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Don H. Johnson
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Don H. Johnson*
Stuart DeGraaf†

Department of Electrical Engineering
Rice University
Houston, Texas 77001

ABSTRACT

A method of improving the bearing-resolving capabilities of a passive array is discussed. This method is an adaptive beamforming method, having many similarities to the minimum energy approach. The evaluation of energy in each steered beam is preceded by an eigenvalue-eigenvector analysis of the empirical correlation matrix. Modification of the computations according to the eigenvalue structure result in improved resolution of the bearing of acoustic sources. The increase in resolution is related to the time-bandwidth product of the computation of the correlation matrix. However, this increased resolution is obtained at the expense of array gain.

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I. Introduction

An array of acoustic sensors is placed in a known spatial pattern to record the acoustic environment. Measurement of the acoustic field by an array offers two basic improvements over the signal-processing capabilities of a single sensor. The first is the determination of the bearing of the acoustic source(s). Bearing cannot be obtained with a single sensor whereas an array offers some bearing resolving capability. This capability is usually measured by the just-detectable separation of two equi-strength sources for a given signal-to-noise ratio at one sensor in the array[6]. The determination of the bearing of a remote source of acoustic energy remains one of the fundamental problems of passive sonar systems. The waveforms recorded by each sensor can be acquired with as much fidelity as desired. In contrast, the size and geometry of the array is usually limited by physical considerations; these limitations restrict the spatial resolution of source bearing. The second improvement is the increase of signal-to-noise ratio. If the noise field is uncorrelated at each sensor location with respect to all other locations, the signal-to-noise ratio in the array output is increased by a factor equal to the number of sensors comprising the array. This factor decreases when the noise field is correlated at the sensor locations. The measure of improvement in the signal-to-noise ratio is array gain: the ratio of the signal-to-noise ratio at the array output to that obtained with a single sensor.

Adaptive beamforming methods (ABF) are known to have superior bearing resolution capabilities when compared to conventional beamformers (in a theoretical sense)[6]. Specifically, the minimum energy method has been analyzed extensively in this regard. However, there is no theoretical basis indicating that this method has the best-possible bearing resolution properties. On the other hand, the array gain provided by this method is optimum: no other beamforming technique can yield a larger increase in signal-to-noise ratio. This report is concerned with a new ABF scheme which is similar in many respects to the minimum energy method. It can demonstrate greatly increased bearing resolution properties but at the expense of array gain. The scheme is based on an eigenvector-eigenvalue decomposition of the empirical correlation matrix, which is then truncated so as to retain only those terms which best contribute to increased bearing resolution. This approach is similar to those described by Schmidt[13] in his MUSIC system, by Owsley[12] in his modal decomposition approach, and by Bienvenu[2, 3]. Analytic results are presented here which contrast the bearing resolving properties of these various eigenvector methods and the minimum energy method.

II. Preliminaries

Let $x_m(t)$ denote the outputs taken from an array of sensors having a known geometry. Beamforming consists of computing the quantity

$$y(t) = \sum_{m=0}^{M-1} a_m x_m(t-\tau_m) \quad (1)$$

where a_m is the amplitude weighting (shading) applied to the m^{th}-sensor output, τ_m is the delay applied to the m^{th}-sensor output, and M is the number of sensors in the array. The parameters $\{a_m\}$ and $\{\tau_m\}$ of a beam are chosen according to some desired criterion (e.g., steering the beam in a particular direction, minimizing sidelobe height, etc.).

Beamforming can also be viewed as a type of multidimensional spectral analysis[1, 9]. Evaluating the Fourier transform of the beam y(t), we have

$$Y(f) = \sum_{m=0}^{M-1} a_m e^{-j2\pi f\tau_m} X_m(f). \quad (2)$$

Therefore, at each temporal frequency f, the Fourier transform of the beam can be written as the dot product of two vectors

$$Y(f) = \underline{A}'\underline{X} \quad (3)$$

where $\underline{A}$ denotes the steering vector consisting of the elements

$$A_m = a_m e^{+j2\pi f\tau_m}$$

and $\underline{X}$ denotes the vector comprised of the Fourier transforms of the sensor outputs. Here, $\underline{A}'$ denotes the conjugate transpose of $\underline{A}$. Assume that we have a linear array of equally-spaced sensors; in this instance, the delays τ_m will be of the form $\tau_m = mT$. Equation (2) becomes

$$Y(f, T) = \sum_{m=0}^{M-1} a_m e^{-j2\pi mfT} X_m(f). \quad (4)$$

Consequently, Y(f, T) is the Fourier transform of the sequence $a_m X_m(f)$. For a linear array, computing a transform along the index m is identical to computing the transform in space across the array. Y(f, T) is, therefore, the spatial transform of $a_m X_m(f)$ evaluated at the spatial frequency $k = fT$. The result of applying a particular shading a_m is to convolve the true (infinite aperture) spatial spectrum with the Fourier transform of a_m, $m=0,\ldots, M-1$, thereby smearing the true spectrum and limiting the resolution that can be obtained.

III. High-Resolution Techniques

One can obtain a set of weights (or, equivalently, a steering vector) to achieve better resolution by adapting them to the particular noise field and signal field impinging on the array. In these adaptive beamforming schemes, the steering vector is the solution to an optimization problem[4, 11]: find the steering vector which minimizes the energy in the beam subject to the constraint $\underline{A}'\underline{Z} = 1$, where $\underline{Z}$ is the constraint vector. The energy contained in a beam can be expressed by the quadratic form $\underline{A}'\mathbf{R}\underline{A}$ where $\mathbf{R}$ denotes the empirical spatial correlation matrix of the Fourier transforms of the sensor outputs. The correlation matrix $\mathbf{R}$ is usually estimated from the sensor outputs by a variation of the Bartlett procedure. The output of each sensor is sectioned and windowed. The Fourier transform of each section is evaluated and the vector $\underline{X}_i(f_o)$ of Fourier transform values at the frequency f_o across the array for the $i^{\underline{th}}$ section is formed. Assuming that K sections are available, $\mathbf{R}$ is computed according to

$$\mathbf{R} = \frac{1}{K}\sum_{i=0}^{K-1} \underline{X}_i(f_o)\underline{X}_i'(f_o) \tag{5}$$

Usually K is taken to be the number of statistically independent terms used in the empirical computation of the correlation matrix $\mathbf{R}$; K is frequently referred to as the time-bandwidth product.

The solution to this optimization problem is

$$\underline{A} = \frac{\mathbf{R}^{-1}\underline{Z}}{\underline{Z}'\mathbf{R}^{-1}\underline{Z}}. \tag{6}$$

The resulting value of the energy in the beam is

$$\underline{A}'\mathbf{R}\underline{A} = (\underline{Z}'\mathbf{R}^{-1}\underline{Z})^{-1}. \tag{7}$$

In the so-called high-resolution[5] or <u>minimum</u> <u>energy</u> scheme, the constraint vector $\underline{Z}$ is a plane-wave direction-of-look vector $\underline{W}$, each element of which is given by $W_m = e^{j2\pi mk}$ where k corresponds to a specific spatial frequency. As $k = fT$ and f is known, each value of k corresponds to a specific per-channel delay. Defining θ to the bearing of the source relative to array-broadside, $T = (d/f\lambda)\sin\theta$. Consequently, spatial frequency k is related to bearing θ as $k = (d/\lambda)\sin\theta$. The constraint $\underline{A}'\underline{W} = 1$ fixes the gain of the steering vector in the direction-of-look $\underline{W}$ to be unity. Forcing the energy to be minimum thereby reduces the contributions from plane-waves arriving from other directions and from the noise field. The energy in the beam corresponding to the spatial frequency k is expressed by

$$S_{ME}(k) = (\underline{W}'\mathbf{R}^{-1}\underline{W})^{-1}. \tag{8}$$

The bearing of the target(s) is determined by finding the spatial frequency(s) at which the quantity in equation (8) achieves maxima. The maximum likelihood spectral estimate is closely related to the minimum energy estimate, but differs from it in an important way. The maximum likelihood uses the noise-only correlation matrix in its evaluation of the beam energy.

$$S_{ML}(k) = (\underline{W}'R_n^{-1}\underline{W})^{-1} \quad (9)$$

Here $R_n = E[\underline{NN}']$, the theoretical correlation matrix of the noise component of $\underline{X}$. Note specifically that the matrix R_n is assumed to be a known quantity and is not computed from empirical data. The maximum likelihood method yields the optimum array gain[6, 7]. The maximum likelihood and minimum energy methods yield the same array gain when the beam is steered toward the acoustic source[6].

IV. Improving Resolution

From the viewpoint presented here, there is great flexibility in choosing the constraint vector $\underline{Z}$. One wonders if there is a particular choice for the constraint vector which can maximize the spatial resolution of source bearing. For example, choose a constraint vector of the form

$$\underline{Z} = C\underline{W} \quad (10)$$

where C is a matrix to be described. The energy in the beam when steered toward the source would then be

$$S(k) = (\underline{W}'C'R^{-1}C\underline{W})^{-1}. \quad (11)$$

Suppose the vector $\underline{W}$ corresponded to an actual plane-wave source. If C were a matrix having the property that this choice of $\underline{W}$ lay in the null space of the matrix ($C\underline{W} = \underline{0}$), the energy in the beam when steered in this direction would be infinite. If the direction vector corresponding to the plane-wave source were the only direction vector that lay in the null space of C, one would therefore obtain a marked indication of the bearing of the source.

While it is theoretically possible to have a perfect indication of source bearing by this approach, the difficulty lies in finding the matrix C. This matrix must have the property that direction vectors lying in its null space correspond only to plane-waves eminating from actual sources. Assuming that the bearing of the sources is not known, construction of the matrix C would seem impossible. However, one can construct a matrix having a null space consisting of vectors which closely resemble direction vectors of source plane-waves. The key idea of this procedure is to analyze carefully the eigenvectors and eigenvalues of the correlation matrix.

IV.A. The Eigenvector Method

The eigenvectors of the matrix R are defined by the property

$$R\underline{V}_i = \lambda_i\underline{V}_i \quad i = 1,\dots,M \quad (12)$$

where λ_i is the eigenvalue associated with the eigenvector $\underline{V}_i$. As correlation matrices are conjugate-symmetric (Hermitian), the eigenvectors form an orthonormal set. Assume that the sensor outputs contain one signal and noise uncorrelated with the signal ($\underline{X} = \underline{S}+\underline{N}$). The result of computing R according to equation (5) for sufficiently large K is

$$R = \sigma_n^2 Q + \sigma_s^2 \underline{S}\underline{S}'. \tag{13}$$

Q is the noise correlation matrix normalized to have a trace equal to the dimension M of the matrix R. $\underline{S}$ is the direction vector of a plane-wave source and has a squared-norm equal to M. The cross-terms involving signal and noise are assumed to be negligible. If the noise correlation matrix equals the identity matrix (i.e., only sensor noise is assumed to be present), the eigenvector corresponding to the largest eigenvalue (hereby referred to as the "largest eigenvector") is equal to the vector $\underline{S}$ with eigenvalue equal to $\sigma_n^2 + M\sigma_s^2$. The remaining M-1 eigenvectors of R consist of those vectors orthogonal to $\underline{S}$ and each has eigenvalue σ_n^2. If p incoherent, linearly-independent signals $\underline{S}_i$, i = 1,...,p, are present so that the correlation matrix is of the form

$$R = \sigma_n^2 I + \sum_{i=1}^{p} \sigma_{s,i}^2 \underline{S}_i \underline{S}_i', \tag{14}$$

the p largest eigenvectors correspond to the signal terms and the M-p smallest are orthogonal to all of the signal direction vectors. Note that the largest eigenvectors are not necessarily equal to the signal direction vectors in this case; these eigenvectors comprise an orthonormal basis for the vector space containing the signal vectors. Consequently, one cannot always inspect the largest eigenvectors and determine the signal vectors directly.

Define C_{EV} to be the sum of the outer products of the M-p smallest eigenvectors*

$$C_{EV} = \sum_{i=1}^{M-p} \underline{V}_i \underline{V}_i'. \tag{15}$$

As the p largest eigenvectors are orthogonal to each of the M-p smallest, $C_{EV}\underline{V}_i = \underline{0}$, i = M-p+1,...,M. As the p largest eigenvectors span the space containing the signal vectors, each signal lies in the null space of C_{EV} and $C_{EV}\underline{S}_i = \underline{0}$. In this manner, perfect resolution of the bearing of multiple sources can be obtained from an eigenvector analysis of the correlation matrix R.

Note that in computing the beam energy (equation 11), the matrix C_{EV} need never be computed. The correlation matrix R can be expressed in terms of its eigenvectors as

$$R = \sum_{i=1}^{M} \lambda_i \underline{V}_i \underline{V}_i' \tag{16}$$

and the inverse of R as

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*By convention, the smallest eigenvector is denoted by the subscript 1, the next smallest by 2, etc.

$$R^{-1} = \sum_{i=1}^{M} \frac{1}{\lambda_i} \underline{V}_i \underline{V}_i'. \qquad (17)$$

Because of the orthogonality property of the eigenvectors of a correlation matrix, the quantity $C_{EV}' R^{-1} C_{EV}$ appearing in equation (11) becomes

$$C_{EV}' R^{-1} C_{EV} = \sum_{i=1}^{M-p} \frac{1}{\lambda_i} \underline{V}_i \underline{V}_i'. \qquad (18)$$

It is this matrix which is computed in the evaluation of the quadratic form of equation (11).

$$S_{EV}(k) = (\underline{W}' C_{EV}' R^{-1} C_{EV} \underline{W})^{-1} = \left[\sum_{i=1}^{M-p} \frac{1}{\lambda_i} |\underline{W}' \underline{V}_i|^2 \right]^{-1} \qquad (19)$$

The following sequence of computations constitute the <u>eigenvector</u> <u>method</u>.

1. Compute the correlation matrix R.
2. Decompose the matrix R into its eigenvectors and eigenvalues.
3. Determine the number p of sources present in the acoustic field.
4. Compute the energy in the beams corresponding to all possible bearings (equation 19).
5. The major peaks in this spectrum correspond to acoustic sources.

The methods of Owsley[12], Bienvenu[2, 3], and Schmidt[13] are similar, but differ somewhat from the eigenvector method just described. The expression for R^{-1} is truncated as in equation (5); however, the small eigenvalues are set to the same value (taken here to be unity). Instead of evaluating the quadratic form of equation (11), the spectral estimate of the MUSIC method can be expressed by

$$S_{MUSIC}(k) = (\underline{W}' C_{EV}' C_{EV} \underline{W})^{-1} = \left[\sum_{i=1}^{M-p} |\underline{W}' \underline{V}_i|^2 \right]^{-1} \qquad (20)$$

This estimate can also be evaluated in the same manner as equation (11) if the matrix C is redefined to be

$$C_{MUSIC} = \sum_{i=1}^{M-p} \sqrt{\lambda_i}\, \underline{V}_i \underline{V}_i'$$

A steering vector can be defined (equation 6) when the latter approach is used to express the MUSIC spectral estimate. Under the conditions just described, this spectral estimate also has the capability of resolving source bearing perfectly.

The presumption of the preceding analysis has been that the noise correlation matrix **Q** is equal to the identity matrix. It is this key assumption which leads to perfect bearing resolution. Imperfection in bearing resolution occurs when this presumption is false. The noise field may contain more than just sensor noise: for example, isotropic noise may also be present. In addition, a finite amount of averaging is used to compute the correlation matrix **R**. Even if the noise field were spatially white, an empirical noise correlation matrix would not be an identity matrix. Either of these situations can reduce the resolution of the eigenvector and MUSIC methods.

The performances of the eigenvector method and of the MUSIC method under these more realistic conditions are analyzed mathematically in the appendix. From this analysis, an approximate lower limit on the energy in the beam when steered on-target using the eigenvector method is found to be

$$S(k) = \frac{\sigma_s^2}{\frac{\sigma_n^2}{\sigma_s^2}\frac{\gamma^2}{M} + \frac{M}{K}} \qquad (21)$$

A similar result is obtained for the MUSIC method. The quantity γ is a measure of the spread of the eigenvalues α_i of **Q**. For the eigenvector method, this quantity is given by

$$\gamma_{EV}^2 = \frac{1}{M}\sum_{i=1}^{M}(\alpha_i - \frac{1}{M}\sum_m \alpha_m^2)^2 \qquad (22)$$

and for the MUSIC method by

$$\gamma_{MUSIC}^2 = \frac{1}{M}\sum_{i=1}^{M}\alpha_i(\alpha_i - \frac{1}{M}\sum_m \alpha_m^2)^2. \qquad (23)$$

γ_{MUSIC}^2 tends to be smaller than γ_{EV}^2. In either case, the quantity γ is an implicit function of the time-bandwidth product K. Generally speaking, γ will decrease with increasing K, tending toward the spread of the eigenvalues of the spatial correlation matrix of the underlying noise process. If the theoretical noise correlation matrix equals the identity matrix (i.e., spatially white noise), the spread of its eigenvalues is zero. Otherwise, the spread is nonzero.

The first term in the denominator of equation (21) determines how large the energy peak will be if an infinite time-bandwidth product were available. The second term describes how the size of the energy peak depends on K. The larger of these will dominate the expression in equation (21). When steered off-target, the energy in the beam produced by the eigenvector method coincides with that produced with the minimum energy approach. The ratio $\mathcal{U}$ of the on-target to off-target beam energy can be used to assess the size of the peak in the beam energy as the direction vector $\underline{W}$ is scanned through all possible bearings. When only sensor noise is present, this quantity is given in the minimum energy method by[10]

$$\mathcal{V}_{ME} = 1 + \frac{M\sigma_s^2}{\sigma_n^2}. \qquad (24)$$

In the eigenvector method and the MUSIC method, this quantity is given by

$$\mathcal{V} = \frac{1}{\frac{\sigma_n^2}{\sigma_s^2}\frac{\gamma^2}{M^2} + \frac{1}{K}} \cdot \frac{\sigma_s^2}{\sigma_n^2}. \qquad (25)$$

From simulation studies, this quantity can be somewhat larger (a few dB) in the MUSIC method than in the eigenvector method. Comparing equations (24) and (25), one concludes tnat when the eigenvector method is used, the array appears to consist of a number of elements $\tilde{M}$ given by

$$\tilde{M} = \frac{1}{\frac{\sigma_n^2}{\sigma_s^2}\frac{\gamma^2}{M^2} + \frac{1}{K}} \qquad (26)$$

In most instances, this quantity is larger than the actual number of sensors M.

IV.B. Simulation Results

Computer simulations were used to evaluate the eigenvector and MUSIC methods and to compare them with the minimum energy method. In these studies, sequences of the data vector $\underline{X}_i = \underline{S} + \underline{N}_i$ were produced. The parameters of the signal vector $\underline{S}$ were defined as described in section II. The noise vector $\underline{N}_i$ consisted of identically-distributed complex Gaussian noise components. The covariance matrix of this random vector could be specified by the user. Each noise vector was generated to be statistically independent of all other noise vectors. The empirical correlation matrix **R** was then computed as in equation (5) and its eigenvectors and eigenvalues computed according to a QL algorithm[8].

With one exception, each step of the procedure outlined above for the eigenvector method was followed. As the number p of sources present in the acoustic field was known by the user, it was supplied by him. In a physical situation, this parameter would not be so readily known. However, the purpose of the simulations was to determine the validity of the theory and to test how well the methods could perform. The effects of inaccurate choices for p are described in a later section.

A comparison of the beam energies produced by the minimum energy method and by the eigenvector method is shown in figure 1a when one source is present in the acoustic field. A similar comparison is found in figure 1b for the MUSIC method. Note that the height of the main peak relative to the background noise level varies with K, the time-bandwidth product, in both methods whereas it does not in the minimum energy method. In this example, sensor noise (i.e., spatially white) is present. For the array length used, the second term in denominator of equation (21) was larger than the first. Simulated and theoretical values of $\mathcal{V}_{EV}$ are compared in Table I. The theoretical prediction of the value of this quantity is close to that obtained from the simulations.

The spectra obtained from the eigenvector method and the MUSIC method differ only slightly in these examples. The latter method tends to produce much flatter off-target spectra. The small eigenvalues of the correlation matrix $\mathbf{R}$ tend to represent the noise field (see the discussion following equation 13) and the equalization of these values will "whiten" the background noise, thereby resulting in a flat spectrum. This effect is further illustrated by considering a case where isotropic noise is present in the acoustic field (figure 2).

IV.C. Resolution of Multiple Targets

Cox[6] derives analytic expressions for the limits to which the minimum energy method applied to a linear array can resolve two equi-strength, incoherent acoustic sources. There, "resolution" is defined as the minimum bearing separation at broadside at which two targets can be distinguished by evaluating beam energy. Here, a slight dip in the beam energy is required when the array is steered between the sources. The critical factors determining the resolution of beams formed by the minimum energy approach are aperture (defined as the spatial extent of the array relative to a wavelength $- \frac{D}{\lambda}$), the number of elements in the array (M), and the sensor signal-to-noise ratio (σ_s^2/σ_n^2). Cox's results are summarized in figure 5 of his paper[6]; an approximation to those results is

$$\frac{M\sigma_s^2}{\sigma_n^2} \;\alpha\; \left[\theta \, \frac{D}{\lambda}\right]^4 \tag{27}$$

where θ is bearing separation of the two targets.

When the eigenvector method is applied in situations such as these, targets are more easily resolved and furthermore, the resolution capabilities of the array are increased. Figures 3 and 4 display typical examples of these cases. A theoretical prediction of the degree to which resolution is increased can be obtained from equation (27). If one substitutes $\mathcal{M}$ evaluated by equation (26) for M, the value of θ thus obtained is the resolution limit of the eigenvector method and the MUSIC method. A comparison of the resolution obtained from some of the simulations with that predicted by the theory is shown in Table II. The degree of agreement between theory and simulation results implied by Table II is valid for all of the simulations.

IV.D. Effect of an Improper Choice for p

The theoretical and simulation results presented thus far are valid only when the parameter p equals the actual number of acoustic sources. In practice, this quantity may not be known and one questions the sensitivity of the eigenvector and MUSIC methods to an incorrect choice of p. This sensitivity was studied through the simulations; no analytic results were obtained on this issue.

An incorrect choice for p has different effects on the eigenvector and MUSIC methods. In both methods, choosing p too small does <u>not</u> result in a beamformer having superior bearing resolution properties to the minimum energy method. In the eigenvector method, the spectra tend to resemble those obtained with the minimum energy method. In particular, by assuming no sources are present (setting p = 0 or the matrix $C_{EV} = I$), the eigenvector method is exactly the minimum energy method. In contrast, the MUSIC method tends to produce a number of spectral peaks equal to p. For example, if a value of zero is chosen for p, a uniformly flat spectrum results. The peaks that result from nonzero choices tend to correspond to the bearings of acoustic sources; however, which sources are thus located is not easily predicted. When p is chosen too large, the bearing resolving capabilities of either method are not greatly reduced. The spectra produced by the eigenvector method tend not to vary from that obtained with the proper value of p. The MUSIC method tends to produce spurious peaks that do not correspond to physical sources. The effects are illustrated in figure 5.

V. Resolution and Array Gain

While these approaches increase the resolution of bearing, this increase in resolution is accompanied by a decrease in the array gain. To show this, assume the correlation matrix R is of the form given in equation (13). The signal-to-noise ratio at each sensor is therefore σ_s^2/σ_n^2. The signal-to-noise ratio in the beam output is the quantity

$$\frac{\sigma_s^2 \underline{A}' \underline{S}\underline{S}' \underline{A}}{\sigma_n^2 \underline{A}' Q \underline{A}} \tag{28}$$

The array gain G is the ratio of these signal-to-noise ratios.

$$G = \frac{\underline{A}' \underline{S}\underline{S}' \underline{A}}{\underline{A}' Q \underline{A}} \tag{29}$$

Recalling that the steering vector in this case is given by

$$\underline{A} = \frac{R^{-1} C \underline{W}}{\underline{W}' C' R^{-1} C \underline{W}} \tag{30}$$

and setting the direction vector to correspond to the source ($\underline{W} = \underline{S}$), the array gain becomes

$$G = \frac{\underline{S}' C' R^{-1} \underline{S}\underline{S}' R^{-1} C \underline{S}}{\underline{S}' C' R^{-1} Q R^{-1} C \underline{S}}. \tag{31}$$

As the matrix Q is given by

$$Q = \frac{1}{\sigma_n^2}(R - \sigma_s^2 \underline{S}\underline{S}'), \tag{32}$$

the denominator of equation (31) becomes

$$\underline{S}'C'R^{-1}QR^{-1}C\underline{S} = \frac{1}{\sigma_n^2}(\underline{S}'C'R^{-1}C\underline{S} - \sigma_s^2\underline{S}'C'R^{-1}\underline{S}\cdot\underline{S}'R^{-1}C\underline{S}) \quad (33)$$

so that we obtain

$$G = \frac{\sigma_n^2}{\dfrac{\underline{S}'C'R^{-1}C\underline{S}}{\underline{S}'C'R^{-1}\underline{S}\cdot\underline{S}'R^{-1}C\underline{S}} - \sigma_s^2} \quad (34)$$

For the eigenvctor method,

$$C'_{EV}R^{-1} = R^{-1}C_{EV} = C'_{EV}R^{-1}C_{EV} \quad (35)$$

and the array gain becomes

$$G_{EV} = \frac{\sigma_n^2\underline{S}'C'_{EV}R^{-1}C_{EV}\underline{S}}{1-\sigma_s^2\underline{S}'C'_{EV}R^{-1}C_{EV}\underline{S}}. \quad (36)$$

In the MUSIC method, the ratio appearing in the denominator of equation (34) can be bounded using the Schwarz inequality.

$$\frac{\underline{S}'C'_{MUSIC}R^{-1}C_{MUSIC}\underline{S}}{\underline{S}'C'_{MUSIC}R^{-1}\underline{S}\cdot\underline{S}'R^{-1}C_{MUSIC}\underline{S}} \geq \frac{1}{\underline{S}'C'_{EV}R^{-1}C_{EV}\underline{S}} \quad (37)$$

Equality occurs only when the M-1 smallest eigenvalues of R are identical (i.e., $Q = I$). This bound can be used in equation (34) to obtain an upper bound on G_{MUSIC} if the bound is not smaller than σ_s^2. As the expression thus obtained equals G_{EV} (equation 36), this condition is satisfied. Consequently,

$$G_{MUSIC} \leq G_{EV} \quad (38)$$

Considering equation (36), G_{EV} is a monotonically increasing function of the quadratic form $\underline{S}'C'_{EV}R^{-1}C_{EV}\underline{S}$. Therefore, whenever one decreases this quadratic form to improve the indication of bearing (equation 19), the array gain decreases in the eigenvector method. Because of the relationship given in equation (38), the array gain obtained with the MUSIC method also decreases. In the limit, perfect indication of bearing (a zero-valued quadratic form) corresponds to zero array gain with either method.

VI. Conclusions

The eigenvector method can enhance the bearing-resolving capabilities of an array. Here, the eigenvectors and eigenvalues of the correlation matrix must be found and the weighted sum of Fourier transforms of the eigenvectors computed. In the minimum energy method, the inverse of the correlation matrix must be found and the quadratic form of equation (8) computed. Roughly speaking, the computational complexities involved in the use of the eigenvector method are not excessive when compared to those required in the minimum energy method.

The degree to which the resolving power is increased is related to the quantity $\tilde{M}$. Because of equation (27), this increase is proportional to $(\tilde{M}/M)^{1/4}$. Consequently, to increase the bearing resolution by a factor of 2 requires the virtual number of sensors $\tilde{M}$ to be 16 times the actual number. Referring to Table I, such large virtual array lengths can be obtained only when large time-bandwidth products are possible. Under these circumstances, enhanced bearing resolution is possible. For a given time-bandwidth product, the smaller the number of elements in the array, the greater the increase in bearing resolution.

The eigenvector method and the MUSIC method produce quite similar results. They differ in at least two respects, however. The first is that a non-zero value of p, the number of assumed sources in the acoustic field, must be chosen in the MUSIC method. If the value of p is not close to the actual value, the spectra thus obtained can differ from that obtained with the proper value: spurious peaks appear and/or peaks can be missed. In contrast, the eigenvector method is less sensitive to the choice of p. Second, the shape of the spectrum of the background "noise" is drastically altered in the MUSIC method. For example, the variations due to low-level sources or to the physical noise spectrum are lost (see figure 2, for example). This portion of the spectrum can also vary as p is changed; this effect is much less pronounced in the eigenvector method.

The increase in bearing resolution is obtained at the expense of array gain. Consequently, if more than bearing information is required, other techniques should probably be used to obtain them. One can conceive of the eigenvector method being used to acquire source bearing and this information being used to steer a beam with the minimum energy method so as to analyze the waveform produced by the source. Note that this two-step procedure need only be sequential in a conceptual manner. Because of the close relationship between the two methods, obtaining the steering vector for the minimum energy beamformer means including more terms in the eigenvector decomposition of **R**.

The decrease of array gain with increasing resolution raises many theoretical issues. The minimum energy method is known to yield the optimal value of array gain. Consequently, any method which has greater resolution capabilities cannot also increase array gain. Can array gain be maintained while increasing resolution or is increased resolution always obtained at the expense of array gain? The present method has the latter property. A theoretical understanding of the limits to which array gain and resolution can be traded against each other would be of interest.

The main issue not addressed in this study is the determination of the number of sources - p. From the analysis presented in section IV, the number of sources corresponds to the number of dominant eigenvalues in the matrix **R**. Determining p in this way can be difficult when small time-bandwidth products are available and isotropic noise is present. Reasonably accurate methods of determining p from the eigenvalues of **R** are not known at this time.

Acknowledgement

The authors are indebted to D. J. Edelblute of the Naval Ocean Systems Center for many fruitful discussions concerning this research.

Appendix

Let the vector $\underline{X}$ of sensor outputs be of the form $\underline{X}=\sigma_s\underline{S} + \sigma_n\underline{N}$ where $\underline{S}$ denotes the source direction vector as before and $\underline{N}$ denotes additive noise. The correlation matrix $\mathbf{R}$ is computed empirically according to equation (5) to yield.

$$\mathbf{R} = \sigma_n^2\mathbf{Q} + \sigma_n\sigma_s\underline{SN}' + \sigma_n\sigma_s\underline{NS}' + \sigma_s^2\underline{SS}' \qquad (A.1)$$

where $\mathbf{Q} = \frac{1}{K}\sum_{i=1}^{K}\underline{N}_i\underline{N}_i'$, a statistical estimate of the noise correlation matrix and $\underline{N} = \frac{1}{K}\sum_{i=1}^{K}\underline{N}_i$, an estimate of the average noise component. The vectors $\underline{N}_i$ are assumed to be statistically independent random vectors; each component of $\underline{N}_i$ has zero mean and unity variance. Consequently, the components of the vector $\underline{N}$ have zero mean and energy 1/K. Define the matrix P to be the noise-related terms in equation (A.1).

$$\mathbf{P} = \mathbf{Q} + \frac{\sigma_s}{\sigma_n}\underline{SN}' + \frac{\sigma_s}{\sigma_n}\underline{NS}' \qquad (A.2)$$

Consequently, the expression for $\mathbf{R}$ in equation (A.1) can be written more simply as

$$\mathbf{R} = \sigma_n^2\mathbf{P} + \sigma_s^2\underline{SS}' \qquad (A.3)$$

The estimate of the energy in the beam pointed in the $\underline{W}$ direction is given by equation (11). Following Cox[6], this expression can be written as

$$S(k) = \frac{\sigma_n^2}{\underline{W}'C'P^{-1}C\underline{W}}\left[\frac{1+(\frac{S}{N})_{MAX}}{1+(\frac{S}{N})_{MAX}\sin^2(C\underline{W},\ \underline{S};\ P^{-1})}\right] \qquad (A.4)$$

where $(\frac{S}{N})_{MAX} = \underline{S}'P^{-1}\underline{S}\ (\sigma_s^2/\sigma_n^2)$ is the signal-to-noise ratio of the beam output obtained with the optimally-chosen steering vector and $\sin^2(C\underline{W},\ \underline{S};\ P^{-1})$ is the sine-squared of the angle between the vectors $C\underline{W}$ and $\underline{S}$ with respect to the matrix P^{-1}. The matrix C_{EV} is given by equation (15) when p, the number of signals in the acoustic field, is assumed to be one. The critical aspect of equation (A.4) is the quantity $C_{EV}\underline{W}$. This quantity can be viewed as the projection of the vector $\underline{W}$ onto the set of eigenvectors orthogonal to the largest eigenvector of $\mathbf{R}$. As indicated earlier, these eigenvectors are approximately orthogonal to the signal vector. To a good approximation, the vector $C_{EV}\underline{W}$ is orthogonal to $\underline{S}$, thereby implying for all $\underline{W}$ that $\sin^2(C_{EV}\underline{W},\ \underline{S};\ P^{-1}) = 1$. Expression (A.4) therefore becomes

$$S(k) = \frac{\sigma_n^2}{\underline{W}' C'_{EV} P^{-1} C_{EV} \underline{W}}. \tag{A.5}$$

When $\underline{W}$ does not correspond to the signal direction vector $\underline{S}$, the matrix C_{EV} has little effect on the vector $\underline{W}$. In this case, one obtains

$$S(k) = \frac{\sigma_n^2}{\underline{W}' P^{-1} \underline{W}},$$

the result obtained with the minimum energy estimate. Consequently, one should expect the eigenvector procedure and the minimum energy procedure to yield the same numerical values when steered off-target. As the beam is steered toward the source, the two estimates will begin to differ as the matrix C_{EV} begins to affect the vector $\underline{W}$.

In the succeeding analysis the inverse of the matrix P is assumed to be approximately equal to the inverse of Q in the computation of the quadratic form appearing in equation (A.5). Inspecting equation (A.2), this approximation will be less accurate as the signal-to-noise ratio (σ_s/σ_n) increases and as the amount of averaging (K) decreases. The energy estimate can be written approximately as

$$S(k) \simeq \frac{\sigma_n^2}{\underline{W}' C'_{EV} Q^{-1} C_{EV} \underline{W}}. \tag{A.6}$$

The quadratic form in equation (A.6) can be interpreted as the squared-length of the vector $C_{EV}\underline{W}$ with respect to the norm induced by the matrix Q^{-1}. We therefore seek an expression for this quantity when $\underline{W}=\underline{S}$.

Assume that $\underline{V}_M$, the largest eigenvector of R, is given by $\underline{V}_M = \underline{S}+\underline{\varepsilon}$ where $\underline{\varepsilon}$ is a vector orthogonal to $\underline{S}$. Consider the vector diagram shown in figure A1. The vector $\underline{a}$ is defined to be orthogonal to the eigenvector $\underline{S}+\underline{\varepsilon}$. What is sought in equation (A.6) is the square of the length L of the vector $\underline{S}$ projected onto $\underline{a}$. As shown below, the length of $\underline{\varepsilon}$ is small compared to $\underline{S}$; therefore, the quantity L will be approximately equal to the length of $\underline{\varepsilon}$. To a good approximation, the energy in the beam steered on-target is given by

$$S(k) = \frac{\sigma_n^2}{\|\underline{\varepsilon}\|^2_{Q^{-1}}} \tag{A.7}$$

where $\|\underline{\varepsilon}\|^2_{Q^{-1}}$ denotes the squared-norm of $\underline{\varepsilon}$ with respect to Q^{-1}.

$$\|\underline{\varepsilon}\|^2_{Q^{-1}} = \underline{\varepsilon}'Q^{-1}\underline{\varepsilon}.$$

The definition of an eigenvector implies that this vector must satisfy

$$\mathbf{R}(\underline{S}+\underline{\varepsilon}) = \lambda_M(\underline{S}+\underline{\varepsilon}). \quad \text{(A.8)}$$

As the vectors $\underline{S}$ and $\underline{\varepsilon}$ are assumed to be orthogonal, the eigenvalue λ_M can be found through the relationship

$$\underline{S}'\mathbf{R}(\underline{S}+\underline{\varepsilon}) = \lambda_M\underline{S}'(\underline{S}+\underline{\varepsilon}) = M\lambda_M$$

Using equation (A.3) for $\mathbf{R}$, we have

$$\lambda_M = \frac{\sigma_n^2\underline{S}'Q\underline{S}}{M} + \sigma_s\sigma_n(\underline{S}'\underline{N}+\underline{N}'\underline{S}+\underline{N}'\underline{\varepsilon}) + M\sigma_s^2. \quad \text{(A.9)}$$

An expression for the vector $\underline{\varepsilon}$ is obtained by evaluating the quantity $\mathbf{R}(\underline{S}+\underline{\varepsilon})-\lambda_M\underline{S}$. After some manipulation and assuming that the length of $\underline{\varepsilon}$ is small compared to the length of $\underline{S}$, we have

$$\underline{\varepsilon} = (\lambda_M I-\sigma_n^2 Q)^{-1}(\sigma_n^2 Q\underline{S} - \sigma_n^2\frac{\underline{S}'Q\underline{S}}{M}\underline{S} + M\sigma_s\sigma_n\underline{N} - \sigma_s\sigma_n\underline{S}'\underline{N}\underline{S}) \quad \text{(A.10)}$$

This expression for the vector ε consists of a matrix $(\lambda_M I-\sigma_n^2 Q)^{-1}$ times the sum of two terms. The first term, denoted by $\underline{B}_1$, is comprised only of the signal-related terms and the matrix Q.

$$\underline{B}_1 = \sigma_n^2 Q\underline{S} - \sigma_n^2\frac{\underline{S}'Q\underline{S}}{M}\underline{S}. \quad \text{(A.11)}$$

The second term contains the terms depending on the average noise vector $\underline{N}$.

$$\underline{B}_2 = M\sigma_s\sigma_n\underline{N} - \sigma_s\sigma_n(\underline{S}'\underline{N})\underline{S}. \quad \text{(A.12)}$$

If one assumes the noise vector $\underline{N}$ to be zero (implying infinite statistical averaging), the vector $\underline{\varepsilon}$ is given by the quantity $\underline{\varepsilon} = (\lambda_M I-\sigma_n^2 Q)^{-1}\underline{B}_1$. Note that if $Q = I$, the quantity $\underline{B}_1 = \underline{0}$ which implies $\underline{\varepsilon} = \underline{0}$. Therefore, the signal vector $\underline{S}$ corresponds to the largest eigenvector of $\mathbf{R}$; this result is consistent with the analysis described while leading to equation (5). The term $\underline{B}_2$ expresses the effect on the eigenvector of the statistical averaging process. Note that if noise field can be described as containing only sensor noise ($Q = I$), the expression for $\underline{\varepsilon}$ is dominated by $\underline{B}_2$. The vector $\underline{B}_2$ can be interpreted as the noise vector which results when all of its components in the direction of the signal vector $\underline{S}$ are eliminated (equation A.12). The squared-magnitude of this vector therefore depends on the "angle" between the vectors $\underline{N}$ and $\underline{S}$. This angle will be a random quantity when the computation of R is completed. This vector will be largest when $\underline{N}$ and $\underline{S}$ are assumed to be orthogonal. In this case, $\underline{S}'\underline{N} = 0$, an the expression for $\underline{B}_2$ becomes

$$\underline{B}_2 = M\sigma_s\sigma_n\underline{N}. \tag{A.13}$$

Define $\underline{E}_1$ ($\underline{E}_2$) to be the product of $(\lambda_M I-\sigma_n^2 Q)^{-1}$ and $\underline{B}_1$ ($\underline{B}_2$). The vector $\underline{\varepsilon}$ is therefore written as

$$\underline{\varepsilon} = \underline{E}_1 + \underline{E}_2 \tag{A.14}$$

An expression for the norm of $\underline{\varepsilon}$ can now be obtained. Let $\underline{U}_i$ denote an eigenvector of the matrix Q and α_i the associated eigenvalue. As these eigenvectors are orthonormal, any vector can be expressed as a linear combination of them. As $\underline{U}_i$ is also an eigenvector of $(\lambda_M I-\sigma_n^2 Q)^{-1}$, $\underline{E}_1$ and $\underline{E}_2$ can be written as

$$\underline{E}_1 = \sigma_n^2 \sum_i \frac{(\alpha_i - \frac{1}{M}\sum_m \alpha_m |\underline{S}'\underline{U}_m|^2)\underline{U}_i'\underline{S}}{\lambda_M-\alpha_i\sigma_n^2}\underline{U}_i \tag{A.15a}$$

$$\underline{E}_2 = M\sigma_s\sigma_n \sum_i \frac{\underline{U}_i'\underline{N}}{\lambda_M-\alpha_i\sigma_n^2} U_i \tag{A.15b}$$

The quantity $\lambda_M-\alpha_i\sigma_n^2$ can be simplified. Using equation (A.9) and assuming both $\underline{\varepsilon}$ and $\underline{N}$ are small compared to $\underline{S}$, we have

$$\lambda_M-\alpha_i\sigma_n^2 = M\sigma_s^2 + (\frac{S'QS}{M} - \alpha_i)\sigma_n^2.$$

The quantities within the parentheses are comparable and further assuming $M\sigma_s^2 > \sigma_n^2$, we have

$$\lambda_M - \alpha_i \sigma_n^2 \simeq M\sigma_s^2. \tag{A.16}$$

The norm of $\underline{\varepsilon}$ depends upon the angle between these two vectors. Based on statistical arguments, a zero-mean vector obtained from averaging ($\underline{B}_2$ in this case) will be nearly orthogonal to any fixed vector. Consequently,

$$||\underline{\varepsilon}||^2_{Q^{-1}} = ||\underline{B}_1||^2_{Q^{-1}} + ||\underline{B}_2||^2_{Q^{-1}} \tag{A.17}$$

Now

$$||\underline{B}_1||^2_{Q^{-1}} = \sigma_n^4 \sum_i \frac{(\alpha_i - \frac{1}{M}\sum_m \alpha_m |\underline{S}'\underline{U}_m|^2)^2}{\alpha_i(\lambda_M - \alpha_i\sigma_n^2)^2} |\underline{S}'\underline{U}_i|^2$$

To evaluate this expression, the relationship between the signal direction vector $\underline{S}$ and the eigenvectors of Q must be specified. If $\underline{S}$ were proportional to an eigenvector of Q, the quantity $||\underline{B}_1||^2_{Q^{-1}}$ would be zero. As the norm of B_1 will appear in the denominator of the expression for beam energy (equation A.7), one can obtain a lower bound on the energy in the beam by assuming the largest-possible value for its length. To approximate the maximal length of $\underline{B}_1$, assume that $\underline{S}$ does not prefer <u>any</u> of the eigenvector directions of Q. A reasonable mathematical description of this situation is that $|\underline{S}'\underline{U}_i|^2 = \alpha_i$. In this instance, we have using equation (A.16) that

$$||\underline{B}_1||^2_{Q^{-1}} = \frac{\sigma_n^4}{\sigma_s^4} \frac{1}{M^2} \sum_i (\alpha_i - \frac{1}{M}\sum_m \alpha_m^2)^2. \tag{A.18}$$

The quantity in the summation depends only on the eigenvalues of the matrix Q. Define the quantity γ_{EV}^2 to be

$$\gamma_{EV}^2 = \frac{1}{M}\sum_i (\alpha_i - \frac{1}{M}\sum_m \alpha_m^2)^2. \tag{A.19}$$

Then, equation (A.18) becomes

$$||\underline{B}_1||^2_{Q^{-1}} = \frac{\sigma_n^4}{\sigma_s^4} \cdot \frac{\gamma_{EV}^2}{M}. \tag{A.20}$$

As the term $\underline{E}_2$ is a random quantity, its squared-norm is defined to be

$$\|\underline{E}_2\|^2_{Q^{-1}} = E\left[\frac{\sigma_n^2}{\sigma_s^2}\sum_i \frac{|\underline{N}'\underline{U}_i|^2}{\alpha_i}\right] \tag{A.21}$$

where E[·] denotes expected value. To a good approximation, $\underline{U}_i$ is an eigenvector of the correlation matrix associated with the random vector $\underline{N}$. Consequently,

$$E[|\underline{N}'\underline{U}_i|^2] = \frac{\alpha_i}{K}$$

so that

$$\|\underline{E}_2\|^2_{Q^{-1}} = \frac{\sigma_n^2}{\sigma_s^2}\cdot\frac{M}{K}. \tag{A.22}$$

Substituting equations (A.18), (A.22), and (A.17) into equation (A.7), we have finally

$$S_{EV}(k) = \frac{\sigma_s^2}{\frac{\sigma_n^2}{\sigma_s^2}\frac{\gamma_{EV}^2}{M} + \frac{M}{K}} \tag{A.23}$$

as an expression for the energy in the beam when steered toward the source.

The analysis for the MUSIC method differs only in detail from that just described. In this method, the spectral estimate is given by

$$S_{MUSIC}(k) = (\underline{W}'C_{EV}'C_{EV}\underline{W})^{-1} \tag{A.24}$$

Off target, $C_{EV}\underline{W}$ approximately equals $\underline{W}$, implying that $S_{MUSIC}(k) = M^{-1}$. When steered on target, the expression for the MUSIC estimate differs little from that given in equation (A.7). The significant difference is that the norm of $\underline{\varepsilon}$ is computed with respect to the identity matrix instead of Q^{-1}. The quantity of interest is therefore

$$\|\varepsilon\|^2 = \|\underline{E}_1\|^2 + \|\underline{E}_2\|^2.$$

The norm of $\underline{E}_2$ with respect to I equals that computed with respect to Q^{-1}; the expression for the norm of $\underline{E}_1$ is

$$\|\underline{E}_1\|^2 = \frac{\sigma_n^4}{\sigma_s^4} \frac{1}{M^2} \sum_i \alpha_i (\alpha_i - \frac{1}{M} \sum_m \alpha_m^2)^2$$

Therefore, the on-target beam energy in the MUSIC method is given by

$$S_{MUSIC}(k) = \frac{\sigma_s^2/\sigma_n^2}{\frac{\sigma_n^2}{\sigma_s^2} \frac{\gamma_{MUSIC}^2}{M} + \frac{M}{K}} \qquad (A.25)$$

where γ_{MUSIC}^2 is defined to be

$$\gamma_{MUSIC}^2 = \frac{1}{M} \sum_{i=1}^{M} \alpha_i (\alpha_i - \frac{1}{M} \sum_m \alpha_m^2)^2 .$$

Table I. Empirical and Theoretical Peak-to-Background Ratios

The results of computer simulations and theoretical predictions of the value of $\mathcal{U}$ are shown. In the simulations, an equally-spaced linear array containing 10 sensors was assumed to be present in an acoustic field containing sensor noise and a plane-wave source. The sensor signal-to-noise ratio was 0 dB. Values of γ were computed in separate simulations from noise-only correlation matrices having the same time-bandwidth product. The source was assumed to be narrowband, with all of its source energy concentrated in one temporal-frequency analysis bin. The sensor spacing is one-half wavelength. Plot of beam energy vs. bearing were obtained and empirical values of $\mathcal{U}$ estimated. The quantities $\hat{\mathcal{U}}_{ME}$ and $\hat{\mathcal{U}}_{EV}$ correspond to these empirical values.

K	M	$\hat{\mathcal{U}}_{ME}$ (dB)	$\hat{\mathcal{U}}_{EV}$ (dB)	$\mathcal{U}_{EV}$ (dB)
20	18	10	13	13
50	46	11	18	17
100	90	12	22	20
200	183	11	27	23
500	461	11	29	27
1000	940	11	32	30
2000	1784	11	34	33
5000	4597	11	39	37

Table II. Empirical and Theoretical Resolution Limits

The results of computer simulations and theoretical predictions of resolution limits are summarized. The array configuration used in Table I was used here. The sources were symmetrically located about broadside (0 degrees). Plots of beam energy vs. bearing were obtained and the separation between the sources reduced until they could just be resolved. The measurements of separation were made in half-degree increments. The angular quantities θ are indicated in degrees.

K	θ_{ME}	$\hat{\theta}_{EV}$	θ_{EV}
20	8	7.5	7.5
50	8	6	5.8
100	8	6	5.4
200	8	4.5	4.6
500	8	3.5	3.7
1000	8	3.5	3.0
2000	8	3	2.6
5000	8	2.5	2.1

Figure Captions

Figure 1. Energy in beams formed by the minimum energy method (ME) and by the eigenvector method (EV) are plotted against bearing for a linear array. Energy is expressed in dB relative to the peak value. Bearing is expressed in degrees with zero corresponding to broadside. The sensors are equally spaced and separated by half a wavelength. The source was assumed to be narrowband, with all of its source energy concentrated in one temporal-frequency analysis bin. In each sub-figure, sensor noise and one source located at 0^{o} are present in the sound field; the sensor signal-to-noise ratio is 0 dB. The results obtained when two time-bandwidth products (K) are used are shown in each sub-figure for each method.

Figure 2. Beam energy is plotted against bearing when one source (located at 0^{o}) and isotropic noise are present in the sound field. The array configuration is similar to that described in figure 1, the only exception being that the sensor spacing is three-eighths of a wavelength. The sensor signal-to-noise ratio is 0 dB. The results of applying the minimum energy (ME), eigenvector (EV), and MUSIC methods are shown. The time-bandwidth product here is 50; the theoretical values of $\mathcal{V}$ corresponding to this situation are $\mathcal{V}_{EV}$ = 15.7 dB and $\mathcal{V}_{MUSIC}$ = 16.4 dB.

Figure 3. Beam energy is plotted against bearing when two sources are present in the acoustic field. The conventions defined in Figure 1 apply to this plot. Here, the sensor signal-to-noise ratio of each source is 0 dB and the the sources are located at -5^{o} and $+5^{o}$.

Figure 4. Beam energy is plotted against bearing when two sources are present in the acoustic field. The conventions defined in Figure 1 apply to this plot. Here, the sensor signal-to-noise ratio of each source is 0 dB and the the sources are located at -3^{o} and $+3^{o}$.

Figure 5. Beam energy is plotted against bearing with the number p of terms truncated from the eignevector expansion of R^{-1} as a parameter. The parametric beam energy functions in each panel is plotted with the same vertical scaling. A linear array of ten equally-spaced sensors (spacing equal to three-eights of a wavelength) is present in an acoustic field. Three incoherent sources are present in the acoustic field: two have unity amplitude and are located at bearings $+5^{o}$ and -5^{o} while the third has amplitude of one-half and bearing -40^{o}. Isotropic noise is also present in the acoustic field; the sensor signal-to-noise ratio (relative to the larger signals) is 0 dB. The time-bandwidth product in both panels is 50. The upper panel displays the result of applying the eigenvector method and the bottom panel illustrates the result of applying the MUSIC method for the same set of data. Note that the proper value of p for these data is p=3.

Figure A1. Relationship of the Signal Vector and the Largest Eigenvector

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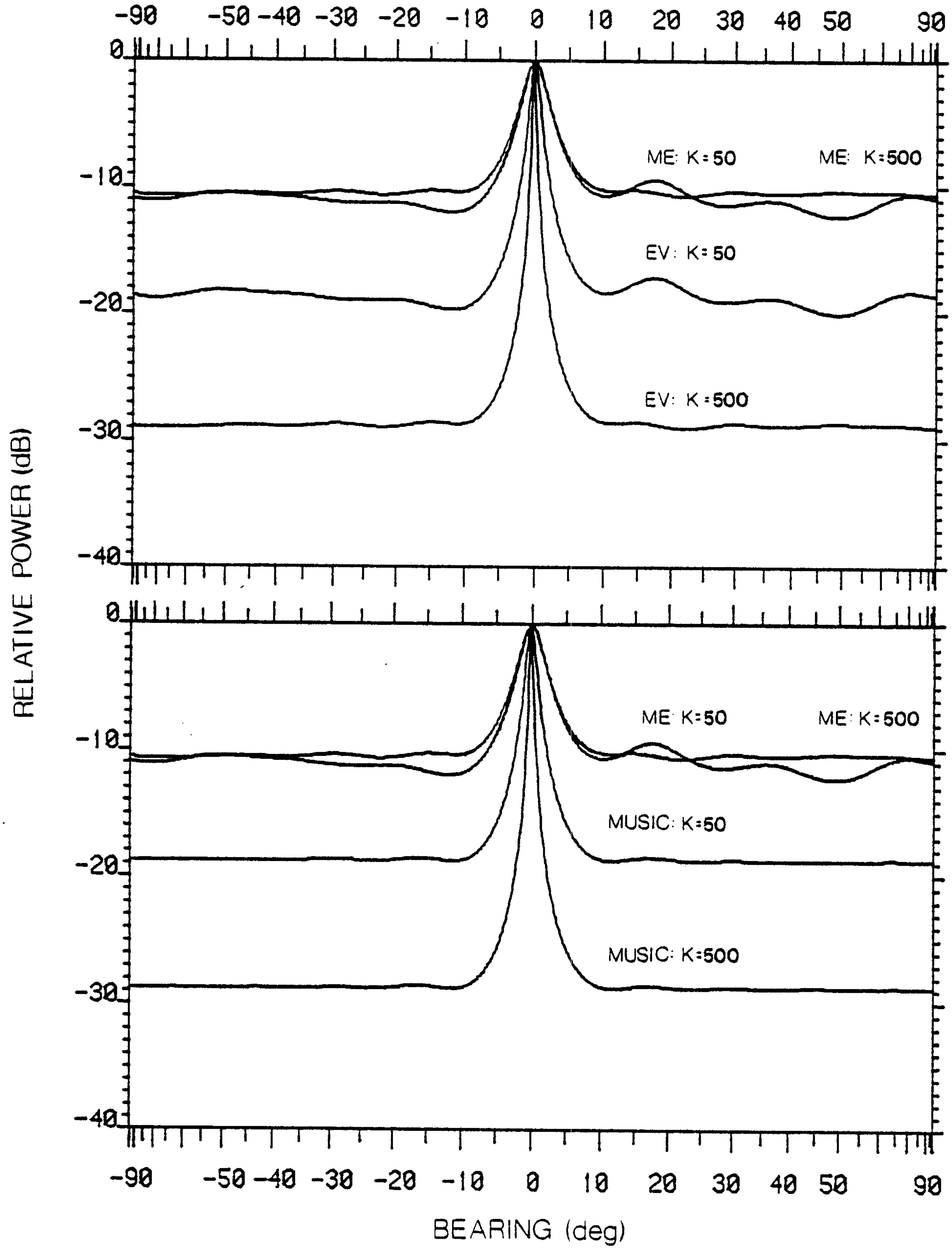
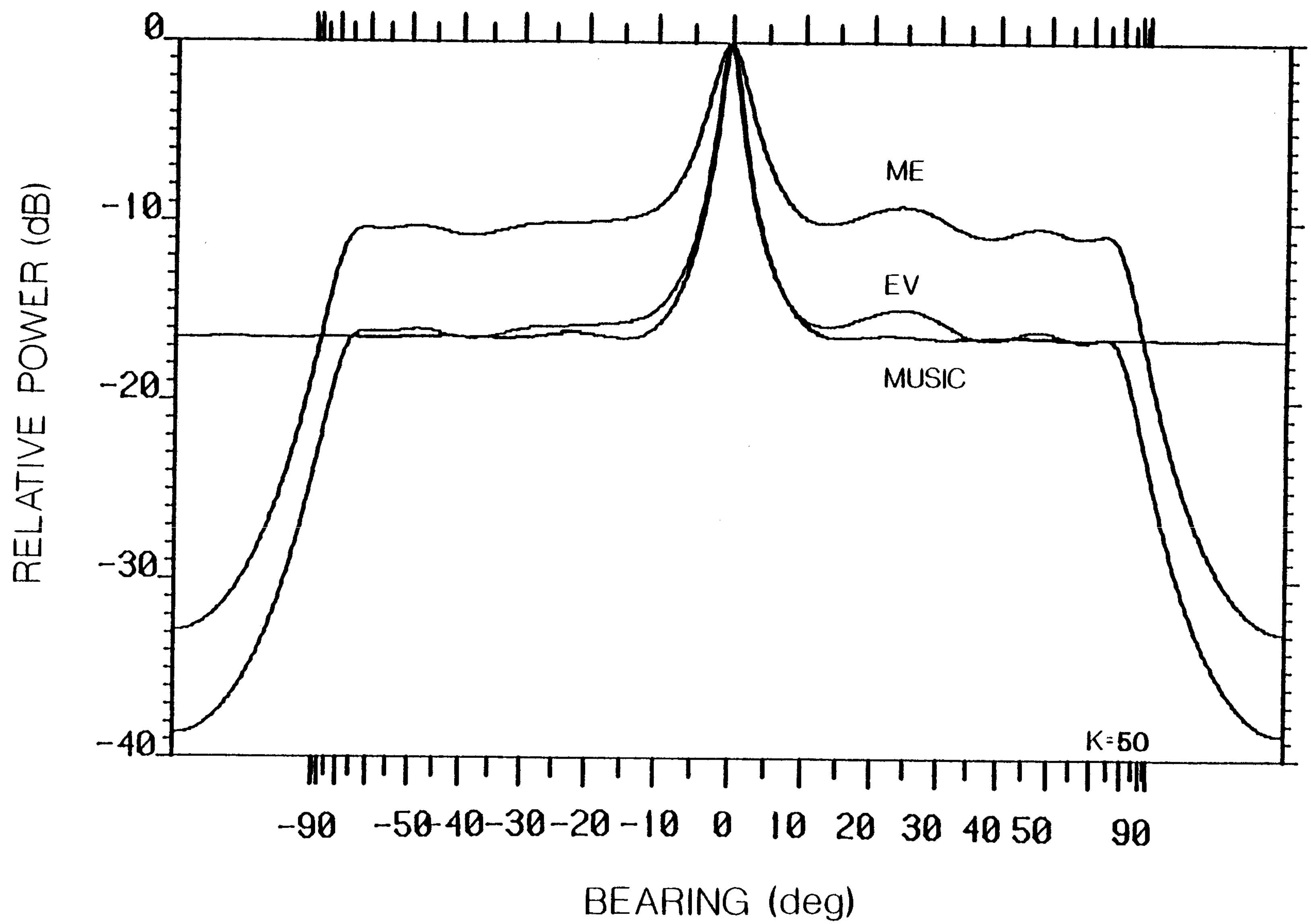
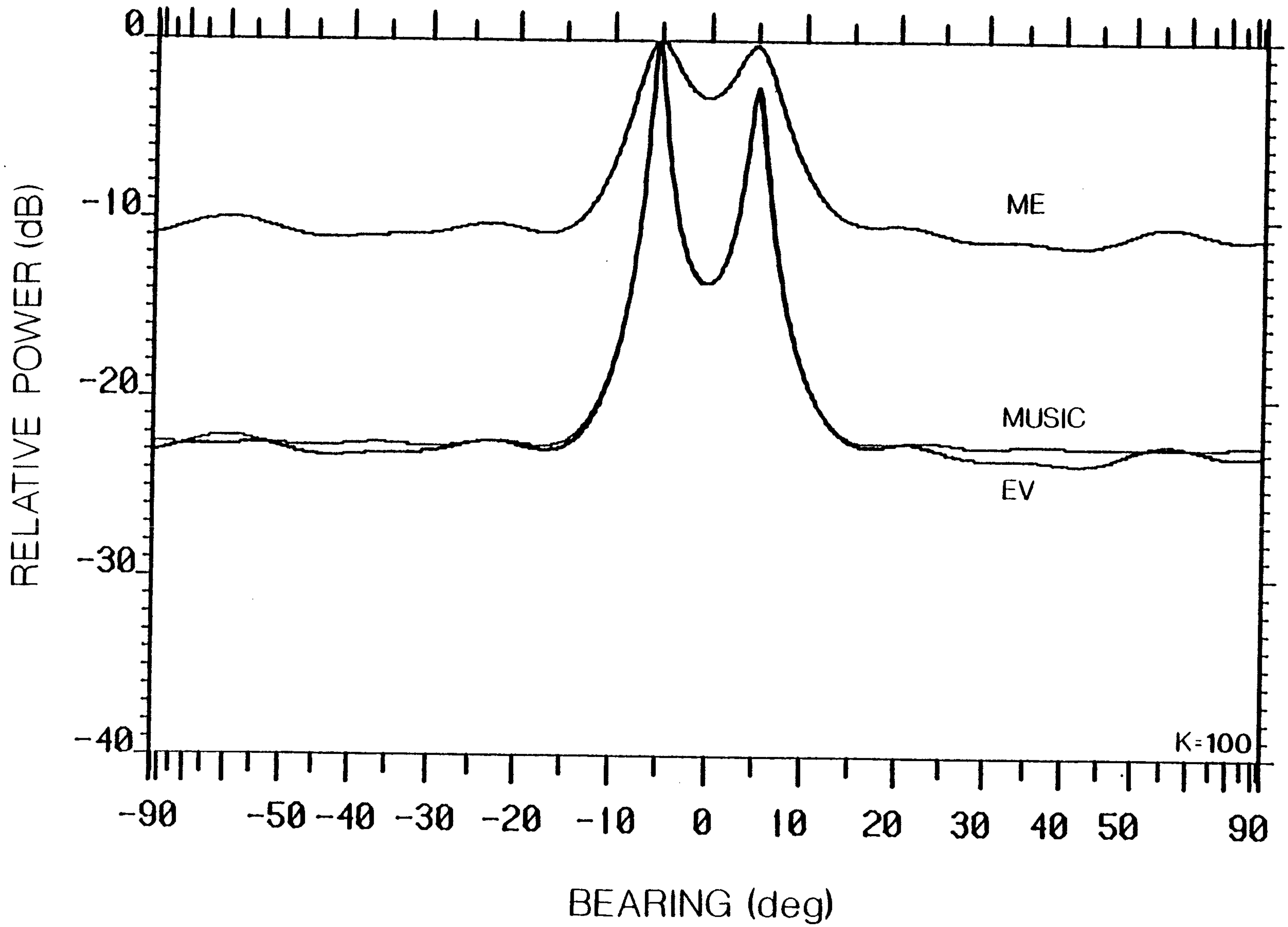


-90 -50 -40 -30 -20 -10 0 10 20 30 40 50 90
0
-10
-20
-30
-40
ME: K=50
ME: K=500
EV: K=50
EV: K=500
MUSIC: K=50
MUSIC: K=500
RELATIVE POWER (dB)
BEARING (deg)


RELATIVE POWER (dB)
0
-10
-20
-30
-40
ME
EV
MUSIC
K=50
-90
-50
-40
-30
-20
-10
0
10
20
30
40
50
90
BEARING (deg)


RELATIVE POWER (dB)
0
-10
-20
-30
-40
ME
MUSIC
EV
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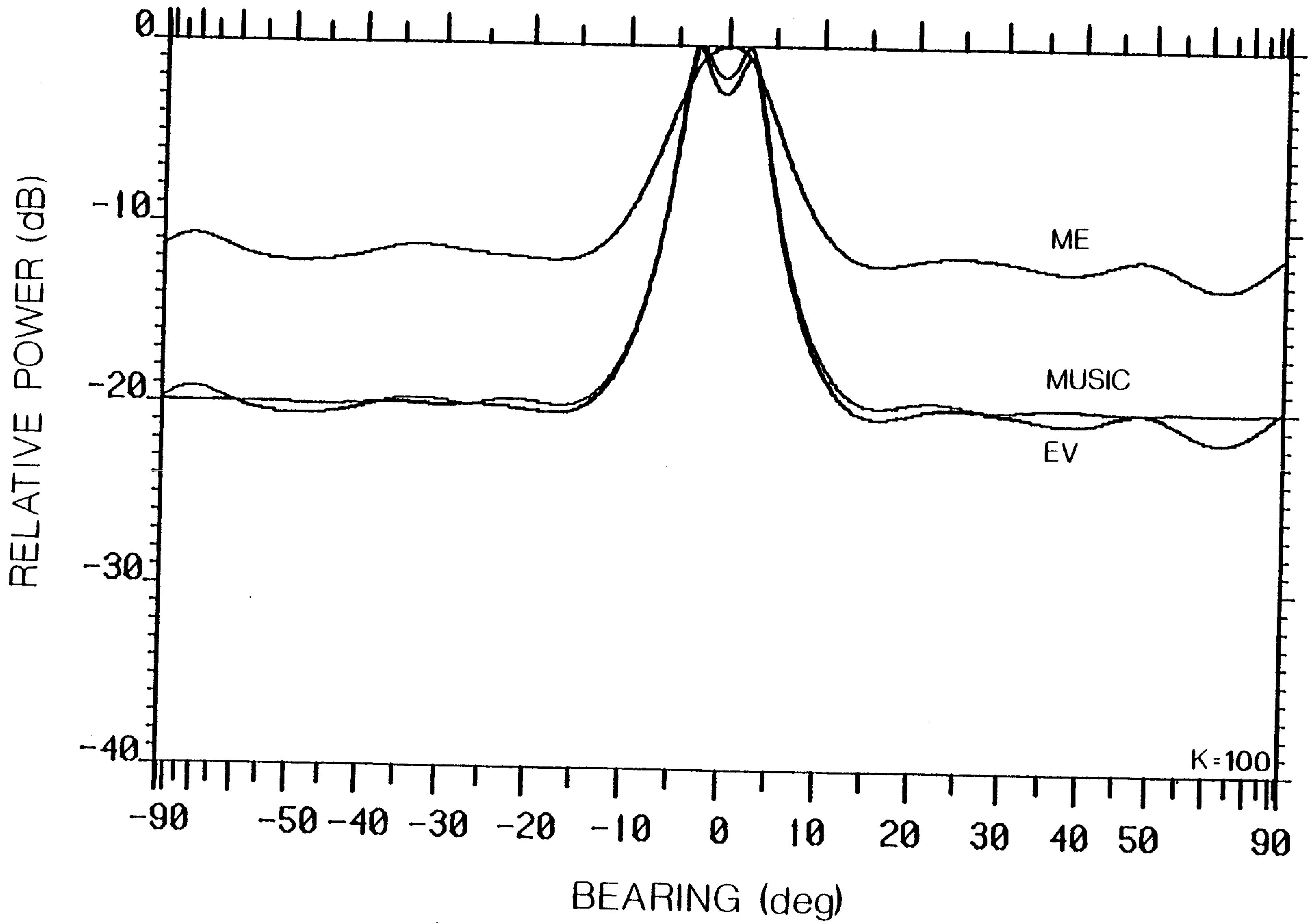
-40
-30
-20
-10
0
10
20
30
40
50
90
BEARING (deg)


RELATIVE POWER (dB)
0
-10
-20
-30
-40
ME
MUSIC
EV
K=100
-90
-50
-40
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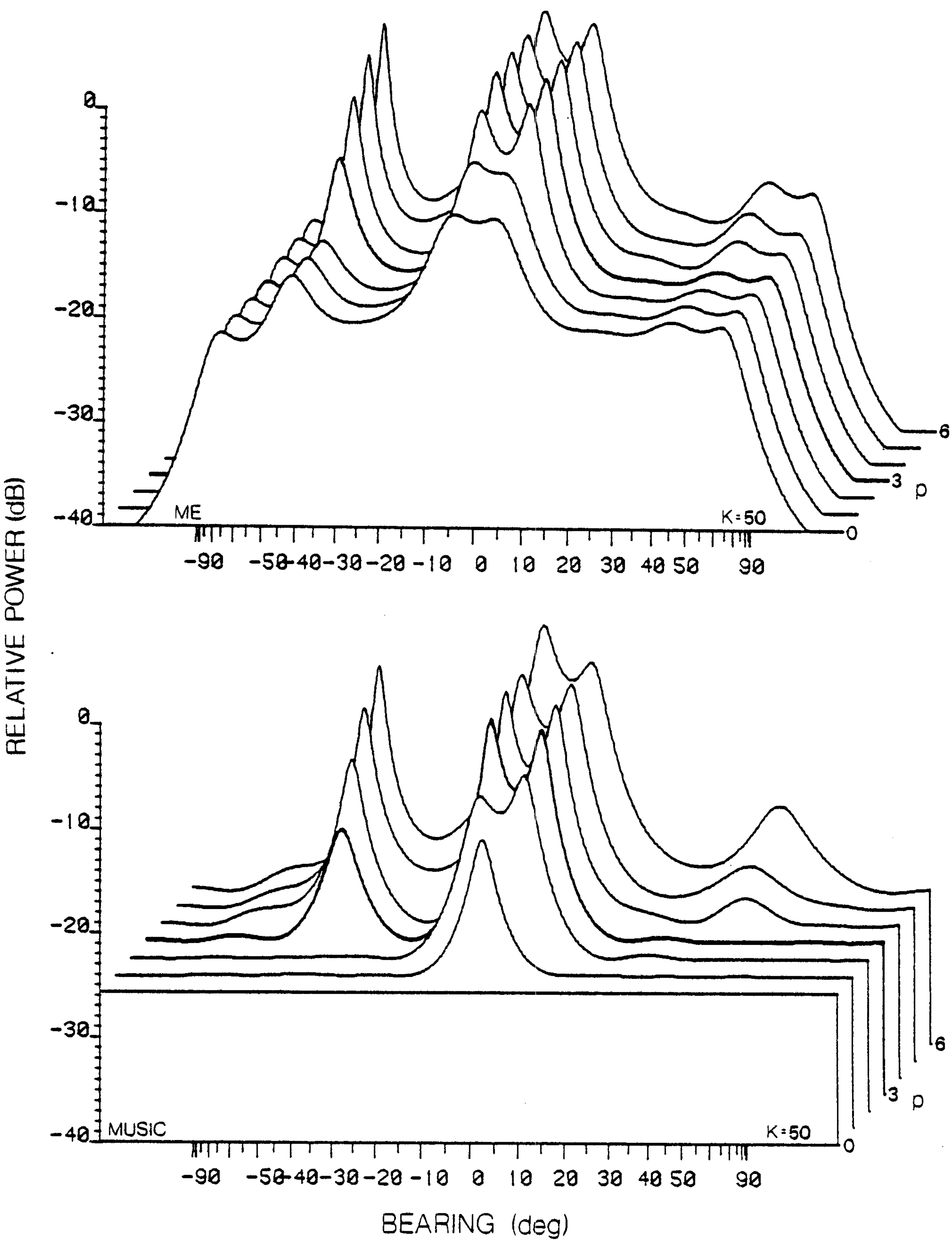
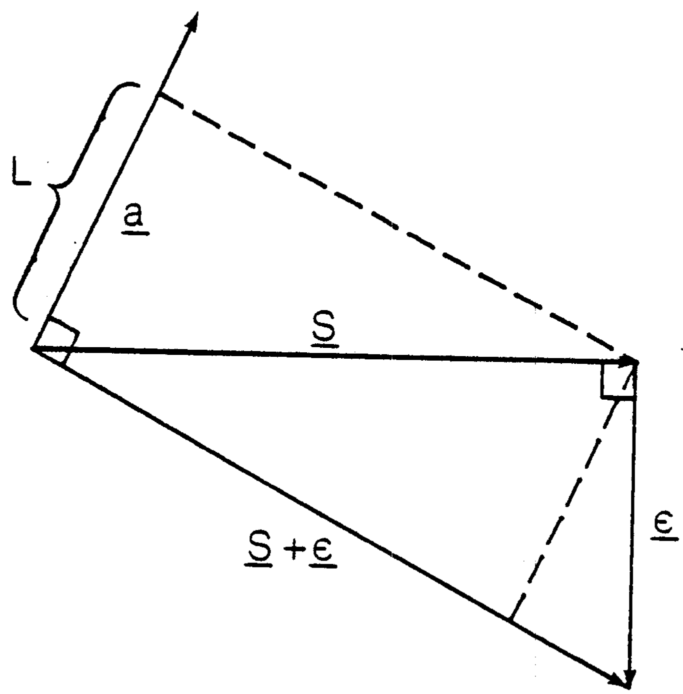
-20
-10
0
10
20
30
40
50
90
BEARING (deg)



RELATIVE POWER (dB)
0
-10
-20
-30
-40
ME
K=50
6
3 p
0
-90 -50-40-30-20-10 0 10 20 30 40 50 90
MUSIC
K=50
6
3 p
0
-90 -50-40-30-20-10 0 10 20 30 40 50 90
BEARING (deg)


L
a
S
S + ε
ε