The Annals of Statistics
1983, Vol. 11, No. 1, 121–128

IMPROVING SOME MULTIPLE COMPARISON PROCEDURES

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Genizi and Hochberg (1978) recommended using Contrast Set Preserving (CSP) procedures in the class of $T(Q)$ procedures for multiple comparisons in general unbalanced designs based on partial results. They did not, however, propose a general method for selecting a specific CSP procedure, or for replacing a given non-CSP procedure with a better CSP one. In this work we identify a certain orthogonal transformation of non-CSP procedures into CSP ones and give a sufficient condition for the uniform dominance (shorter confidence intervals for all contrasts) of the latter over the former. Two important implications of the given condition are: (i) Applying the given transformation to Spjøtvoll and Stoline's (1973) T'-procedure in any unbalanced ANOVA gives a uniformly improved procedure. (ii) In any arbitrary design, our transformation gives uniform improvement if the original procedure is "nearly CSP."

1. Introduction. Consider an arbitrary experimental design that produced a vector of estimates $\hat{\boldsymbol{\theta}}$ for the vector of means $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_k)'$, and an estimator s^2 of the experimental variance σ^2. We assume that $\hat{\boldsymbol{\theta}}$ has a multivariate normal distribution with mean vector $\boldsymbol{\theta}$ and variance matrix $\sigma^2\mathbf{B}$ and that $\nu s^2/\sigma^2$ is independently distributed as a Chi squared with ν d.f.

If the variances of the pairwise comparisons $\hat{\theta}_i - \hat{\theta}_j$ are not all equal, then we call the design—and the matrix **B**—*unbalanced.* Tukey's T-method of multiple comparisons can be used only in balanced designs, c.f. Genizi and Hochberg (1978).

For the unbalanced one-way layout, Spjøtvoll and Stoline (1973) proposed their T'-method. The T'-procedure was extended to any design by Hochberg (1975). From that extension one actually obtains an infinite family of Generalized T (GT) procedures for any given design. Thus, corresponding to any matrix $\mathbf{Q}$ satisfying $\mathbf{QQ}' = \mathbf{B}$ we have a GT procedure given by the simultaneous probability statement:

$$\Pr\{|\ell'(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})| \le s\tilde{q}^{(\alpha)}_{k,\nu} M(\mathbf{Q}'\ell), \forall \ell \in E^k\} = 1 - \alpha, \tag{1.1}$$

where $\tilde{q}^{(\alpha)}_{k,\nu}$ is the upper αth quantile of the Studentized Augmented Range distribution (recently tabulated by Stoline, 1978) and $M(\mathbf{x})$ is the larger of the sum of positive elements and the sum of absolute values of the negative elements in the vector $\mathbf{x}$; that is

$$M(\mathbf{x}) \equiv M(x_1, \cdots, x_k) = \max(\textstyle\sum_{i=1}^k x_i^+, \sum_{i=1}^k x_i^-),$$

where for a scalar ξ we define $\xi^+ = \max(\xi, 0)$ and $\xi^- = \max(-\xi, 0)$.

We denote a specific GT procedure as in (1.1) by $T(\mathbf{Q})$. Note that Spjøtvoll and Stoline's T'-procedure is a $T(D)$ procedure, where $\mathbf{D}$ is the diagonal matrix whose (i, i)th element is the positive square root of the reciprocal of the ith treatment's sample size in a one-way layout.

Since the GT procedures in (1.1) are not invariant under $\mathbf{Q}$'s, the problem of choosing an optimal $\mathbf{Q}$ arises. Genizi and Hochberg (1978) recommend the use of Contrast Set Preserving (CSP) $T(\mathbf{Q})$ procedures, i.e. $\mathbf{c}'\mathbf{Q1} = 0$ for any contrast vector $\mathbf{c}$. This recommendation was supported by the following:

Received July 1980; revised August 1982.

AMS 1980 *subject classifications.* Primary 62F99,62F25.

Key words and phrases. Generalized t-method, unbalanced designs, contrasts, orthogonal transformations.

(i) The optimal $T(Q)$ procedure obtained by Genizi and Hochberg (1978) for the one-way layout with only two different sample sizes is a CSP procedure.
(ii) Uniform dominance of any CSP procedure over the T' method was proved by Genizi and Hochberg for $k = 3$ and a sufficiently high degree of imbalance.
(iii) If a given GT procedure is CSP and the contrasts are the only linear combinations (of the means) which are of interest, then $\tilde{q}_{k,\nu}^{(\alpha)}$ in (1.1) can be replaced by the lower value of $q_{k,\nu}^{(\alpha)}$ the upper αth quantile of the Studentized Range Distribution.

In this paper we examine a certain orthogonal transformation of given non-CSP $T(\mathbf{Q})$ procedures into other procedures which are CSP—say, $T(\mathbf{QR})$. In Section 2 we describe the transformation and give a simple condition which guarantees its dominance over the original non-CSP procedure uniformly for all contrasts. In the one-way ANOVA, this condition is easily seen to hold for the Spjøtvoll and Stoline T'-procedure and thus, a uniform improvement of T' exists. In other cases a uniform improvement is also achieved when the original $T(\mathbf{Q})$ procedure is "nearly CSP."

In Section 3 we give the details for constructing the transformed $T(\mathbf{QR})$ procedure and exemplify it with the one-way ANOVA design and the T'-procedure. Appendix I includes an example which shows that there are cases where our transformation does not give a uniform improvement. In Appendix II we show that for any k and any design the transformation always improves on an original non-CSP procedure for a certain one dimensional linear subspace of contrasts.

2. The T(QR) procedure and a sufficient condition for its uniform dominance over T(Q). Consider a given $T(\mathbf{Q})$ procedure in an arbitrary design, $\mathbf{QQ}' = \mathbf{B}$. The $k - 1$ dimensional contrast subspace C has the vector $\mathbf{e} = (1/\sqrt{k})\mathbf{1}$ as its normal and the $k - 1$ dimensional subspace $\mathbf{Q}(C) \equiv \{\mathbf{Q}'\mathbf{c} : \mathbf{c} \in C\}$ has the vector $\mathbf{a} = \mathbf{Q}^{-1}\mathbf{1}/\|\mathbf{Q}^{-1}\mathbf{1}\|$ as its normal. Consider the set of vectors $Y = C \cap \mathbf{Q}(C)$.

PROPOSITION 1. *If* $\mathbf{a} \neq \mathbf{e}$ *then* Y *is a linear subspace of dimension* $k - 2$.

PROOF. Y is clearly a linear subspace. If $\mathbf{a} \neq \mathbf{e}$ then any vector $\mathbf{x}$ in the two-dimensional subspace spanned by $\mathbf{a}$ and $\mathbf{e}$ is orthogonal to all $\mathbf{y} \in Y$. Hence Y is *at most* of dimension $k - 2$ (since its orthogonal complement is at least of dimension 2). Also, any $\mathbf{y}$ which is orthogonal to both $\mathbf{a}$ and $\mathbf{e}$ is in Y. Since the space of all such vectors is of dimension $k - 2$ it follows that Y is of dimension *at least* $k - 2$.

We now consider a rotation of E^k that takes $\mathbf{Q}(C)$ into C pivoting around Y as axis. This rotation is defined by a linear transformation $\mathbf{R}'\mathbf{z}$ (where $\mathbf{z} \in E^k$ and $\mathbf{R}'$ is a square matrix of real elements). We want this transformation to satisfy:

$$\mathbf{R}'\mathbf{y} = \mathbf{y} \quad \text{for all} \quad \mathbf{y} \in Y, \quad \text{and} \quad \mathbf{R}'\mathbf{a} = \mathbf{e} \tag{2.1}$$

PROPOSITION 2. *The transformation* (2.1) *can always be obtained by an orthogonal matrix* $\mathbf{R}$.

PROOF. Let $\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_{k-2}$ be an orthogonal basis of Y. Let $\mathbf{d}$ be a vector of unit length proportional to the projection of $\mathbf{a}$ into C and let $\mathbf{u}$ be the unit vector in $\mathbf{Q}(C)$ orthogonal to $\mathbf{a}$ and to $\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_{k-2}$. The matrix $\mathbf{A} = [\mathbf{y}_1, \cdots, \mathbf{y}_{k-2}, \mathbf{a}, \mathbf{u}]$ is one orthogonal basis of E^k and the matrix $\mathbf{B} = [\mathbf{y}_1, \cdots, \mathbf{y}_{k-2}, \mathbf{e}, \mathbf{d}]$ is another orthogonal basis of E^k. Thus, the matrix $\mathbf{R}'$ which transforms $\mathbf{A}$ to $\mathbf{B}$ ($\mathbf{R}'\mathbf{A} = \mathbf{B}$) is orthogonal and will satisfy (2.1). If $\mathbf{R}'$ is orthogonal so is R.

The vector $\mathbf{a}$ can be expressed as

$$\mathbf{a} = \lambda\mathbf{e} - \nu\mathbf{d}, \tag{2.2}$$

where $\lambda = \mathbf{a}'\mathbf{e}$, $-\nu = \mathbf{a}'\mathbf{d}$ and $\lambda^2 + \nu^2 = 1$. Consider any $\mathbf{x} \in \mathbf{Q}(C)$. Projecting $\mathbf{x}$ on $\mathbf{e}$, $\mathbf{d}$ and Y gives

$$\mathbf{x} = (\mathbf{x}'\mathbf{e})\mathbf{e} + (\mathbf{x}'\mathbf{d})\mathbf{d} + \mathbf{y}, \tag{2.3}$$

where $\mathbf{y}$ is the projection of $\mathbf{x}$ on Y. Since $\mathbf{a}'\mathbf{x} = 0$ it follows from (2.2) and (2.3) that $\lambda(\mathbf{x}'\mathbf{e}) = \nu(\mathbf{x}'\mathbf{d})$ and hence (2.3) can be written as

$$\mathbf{x} = \gamma(\nu\mathbf{e} + \lambda\mathbf{d}) + \mathbf{y}, \tag{2.4}$$

where $\gamma = (\mathbf{x}'\mathbf{e})/\nu$.

PROPOSITION 3. *The transformation described by* (2.1) *will take any* $\mathbf{x}$ *as in* (2.4) *into*

$$\mathbf{R}'\mathbf{x} = \gamma\mathbf{d} + \mathbf{y}. \tag{2.5}$$

PROOF. The vector $\nu\mathbf{e} + \lambda\mathbf{d}$ (of unit length) is obviously the vector $\mathbf{u}$ (introduced in the proof of Proposition 2). This so because $\mathbf{y}$ is the projection of $\mathbf{x}$ on Y and $\mathbf{x} \in \mathbf{Q}(C)$. We saw that under $\mathbf{R}'$ the vector $\mathbf{u}$ is transformed to $\mathbf{d}$ (and any $\mathbf{y} \in Y$ is unchanged).

In the following we will show that under a certain condition (Theorem 1) and if $\nu > 0$ then

$$M(\mathbf{R}'\mathbf{x}) \leq M(\mathbf{x}), \quad \text{for all} \quad \mathbf{x} \in \mathbf{Q}(C) \quad \text{s.t.} \quad \mathbf{x}'\mathbf{e} > 0. \tag{2.6}$$

Choosing $\nu > 0$ is no loss in generality since we can always define (2.2) in terms of $\mathbf{d}$ or $-\mathbf{d}$. If (2.6) holds then since $M(\mathbf{z}) = M(-\mathbf{z})$ for any $\mathbf{z}$ it will imply $M(\mathbf{R}'\mathbf{x}) \leq M(\mathbf{x})$ for *any* $\mathbf{x} \in \mathbf{Q}(C)$. Thus, without loss in generality we now take $\nu > 0$ and $\gamma > 0$.

The condition of uniform improvement in $T(\mathbf{QR})$ over $T(\mathbf{Q})$ is equivalent to $M(\mathbf{y} + \gamma(\nu\mathbf{e} + \lambda\mathbf{d})) \geq M(\mathbf{y} + \gamma\mathbf{d})$, and after dividing by γ and denoting $\mathbf{y}/\gamma$ also by $\mathbf{y}$ it becomes

$$M(\mathbf{y} + \nu\mathbf{e} + \lambda\mathbf{d}) \geq M(\mathbf{y} + \mathbf{d}), \qquad \forall\, \mathbf{y} \in Y. \tag{2.7}$$

THEOREM 1. *A sufficient condition for* (2.7) *is*

$$\lambda \geq \frac{kd_{\max}^2 - 1}{kd_{\max}^2 + 1}, \qquad d_{\max} = \max_{1\leq i\leq k}\{d_i\}. \tag{2.8}$$

PROOF. Given $\mathbf{y} \in Y$, let p be the number of positive coordinates of the contrast vector $\mathbf{y} + \mathbf{d}$, and let $P = \{i \mid 1 \leq i \leq k, y_i + d_i > 0\}$. Thus, $|P| = p$ and $1 \leq p \leq k - 1$. Since the M-norm of a vector always majorizes any partial sum of its coordinates, we have

$$M(\mathbf{y} + \nu\mathbf{e} + \lambda\mathbf{d}) \geq \textstyle\sum_{i\in P} (y_i + \nu e_i + \lambda d_i) = \sum_{i\in P}(y_i + d_i) + \left\{\dfrac{\nu p}{\sqrt{k}} - (1 - \lambda) \sum_{i\in P} d_i\right\}$$

$$= M(\mathbf{y} + \mathbf{d}) + \Delta(P),$$

where $\Delta(P)$ is the expression in the square brackets. Now, (2.7) will be satisfied whenever $\Delta(P) \geq 0$ for all P as above; since

$$\frac{\Delta(P)}{p} \geq \frac{\nu}{\sqrt{k}} - (1 - \lambda)\, d_{\max} = \frac{\sqrt{1 - \lambda^2}}{\sqrt{k}} - (1 - \lambda)\, d_{\max}$$

(recall that $\nu \geq 0$). From this we get the sufficient condition (2.8).

COROLLARY 1. *If* $\mathbf{a} \geq \mathbf{0}$ (e.g., *as in the* T'*-procedure*) *then* (2.8) *always holds.*

PROOF. Assume that $a_i = \lambda/\sqrt{k} - \nu d_i \geq 0$ for $i = 1, \cdots, k$. Then $d_{\max} \leq \lambda/(\nu\sqrt{k})$, hence, since $\lambda^2 + \nu^2 = 1$, $\lambda^2 \geq kd_{\max}^2/(kd_{\max}^2 + 1)$, which implies (2.8) (note that $\lambda \geq 0$, since $\nu \geq 0$ and $\mathbf{d}$, being a contrast has at least one non-positive coordinate).

COROLLARY 2. *If* $\lambda \geq (k - 2)/k$ (i.e., *the original* $T(\mathbf{Q})$ *procedure is "near* CSP") *then the transformed procedure gives uniform improvement.*

PROOF. For any normalized contrast $\mathbf{d}$, $d^2_{\max} \leq (k-1)/k$. Substitution in (2.8) proves this corollary.

We note that the inequality used in proving Theorem 1 ($M(\mathbf{x}) \geq \sum_{i \in P} x_i$) is a very crude one and, moreover, the fact that $\mathbf{y}$ is constrained to a certain $k-2$ dimensional subspace, did not play any role at all. Nevertheless, even with this crude lower bound we see that a uniform improvement is often obtained. However, this is not always the case as suggested by a referee who provided the counterexample in Appendix I.

REMARK. Another way of viewing $\mathbf{R}$ is as follows. For a vector $\mathbf{z}$ not proportional to $\mathbf{e}$, define

$$\mathbf{z}^* = \frac{\| \mathbf{z} \|}{\| \mathbf{z} - \bar{z}\mathbf{1} \|} (\mathbf{z} - \bar{z}\mathbf{1}),$$

where $\bar{z} = \sum_i^k z_i/k$. The vector $\mathbf{z} - \bar{z}\mathbf{1}$ is the projection of $\mathbf{z}$ on C and since $\| \mathbf{z} \| = \| \mathbf{z}^* \|$ we may regard $\mathbf{z}^*$ as the "orthogonal rotation of $\mathbf{z}$ into C." We can now describe the rotation $\mathbf{R}'\mathbf{x}$ as the direct sum of the identity on Y and the $*$-transformation on $Y^{\perp}$.

3. Details for using the transformed procedure and the uniform improvement of the T'-method. By (2.5) $\mathbf{R}'\mathbf{Q}'\mathbf{c} = \mathbf{y} + \gamma\mathbf{d}$ and by (2.4) $\mathbf{Q}'\mathbf{c} = \mathbf{y} + \gamma(\nu\mathbf{e} + \lambda\mathbf{d})$. We get

$$\mathbf{R}'\mathbf{Q}'\mathbf{c} = \mathbf{Q}'\mathbf{c} - \gamma(\nu\mathbf{e} + \lambda\mathbf{d}) + \gamma\mathbf{d} = \mathbf{Q}'\mathbf{c} + \beta(\mathbf{e} + \mathbf{a}), \tag{3.1}$$

with $\beta = -\gamma(1-\lambda)/\nu$. Using the fact that (3.1) is a contrast, we find

$$\beta = \frac{-\mathbf{c}'\mathbf{Q}\mathbf{1}}{(k^{1/2} + \sum a_i)}, \tag{3.2}$$

where $\mathbf{a} = (a_1, \cdots, a_k)'$.

On letting $\delta = 1/(k^{1/2} + \sum a_i)$ we see that (3.1) can be expressed by the linear transformation

$$\mathbf{R}'\mathbf{Q}'\mathbf{c} = \mathbf{L}\mathbf{Q}'\mathbf{c}, \qquad \mathbf{c} \in C, \tag{3.3}$$

where

$$\mathbf{L} = \mathbf{I} - \delta(\mathbf{e} + \mathbf{a})\mathbf{1}'. \tag{3.4}$$

To exemplify the use of the new method and the level of improvement it achieves, we now consider its application to the T'-method. Let n_i be the sample size for treatment i in a one way layout, $i = 1, \cdots, k$. The T'-method is based on $Q = \text{diag}(n_1^{-1/2}, \cdots, n_k^{-1/2})$. Corollary 1 implies that T' can be uniformly improved.

The normal $\mathbf{a}$ here is

$$\mathbf{a} = \frac{1}{(\sum n_i)^{1/2}} (n_1^{1/2}, \cdots, n_k^{1/2})', \quad \text{and} \quad \delta = \frac{(\sum n_i)^{1/2}}{(k \sum n_i)^{1/2} + \sum n_i^{1/2}},$$

and $\mathbf{R}'\mathbf{Q}'\mathbf{c}$ can easily be computed from (3.1).

EXAMPLE. Consider $k = 3$ and $(n_1, n_2, n_3) = (1, 4, 9)$ then

$$\mathbf{a} = \frac{1}{\sqrt{14}} (1, 2, 3)' \quad \text{and} \quad \beta = -\frac{\sqrt{14}}{\sqrt{42} + 6} \mathbf{c}'\mathbf{Q}\mathbf{1}.$$

For $\mathbf{c} = (1, -1, 0)'$ we have $\mathbf{c}'\mathbf{Q} = (1, -\tfrac{1}{2}, 0)$ and the transformed vector $\mathbf{c}'\mathbf{Q}\mathbf{R} \cong (0.8734, -0.6667, -0.2067)$. Thus, $M(\mathbf{Q}'\mathbf{c}) = 1 > M(\mathbf{R}'\mathbf{Q}'\mathbf{c}) \cong 0.8734$.

In Table 1 we give the percentage of $M(\mathbf{R}'\mathbf{Q}'\mathbf{c}_{ij})$ versus that of the T'-method for all pairwise comparisons in some unbalanced designs. Table 1 shows that the improvement

TABLE 1
Relative length of confidence intervals in using the new method

k	q_1	q_2	q_3	q_4	q_5	Types of pairwise comparisons (i, j)	$100\frac{M(\mathbf{R}'\mathbf{Q}'\mathbf{c}_{ij})}{M(\mathbf{Q}'\mathbf{c}_{ij})}$
3	1	1	.4			(1, 3)	83.2
	1	1	.6			(1, 3)	87.9
	1	1	.8			(1, 3)	93.6
	1	.4	.4			(1, 2)	84.9
	1	.6	.4			(1, 2) (1, 3) (2, 3)	89.4 84.1 89.1
	1	.6	.6			(1, 2)	88.7
	1	.8	.4			(1, 2) (1, 3) (2, 3)	94.5 83.5 85.0
	1	.8	.6			(1, 2) (1, 3) (2, 3)	94.1 88.2 91.8
	1	.8	.8			(1, 2)	93.8
4	1	1	1	.4		(1, 4)	86.9
	1	1	1	.7		(1, 4)	92.9
	1	1	.4	.4		(1, 3)	88.1
	1	1	.7	.4		(1, 3) (1, 4) (3, 4)	93.7 87.3 89.5
	1	1	.7	.7		(1, 3)	93.2
	1	.4	.4	.4		(1, 2)	88.9
	1	.7	.4	.4		(1, 2) (1, 3) (2, 3)	94.2 88.4 90.5
	1	.7	.7	.4		(1, 2) (1, 4) (2, 4)	93.9 87.7 89.8
	1	.7	.7	.7		(1, 2)	93.4
5	1	1	1	1	.4	(1, 5)	89.3
	1	1	1	1	.7	(1, 5)	94.2
	1	1	1	.4	.4	(1, 4)	90.1
	1	1	1	.7	.4	(1, 4) (1, 5) (4, 5)	94.8 89.6 91.3
	1	1	1	.7	.7	(1, 4)	94.4
	1	1	.4	.4	.4	(1, 3)	90.7
	1	1	.7	.4	.4	(1, 3) (1, 4) (3, 4)	95.2 90.3 92.1
	1	1	.7	.7	.4	(1, 3) (1, 5) (3, 5)	94.9 89.9 91.5
	1	1	.7	.7	.7	(1, 3)	94.6
	1	.4	.4	.4	.4	(1, 2)	91.2
	1	.7	.4	.4	.4	(1, 2) (1, 3) (2, 3)	95.5 90.9 92.6
	1	.7	.7	.4	.4	(1, 2) (1, 4) (2, 4)	95.3 90.5 92.2
	1	.7	.7	.7	.4	(1, 2) (1, 5) (3, 5)	95.0 90.1 91.8
	1	.7	.7	.7	.7	(1, 2)	94.8

increases with the degree of imbalance. If imbalance would be measured by average q_i/q_j rather than by $\max(q_i)/\min(q_i)$, then the improvement would be practically uninfluenced by k when controlling for degree of imbalance. The percent improvement would be near zero (for nearly balanced designs) and can go substantially higher than the numbers in Table 1 for extremely unbalanced designs. For example, let $(q_1, q_2, q_3) = (1, 1, \varepsilon)$; then $\lim_{\varepsilon\to 0}\{M(\mathbf{R}'\mathbf{Q}'\mathbf{c}_{23})/M(\mathbf{Q}'\mathbf{c}_{23})\} \cong 0.7288$.

4. Discussion. The T'-method was first discussed by Tukey (1953, Chapter 29). He called it the "transformation method" and compared it with another method which he called the "approximation method." The approximation method was published later by Kramer (1956, 1957). Dunnett (1980) and Stoline (1981) refer to it as the Tukey-Kramer (TK) method. Tukey recommended the approximation method for the one way layout. For this design he and his student Kurtz (1956) found out that for $k = 3$ (and for other cases with ratios of sample sizes tending to ∞ or 0) the approximation method is on the conservative side. His basic argument against the transformation method was that it does not transform a simple comparison into a contrast. Regarding other designs (with possibly correlated estimates) Tukey (1953, Chapter 29) did not recommend the approximation method (nor did he recommend the transformation method). He writes, "... we can copy the approximation solution ... for ... general case. The properties ... here are ... less clear."

Since then the following results were established.

I. There exist transformations which take contrasts into contrasts and indeed some of these CSP transformation procedures will do uniformly better than non-CSP ones.

II. Genizi and Hochberg (1978) proved that the minimum confidence interval length achieved by *any* transformation method for a *given* pairwise comparison is that obtained by the TK method.

III. For the one-way layout, simulation work by Dunnett (1980) and some analytic results by Brown (1979) further supported the conservative nature of the TK procedure.

Note that the TK method is giving simultaneous confidence intervals for pairwise comparisons only. The method can be extended to give confidence intervals for *all contrasts* (cf. the extension of GT2 to all contrasts in Hochberg, 1974). If the TK procedure is conservative for a certain design then, for some arbitrary contrasts, the exact $1 - \alpha$ transformation procedures will give shorter confidence intervals.

Based on the above we can summarize as follows:

1. If (i) interest is confined to *pairwise* comparisons only and (ii) the design is a one-way layout and (iii) k is small or extreme imbalances prevail, then the TK method should be preferred over any transformation method. However, it is not clear then that the TK should be the recommended procedure. For example Spurrier (1981) gave a better procedure for $k = 3$ and large imbalanced designs.
2. In arbitrary designs, if the researcher has potential interest in contrasts other than pairwise, then the procedures discussed here as well as Schaffe's S-method can be used. Both these procedures are exact $1 - \alpha$ procedures. The transformation procedure will generally give shorter confidence intervals for pairwise (and hence longer intervals for some arbitrary contrasts).

Based on the apparent conservativeness of the TK in *some* one-way layouts, one might be willing to use it and extend it to all contrasts. As noted above, if it is conservative, then for some arbitrary contrasts it will give longer confidence intervals than the transformation method.

Finally, when considering arbitrary designs, the TK cannot be safely used. Except for the case $k = 3$ where the situation is equivalent to a one-way layout (Brown 1982) the properties of the TK method for arbitrary correlated estimators are not clear yet. Thus, the pursuit of optimal transformation procedures is still justified. We feel that the different opinion expressed by Stoline (1981) is based on the assumptions that the TK procedure is *always* conservative and that interest is confined to pairwise comparisons only. The first assumption might be correct but has not been proved yet. The second assumption is often contradicted by researchers who "snoop" at their data and dig out contrasts other than pairwise.

APPENDIX I

An example where the rotation $\boldsymbol{R}$ *does not reduce the M-norm.* This example was suggested by an anonymous referee. Let $k = 18$, and let

$$\mathbf{d} = \Big(\underbrace{-\frac{1}{6}, \cdots, -\frac{1}{6}}_{12}, \underbrace{\frac{1}{3}, \cdots, \frac{1}{3}}_{6}\Big)', \qquad \mathbf{y} = (11, \underbrace{-1, \cdots, -1}_{11}, -5, \underbrace{1, \cdots, 1}_{5})/\sqrt{18},$$

then $\| \mathbf{d} \| = 1$, $\mathbf{e}'\mathbf{d} = \mathbf{e}'\mathbf{y} = \mathbf{d}'\mathbf{y} = 0$. We have

$$M(\mathbf{y} + \mathbf{d}) = \left(\frac{11}{\sqrt{18}} - \frac{1}{6}\right) + 5\left(\frac{1}{\sqrt{18}} + \frac{1}{3}\right) = \frac{16}{\sqrt{18}} + \frac{3}{2}.$$

Next, let $\nu, \lambda > 0$, $\nu^2 + \lambda^2 = 1$, and $\mathbf{x} = \mathbf{y} + \nu\mathbf{e} + \lambda\mathbf{d}$. As λ converges to zero, we obtain

$$M(\mathbf{x}) = \left(\frac{11+\nu}{\sqrt{18}} - \frac{\lambda}{6}\right) + 5\left(\frac{1+\nu}{\sqrt{18}} + \frac{\lambda}{3}\right) \to \frac{22}{\sqrt{18}}.$$

Therefore, for λ small enough, $M(\mathbf{x}) < M(\mathbf{y} + \mathbf{d})$, and (2.7) is not satisfied.

APPENDIX II

The dominance of $T(\mathbf{QR})$ over $T(\mathbf{Q})$ for a sub-class of contrasts. We will show here that the *-transformation (see the Remark in Section 2) always reduces the M-norm. This implies that (2.7) holds for $\mathbf{y} = \mathbf{0}$ (i.e., for $\mathbf{x} \in Y^{\perp}$).

THEOREM 2. *$M(\mathbf{x}) > M(\mathbf{x}^*)$ for all $\mathbf{x} \notin C$ (recall that $\mathbf{x} = \mathbf{x}^*$ for $\mathbf{x} \in C$).*

PROOF. Without loss of generality, let $\mathbf{x} - \bar{x}\mathbf{1} = (\alpha_1, \cdots, \alpha_p, -\beta_1, \cdots, -\beta_q)'$, where $\alpha_1, \cdots, \alpha_p \geq 0$, $0 < \beta_1 \leq \beta_2 \leq \cdots \leq \beta_q$, $\sum_{i=1}^{p} \alpha_i = \sum_{j=1}^{q} \beta_j = 1$, and $\bar{x} \equiv \xi > 0$. Then $M(\mathbf{x} - \bar{x}\mathbf{1}) = 1$ and

$$\| \mathbf{x} - \bar{x}\mathbf{1} \|^2 = \textstyle\sum_{i=1}^{p} \alpha_i^2 + \sum_{j=1}^{q} \beta_j^2 \geq \dfrac{1}{p} + \dfrac{1}{q} = k/pq.$$

Now $\mathbf{x} = (\xi + \alpha_1, \cdots, \xi + \alpha_p, \xi - \beta_1, \cdots, \xi - \beta_q)'$; let r be such that $\beta_r \leq \xi \leq \beta_{r+1}$ (we put $\beta_0 \equiv 0$ and $\beta_{q+1} \equiv \infty$), then

$$M(\mathbf{x}) = (p + r)\xi + \textstyle\sum_{i=1}^{p} \alpha_i - \sum_{j=1}^{r} \beta_j$$

and

$$\| \mathbf{x} \|^2 = k\xi^2 + \| \mathbf{x} - \bar{x}\mathbf{1} \|^2.$$

The inequality $M(\mathbf{x}) > M(\mathbf{x}^*)$ thus reduces to:

$$\{M(\mathbf{x})\}^2 > \frac{k\xi^2 + \| \mathbf{x} - \bar{x}\mathbf{1} \|^2}{\| \mathbf{x} - \bar{x}\mathbf{1} \|^2} = 1 + \frac{k\xi^2}{\| \mathbf{x} - \bar{x}\mathbf{1} \|^2}$$

and it suffices to prove that

$$\{M(\mathbf{x})\}^2 > 1 + pq\xi^2.$$

CASE 1. $q\xi \leq 1$. Then $M(\mathbf{x}) \geq p\xi + 1$ (since $\xi - \beta_j \geq 0$ for all $1 \leq j \leq r$), and $(p\xi + 1)^2 > p\xi + 1 \geq pq\xi^2 + 1$.

CASE 2. $q\xi \geq 1$. Here we have $M(\mathbf{x}) = (p + r)\xi + \sum_{j=r+1}^{q} \beta_j \geq (p + r)\xi + (q - r)\xi = (p + q)\xi$, and $[(p + q)\xi]^2 > pq\xi^2 + q^2\xi^2 \geq pq\xi^2 + 1$, completing our proof.

REMARK. Theorem 2 actually proves a Min-Max property of any CSP procedure, a property that was conjectured by Genizi and Hochberg (1978). That is: the minimum (over all $\mathbf{Q}$ matrices) of

$$\max\{M(\mathbf{Q}'\mathbf{c}) : \mathbf{c} \in C, \| \mathbf{Q}'\mathbf{c} \| = 1\}$$

is achieved only by a CSP $\mathbf{Q}$, and *any* CSP type $\mathbf{Q}$ achieves that minimum. This follows from the fact that if $\mathbf{Q}$ is not CSP, then $M(\mathbf{Q}'\mathbf{c}) > M((\mathbf{Q}'\mathbf{c})^*)$, thus, the maximal M-norm on $\mathbf{Q}(C)$ is always larger than on C.

Acknowledgment. We are grateful to a referee whose counterexample (Appendix I) brought about a revision of our earlier draft wherein we conjectured that the transformation will always uniformly improve any non-CSP procedure.

REFERENCES

BROWN, L. D. (1979). A proof that Kramer's multiple comparison procedure for differences between treatment means is level $-\ \alpha$ for 3, 4 or 5 treatments. Unpublished manuscript.

BROWN, L. D. (1982). Personal communication.

DUNNETT, C. W. (1980). Pairwise multiple comparisons in the homogeneous variance, unequal sample size case. *J. Amer. Statist. Assoc.* **75** 789–795.

GENIZI, A. and HOCHBERG, Y. (1978). On improved extensions of the *T*-method of multiple comparisons for unbalanced designs. *J. Amer. Statist. Assoc.* **73** 879–884.

HOCHBERG, Y. (1974). Some generalizations of the *T*-method in simultaneous inference. *J. Multivariate Anal.* **4** 224-234.

HOCHBERG, Y. (1975). An extension of the *T*-method to general unbalanced models of fixed effects. *J. Roy. Statist. Soc. Ser. B* **37** 426–433.

KURTZ, T. E. (1956). An extension of a multiple comparisons procedure. Unpublished Ph.D. thesis, Princeton University.

SPJØTVOLL, E. and STOLINE, M. R. (1973). An extension of the *T*-method of multiple comparison to include the cases with unequal sample sizes. *J. Amer. Statist. Assoc.* **68** 975–978.

SPURRIER, J. D. (1981). An improved GT2 method for simultaneous confidence intervals on pairwise differences. *Technometrics* **23** 189–192.

STOLINE, M. R. (1978). Tables of the Studentized augmented range and applications to problems of multiple comparisons. *J. Amer. Statist. Assoc.* **73** 656–660.

STOLINE, M. R. (1981). The status of multiple comparisons: simultaneous estimation of all pairwise comparisons in one way ANOVA designs. *Am. Statist.* **35(3)** 134–141.

TUKEY, J. W. (1953). The problem of multiple comparisons. Unpublished Manuscript.

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