}{(2k-r)!(r-k)!}$$

The fact that

$$(k+s)!(k+s-1),\ldots,(k+1)(k!)(r-k)! >$$
$$2(r-k+s)(r-k+s-1),\ldots,(r-k+1)(r-k)!k!$$

implies

$$(k+s)!(r-k)! > (2k)!(r-k+s)!.$$

Thus

$$\frac{(k+s)!}{(r-k+s)!} > \frac{(2k)!}{(r-k)!}.$$

Therefore $C^{k+s}_{r-k+s} \geq 2C^k_{r-k}$. and $H(0,r) \geq H(s,r)$ for $k \leq r < k+s$

Case 4: $k \leq r \leq 2k-2$.

The fact that

$$C^{k+s}_{r-k+s} + C^{k-s}_{r-k-s} \geq C^{k+s}_{r-k+s} \geq 2C^k_{r-k}$$

implies $H(0,r) \geq H(s,r)$. □

Recall from Chapter III that, if $S = \{e_1, e_2, \ldots, e_k\} \subseteq E(G)$ is the the set of edges which receive a new probability assignment, namely $S^* = \{(e_1, \Delta_1), (e_2, \Delta_1), \ldots, (e_k, \Delta_1)\}$, then $R(G, \Gamma) = G(G, S^*, q)$.

Lemma 8 *Let G be the graph constructed from the vertex set $\{x, y, z\}$ by joining $k+s$ edges of the form xy and $k-s$ edges of the form yz (see Figure 4.4). If m_{2k-r} is the number of induced connected subgraphs of size $2k-r$ in G then $m_{2k-r} = H(s,r)$, where $H(s,r)$ is the function defined in Lemma 7.*

Proof: The fact that any induced connected subgraph of size $2k-r$ has to use at least one edge of the form xy, and another edge of the form yz together with Lemma 7 implies the result. □

Remark 10 *Given two (n,m)-graphs G_1 and G_2, if $m_{N-r}(G_1) > m_{N-r}(G_2)$ for all $r = 0, 1, 2, ..., N$ where $m_{N-r}(G_1)$ and $m_{N-r}(G_2)$ are the numbers of the induced connected subgraphs of G_1 and G_2 respectively, then $R(G_1, q) > R(G_2, q)$.*

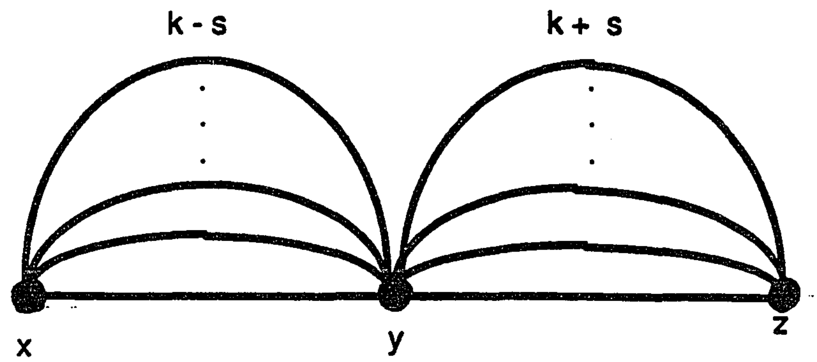

Figure 4.4

where,

$$\Gamma_1\{(e_1, p(e_1)), (e_2, p(e_2)), \ldots, (e_{|E(G_1)|}, p(e_1))$$

and

$$\Gamma_2\{(e'_1, p(e'_1)), (e'_2, p(e'_2)), \ldots, (e'_{|E(G_2)|}, p(e'_1))$$

are the probability assignments of $E(G_1)$ and $E(G_2)$ respectively.

Proof: Let m_{N-r} be the number of connected subgraphs with size $N-r$, where $N = n + 2k$.

In G_1, without loss of generality, let $e_1 = xy$ and $e_2 = yz$ be the two edges which receive k extra edges each. Let T_x, T_y and T_z be the components in $T - \{e_1, e_2\}$ which contain x, y and z respectively. If r is number of the edges which are in the failed state in each of G_1 and G_2, then

$$m_{N-r}(G_1) = H(0, r) \;\; and \;\; m_{N-r}(G_2) = H(s, r)$$

where $H(x, y)$ is the function described in Lemma 7. The fact that $m_{N-r}(G_1) \geq$

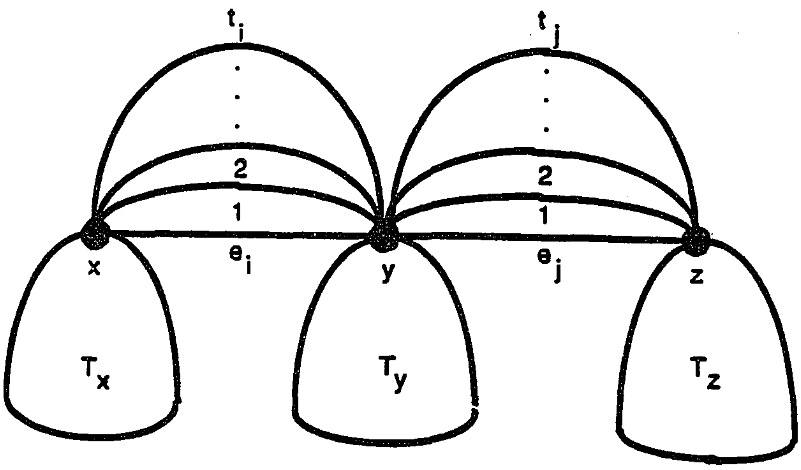

Figure 4.5

$m_{N-r}(G_2)$ for all r, follows from the same lemma; therefore:

$$\sum_{r=0}^{N} m_{N-r}(G_1)p^{N-r} > \sum_{r=0}^{N} m_{N-r}(G_2)p^{N-r}.$$

Thus, $R(G_1,\Gamma_1) \geq R(G_2,\Gamma_2)$. □

Definition 7 *The graph* $T_{n+1}(\overbrace{t+1,t+1,\ldots,t+1}^{r},t,\ldots,t)$ *will be denoted by* $T_{n+1}\{(t+1,r)\}$, *where* r *is the number of edges in* T_{n+1} *which receive* $t+1$ *extra edges.*

Theorem 4.35 *Among all enhanced trees in* $T_{n+1}[k]$, *where* $k = tn + r$, $0 \leq r < n$, *the most reliable graph is* $G_0 = T_{n+1}\{(t+1,r)\}$.

Proof: Let $T_{n+1}(t_1,t_2,\ldots,t_n) \in T_{n+1}[k]$ and $T_{n+1}\{(t+1,r)\} \in T_{n+1}[k]$. Observe that

$$A = \{t_1,t_2,\ldots,t_n\}$$

$$B = \{\overbrace{t+1,t+1,\ldots,t+1}^{r},t,\ldots,t\}$$

Proof: Let $T_{n+1}(t_1, t_2, \ldots, t_n) \in T_{n+1}[k]$ and $T_{n+1}\{(t+1, r)\} \in T_{n+1}[k]$. Observe that

$$A = \{t_1, t_2, \ldots, t_n\}$$

$$B = \{\overbrace{t+1, t+1, \ldots, t+1}^{r}, t, \ldots, t\}$$

are two partitions to k. If $A \neq B$, then there exist t_i and t_j in A such that $|t_i - t_j| \geq 2$. Without loss of generality, let $t_i > t_j$. Assume $e_i = xy$ and $e_j = yz$ are two incident edges in G_1 (see Figure 4.5). Let G_2 be the graph obtained from G_1 by moving one xy edge to yz edge.

Claim: $R(G_1, q) > R(G_2, q)$.

Let $m_{N-r}(G_1)$ and $m_{N-r}(G_2)$ be the count of the induced connected subgraphs with size $N - r$ in G_1 and G_2 respectively, where r is the number of failed edges in both G_1 and G_2. The number $m_{N-r}(G_1)$ can be written as $x_1(r) + x_2(r)$ where $x_1(r)$ is the number of the induced connected subgraphs when the failed edges are of the form xy or yz and $x_2(r)$ is the number of the induced connected subgraph when the failed edges are not of the form xy or yz. Similarly $m_{N-r}(G_2)$ can be written as $y_1(r) + y_2(r)$ where $y_1(r)$ is the number of failed edges of the form xy or yz, and $y_2(r)$ is the number of edges when the failure edges are not of the form xy or yz. Necessarily $x_1(r) = y_2(r)$ and by using Theorem 4.35, we have $y_2(r) > x_2(r)$ for all $r = 0, 1, \ldots, 2k$. Therefore, the graph G_1 is not an optimally reliable graph in $T_{n+1}[k]$, with respect to the global reliability. □

The above result shows that a tree T with k extra edges is more reliable when the edges are distributed evenly.

Given two trees T_1 and T_2, if $|V(T_1)| < |V(T_2)|$ then $R(T_1, q) > R(T_2, q)$. This follows from the fact that $p^x > p^y$ if $x < y$, for all $0 < p < 1$.

We will denote the change in global reliability after multiple edge replacement or improvement in the optimal way by $\Delta R(G, S^*)$, where S^* is the probability assignment of the set S which gives the optimal reliability.

Theorem 4.36 *Let T_1 and T_2 be trees of order n_1 and n_2, respectively and let m be the number of extra edges. For $S \subseteq E(T_1)$ and $I \subseteq E(T_2)$ with $|S| = |I|$, the following is true: If $n_2 > n_1$, then $\Delta R(T_1, S^*) \geq \Delta R(T_2, I^*)$.*

Proof: Let $m = nk + r$

$$\begin{aligned} \Delta R(T_1, S^*) &= p_{n+1}^r \cdot p_n^{k-r} \cdot p^{n_1-k} - p^{n_1} \\ \Delta R(T_2, I^*) &= p_{n+1}^r \cdot p_n^{k-r} \cdot p^{n_2-k} - p^{n_2} \end{aligned}$$

where $p_i = 1 - (1-p)^{i+1}$

$$\begin{aligned} &\Delta R(T_1, S^*) - \Delta R(T_2, I^*) = \\ &\quad p_{n+1}^r \cdot p_n^{k-r}[p^{n_1-k} - p^{n_2-k}] - [p^{n_1} - p^{n_2}] \\ &= p_{n+1}^r \cdot p_n^{k-r}[p^{n_1-k} - p^{n_2-k}] - p^k[p^{n_1-k} - p^{n_2-k}]. \end{aligned}$$

Since $p_{n+1}^r \cdot p_n^{k-r} > p_n^k$ implies $\Delta R(T_1, S^*) > \Delta R(T_2, I^*)$. □

Enhancing tree networks, can be done by replacing the edges of T by more reliable edges. Let $S = \{e_1, e_2, \ldots, e_k\}$ be the set of edges to be improved in T, and

$$S^* = \{(e_1, \Delta p_1)\}.(e_2, \Delta p_2), \ldots, (e_k, \Delta p_k)\}.$$

Let $\Delta p = \sum_{i=1}^k \Delta p_i$. The next result shows that the best distribution of Δp is the one when $\Delta p_i = \frac{\Delta p}{k}$, for all $i = 1, 2, \ldots, k$. As in Chapter III, let $T_n[\Delta p]$ denote the class of trees of order n and a total increase in edge reliability of Δp.

Theorem 4.37 *Let $\Delta p = \sum_{i=1}^k \Delta p_i$ where $k \leq (n-1)$. The graph $T_n(\Delta p_1, \Delta p_2, \ldots, \Delta p_k, 0, \ldots, 0)$ has maximum global reliability when $\Delta p_i = \frac{\Delta p}{k}$, for all $i = 1, 2, \ldots, k$.*

Proof: Let $T_n(\Delta p_1, \Delta p_2, \ldots, \Delta p_k, 0, \ldots, 0)$ be a graph in $T_n[\Delta p]$. Without loss of generality, let $\Delta p_1 > \Delta p_2$, we will show that G_1 can be modified to a more reliable graph G_2, without changing the value Δp. Let $\Delta p_1 + \Delta p_2 = x$ and consider the graph

$$G_t = T_n(tx, (1-t)x, \Delta p_3, \ldots, \Delta p_k, 0, \ldots, 0).$$

Let

$$H(t) = R(G_t) = p^{n-k} \cdot (p + \Delta p_3)(p + \Delta p_4) \dots (p + \Delta p_k) \cdot (p + tx)(p + (1 - t)x).$$

The function $H(t)$ has maximum value at $t = t_0$ if and if only if the function $h(t) = (p + tx)(p + (1 - t)x)$ has maximum value at $t = t_0$. But latter one is just a quadratic function of t which has a maximum value at $t = \frac{1}{2}$. Therefore the function $H(t)$ has a maximum value at $t = \frac{1}{2}$. Letting $G_2 = G_{1/2} = T_n(x/2, x/2, \Delta p_3, \dots, \Delta p_k, 0, \dots, 0)$ implies that G_2 is more reliable than G_1. This implies that any partition of Δp into k distinct equal numbers will result in a graph which is not the most reliable. Hence Δp must be partitioned into k equal numbers, namely

$$\{\frac{\Delta p_k}{k}, \frac{\Delta p_k}{k}, \dots, \frac{\Delta p_k}{k}\}.$$

□

In the above theorem, if the set S of edges which are to be improved by a total of Δp is $E(T)$, then it is always better to improve all edges in $E(T)$, and the distribution of Δp among $E(T)$ must be done evenly.

4.2 Ring Networks

The ring network has been one of the most commonly used network topologies in the design and implementation of local area networks. This is due to its simplicity and expandability. The ring network can be modeled by the cycle C_n of n vertices. In this section we analyze the enhancing of ring networks by the two methods mentioned in Section 4.1.

Given a positive integer k and a cycle C_n, the class of graphs obtained from C_n by enhancing the edges in $E(C_n)$ by k multiple edges, is denoted by $C_n[k]$. In this chapter we show that $G \in C_n[k]$ is an optimally reliable graph with respect to the global reliability, if k extra edges are distributed evenly among $E(C_n)$.

Definition 8 *Let $A = \{G_1, G_2, ..., G_n\}$ be a set of graphs. A graph G_0 in A is called optimally reliable in A, if G_0 is the most reliable graph in A with respect to the global reliability.*

Throughout this discussion, the term optimally reliable graph refers to the global reliability.

Lemma 9 *Assume C_n is a cycle of order n. Then $R(C_n, q) = p^n + np^{n-1}q$, where $p = 1 - q$ and q is the probability of failure of the edges in C_n.*

Proof: By definition of global reliability $R(C_n, q) = \sum_{S\in\Omega} f(S) \cdot p(S)$. Since any two edges in $E(C_n)$ will disconnect C_n therefore, $f(S) = 0$ for all $|S| < n - 1$. This leaves only two cases to consider. If $|S| = n$ then there is only one induced connected subgraph. On the other hand if $|S| = n - 1$ then there are n different ways to chose S so that the induced subgraphs of S is connected. Therefore $R(C_n, q) = 1p^n + (n)p^{n-1}q$. □

Remark 11 *If $\Gamma = \{p_1, p_2, \ldots, p_n\}$ is the probability assignment of $G(C_n)$, then*

$$R(C_n, \Gamma) = \prod_{i=1}^{n} p_i + \sum_{i=1}^{n}(1 - p_i) \prod_{j\neq i}^{n} p_j.$$

In $C_n[k]$, if the edges in C_n are labeled $e_1, e_2, \ldots, e_n$ then the graph $C_n(t_1, t_2, \ldots, t_n)$, denotes the cycle C_n with extra t_i edges are added to the edge e_i, $i = 1, 2, \ldots, n$.

Theorem 4.38 *If γ is any permutation on the set $\{t_1, t_2, \ldots, t_n\}$ then the two graphs*

$$C_n(t_1, t_2, \ldots, t_n)$$

$$C_n(t_{\gamma(1)}, t_{\gamma(2)}, \ldots, t_{\gamma(n)})$$

have the same reliability.

Proof: Let $G_1 = C_n(t_1, t_2, ..., t_n)$ and $G_2 = C_n(t_{\gamma(1)}, t_{\gamma(2)}, ..., t_{\gamma(n)})$. By definition if $uv = e_i \in E(G_1)$, then there are $t_i + 1$ edges of the form uv. This implies that $p(e_i) = 1 - (1-p)^{t_i}$, we simply denote $p(e_i)$ by p_i. By remark 11, the global reliability of G_1 is

$$R(G_1, \Gamma_1) = \prod_{i=1}^{n} p_i + \sum_{i=1}^{n} (1-p_i) \prod_{j \neq i}^{n} p_j$$

where Γ_1 is the probability assignment of $E(G_1)$. Moreover,

$$R(G_2, \Gamma_2) = \prod_{i=1}^{n} p'_i + \sum_{i=1}^{n} (1-p'_i) \prod_{j \neq i}^{n} p'_j$$

where $p'_i = 1 - (1-p)^{\gamma(t_i)+1}$ and $p'_j = 1 - (1-p)^{\gamma(t_j)+1}$ for all $1 \leq i, j \leq n$. Γ_2 is the probability assignment of $E(G_2)$. Since γ is a permutation on the set $\{t_1, t_2, ..., t_n\}$, this implies that for every edge $e_i \in E(G_1)$, there exist an edge $e'_i \in E(G_2)$, such that $p(e_i) = p(e'_i)$ or $p_i = p'_i$. Therefore

$$\prod_{i=1}^{n} p_i = \prod_{i=1}^{n} p'_i$$

and

$$\sum_{i=1}^{n} (1-p_i) \prod_{j \neq i}^{n} p_j = \sum_{i=1}^{n} (1-p'_i) \prod_{j \neq i}^{n} p'_j.$$

Thus, $R(G_1, \Gamma_1) = R(G_2, \Gamma_2)$. □

The formula 4.2 can be modified to

$$R(G, \Gamma) = \sum_{r=0}^{N} m_{N-r}(G) \cdot p^{N-r}(1-q)^r$$

where $m_{N-r}(G)$ is the number of induced connected subgraphs in G after the failure of r edges in G.

Theorem 4.39 *$R(C_n(k, k, 0, \ldots, 0)) \geq R(C_n(k+s, k-s, 0, \ldots, 0))$, for all $s = 0, 1, \ldots, k$.*

Proof: Suppose $G_1 = (C_n(k, k, 0, \ldots, 0))$ and $G_2 = (C_n(k+s, k-s, 0, \ldots, 0))$ and let $m_{N-r}(G_1)$ and $m_{N-r}(G_2)$ be the number of connected induced subgraphs in G_1 and G_2 with size $N-r$ respectively, $N = n+2k$ and r is the number of failed edges.

By simple combinatorial analysis we have the following:

$$m_{N-r}(G_1) = (n-2)C_{r-1}^{2k+2} + C_r^{2k+2}$$

and

$$\begin{aligned} m_{N-r}(G_2) &= (n-2)\sum_{i=0}^{k-s}[C_i^{k-s+1} \cdot C_{r-i-1}^{k+s+1}] + C_r^{2k+2} \\ &= (n-2)\sum_{i=0}^{k-s+1}[C_i^{k-s+1} \cdot C_{r-i-1}^{k+s+1} - C_{k-s+1}^{k-s+1} \cdot C_{r+s-k+2}^{k+s+1}] + C_r^{2k+2} \\ &= (n-2)[C_{r-1}^{2k+2} - C_{r+s-k+2}^{k+s+1}] + C_r^{2k+2}. \end{aligned}$$

Since $m_{N-r}(G_1) \geq m_{N-r}(G_2)$ for all r, therefore $R(G_1) \geq R(G_2)$. □

Theorem 4.40 *If $G = C_n(t_1, t_2, ..., t_n) \in C_n[k]$ is an optimally reliable graph, then $|t_i - t_j| \leq 1$.*

Proof: Let $G \in C_n[k]$, with $G = C_n(t_1, t_2, \ldots, t_n)$. Assume that there exist t_i and t_j such that $t_i - t_j > 1$.

Claim: G_1 is not optimal reliable graph in $C_n[k]$.

In order to show this, let $u_i u_{i+1}$ be the edge e_i and $u_{i+1}u_{i+2}$ be the edge e_j (see Figure 4.6). Let G' be a graph in $C_n[k]$ constructed from G_1 by taking an edge of the form $u_i u_{i+1}$ and placing it in parallel between the vertices u_{i+1} and u_{i+2}. Suppose that A_1 and A_2 are the sets of edges of the form $u_i u_{i+1}$ or $u_{i+1}u_{i+2}$ in G_1 and G_2 respectively. Let $f_j(i)$ represents the count of the induced connected graphs in $G_j - A_j$ with $|E(G_j - A_j)| - i$ edges, $j = 1, 2$ and let $g_j(r-i)$ represents the count of the induced subgraphs with $|A_j| - (r-i)$ edges in $G_i - E(G_j - A_j), j = 1, 2$. Define $m_{N-r}(G_j)$ for $j = 1, 2$ to be the count of a connected subgraphs in G_j with size $N - r$, where $N = n + k$. One can see the following:

$$m_{N-r}(G_1) = \sum_{i=0}^{r} f_1(i) \cdot g_1(r-i)$$

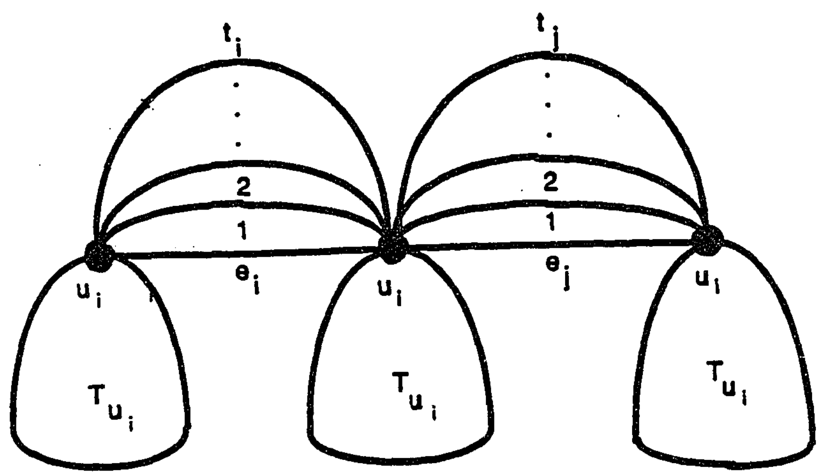

Figure 4.6

and

$$m_{N-r}(G_2) = \sum_{i=0}^{r} f_2(i).g_2(r-i).$$

Observe that

$$G_1 - A_1 \cong G_2 - A_2.$$

Therefore $f_1(i) = f_2(i)$ for all $i = 1, 2, \ldots, r$. Thus, in order to prove that $m_{N-r}(G_1) \geq m_{N-r}(G_2)$ it is enough to show that $\sum_{i=0}^{r} g_1(r-i) \geq \sum_{i=0}^{r} g_2(r-i)$, for all i. If $i \geq 1$ the result follows from Lemma 8. Therefore $m_{N-r}(G_1) \geq m_{N-r}(G_2)$ for all r. □

We will denote the optimal ring in $C_n[k]$ by $C_n\{(r,t)\}$ which means that there are r edges in C_n which receive an extra $t+1$ edges and $n-r$ edges in C_n which receive t extra edges. Let Γ be the probability assignment of $R(C_n\{(r,t)\})$. Since the notation $C_n\{(r,t)\}$ describes Γ uniquely, one can replace $R(G,\Gamma)$ (the global reliability of G) by $R(G)$. By convention, if $r > n$ then $R(C_n\{(r,t)\}) = 0$.

Remark 12 *For the ring $C_n\{(r,t)\}$, the global reliability is*

$$R(C_n(r,t)) = p_{t+1}^r p_t^{n-r} + C_0^r p_{t+1}^r p_t^{n-r-1}(1-p_t) + C_1^r p_{t+1}^{r-1}(1-p_{t+1})p_t^{n-r}.$$

where $p_{t+1} = 1-(1-p)^{t+2}$ *and* $p_t = 1-(1-p)^{t+1}$

Lemma 10 *If C_n and C_m are two cycles and $n < m$, then $R(C_n, q) > R(Cm, q)$.*

Proof: Observe that

$$\begin{aligned} R(C_n, q) &= p^n + np^{n-1}(1-p) \text{ and} \\ R(C_m, q) &= p^m + mp^{m-1}(1-p). \end{aligned}$$

Letting $m - n = k > 0$ then

$$\begin{aligned} R(C_n, q) - R(C_m, q) &= p^n[1-p^k] + (1-p)[n - mp^k]p^{n-1}p \\ &= (1-p)p^{n-1}[p + p^2 + p^3 + \ldots + p^k + n - mp^k] \\ &= [(1-p)p^{n-1}[p + p^2 + p^3 + \ldots + p^{k-1} + n - (m-1)p^k]. \end{aligned}$$

but

$$\begin{aligned} n + p + p^2 + \ldots + p^{k-1} &> \\ & (k-1)p^{k-1} + np^{k-1} \\ &= (k+n-1)p^{k-1} \\ &= (m-1)p^{k-1} \\ &> (m-1)p^k. \end{aligned}$$

Therefore $R(C_n, q) - R(C_m, q) > 0$. □

Let $C_n[\Delta p]$ be the class of all cycles which have a total of Δp increase in the reliability of the edges of C_n. The following result show that $G \in C_n[\Delta p]$ is an optimally reliable graph if and only if $G = C_n[\Delta p/n, \ldots, \Delta p/n]$.

Theorem 4.41 *The graph $G = C_n[\Delta p/n, \ldots, \Delta p/n]$ is the optimally reliable graph in $C_n[\Delta p]$.*

Proof: Let $G = C_n(\Delta p_1, \Delta p_2, \ldots, \Delta p_n)$ be a graph in $C_n[\Delta p]$. Without loss of generality let $\Delta p_1 > \Delta p_2$.

Claim: G is not an optimally reliable graph.

Let $x = \Delta p_1 + \Delta p_2$ and let $G_t = C_n(tx, (1-t)x, \Delta p_3, \ldots, \Delta p_n)$. First we find the optimal reliable graph in $\{G_t, t \in [0, x]\}$.

$$\begin{aligned} R(G_t) = H(t) &= [p+tx][p+(1-t)x]\prod_{i=3}^{n}[p+\Delta p_i] + \\ &\quad [p+tx][1-p-(1-t)x]\prod_{i=3}^{n}[p+\Delta p_i] + \\ &\quad [p+(1-t)x][1-p-tx]\prod_{i=3}^{n}[p+\Delta p_i] + \\ &\quad [p+tx][p+(1-t)x]\sum_{i=0}^{n}[1-p-\Delta p_i]\prod_{j\neq i}[p+\Delta p_j]; \\ H(t) &= \\ &\quad [x^2(1-2t)]\prod_{i=3}^{n}[p+\Delta p_i] + \\ &\quad [2tx^2 - x^2 + x]\cdot\prod_{i=3}^{n}[p+\Delta p_i] + \\ &\quad [2tx^2 - x^2 - x]\prod_{i=3}^{n}[p+\Delta p_i] + \\ &\quad [x^2(1-2t)]\sum_{i=1}^{n}\{[1-p-\Delta p_i]\prod_{j\neq i, i\geq 3}[p+\Delta p_j]\}. \end{aligned}$$

Differentiating with respect to t, we get :

$$\begin{aligned} H'(t) &= [x^2(1-2t)][\prod_{i=3}^{n}[p+\Delta p_i] + \\ &\quad \sum_{i}\{[1-p-\Delta p_i]\prod_{j\neq i, i\leq 3}[p+\Delta p_j]\} - \\ &\quad 2x^2[1-2t]\prod_{i=3}^{n}[p+\Delta p_i]. \end{aligned}$$

Solving for t and using the second derivative test, one can show that the function $H(t)$ has a mimum value at $t = \frac{1}{2}$. Therefore, the graph $G_{\frac{1}{2}}$ is the optimally reliable graph in $\{G_t\}$. Observe that $G \in \{G_t\}$, hence $R(G_{\frac{1}{2}}) > R(G)$ and G is not a optimally reliable graph in $C_n[\Delta p]$. □

4.3 Unicyclic Networks

Definition 9 *A graph G is called a unicyclic graph if G contains exactly one cycle.*

Here we analyze the improvement of unicyclic networks by adding multiple edges.

Given a unicyclic graph G, the edges in $E(G)$ can be partitioned into two sets: set A, the edges located on the cycle and set B, the edges in $E(G) - A$. The induced subgraph Ind(A) from A is the cycle C, and the induced subgraph Ind(B) is a forest F. We denote the family of unicyclic graphs with $|A| = n$ and $|B| = m$ by $\{C_n \cdot F_m\}$ with a cycle C_n and forest F_m.

Theorem 4.42 *If* $G \in \{C_n \cdot F_m\}$ *then*

$$R(G,q) = p^{n+m} + np^{n+m-1}(1-p).$$

Proof: The graph G has order $N = n + m$. If m_{N-r} is the number of induced connected subgraph $< S >$ of size N, then

$$R(G,q) = \sum_{r=0}^{N} m_{N-r}p^{N-r}(1-q)^r.$$

If $|S| \leq n + m - 2$, then $f(S) = 0$. Therefore, the only cases which need to be considered are $|S| > n+m-2$. If $|S| = n+m$, then $m_N = 1$ and $R(S) = p^{n+m-1}$. If $|S| = n+m-1$, then $m_{N-r} = n$ and $R(S) = p^{n+m-1}(1-p)$. Substituting these values in the formula of $R(G,q)$ implies the result. □

Theorem 4.43 *Among all an (N,N)-graphs the cycle C_N is the most reliable graph.*

Proof: Let $G = C_n \cdot f_m$ be an (N,N)-graph, where C_n is the cycle in G and F_m is the induced subgraph $Ind(E(G) - E(C_n))$

$$R(G,q) = p^N + np^{N-1}(1-p)$$

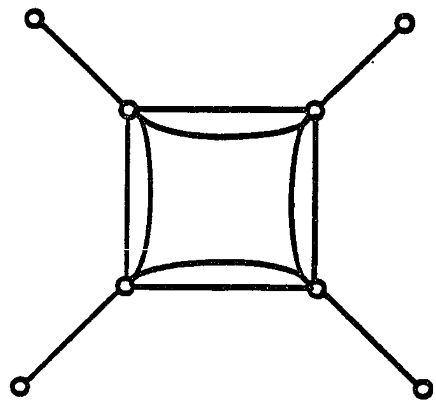

Figure 4.7

and

$$\begin{aligned} R(C_N,q) &= p^N + Np^{N-1}(1-p) \\ R(C_N,q) - R(G,q) &= p^{N-1}(1-p)[N-n] \\ &\geq 0. \end{aligned}$$

for all $N \geq n$. Therefore C_N is the most reliable (N,N)-graph. □

Next we improve the unicyclic graph by the method of adding multiple edges. Given a unicyclic graph $G = C_n \cdot F_m$. We denote the family of graphs $C_n \cdot F_m$ which have k extra edges by $C_n \cdot F_m[k]$.

Example: Let $G = C_4 \cdot F_4$. Figure 4.7 and Figure 4.8 show two graphs G_1 and G_2 respectively from the family $C_4.F_4[4]$. The global reliability of G_1 and G_2 are

$$R(G_1,q) = p_1^4p^4 + 4p^4p_1^3(1-p)$$

and

$$R(G_2,q) = p_1^4p^4 + 4p_1^4p^3(1-p),$$

where $p_1 = 2p - p^2$. Since $p_1 > p$, it is not hard to see that G_2 is more reliable than G_1.

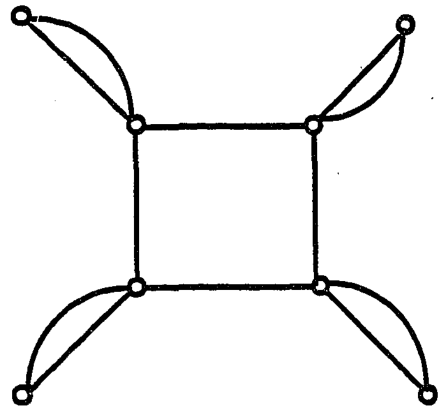

Figure 4.8

Lemma 11 *Let G be a graph constructed by identifying one vertex from a graph G_1 with a vertex of another graph G_2. Then $R(G,q) = R(G_1,q) \cdot R(G_2,q)$.*

Proof: Let the graph G be in state $S = S_1 \cup S_2$ where $S_1 = S \cap E(G_1)$ and $S_2 = S \cap E(G_2)$. $< S >$ is connected if and only if $< S_1 >, < S_2 >$ are connected.

$$\begin{aligned} R(G,q) &= \sum_{S \in \Omega} f_G(S) \cdot p(S) \\ R(G_i,q) &= \sum_{S \in \Omega_i} f_{G_i}(S) \cdot p(S) \;\; for \;\; i = 1,2 \end{aligned}$$

where,

$$f_G(S) = \begin{cases} 1 & \text{if } < S > \text{ is connected in } G, \\ 0 & \text{otherwise.} \end{cases}$$

Note that, for $i = 1,2$ we have:

$$f_{G_i}(S) = \begin{cases} 1 & \text{if } < S > \text{ is connected in } G_i. \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $f_G(S) = f_{G_i}(S \cap E(G_i)) \cdot f_{G_2}(S \cap E(G_2))$. Moreovere $p(S) = p(S \cap E(G_1)) \cdot p(S \cap E(G_2))$. Thus it follows that $R(G,q) = R(G_1,q) \cdot R(G_2,q)$. □

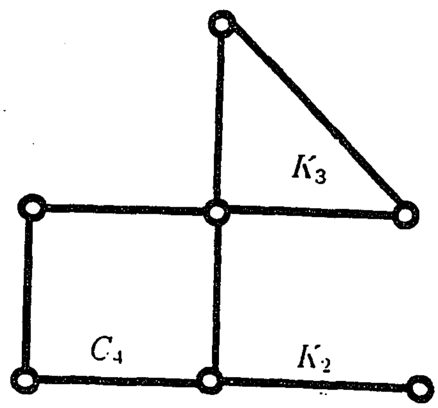

Figure 4.9

Definition 10 *Given a set of graphs $A = \{G_1, G_2, \ldots, G_k\}$. The graph G is said to be a series connection from A, if G is constructed from A in the following way:*
(1) No two graphs have more than one point in common,
(2) G is a connected graph and has a number of cycles equal to the number of cycles in $\cup_{i=1}^{k} G_i$.
Figure 4.9 shows a series connection graph constructed from the set
$A = \{K_3, C_4, K_2\}$.

Corollary 14 *If G is in series connection from $A = \{G_1, G_2, \ldots, G_k\}$, then*

$$R(G,q) = \prod_{i=1}^{k} R(G_i, q).$$

Lemma 11 and Corollary 14 can be extended to any probabilistic graph G.

Given a unicyclic graph $G = C_n \cdot F_m$ let k be the number of extra edges. If we label the edges of the cycle C_n by $\{e_1, e_2, ..., e_n\}$ and the edges of the forest by F_m by $\{e'_1, e'_2, ..., e'_n\}$, then the graph

$$G_0 = C_n(t_1, t_2, \ldots, t_n) \cdot F_m(s_1, s_2, \ldots, s_m)$$

denotes the graph with t_i extra edges on the edge e_i in the circle, and s_i extra edges on the edge e'_i of the forest. If C_n or F_m has no extra edges, then $C_n(0,0,...,0)$ and $F_m(0,0,...,0)$ will simply be replaced by $C_n and\ F_m$, respectively..

Lemma 12 *If* $G_1 = C_1(1,0,\ldots,0)\cdot F_m$ *and* $G_2 = C_1 \cdot F_m(1,0,\ldots,0)$, *then* $R(G_1) < R(G_2)$.

Proof: It follows from Corollary 14 that

$$\begin{aligned} R(G_1) &= p^m[p^n + p^{n-1}(1-p_1) + (n-1)p^{n-2}\cdot p_1(1-p)] \\ R(G_2) &= p^{m-1}\cdot p_1[p^n + np^{n-1}(1-p)] \end{aligned}$$

where $p_1 = 2p - p^2$

$$\begin{aligned} R(G_2) - R(G_1) &= p^{m-1}p^{n-1}[n + npp_1 + p_1 + pp_1 - np_1 - np - p - pp] \\ &= p^{m-1}p^{n-1}[n(1 + pp_1 - p_1 - p) + (p_1 - p) + (pp_1 - pp)] \\ &= p^{m-1}p^{n-1}[n(1-p)(1-p_1) + (p_1 - p) + p(p_1 - p)] \\ &> 0 \end{aligned}$$

for all $n \geq 3$, $m \geq 1$, $0 < p < 1$. □

Corollary 15 *Given a graph* $G = C_n[k_1]\cdot F_m[k_2]$, *if* Γ *is the probability assignment of* G, *then*

$$R(G,\Gamma) \leq R(C_n\{(t_1,r_1)\}\cdot F_m\{(t_2,r_2)\}$$

where $k_1 = nt_1 + r_1$ *and* $k_2 = mt_2 + r_2$.

Proof: In the graph G, let Γ_1 and Γ_2 be the probability assignment of the edges in C_n and F_m, respectively. We have $\Gamma = \Gamma_1 \cup \Gamma_2$, therefore

$$\begin{aligned} R(G,\Gamma) &= R(C_n,\Gamma_1)\cdot R(F_m,\Gamma_2) \\ R(C_n,\Gamma_1) &\leq R(C_n\{(t_1,r_1)\} \\ \textit{and}\ R(F_m,\Gamma_2) &\leq R(F_m\{(t_2,r_2)\}). \end{aligned}$$

Therefore,

$$R(G,\Gamma) \leq R(C_n\{(t_1,r_1)\} \cdot F_m\{(t_2,r_2).$$

□

Lemma 13 *Let $G = C_n(t_1,t_2,\ldots,t_n) \cdot F_m(s_1,s_2,\ldots,s_m)$ and let P_{m+1} be the path of size m. If $\Gamma_1 = \{t_1,t_2,\ldots,t_n\}$ and $\Gamma_2 = \{s_1,s_2,...,s_m\}$ are the probability assignments of the edges in C_n and F_m respectively, then*

$$R(G,\Gamma) = R(C_n,\Gamma_1) \cdot R(P_{m+1},\Gamma_2).$$

Proof: $R(G,\Gamma) = R(C_n,\Gamma_1) \cdot R(F_m,\Gamma_2)$. Let $F = T_1 \cup T_2 \cup \ldots \cup T_l$, where T_i is a nontrivial component in $G - E(C_n)$.

$$R(G,\Gamma) = R(C_n,\Gamma_1) \cdot [\prod_{i=1}^{l} R(T_i,\Gamma_i')]$$

where Γ_i' is the probability assignment of the edges in T_i with order exactly the same as in F_m. Since

$$\prod_{i=1}^{l} R(T_i,\Gamma_i') = R(P_{m+1},\Gamma_2).$$

we have

$$R(G,\Gamma) = R(C_n,\Gamma_1) \cdot R(G,\Gamma_2).$$

□

By using the above lemma we can find the global reliability of graph

$$G = C_n(t_1,t_2,\ldots,t_n) \cdot F_m(s_1,s_2,\ldots,s_m)$$

by simply considering the graph

$$G' = C_n(t_1,t_2,\ldots,t_n) \cdot P_{m+1}(s_1,s_2,\ldots,s_m).$$

That is, by replacing the forest F_m with the path P_{m+1}. This replacement will make the analysis of improving unicyclic graphs much easier.

If the variable in $R(G,\Gamma)$ is the number of extra edges k,then $R(G,\Gamma)$ is defined to be $R(G,\Gamma(t))$, where $t = 0,1,\ldots,k$. The function $R(G,\Gamma(t))$ is not continuous on $[0,k]$. In the coming discussion we will allow this function to take any value between$[0,k]$. The modified function $R(G,\Gamma(t))$ is continuous and differentiable for all values of t. If the modified function $R(G,\Gamma(t))$ is monotonic on $[n,n+1]$ for all $n = 0,1,\ldots,k$, then $R(G,\Gamma(t))$ has maximum or minimum value at t_0, then the original function has maximum or minimum value at $\lfloor t_0 \rfloor$ or $\lceil t_0 \rceil$.

Remark 13 $R(C_n\{(0,s)\}) = p_1^s p^{n-s} + (n-s)p_1^s p^{n-s-1} + s p_0^{s-1}(1-p)^2 \cdot p^{n-s}$.

Theorem 4.44 *Given a graph $G = C_n \cdot F_m$, let k be the number of edges to be used in the enhancement of G. If $k \leq m$, then $C_n \cdot F_m\{(0,k)\}$ is the optimally reliable graph in $C_n \cdot F_m[k]$.*

Proof: By the previous lemma, it is enough to show that $C_n \cdot P_{m+1}\{(0,k)\}$ is the optimal graph in $C_n \cdot P_m[k]$. Let t be the count of the enhanced edges used to improve P_m. By using Corollary 15, the optimal graph in

$$C_n[t] \cdot P_{m+1}[k-t]$$

is

$$C_n\{(0,t)\} \cdot P_{m+1}\{(0,k-t)\}.$$

Let

$$A = \{G_t | G_t = C_n\{(0,t)\} \cdot P_{m+1}\{(0,k-t)\}\}.$$

be the set of all graphs with t and $k-t$ extra edges distributed evenly among $E(C_n)$ and $E(P_{m+1})$ respectively. Next we show that G_0 is the optimally reliable graph in A.

$$\begin{aligned} R(G_t) &= R(C_n\{(0,t)\}) \cdot R(P_{m+1}\{(0,k-t)\}) \\ &= [p_1^t \cdot p^{n-t} + (n-t)(1-p)p^{n-t-1} \cdot p_1^t + t(1-p_1)p_1^{t-1} \cdot p^{n-t}] \cdot p_1^{K-t} \cdot p^{m-K-t} \\ &= p_1^k \cdot p^{n+m-K} + (n-t)(1-p)p^{n+m-K-1} \cdot p_1^K + t(1-p_1)p_1^{K-1} \cdot p^{m+n-K}. \end{aligned}$$

This is a linear function of t which has a maximum value at $t = 0$. Therefore G_0 is the optimally reliable graph in A, and hence in $C_n \cdot F_m[K]$. □

The above result shows an efficient way to improve a given unicyclic graph $C_n \cdot F_m$, but only if the number of additional edges is not greater than m.

Corollary 16 *In the previous theorem, if $k > m$ and no more than two edges are allowed between two vertices, then the set*

$$A = \{G_t | G_t = C_n\{(0,t)\} \cdot P_{m+1}\{(0, k-t)\}\}$$

is the optimal reliable graph, when $t = k - m$.

Proof: Using Theorem 4.44 we have

$$R(G_t) = p_1^k \cdot p^{n+m-k} + (n-t)(1-p)p^{n+m-k-1} \cdot p_1^k + t(1-p_1)p_1^{k-1} \cdot p^{m+n-k}.$$

$R(G_t)$ is a function of t. By finding the second derivative of $R(G_t)$, we have $R(G_t)'' = 0$ for all $t \in [k-m, n]$. Therefore the maximum of $R(G_t)$ in $[k-m, n]$ is at the end points. By comparing $R(G_n)$ and $R(G_{k-m})$, one can see that $R(G_{k-m})$ is the maximum value. Therefore, G_{k-m} is the optimally reliable graph in A. □

If a and b are two vertices connected by L edges, then the probability of having a and b connected will be denoted by p_{L-1}. Notice that $p_{L-1} = 1 - (1-p)^L$. By convention $p_0 = p$. Now, we study the case when the number of extra edges available to enhance the network is k where $n + m < k \leq 2n + 2m$.

If each edge in the graph G is enhanced by $k - 1$ edges then the new graph is denoted by G^k.

Theorem 4.45 *Let $k \leq n$ be the number of extra edges. The graph $C_n[t] \cdot P^2_{m+1}[k-t]$ is the optimally reliable graph when $t = k$.*

Proof: It is enough to find the optimally reliable graph for the following set of graphs

$$G = C_n\{(0,t)\} \cdot P^2_{m+1}\{(0, k-t)\}.$$

$$\begin{aligned} R(G) &= H(t) \\ &= p_2^{k-t} \cdot p_1^{m-k+2t-1} \cdot p^{n-t-1} \cdot [p_1 p + t(1-p)^2 \cdot p + (n-t)(1-p)p - 1]. \end{aligned}$$

In order to find the maximum value for $H(t)$, we will first find all critical values of $H(t)$. Let $H'(t)$ be the derivative of $H(t)$ then

$$\frac{H'(t)}{H(t)} = -D\log p^2 + 2D\log p_1 - D\log p + p(p-1)^3 = 0$$

where

$$D = p_1 p + t(1-p)^2 \cdot p + (n-t)(1-p)p_1$$

solving for t we have the following

$$t = n + n(1-p) + p(\frac{2-p}{1-p}) + \frac{(p-1)^2}{s} = t_0$$

where $s = \log p_2 + \log p_1 + \log p$. By using the second derivative test, one can show that $H''(t) < 0$, therefore, $H(t)$ has a maximum value at t_0. For $0 < p < 1$, it can be shown that $t_0 \geq n$, therefore, the function $H(t)$ is increasing in $[0, t_0]$, which implies that $H(t)$ has maximum value at $t = k$. □

4.4 Multi-Ring Graphs

Given a set of cycles $\{C_{t_1}, C_{t_2}, ..., C_{t_n}\} = A$, where t_i is the order of the cycle C_{t_i}, the graph G constructed from A by series connections (see Chapter I) is called a *multi-ring graph.* Given a set of cycles A, in order to identify the graphs G which result from series connection from A, we need to introduce some new notation. If G is a multi-ring graph of n cycles, we construct a labeled tree T of order n from G in the following way: each cycle C_i in G will be replaced by a vertex v_i and two vertices are adjacent in T if the corresponding circles are share a vertex. The order of T is equal to the number of cycles in A. The tree T uniquely describes G, and G will be denoted by $T(A)$, where A is the set of cycles and T is the labeled tree which describes the connection between the cycles in G.

Remark 14 *Let $A = \{C_{t_1}, C_{t_2}, ..., C_{t_n}\}$ and let T be any tree of order n, then*

$$R(T(A), q) = \prod_{C_i \in A} R(C_i, q).$$

Next we show how to best improve multi-ring networks. Let $T(A)[k]$ be the class of graphs obtained from $T(A)$ by adding k multiple edges.

Theorem 4.46 *The set* $B = \{G_t = C_n\{(0, m-t)\} \cdot C_{n+K}\{(0,t)\} \;\; t \in [0,m]\}$ *has an optimally reliable graph when*

$$t = \frac{m}{2} + \frac{K}{2}p$$

where m *is the number of the extra edges.*

Proof: Let G_t be the graph $T(A)$

$$R(G_t) = R(C_n\{(0, m-t)\}) \cdot R(C_{n+K}\{(0,t)\})$$

$$\begin{aligned}
&R(C_n\{(0, m-t)\}) \\
&= p_1^{m-t} \cdot p^{n-m+t} + (n-m+t)p_1^{m-t} \cdot p^{n-m+t-1} \cdot (1-p) \\
&+ (m-t)p_1^{m-t}(1-p)^2 \cdot p^{n-m+t} R(C_{n+K}\{(0,t)\}) \\
&= p_1^t \cdot p^{n+K-t} + (n+K-t) \cdot p_1^t \cdot p^{n+K-t-1} + tp_1^t(1-p)^2 \cdot p^{n+K-t}.
\end{aligned}$$

Let

$$H_1 = p_1 p + (n-m+t)(1-p)p_1 + (m-t)(1-p)^2 p$$

and

$$H_2 = p_1 p + (n+K-t)(1-p)p_1 + t(1-p)^2 p.$$

By Remark 14

$$R(G_t) = P_1^{m-2} \cdot P^{2n+K-m-2} \cdot [H_1(t)][H_2(t)].$$

Maximizing the function $R(G_t)$ is equivalent to maximizing $[H_1(t)][H_2(t)]$. Let $H(t) = [H_1(t)][H_2(t)]$, then

$$\begin{aligned}
H'(t) &= H_1(t)[(1-p)^2 p - p_1(1-p)] + H_2(t)[p_1(1-p) - p(1-p)] \\
&= [H_1(t) - H_2(t)][(1-p)(p(1-p) - p_1)] \\
&= [H_1(t) - H_2(t)](1-p)(-p).
\end{aligned}$$

Letting $H'(t) = 0$ and solve for t we have that $H_1(t) - H_2(t) = 0$ or

$$2t[p_1 - p(1-p)] = (m+K)p_1 - mp(1-p)$$

where $p_1 = 1 - (1-p)^2 = 2p - p^2$

$$\begin{aligned} t &= \frac{m + 2K - Kp}{2} \\ &= \frac{m}{2} + K(1 - \frac{p}{2}) \\ t &= \frac{m}{2} + \frac{K}{2} \cdot p. \end{aligned}$$

Calculating $H''(t)$, we find

$$H''(t) = -(1-p)p[p_1(1-p) - p(1-p) - (1-p)^2 p + P_1(1-p)] = -2p^2(1-p)^2 < 0$$

for all $0 < p < 1$. Therefore the function $H(t)$ has maximum value at

$$t = \frac{m}{2} + \frac{K}{2} \cdot p = t_0$$

□

In the above theorem, note that t_0 is a function of m, K and p. If p is small, then the effect of K becomes small and edges are evenly distributed.

If the two cycles have the same order, the function $H(t)$ has maximum value at $t = \frac{m}{2}$. We will generalize this for any set of cycles of the same order having a series connection.

Theorem 4.47 *Given a set of cycles* $A = \{C_{n_1}, C_{n_2}, ..., C_{n_k}\}$, *where* $n_i = n_j$ *for all* $1 \leq i,j \leq k$, *let* m *be the number of extra edges, where* $m \leq \sum_{i=1}^{k} n_i$. *Let* $\{m_1, m_2, \ldots, m_k\}$ *be a partition to* m *such that* $|m_i - m_j| \leq 1$ *for all* $1 \leq i$, $j \leq k$. *For any tree* T *of order* k, *the class* $T(A)[k]$ *has*

$$G_0 = C_{n_1}\{(0,m_1)\} \cdot C_{n_2}\{(0,m_2)\} \cdots C_{n_k}\{(0,m_k)\}$$

as an optimally reliable graph.

Proof: The proof is by induction on $k = |A|$. The result is true for $k = 1$. Assume the result is true for $|A| \leq k$. Let $A = \{C_{n_1}, C_{n_2}, ..., C_{n_{k+1}}\}$ Let $G \in T(A)[m]$. First we need to show that if G is optimal, then

$$G = T(A'), \quad where \quad A' = \{C_{n_1}\{(0, \frac{m}{k+1})\}, ..., C_{n_{k+1}}\{(0, \frac{m}{k+1})\}\}.$$

Consider the graph

$$G_t = C_{n_1}\{(0, t_1)\} \cdots C_{n_k}\{(0, t_k)\} \cdot C_{n_{k+1}}\{(0, t_{k+1})\}$$

where $\{t_1, t_2, \ldots, t_k, t_{k+1}\}$ is any partition to m.

$$R(G_t) = R(C_{n_{k+1}}\{(0, t_{k+1})\} \prod_{i=1}^{k} R(C_{n_i}\{(0, t_i)\}$$

The function $R(G_t)$ has maximum value when both $\prod_{i=1}^{k} R(C_{n_i}\{(0, t_i)\}$ and $R(C_{n_{k+1}}\{(0, t_{k+1})\}$ are maximum. Assume $t = \sum_{i=1}^{k} t_i$. By induction hypothesis, the t units should be distributed evenly among the k cycles. It is enough to consider the graph

$$G_{t_0} = C_{n_1}\{(0, \frac{t}{k})\} ... C_{n_k}\{(0, \frac{t}{k})\} \cdot C_{n_{k+1}}\{(0, m - t)\}.$$

$$\begin{aligned} R(G_{t_0}) &= [p_1^{\frac{t}{k}-1} \cdot p^{n-\frac{t}{k}-1}[p_1 p + (n - \frac{t}{k})p_1(1-p) + \frac{t}{k}(1-p)^2 p]^k] \cdot \\ &\quad [p_1^{m-t-1} \cdot p^{n-m+t-1}[p_1 p + (n - m + t)p_1(1-p) + (m-t)(1-p)^2 p]]. \end{aligned}$$

$$\begin{aligned} If \ H_1(t) &= p_1 p + n - \frac{t}{k} p_1(1-p) + \frac{t}{k}(1-p)^2 p. \\ and \ H_2(t) &= p_1 p + (n - m + t)(p_1(1-p) + (m-t)(1-p)^2 p, \end{aligned}$$

$$then \quad R(G_t) = p_1^{m-(k+1)} \cdot p^n (k+1) \cdot [H_1(t)]^k \cdot H_2(t).$$

The function $R(G_t)$ is maximum at t_0, if and only if the function

$$H(t) = [H_1(t)]^k \cdot H_2(t)$$

has a maximum at t_0. Differentiating $H(t)$ with respect to t, we have

$$H'(t) = k[H_1(t)]^{k-1} \cdot H_1'(t) \cdot H_2(t) + [H_1(t)]^k \cdot H_2'(t).$$

Observe that $H_1(t) \neq 0$ for $0 < p < 1$. Therefore $H'(t) = 0$ implies

$$kH_2(t)H_1'(t) + H_2'(t)H_1(t) = 0$$

or

$$H_2(t)[p(1-p) - p_1] - H_1(t)[p(1-p) - p_1 = 0.$$

The fact that $p(1-p) - p_1 \neq 0$ for $0 < p < 1$ implies $H_1(t) = H_2(t)$. Namely

$$(n - t/k)p_1(1-p) + t/k(1-p)^2 p =$$

$$(n - m + t)p_1(1-p) + (m-t)p(1-p)^2$$

or $\frac{t}{k}(-1-k) = -m$; solving for t, we get

$$t = \frac{km}{1+k} = \frac{k}{1+k}m$$

and

$$m - t = \frac{1}{1+k} \cdot m.$$

The second derivative test shows that at this point $H(t)$ has maximum value.

Since the number of edges takes on only integer value, it follows that the maximum value of $H(t)$ is $\lfloor \frac{m}{1+k} \rfloor$ or $\lceil \frac{m}{1+k} \rceil$ (i.e if $m = t(k+1) + r$ then r cycles will receive only $t+1$ edges, and $k+1-r$ cycles will receive only t edges). Therefore, if m_i represent the number of extra edges, the cycles C_{n_i} will receive, then for all $1 \leq i, j \leq k+1$, $|m_i - m_j|$ the graph $T(A')$ is the optimal reliable graph in $T(A)[m]$. □

Definition 11 *A graph G is said to be n-cyclic graph if G has only n cycles and the induced subgraph on those cycles is a multi-ring graph.*

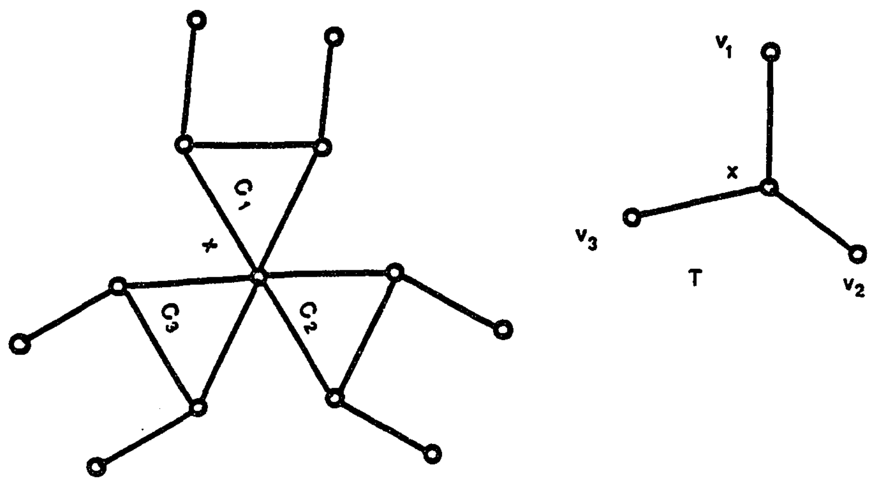

Figure 4.10

Example: The graph in Figure 4.10 shows 3-cycles graph of order 13. If G is an n-cyclic graph then G can be written in the form $T(A) \cdot F_m$, where T is the labeled tree which describes the connection between the cycles. The tree T is constructed as follows: If C_{n_1} and C_{n_2} are two cycles in G having the vertex x in common, then we will replace the cycle C_{n_1} by the vertex v_{n_1} and the cycle C_{n_2} by the vertex v_{n_2} and we join the vertex x to the vertices v_{n_1} and v_{n_2} by two edges. The tree corresponding to the graph mention above is shown in Figure 4.10.

Lemma 14 *If $T(A) \cdot F_m$ is an n-cyclic graph, then*

$$R(T(A) \cdot F_m, q) = p^m t \prod_{i=1}^{k} R(C_{n_i}, q)$$

where $A = \{C_{n_1}, C_{n_2}, \ldots, C_{n_k}\}$.

Remark 15 *If $T(A) \cdot F_m$ is an n-cyclic graph, then*

$$R(T(A) \cdot F_m, \Gamma) = R(T(A) \cdot P_m, \Gamma)$$

where P_m is the path of length m.

Theorem 4.48 *Let $G = T(A) \cdot F_m$ be an n-cyclic graph with $A = \{C_{n_1}, C_{n_2}, \ldots C_{n_k}\}$. Let M be the number of the extra edges with $M \leq m$. $T(A) \cdot F_m\{(0, M)\}$ is the optimally reliable graph in $G[m]$.*

Proof: We will use induction on k the number of the cycles in G. The result is true for $k = 1$ (see Theorem 4.44). Assume the result is true for a graph with a number of cycles less than or equal to k. Let $G = T(A) \cdot F_m$ be (k+1)-cyclic graph, where $A = \{C_{n_1}, C_{n_2}, \ldots, C_{n_{k+1}}\}$. Suppose $B = \{e_0, t_1, \ldots, t_k, t_{k+1}\}$ is any partition to M. By Remark 15, it is enough to consider the graph $G_0 = T(A) \cdot P_m$, where P_m is the path of length m. If $t = \sum_{i=1}^{k} t_i$, then $t_{k+1} = M - t$. We will consider the graph

$$\begin{aligned} G_A &= P_m\{(0,t_0)\} \cdot C_{n_1}\{(0,t_1)\} \cdot C_{n_2}\{(0,t_2)\} \cdots C_{n_k}\{(0,t_k)\} \cdot C_{n_{k+1}}\{(0,t_1)\}. \\ R(G_A) &= R(P_m\{(0,t_0)\} \cdot R(C_{n_{k+1}}\{(0,M-t)\}) \cdot \prod_{i=1}^{k} R(C_{n_i}\{(0,t_i)\}). \end{aligned}$$

By the induction hypothesis

$$R(P_m\{(0,t_0)\} \cdot \prod_{i=1}^{k} R(C_{n_i}\{(0,t_i)\})$$

has maximum value when $t_0 = t$. Observe that $C_{n_i}\{(0,0)\}$ is the cycle C_{n_i} without any change in the reliability of the edges. Thus, we consider the graph

$$\begin{aligned} G'_A &= P_m\{(0,t)\} \cdot C_{n_{k+1}}\{(0,M-t)\} \cdot C_{n_1} \cdot C_{n_2} \ldots C_{n_k} \\ R(G'_A) &= R(P_m\{(0,t)\} \cdot R(C_{n_{k+1}}\{(0,M-t)\}) \cdot \prod_{i=1}^{k} R(C_{n_i}). \end{aligned}$$

This is a function of t and the maximum of $R(G'_A)$ is independent of $\prod_{i=1}^{k} R(C_{n_i}\{(0,t_i)\})$. Therefore

$$H(t) = R(P_m\{(0,t)\} \cdot R(C_{n_{k+1}}\{(0,M-t)\}).$$

By an argument analogous to the one used in the proof of Theorem 4.44 we can show that $H(t)$ has maximum value at $t = M$. Therefore $T(A) \cdot F_m\{(0, M)\}$ is an optimally reliable graph in $T(A) \cdot F_m[M]$. □

CHAPTER V

IMPROVING K-TERMINAL RELIABILITY

5.1 Improving K-Terminal Reliability I

Recall from Chapter I that the K-terminal reliability of a graph G is the probability that the vertices in K are connected. We denote this by $R_K(G,q)$ where q is probability of edge failure. The function $R_K(G,q)$ can be written as :

$$R_K(G,q) = \sum_{S \in \Omega} f(S) \cdot R(S)$$

where $R(S)$ is the probability of having G in state S and

$$f(S) = \begin{cases} 1 & \text{if } < K > \text{ is connected in } < S >, \\ 0 & \text{otherwise} \end{cases}$$

In case of having only two vertices in K, we call such reliability, st-reliability and $R_K(G,q) = R_{s,t}(G,q)$. If $E(G)$ has Γ as its probability assignment, then the st-reliability will be denoted by $R_{s,t}(G,\Gamma)$. A graph G is called a multistage graph if $V(G)$ can be partitioned into $V_1, V_2, \ldots, V_k$, such that if $e = xy$ is an edge in $E(G)$ then x and y must be located in two consecutive sets, namely V_i and V_{i+1}, for some $i, 1 \leq i \leq k-1$. The sets $V_1, V_2, \ldots, V_k$ are called stages.

Example: The graph G in Figure 5.1 shows a multistage graph of order 6. The set $V(G)$ is partitioned into four different sets, V_1, V_2, V_3 and V_4. If $|V_1| = |V_{k+1}| = 1$, then such a graph is called an st-multistage graph, and it is denoted by $M_{s,t}(G)$.

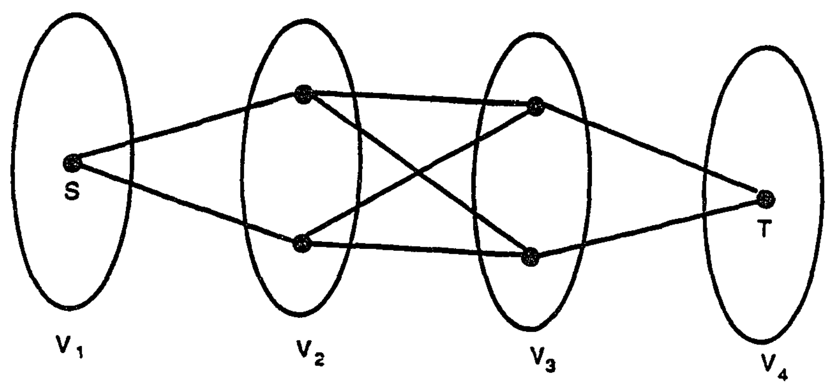

Figure 5.1

Since the st-reliability measures the probability that s is connected to t, it follows that $R_{s,t}(G,q)$ is the probability of having at least one st-path in G. If q (the probability of edge failure) is known, we drop q and simply write $R_{s,t}(G)$ instead of $R_{s,t}(G,q)$. In the st-multistage graph, if all st-paths are vertex disjoint, then G is denoted by $M_{s,t}(k,l)$, where $k = dist(s,t)$ and l is the number of vertex disjoint st-path in G.

Theorem 5.49 *If $G = M_{s,t}(k,l)$ is an st-multistage graph, then*

$$R_{s,t}(G,q) = 1 - (1 - p^{k-1})^l.$$

Proof: In the graph G, the two vertices s and t are connected if and only if there exists at least one st-path in G. The probability of having an st-path P in G is equal to p^{k-1}. Hence, the probability of not having an st-path P is $1 - p^{k-1}$. If $P_1, P_2, \ldots, P_l$ are the st-paths in G, then the probability of having no st-path is

$(1-p^{k-1})^l$. Therefore, the probability of having an st-path in G is

$$R_{s,t}(G,q) = 1 - (1-p^{k-1})^l.$$

□

As in Chapter IV, we improve the st-reliability of a graph G by either adding multiple edges or by replacing the edges in G by more reliable ones.

Remark 16 *If s and t are the two end vertices of the path P_{n+1}, then*

$$R_{s,t}(P_{n+1},q) = R(P_{n+1},q) = p^n,$$

where $R(P_{n+1},q)$ is the global connectivity of the path P_{n+1}.

Proof: If s and t are the two end vertices of the path P_{n+1}, then the probability of having an st-path is the same as the probability of P_{n+1} being connected. □

As in Chapter IV, $G[k]$ denotes the set of all graphs obtained from G by adding k new multiple edges. The graph $G_0 \in G[k]$ is called an optimal st-reliable graph if $R_{s,t}(G_0,q) \geq R_{s,t}(G',q)$ for all $G' \in G[k]$.

Remark 17 *If s and t are the end vertices of the path P_{n+1}, then the the optimal st-reliability graph is obtained by evenly distributing the k extra edges.*

Proof: This follows from the fact that $R_{s,t}(G,q) = R(G,q)$ together with the results in Chapter IV. □

Given a multistage graph $G = M_{s,t}(k,l)$, let $G[k]$ be all graphs obtained from G by adding k new multiple edges to G. Assume the st-paths of G are labeled $P_1, P_2, \ldots, P_l$. Let t_i be the number of extra edges added on the path P_i of G, $i = 1,2,...,l$; then the new graph is denoted by $G\{t_1,t_2,...,t_l\}$, where $\sum_{i=1}^{l} t_i = k$. We investigate the question of how to partition k into l numbers $t_1, t_2, ..., t_l$, so that the graph $G\{t_1,t_2,...,t_l\}$ is an optimal st-reliable graph in $G[k]$. Throughout our discussion we assume that the t_i edges assigned to the path P_i are distributed evenly on the edges of P_i.

If the variable in $R_{s,t}(G,\Gamma(t))$ is the number of extra edges, then $R_{s,t}(G,\Gamma(t))$ takes values only at $t = 0, 1, \ldots k$, where k is the number of the extra edges. As in Chapter IV we will allow the function $R_{s,t}(G,\Gamma(t))$ to take any values of t between 0 and k.

Theorem 5.50 *Among all multistage graphs $M_{s,t}(k,2)$ with m extra edges, where $m \leq k$, the graph $G\{m,0\}$ is the optimal st-reliable graph.*

Proof: We will consider the graph $G\{m-t,t\}$. The st-reliability $G\{m-t,t\}$ is

$$R_{s,t}(G\{m-t,t\}) = 1 - [1 - p_1^{m-t}p^{k-m+t}][1 - p_1^t p^{k-t}]$$

where $p_1 = 2p - p^2$ (the probability that two vertices are connected, when two edges between them are present). Let $R(G\{m-t,t\}) = H(t)$, and consider the function

$$H_0(t) = [1 - p_1^{m-t}p^{k-m+t}][1 - p_1^t p^{k-t}].$$

The function $H(t)$ has maximum value at t if and only if $H_0(t)$ has minimum value at t. If $H_0'(t)$ represents the first derivative of $H_0(t)$, then

$$H_0'(t) = p_1^{m-t}p^{k-m+t}lnp_1 - p_1^{m-t}p^{k-m+t}lnp - p_1^t p^{k-t}lnp_1 + p_1^t p^{k-t}lnp =$$

$$p_1^{m-t}p^{k-m+t}[lnp_1 - lnp] - p_1^t p^{k-t}[lnp_1 - lnp].$$

Since $lnp_1 - lnp \neq 0$, this implies $H_0'(t) = 0$ if and only if

$$p_1^{m-t}p^{k-m+t} = p_1^t p^{k-t}.$$

Now solving for t, we get

$$\begin{aligned}(k-m+t)lnp + (m-t)lnp &= tlnp_1 + (k-t)lnp \\ lnp(2t-m) &= lnp_1(2t-m) \\ (2t-m)(lnp_1 - lnp) &= 0.\end{aligned}$$

Since $lnp_1 - lnp \neq 0$ for $0 < p < 1$, implies $t = \frac{m}{2}$. By using the second derivative test, one can show that $H_0''(t) < 0$. Therefore, $H_0(\frac{m}{2})$ is maximum in $(0,m)$. Since

$H_0(t)$ has a maximum value in $(0, m)$, this implies that $H_0(t)$ has a minimum value at the end points of $[0, m]$. By comparing the two values

$$H_0(0) = [1 - p_1^m p^{k-m}][1 - p^k]$$

$$and \;\; H_0(m) = [1 - p^k][1 - p_1^m p^{k-m}]$$

one can see that the minimum of $H_0(t)$ occurs at the end points of $[0, m]$. Therefore, $H(t)$ has maximum value at $t = 0$, or $t = m$, namely the optimal reliable graph in $G\{m - t, t\}$ is $G\{0, m\}$ or $G\{m, 0\}$.□

In the process of improving $R_{s,t}[k, 2]$, the above result shows that it is always better to improve one path rather than two paths.

In a graph G, two vertices s and t are connected in parallel if all the st-paths in G are edge disjoint.

Example: Assume G is the circle $C_n, n \geq 3$, then any two vertices in G are connected in parallel.

The next results show how to improve st-reliability when the two vertices s and t are in parallel connection. If G is a graph consisting of two vertices s and t together with a set of st-paths $\{P_1, P_2, ..., P_l\}$, then G is denoted by $M_{s,t}[P_1, P_2, ..., P_l]$. Sometimes, we refer to such a graph as an st-parallel connection graph. If all paths in A have the same length k, then G is just the multistage graph $M_{s,t}(k, l)$.

As in the multistage graph, if G is the graph $M_{s,t}[P_1, P_2, ..., P_l]$, then $G\{t_1, t_2, ..., t_l\}$ denotes the graph obtained from $M_{s,t}[P_1, P_2, ..., P_l]$ by adding t_i new edges on the path P_i, $i = 1, 2, ..., l$.

The following summary should illustrate the meaning of the different notation used in this chapter.

(1)- $G[m]$: a graph with m multiple edges.

(2)- $M_{s,t}(k, l)$: an st-multistage graph with l disjoint st-paths, each of length k.

(3)- $M_{s,t}[P_1, P_2, \ldots, P_l]$: The st-multistage graph with l different st-paths, namely $P_1, P_2, \ldots, P_l$. Sometimes we refer to such a graph as an st-parallel connection graph.

(4)- $G\{t_1, t_2, \ldots, t_l\}$: The graph obtained from $M_{s,t}[P_1, P_2, \ldots, P_l]$, by adding t_i multiple edges on the path P_i, for all $i = 1, 2, \ldots, l$. Here we assume that the t_i multiple edges are distributed evenly on to the path P_i, for $i = 1, 2, \ldots, l$.

(5)- $R_{s,t}(G, q)$: The st-reliability of the graph G, when all the edges have the probability of failure equal to q.

(6)- $R(G, q)$: The global reliability of the graph G, when all the edges have the probability of failure equal to q.

Theorem 5.51 *Let $G = M_{s,t}[P_{l+1}, P_{k+1}]$ and $l \leq k$. For $m \leq l$, the graph $G(m, 0)$ is an an optimally reliable graph in $G[m]$.*

Proof: Since G contains only two st-paths P_{l+1}, P_{k+1}; we will consider the graph $G\{t, m-t\}$ where $t \in [0, m]$. The st-reliability of $G\{t, m-t\}$ is

$$\begin{aligned} H(t) = G\{t, m-t\} &= 1 - [1 - p_1^{m-t} p^{k-m+t}][1 - p_1^t p^{l-t}] \\ &= p_1^{m-t} p^{k-m+t} + p_1^t p^{l-t} - p_1^m p^{l+k-m} \end{aligned}$$

where $p_1 = 2p - p^2$. If $H'(t)$ denotes the derivative of $H(t)$, then

$$H'(t) = \ln p_1 [p_1^t p^{l-t} - p_1^{m-t} p^{k-m+t}] - \ln p [p_1^t p^{l-t} - p_1^{m-t} p^{k-m+t}].$$

If the function $H(t)$ has a critical point at t then $H'(t) = 0$. This implies that

$$[p^t p_1^{l-t} - p_1^{m-t} p^{k-m+t}][\ln p_1 - \ln p] = 0.$$

Since $\ln p_1 - \ln p \neq 0$ for all $0 < p < 1$ this implies that

$$p_1^t p^{l-t} = p_1^{m-t} p^{k-m+t}.$$

Solving for t, we have the following:

$$t = t_0 = \frac{ln p_1[m] - \ln p[m + l - k]}{2[\ln p_1 - \ln p]}$$

where t_0 is the critical point for $H(t)$. By using the second derivative test we have

$$H''(t) = (p_1^t p^{l-t}) p_1^{m-t} p^{k-m+t} (lnp_1 - lnp)^2.$$

Since all factors are positive in $H''(t)$ for $t \in [0, m]$, then $H''(t) > 0$. The function $H(t)$ has minimum value at t_0. Thus, $H(t)$ has maximum value at the end points of $[0, m]$. By inspection:

$$\begin{aligned} H(0) &= p_1^m p^{k-m} + p^l - p_1^m p^{l+k-m} \\ H(m) &= p_1^m p^{l-m} + p^k - p_1^m p^{l+k-m} \\ H(m) - H(0) &= p^m [p^l - p^k][p_1^m - p^m]. \end{aligned}$$

For $k \geq l$, the difference $H(m) - H(0) \geq 0$, which implies $H(m)$ has a maximum value for $k > l$; namely it is more efficient to use all m edges to improve the shortest path P_{l+1}.□

The next result shows that it is always best to improve the shortest st-path for the st-multistage graph $M_{s,t}[P_{l_1+1}, P_{l_2+1}, ..., P_{l_n+1}]$.

Theorem 5.52 *Given $G = M_{s,t}[P_{l_1+1}, P_{l_2+1}, ..., P_{l_N+1}]$ with $l_1 \leq l_2 \leq ... \leq l_N$, where $m \leq l_1$, the graph $G\{m, 0, 0, ..., 0\}$ is the optimal st-reliable graph in $G[m]$.*

Proof: We use induction on N (the number of st-paths). The result is true for $N = 1$. Assume the result is true for all graphs with the number of st-paths less than or equal to N.

Let G be the graph

$$M_{s,t}[P_{l_1+1}, P_{l_2+1}, ..., P_{l_{N+1}+1}]$$

(see Figure 5.2). The graph G can be decomposed into two subgraphs, G_1 and G_2. G_1 is the subgraph induced by the first N paths and G_2 is the graph induced by the path $P_{l_{N+1}+1}$. Let $\{t_1, t_2, \cdots, t_N, t_{N+1}\}$ be a partition to m and let $t = \sum_{i=1}^N t_i$. Consider the graph $G'_t = G\{t_1, t_2, \ldots, t_{N+1}\}$. To maximize the st-reliability of

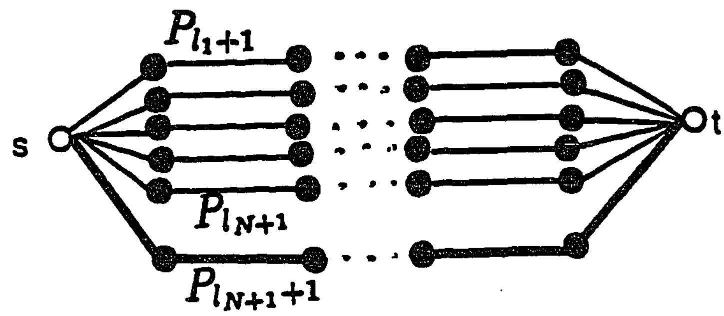

Figure 5.2

is distributed evenly among the edges of $P_{l_{N+1}+1}$. Therefore, we will consider the graph $G_t = G\{t, 0, 0, \ldots, m-t\}$.

$$R_{s,t}(G_t) = 1 - (1 - p_1^t p^{l_1 - t})(1 - p_1^{m-t} p^{l_{N+1}-m+t}) \prod_{i=2}^{N} (1 - p^{l_i}).$$

Let

$$H_0(t) = (1 - p_1^t p^{l_1 - t})(1 - p_1^{m-t} p^{l_{N+1}-m+t}) \cdot \prod_{i=2}^{N} (1 - p^{l_i}).$$

The function $R_{s,t}(G_t)$ has a maximum value at t if and only if the function $H_0(t)$ has a minimum value at t. Since $\prod_{i=2}^{N}(1 - p^{l_i})$ is constant, one can drop it from $R_{s,t}(G_t)$, for the calculation of the minimum value of $H_0(t)$. Therefore we only consider the function

$$H(t) = 1 - p_1^t p^{l_1 - t} - p_1^{m-t} p^{l_{N+1}-m+t} + p_1^m p^{l_1 + l_{N+1}}.$$

Let $H'(t)$ be the derivative of $H(t)$, then

$$H'(t) = -[p_1^t p^{l_1 - t} + p_1^{m-t} p^{l_{N+1}-m+t}]'.$$

As in the proof of the previous theorem, $H(t)$ has a critical point at

$$t = t_0 = \frac{\ln p_1[m] - \ln p[m + l_1 - l_{N+1}]}{2[\ln p_1 - \ln p]}.$$

As in the proof of the previous theorem, $H(t)$ has a critical point at

$$t = t_0 = \frac{\ln p_1[m] - \ln p[m + l_1 - l_{N+1}]}{2[\ln p_1 - \ln p]}.$$

By testing the second derivative $H''(t)$, one can see that $H''(t) < 0$ for all $t \in (0, m)$. Therefore, $H(t_0)$ is a minimum value, which implies that $R_{s,t}(G_t)$ has maximum value at the end points of $[0, m]$. By inspection, we have $H(m) < H(0)$. Therefore, the m edges should be used to improve p_{l_1}, the shortest st-path.□

Next we generalize the above result to cover the case when only the probability of the paths $P_1, P_2, ..., P_l$ are given. Consider the graph $G = M_{s,t}[P_1, P_2, ..., P_l]$. Assume that edges in each path of G has the same reliability, but edges in different paths may has different reliability. Let p_i denote the probability that the path P_i to be connected (up state). Suppose Δp represents the total increase allowed to be used in improving the st-reliability of the paths. The next result shows how to distribute Δp among the paths of the graph $G = M_{s,t}[P_1, ..., P_l]$, where $\Delta p < 1 - p_0$, and

$$p_0 = max\{p(P_i) \mid i = 1, 2, ..., l\}.$$

Let $G[\Delta p]$ represent all graphs obtained from G by increasing the reliability of the paths in G by a total amount of Δp.

Theorem 5.53 *Given a graph $G = M_{s,t}[P_1, P_2, ..., P_l]$ with*

$$p(P_1) \leq p(P_2) \leq ... \leq p(P_l)$$

and $\Delta p < 1 - p(P_l)$, the graph $G\{0, 0, .., \Delta p\}$ is an optimal st-reliable graph in $G[\Delta p]$.

Proof: We use induction on l (the number of st-paths in G). The result is true for $l = 1$. Assume it is true for l, where $l \leq N$. Let $G = M_{s,t}[P_1, P_2, .., P_{N+1}]$ with

$$\begin{aligned} p(P_1) &\leq p(P_2) \leq ... \leq p(P_{N+1}) \\ and\ \Delta p &\leq 1 - p(P_{N+1}). \end{aligned}$$

Let t be the total increase in the reliability of the paths $P_1, P_2, ..., P_N$, and $\Delta p - t$ be the total increase in the reliability of P_{N+1}, where $0 \leq t \leq \Delta p$. By the induction hypothesis t must be used to increase the reliability of P_N. Thus, the reliability of G after increasing $p(P_N)$ by t and $p(P_{N+1})$ by $\Delta p - t$ is

$$H(t) = R_{s,t}(G\{0, 0, ..., t, \Delta p - t\}) =$$

$$1 - [1 - (p(P_N) + t)][1 - (p(P_{N+1}) + \Delta p - t)] \prod_{i=1}^{N-1} (1 - p(P_i)).$$

The function $H(t)$ has maximum value at t_0 if and only if

$$H_0(t) = [1 - (p(P_N) + t)][1 - (p(P_{N+1}) + \Delta p - t)]$$

has minimum value at $t = t_0$. Differentiating $H_0(t)$ with respect to t, we get

$$H_0'(t) = [1 - (p(P_N) + t)] - [1 - ((p(P_{N+1}) - \Delta p - t))].$$

For $H_0'(t) = 0$,

$$t = \frac{\Delta p}{2} + \frac{p(P_{N+1}) - p(P_N)}{2} = t_0.$$

Since $H_0''(t) = -2 < 0$, it implies that $H_0(t_0)$ is maximum in $[0, \Delta p]$. Therefore, $H_0(t_0)$ has a minimum value at the end points in $[0, \Delta p]$. By inspection, $H_0(0) < H_0(\Delta p)$. Therefore, the function $H(t)$ has a maximum value when $t = 0$. Hence Δp should be used to improve the most reliable path, namely P_{N+1}.□

Theorem 5.54 *Let p_1 and p_2 denote the reliability of the edges in the two st-paths, say p_1 and p_2 of the graph $G = M_{s,t}[P_1, P_2]$ respectively. If $p_1 \geq p_2$, then the graph $G\{m, 0\}$ is an optimal reliable graph in $G[m]$, $0 \leq m \leq k$.*

Proof: Consider the graph $G\{t, m - t\}$, where $0 \leq t \leq m$, If an edge $e = uv$ in $E(P_1)$ receives an extra edge from the enhanced set, then the probability of having u and v connected is $1 - (1 - p_1)(1 - p) = p_{01}$. Similarly, if an edge $e = wz$ receives an extra edge, the new reliability is $1 - (1 - p_2)(1 - p) = p_{02}$. Observe that $p_{01} > p_{02}$, for $p_1 > p_2$. The st-reliability of G after the above assignment is

$$\begin{aligned} R_{s,t}(G\{t, m-t\}) &= \\ H(t) &= 1 - [1 - p_{01}^t p_1^{k-t}][1 - p_{02}^{m-t} p_2^{k+t-m}]. \end{aligned}$$

Suppose

$$\begin{aligned} H_0(t) &= [1 - p_{01}^t p_1^{k-t}][1 - p_{02}^{m-t} p_2^{k+t-m}] \\ &= p_{01}^t p_1^{k-t} p_{02}^{m-t} p_2^{k+t-m} - p_{01}^t p_1^{k-t} p_{02}^{m-t} p_2^{k+t-m}. \end{aligned}$$

Then the function $H(t)$ has a maximum value at t_0 if and only if $H_0(t)$ has a maximum or a minimum value at t_0. Let $H_0'(t)$ be the derivative of $H_0(t)$; then

$$\begin{aligned} H_0'(t) &= p_{01}^t p_{02}^{m-t} p_1^{k-t} p_2^{k+t-m}[lnp_{01} + \ln p_2 - \ln p_1 - \ln p_{02}] \\ &- p_{01}^t p_1^{k-t}[\ln p_{01} - \ln p_1] - p_{02}^{m-t} p_2^{k+t-m}[\ln p_2 - \ln p_{02}]. \\ H_0''(t) &= p_{01}^t p_{02}^{m-t} p_1^{k-t} p_2^{k+t-m}[\ln p_{01} + \ln p_2 - \ln p_1 - \ln p_{02}]^2 \\ &- p_{01}^t p_1^{k-t}[\ln p_{01} - \ln p_1]^2 - p_{02}^{m-t} p_2^{k+t-m}[\ln p_2 - \ln p_{02}]^2. \end{aligned}$$

Observe that

$$\begin{aligned} &[\ln p_{01} + \ln p_2 - \ln p_1 - \ln p_{02}]^2 \\ &\qquad = [(\ln p_{01} - \ln p_1) - (\ln p_{02} - \ln p_2)]^2 \\ &\qquad < [\ln p_{01} - \ln p_1]^2 + [\ln p_{02} - \ln p_2]^2. \end{aligned}$$

Moreover the following is true:

$$p_{01}^t p_{02}^{m-t} p_1^{k-t} p_2^{k+t-m}[(lnp_{01} - lnp_1) - (lnp_{02} - lnp_2)]^2 <$$

$$p_{01}^t p_{02}^{m-t} p_1^{k-t} p_2^{k+t-m}[lnp_{01} - lnp_1]^2 + p_{01}^t p_{02}^{m-t} p_1^{k-t} p_2^{k+t-m}[lnp_{02} - lnp_2]^2 <$$

$$p_{01}^t p_1^{k-t}[lnp_{01} - lnp_1]^2 + p_{02}^{m-t} p_2^{k+t-m}[lnp_2 - lnp_{02}]^2.$$

This shows that $H_0''(t) < 0$, for all $t \in [0, m]$. Hence, the function $H_0(t)$ has a minimum value at the end points of the interval $[0, m]$. By inspection, it can be shown that $H_0(m) < H_0(0)$. Therefore the function $H(t)$ has a maximum value at $t = m$, namely the graph $G\{(m, 0)\}$ is the optimally reliable graph in $G[m]$.□

Corollary 17 *Given* $G = M_{s,t}[P_{l_1}, P_{l_2}, ..., P_{l_N}]$, *with* $l_1 \leq l_2 \leq ... \leq l_N$, *if* $\Delta p \leq 1 - p^{l_1}$, *then the graph*

$$G\{\Delta p, 0, 0, ..., 0\}$$

is the optimal st-reliable graph in $G[\Delta p]$.

Proof: For each st-path P_{i+1} in G the probability of having P_{i+1} connected is $p(P_i) = p^i$. Observe that if $l_1 \leq l_2$, then $p(P_{l_1}) \geq p(P_{l_2})$. The result follows immediately by using this observation together with Theorem 5.53 □

Given a set of graphs $A = \{G_1, G_2, ..., G_n\}$ where G_i is the st-parallel connection graph

$$M_{s_i,t_i}[P_{i1}, P_{i2}, ..., P_{ir_i}]$$

$i = 1, 2, \ldots, n$, we construct a graph G by identifying the vertex t_i in G_i with s_{i+1} in G_{i+1} for all $i = 1, 2, \ldots, n-1$. The resulting graph G is in series connection from A, and is denoted by $G_1 \cdot G_2 \cdots G_n$. The st-reliability of G is defined to be the probability of having an $s_1 t_n$-path in G.

The following result will be used in the analysis of improving st-reliability for the series connection of the graphs in A. In the next lemma, we assume that Δp is used only to improve one edge in the given graph. Given a path P_{n+1}, let $E(P_{n+1}) = (e_1, e_2, \ldots, e_n)$ be the labeling of the edges in P_{n+1}. Recall from Chapter IV that $P_{n+1}(\Delta p_1, \Delta p_2, \ldots, \Delta p_n)$ denotes the path obtained from P_{n+1} by improving the reliability of the edge e_i by Δp_i, for all $i = 1, 2, \ldots, n$.

Lemma 15 *Let* P_{n+1} *be an st-path and* $E(P_{n+1}) = (e_1, e_2, ..., e_n)$ *be the labeling of the edges of* P_{n+1}. *Let* $p(e_i)$ *denote the reliability of the edge* e_i *and suppose*

$$p(e_1) \leq p(e_2) \leq ... \leq p(e_n).$$

If only one edge is to be improved and $\Delta p \leq 1 - p(e_n)$, *then* $P_{n+1}(\Delta p, 0, ..., 0)$ *is the optimal st-reliable graph in* $P_{n+1}[\Delta p]$.

Proof: Let e_i be the edge in $E(P_{n+1})$ which receives an increase of Δp in its reliability. Since $R_{s,t}(P_{n+1}, \Gamma) = R(P_{n+1}, \Gamma)$, where Γ is the probability assignment of $E(P_{n+1})$, it follows that :

$$\begin{aligned} R(P_{n+1}(0,0,...,\Delta p,0,...,0)) &= \prod_{j\neq i} p(e_j)][p(e_i)+\Delta p] \\ &= \prod_{j=i}^{n} p(e_j) + \Delta p \cdot \prod_{j\neq i} p(e_j) \\ &= H(p(e_i)). \end{aligned}$$

This function of $p(e_i)$ which has a maximum when $\prod_{j\neq i} p(e_i)$ is maximum. The fact that $\prod_{j\neq i} p(e_i)$ is maximum when $p(e_i)$ is minimum implies that $H(p(e_1))$ is the maximum value. Therefore Δp should be used to improve the least reliable edge in $E(P_{n+1})$.□

If Δp is used to improve more than one edge of $E(P_{n+1})$, then the previous result is not true, as can be seen in the following example:

Consider the path P_3 with $E(P_3) = \{e_1, e_2\}$. Let $p(e_1) = p_1$, and $p(e_2) = p_2$. Suppose edges e_1 and e_2 are improved by $\Delta p - t$ respectively, where $\Delta p < 2 - p(e_1) - p(e_2)$. Let s and t be the end vertices in P_3 and let $p(s \sim t)$ denote the probability of having an st-path. Then

$$\begin{aligned} p(s\sim t) &= (P_1+t)(P_2+\Delta p - t) \\ &= p_1p_2 + p_1\Delta p + tp_2 + t\Delta p - p_1 t - t^2 \\ &= H(t). \end{aligned}$$

Let $H'(t)$ be the derivative of $H(t)$, then

$$H'(t) = p_2 + \Delta p - p_1 - 2t = 0.$$

Solving for t gives us

$$t = \frac{p_2 - p_1}{2} + \frac{\Delta p}{2} = t_0.$$

By testing $H''(t)$, one can show that $H(t_0)$ is maximum. Therefore, it is better to improve both edges $\{e_1, e_2\}$, rather than just one. The best distribution of Δp

among all the edges on the path P_{n+1} becomes tedious when n is large and the reliability is not the same for all edges.

Given st-multistage graph $G = M_{s,t}[P_1, P_2, \ldots, P_n]$, let $G[[\Delta p]]$ denote the set of all graphs obtained from G by increasing the st-reliability of the st-paths in G by a total of Δp. If the path P_i receives an increase Δp_i in its st-reliability, then $\sum_{i=1}^{n} \Delta p_i = \Delta p$. The resulting st-reliability depends on the values of Δp_i's. Observe that a graph G_0 in $G[[\Delta p]]$ is called an optimal st-reliable graph if $R_{s,t}(G_0, \Gamma_0) \geq R_{s,t}(G', \Gamma')$ for all $G' \in G[[\Delta p]]$, where Γ_1 and Γ_2 are the probability assignments of G_0 and G', respectively.

Assume $A = \{G_1, G_2, ..., G_n\}$, where each G_i is the s_it_i-parallel connection graph $M_{s_i,t_i}(P_{i1}, ..., P_{ir_i})$. Let $G = G_1 \cdot G_2 \ldots \cdot G_n$ be a graph obtained from A by series connection. We will study the optimally reliable graph in $G[[\Delta p]]$.

We will assume that, if Δp_i is the portion of Δp to be used on the graph G_i, then Δp_i should be used in an efficient way to improve the s_it_i-reliability of G_i. Namely, we will consider the optimal s_it_i-reliable graph in $G_i[\Delta p_i]$.

Theorem 5.55 *Let G be the graph defined above and let $R_{s_i,t_i}(G_i) \leq R_{s_i,t_i}(G_{i+1})$ for all $i = 1, 2, ..., n-1$. Assume $\Delta p < R_{s_i,t_i}(G_n)$, if Δp is allowed to be used only on the edges of one subgraph G_i, $i = 1, 2, ..., n$ assume then the optimal graph in $G[[\Delta p]]$ is the one in which G_1 has been optimized.*

Proof: The graph G can be converted to a path P_{n+1} in the following way. Each graph G_i in G is replaced by an edge $e_i = s_it_i$ with $p(e_i) = R_{s_i,t_i}(G_i)$. The new s_1tn-path has $p(e_1) \leq p(e_2) \leq ... \leq p(e_n)$, and the s_1t_n-reliability is the same as the s_1t_n-reliability of G. By using Lemma 15, we should use Δp to improve $p(e_1)$. Therefore, the best subgraph in G to improve is G_1 in G.□

5.2 Improving K-terminal Reliability II

In section 5.1, the study of improving st-reliability was restricted to the case of having at most double edges between the adjacent vertices. In this section we allow any number of multiple edges.

Given a multistage graph $G = M_{s,t}(k,l)$, with m multiple edges, where $m \geq k$, one can ask: What is the optimal st-reliable graph in $G[m]$?. The following example illustrates how the distribution of the m edges changes the value of the st-reliability of the graph G.

Example: Consider the graph $G = M_{s,t}(4,2)$. Let $k = 6$ be the number of extra edges. Define the following: $G_1 = G(4,2)$, $G_2 = G(5,1)$ and $G_3 = G(6,0)$. By previous result the edges in each path must be distributed evenly. Consider the following functions:

$$\begin{aligned}
R_1(p) &= R_{s,t}(G_1) = 1-[1-[1-(1-p)^2]^4][1-[1-(1-p)^2]^2.p^3] \\
R_2(p) &= R_{s,t}(G_2) = 1-[1-[1-(1-p)^3]][1-(1-p)^2]^3][1-(1-p)^2]p^3] \\
R_3(p) &= R_{s,t}(G_3) = 1-[1-[1-(1-p)^3]^2][1-[1-(1-p)^2]^2][1-p^4].
\end{aligned}$$

The graph of the three functions $R_1(p), R_2(p)$ and $R_3(p)$ are shown in Figure 5.3. For $0 \leq p \leq 1$, the graph shows that $R_3(p)$ are always greater than R_1 and R_2. This implies that the best distribution of the 6 extra edges occurs when we use them to improve one path. In fact this result turns out to be true in general.

Let $P_{k+1}(l)$ denote the path of length k, with l multiple edges between any two adjacent vertices.

Example: The graph $G = P_3(2)$ is shown in Figure 5.4.

Theorem 5.56 *Let $x + y = l$ be a positive integer. The graphs in the set $A = \{G_x | G_x = M_{s,t}[P_{k+1}(x), P_{k+1}(l-x)]$ with $x + y = l\}$ have a st-optimal reliability graph when $x = l$.*

Proof: The proof is by contradiction. Suppose $x \neq y$ and assume $x \geq y \geq 2$; we will show that the graph $G = M_{s,t}[P_{k+1}(x), P_{k+1}(y)]$ is not an st-optimal reliable

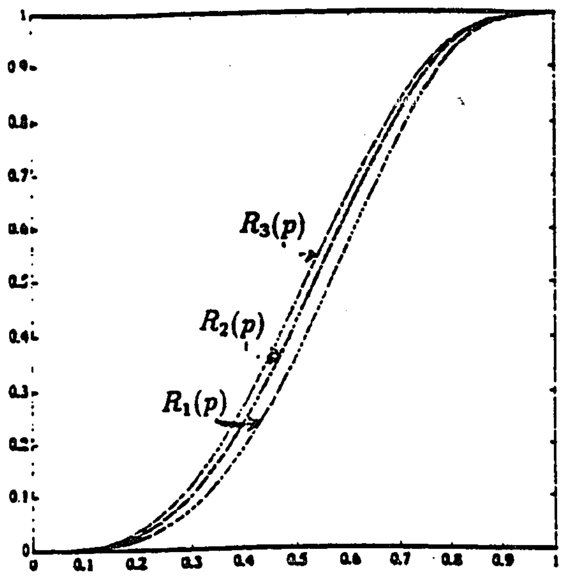

Figure 5.3

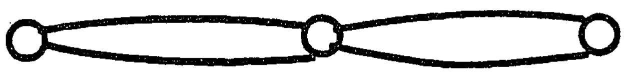

Figure 5.4

graph in A. For the path $P_{k+1}(x)$, the reliability between any two adjacent vertices is $p_{x-1} = 1-(1-p)^x$ and for path $P_{k+1}(y)$, the reliability between any two adjacent vertices is $p_{y-1} = 1-(1-p)^y$. For $x \geq y$ we have $p_{x-1} \geq p_{y-1}$. Now construct the graph $G' = M_{s,t}[P_{k+1}(x+1), P_{k+1}(y+1)]$ by removing k edges from $P_{k+1}(y)$ to enhance the path $P_{k+1}(x)$. Theorem 5.54 implies that the graph G' is more reliable with respect to the st-reliability than G, which is the required contradiction.□

Theorem 5.57 *If $G = M_{s,t}[k,2]$, then for any positive integer m, $G(0,m)$ is an st-optimal reliable graph in $G[m]$.*

Proof: Let $A = \{G_i | G_i = G(i, m-i)\}$, we show that G_0 is an st-optimally graph in A. To the contrary, assume there exists a positive integer t such that G_t is an st-optimal reliable graph in A. Let $t = kl_1 + x$ and $m - t = kl_2 + y$; we proceed by case analysis.

Case 1: $l_1 = l_2$.

Subcase 1: $x = y = 0$. In this case $G_t = M_{s,t}[P_{k+1}(l_1+1), P_{k+1}(l_1+1)]$. Since m is positive integer, it follows that $l_1 \geq 1$. By using a similar arguments to those in Theorem 5.56, we can show that the graph $G_0 = M_{s,t}[P_{k+1}(l_1+2), P_{k+1}(l_1)]$ is more reliable than G_t with respect to st-reliability. Therefore, G_t is not an st-optimal reliable graph, which is a contradiction.

Subcase 2: $x + y \neq 0$. We consider two possibilities:

(1) If $x \neq 0$, then by Theorem 5.54, the graph $G_0 = G(t-x, m-t+x)$ is more reliable with respect to the st-reliability than G_t, which again is a contradiction.

(2) $x \neq 0$. Construct the graph G' by taking k edges from the path $P_{k+1}\{(l,x)\}$ and use them to improve the path $P_{k+1}\{(l,y)\}$. The fact that the probability of having two adjacent vertices connected in the path $P_{k+1}\{(l,y)\}$ is equal to the probability of having adjacent vertices connected in the path $P_{k+1}\{(l,x)\}$, together with Theorem 5.54, implies that G' is more reliable than G_t, which is a contradiction.

Case 2: $l_1 \neq l_2$; Without loss of generality, let $l_1 \neq l_2$. For this case the probability that two adjacent vertices are connected in the path $P_{k+1}\{(l_2,x)\}$ is less than the probability that two adjacent vertices are connected in $P_{k+1}\{(l_2,y)\}$. Construct the graph $G' = M_{s,t}[P_{k+1}\{(l_1+1,x)\}, P_{k+1}\{(l_2-1,y)\}$ by taking k edges from $P_{k+1}\{(l_2,y)\}$ and use them to construct $P_{k+1}\{(l_1+1,x)\}$. By using Theorem 5.54, it can be shown that G' is more reliable than G_t, which is again a contradiction.□

Lemma 16 *The graph $P_{k+1}(2)$ is more reliable than $M_{s,t}[k,2]$.*

Proof: Consider the graph $P_{k+1}(2)$ with

$$R(P_{k+1}(2)) = (2p-p^2)^k = p^k(2-p)^k$$

and the graph $M_{s,t}[k,2]$ with

$$R(M_{s,t}[k,2]) = 1-[1-p^k]^2 = p^k[2-p^k].$$

Observe that

$$R(P_{k+1}(2)) - R(M_{s,t}[k,2]) = p^k[(2-p)^k - (2-p^k)].$$

The function $H(x) = (2-p)^x - (2-p)^x$ is a positive and increasing function, for all $0 < p < 1$. Therefore $R(P_{k+1}(2)) - R(M_{s,t}[k,2]) > 0$ for all p and for all k. Hence $P_{k+1}(2)$ is more reliable than $M_{s,t}[k,2]$. □

Corollary 18 *Given a graph $G = M_{s,t}[P_1,P_2]$, let p_1 and p_2 be the probability of the edges on the paths P_1 and P_2, respectively. Let P be the path whose vertices are connected by two multiple edges, one with probability p_1 and the other with p_2, then $R(P) > R(G)$ (see Figure 5.5).*

Theorem 5.58 *The graph $P_{k+1}(l)$ is more reliable than $M_{s,t}[k,l]$.*

Proof: The proof is by induction on l. The result is true for $l=2$. Assume the result is true up to N. Suppose $G = M_{s,t}[k,N+1]$ and let $P_1, P_2, \ldots, P_N, P_{N+1}$ be the set of st-paths of G. Let G_1 be the induced subgraph by $P_1, P_2, \ldots, P_N$.

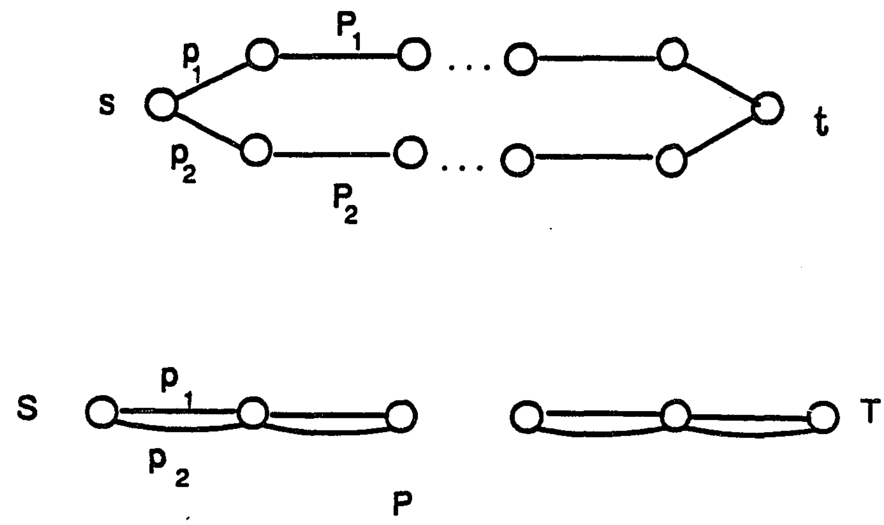

Figure 5.5

Construct a graph G' from G by replacing G_1 by the paths $P_1, P_2, \ldots, P_N$. By the induction hypothesis, the st-reliability of the graph G' is more reliable than G_1. Therefore the graph G' is more reliable than G, with respect of the st-reliability. Since the graph G' consists of two paths, one has edge reliability equal to p_{N-1} and the other has edge reliability p. By using Corallary 18 the graph $P_{k+1}(N+1)$ is more reliable than G'. Therefore, $P_{k+1}(N+1)$ is more reliable than G.

We next study improvement the multistage graphs $M_{s,t}[k,l]$ when m (the number of extra edges) is more than k.

Theorem 5.59 *Let $G = M_{s,t}[k,l]$; the graph G_1 in $G[m]$ is an optimal st-reliable graph, when the m edges are used to improve only one path.*

Proof: The result is true for $k = 1$ or $l = 1$. That can be seen by observing that the graph $M_{s,t}[1,l]$ has just l multiple edge between s and t, and the graph $M_{s,t}[k,1]$ is just the path P_{k+1}. Assume $k \geq 2$ and $l \geq 2$. We use induction on l (the number of the disjoint st-path). Assume the result is true for all $l \leq$

N. Consider the graph $G = M_{s,t}[k, N+1]$. Let $m = rk + d$; we show that the graph $G[P_1\{(r,d)\}, P_2, \ldots, P_{N+1}]$ is an optimally reliable graph in $G[m]$. Let $\{t_1, t_2, \ldots, t_N, m-t\}$ be a partition of m which produces the st-optimally reliable graph, where $t = \sum_{i=1}^{N} t_i$. Let G' be the st-optimally reliable graph obtained from G by adding t_i edges to the path P_i, for all $i = 1, 2, \ldots, N$ and $m - t$ edges to the path P_{N+1}. We show that G' is not an optimally reliable graph. Consider the graph G'_1, constructed by adding t extra edges to the path P_1 and $m - t$ extra edges to the path P_{N+1}. By the induction hypothesis, the graph G'_1 is more reliable than G', which is a contradiction. □

Let ΔR_m denote the increase in st-reliability after the improvement process of the graph; where m is the number of extra edges. The following result measures ΔR_m, when m is given.

Remark 18 *Given a path P_{n+1}, let m be the number of extra edges with s and t the two end vertices of P_{n+1}. If $m < n$, then the increase in the st-reliability of P_{n+1} after the enhancement is*

$$\Delta R_m(P_{n+1}) = p^n[(2-p)^m - 1].$$

Proof: The result follows from the fact that the

$$R_{s,t}(P_{n+1}(0,m)) = p_1^m p^{n-m}$$

and $R_{s,t}(P_{n+1}) = p^n$, where $p_1 = 2p - p^2$. □

Given a graph $G_l = P_{n+1}(l)$, we study the increase in ΔR_m on G_l for different values of l.

Remark 19 *For $m < n$, $\Delta R_m(G_l) = [1-(1-p)^l]^n([\frac{1-(1-p)^{l+1}}{1-(1-p)^l}]^m - 1)$.*

Proof: Let $G_l = P_{n+1}(l)$, then

$$\begin{aligned} R_{s,t}(G_l) &= [1-(1-p)^l]^n \\ R_{s,t}(G_l(m)) &= [1-(1-p)^{l+1}]^m.[1-(1-p)^l]^{n-m}. \end{aligned}$$

By taking the difference of $R_{s,t}(G_l)$ and $R_{s,t}(G_l(m))$ we get the result.□

Given a graph $G = M_{s,t}[P_{l_1}, P_{l_2}]$, let $R_{s,t}(P_{l_1}) = p_1$ and $R_{s,t}(P_{l_2}) = p_2$ be the probability that the two end vertices of P_{l_1} and P_{l_2} are connected. Let Δ_1 be the increase in $R_{s,t}(P_{l_1})$ when we improve P_{l_1} and Δ_2 be the increase in $R_{s,t}(P_{l_2})$ when we improve P_{l_2}. Suppose $p_1 > p_2$ and $\Delta_1 > \Delta_2$. If we have only the choice to improve P_{l_1} or P_{l_2} but not both, we can ask the question: What is the best choice so as to increase the st-reliability of G the most ? For the st-reliability, if we choose P_{l_1}, we have

$$R_{s,t}(P'_{l_1}) = p_1 + \Delta_1 + p_2 - p_2(p_1 + \Delta_1)$$

where P'_{l-1} is the path obtained from P_{l_1} by increasing its reliability by Δ_1. Similarly,

$$R_{s,t}(P'_{l_2}) = p_2 + \Delta_2 + p_1 - p_1(p_2 + \Delta_2)$$

where P'_{l_2} is the path obtained fromP_{l_2} by increasing its reliability by Δ_2. Observe that $R_{s,t}(P'_{l_1}) - R_{s,t}(P'_{l_2}) = \Delta_1(1 - p_2) - \Delta_2(1 - p_1)$. Since $\Delta_1 > \Delta_2$ and $1 - p_2 > 1 - p_1$, it follows that the choice of improving P_{l_2} is better to improve $R_{s,t}(G)$. In improving the multistage graph $G = M_{s,t}[k, 2]$, let G contain the two paths $P_{k+1}(l), P_{k+1}(1)$. The results in this section show that $\Delta R_m(G)$ is maximum when all the m edges are used to improve the path $P_{k+1}(l)$ but not $P_{k+1}(1)$ for $l \geq 1$.

CHAPTER VI

CONCLUSION AND FUTURE DIRECTIONS

6.1 Traffic vectors

For simplicity we restrict our discussion to a graph G, or digraph D which is free of loops and parallel edges.

Let G be an (n,m)-graph. Recall from Chapter II that the traffic vector of a set S of edges in $E(G)$ is a sequence which describes the number of paths of length i which contain S, for all $i = 1, 2, \ldots, n-1$. We extend the definition to a case of digraph D. A digraph D is called (n,m)-digraph, if the order of D is n and the size is m. Given an (n,m)-digraph D and $e \in E(D)$, the traffic vector of e is $TV(e) = (\pi_1(e), \pi_2(e), \ldots, \pi_{n-1}(e))$, where $\pi_i(e)$ is the number of the directed paths which contain e. Note that the direction of the paths is determined by the direction of the edge e.

Example:

Consider the digraph in Figure 6.1. The traffic vector of the edge e is $TV(e) = (1, 3, 1)$.

As in the definition of the dominant edge for the graph, the edge $e_0 \in E(D)$ is a dominant edge, if for a different edges $e \in E(D)$ and for all $j = 1, 2, \ldots, n-1$ it implies that

$$\sum_{i=1}^{j} \pi_i(e_0) \geq \sum_{i=1}^{j} \pi_i(e).$$

The study of characterizing the set of the dominant edges in a graph or a digraph in general can be interesting problem.

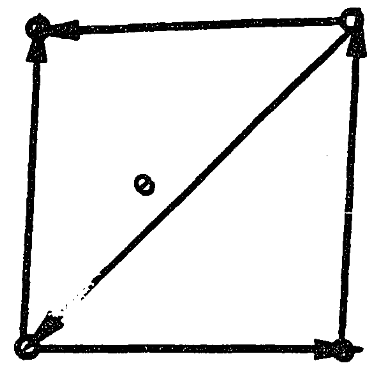

Figure 6.1

For the case of directed trees, several results have been obtained but are not included in this dissertation.

A digraph D is called a probabilistic digraph if the elementing in $V(D)$ and $E(D)$ are assigned positive real numbers which represent the probability that a given element exists in the set $V(D)$ or $E(D)$. If v_i is assigned the number $p(v_i)$, then the probability of having v_i in $V(D)$ is $p(v_i)$. Observe that $0 \leq p(v_i) \leq 1$. Similarly, $p(e_i)$ denotes the probability of having e_i in $E(D)$. For the following discussion, we assume that the probability assigned to the vertices in D is always 1 (the vertices of D are absolutely reliable). Two vertices in $V(D)$ are connected if there exists a directed path from u to v. Define the pair-connected reliability of the digraph D to be the expected number of connected vertices in D. The problem of finding the pair-connected reliability of D in general is an open problem. Moreover the study of improving the pair-connected reliability for a general digraph by the methods outlined in this dissertation can be cited as a good research problem. For the case of a directed tree $D(T)$, several results are obtained by using the directed traffic vectors.

6.2 K-Terminal Reliability

The study in this dissertation is done only for special classes of graphs, which are commonly used in computer networks and other applications. One may extend the study to other types of networks. For complete graphs and bipartite graphs the study of improving different types of reliability measures is still needed.

Consider for example a graph G of order n, and positive integer $k \leq n$. To find the subset $S \in E(G)$ with $|S| = k$ to be improved in G so the increase in the K-terminal reliability is maximal has not been analyzed as the complexity goes.

There are many types of reliability measure for networks that can be improved by the two methods mentioned on this dissertation.

To improve network reliability for a probabilistic graph $G = (V, E)$ we assumed that vertices were fail-safe, but each edge $e \in E(G)$ is down (that is, in failed state) independently with probability q, $0 < q < 1$. Moreover, assume that the node failures are equal and independent. A natural question is ask: How does one improve the reliability measures associated with G by using the two methods used in this dissertation, where both vertices and edges are subject to failure.

6.3 General Reliability in Probabilistic Graphs

A method of studying reliability in general for G can be extended to find the probability or the expected value of having any property in G. Namely, G contains a complete subgraph of order k, where $k \leq |V|$; or G has a set of independent number of edges. To improve any of the network reliability measures mentioned above using the methods outlined in this dissertation can be interesting to study.

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