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Research Article

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Impulsive Caputo-Fabrizio fractional differential equations in *b*-metric spaces

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Abstract: We deal with some impulsive Caputo-Fabrizio fractional differential equations in b-metric spaces. We make use of α-ϕ-Geraghty-type contraction. An illustrative example is the subject of the last section.

Keywords: fractional differential equation, Caputo-Fabrizio integral of fractional order, Caputo-Fabrizio fractional derivative, instantaneous impulse, b-metric space, α-ϕ-Geraghty contraction, fixed point

MSC 2020: 47H10, 54H25

1 Introduction and preliminaries

In the last two decades, differential equations of fractional order (fractional differential equations) take the great interest of the researchers due to wide application potential in various disciplines, see e.g. [1–9]. Indeed, differential equations subject to impulses have various applications [10–12]. Major developments are considered in the books [1,12], the papers [1,2,13–16], and references therein.

On the other hand, the fixed point theory has made serious progress in the last few decades. One of the most improvements is to show the validity of the fixed point theorem in the setting of a b-metric space that is a natural extension of standard metric space. Roughly speaking, by replacing the triangle inequality axiom of the metric notion, Czerwik [17,18] observed this new structure. Several authors reported interesting fixed point results in the framework of complete b-metric spaces, see e.g., [19–36].

In this manuscript, we shall investigate the Cauchy problem of Caputo-Fabrizio impulsive fractional differential equations

$$\begin{cases} \left({}^{\mathcal{CF}}\mathfrak{D}^r_{\vartheta_k}\omega\right)(\vartheta) = f(\vartheta, \omega(\vartheta)), \quad \vartheta \in \mathbf{I}_k,\ k = 0, \ldots, m, \\ \omega(\vartheta_k^+) = \omega(\vartheta_k^-) + L_k(\omega(\vartheta_k^-)), \quad k = 1, \ldots, m, \\ \omega(0) = \omega_0, \end{cases} \tag{1}$$

where $\mathbf{I}_0 = [0, \vartheta_1]$, $\mathbf{I}_k = (\vartheta_k, \vartheta_{k+1}]$, $k = 1, \ldots, m$, $0 = \vartheta_0 < \vartheta_1 < \cdots < \vartheta_m < \vartheta_{m+1} = T$, $\omega_0 \in \mathbb{R}$, $f : \mathbf{I}_k \times \mathbb{R} \to \mathbb{R}$, $k = 0, \ldots, m$, $L_k : \mathbb{R} \to \mathbb{R}$, $k = 1, \ldots, m$ are given continuous functions, ${}^{\mathcal{CF}}\mathfrak{D}^r_{\vartheta_k}$ is the Caputo-Fabrizio derivative of order $r \in (0, 1)$. Indeed, we aim to initiate a study of problem (1) in the framework of b-metric spaces.

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Let $\mathbf{I} := [0, T]$, $T > 0$, and the Banach space $C := \{f : \mathbf{I} \to \mathbb{R}, f \text{ continuous}\}$ with the norm

$$\|u\|_\infty = \sup_{\vartheta \in \mathbf{I}} |u(\vartheta)|.$$

$L^1(\mathbf{I}, \mathbb{R})$ denote the Banach space of measurable functions u that are Lebesgue integrable, with the norm

$$\|u\|_{L^1} = \int_0^T |u(\vartheta)| \mathrm{d}\vartheta.$$

Definition 1.1. For a function $\varphi \in L^1(\mathbf{I})$, the Caputo-Fabrizio fractional integral of order $0 < r < 1$ is

$$\left({}^{CF}I^r \varphi\right)(\vartheta) = \frac{2(1-r)}{M(r)(2-r)}\varphi(\vartheta) + \frac{2r}{M(r)(2-r)} \int_0^\vartheta \varphi(s)\mathrm{d}s, \quad \vartheta \geq 0,$$

where $M(r)$ is a normalization constant depending on r.

Analogously, for a function $\varphi \in C^1(\mathbf{I})$, the Caputo-Fabrizio fractional derivative of order $0 < r < 1$ is

$$\left({}^{CF}\mathfrak{D}^r \varphi\right)(\vartheta) = \frac{(2-r)M(r)}{2(1-r)} \int_0^\vartheta \exp\left(-\frac{r}{1-r}(\vartheta - s)\right)\varphi'(s)\mathrm{d}s, \quad \vartheta \in \mathbf{I}.$$

Note that $\left({}^{CF}\mathfrak{D}^r\right)(\varphi) = 0$ if and only if φ is a constant function.

Example 1.2. [7]

(1) For $\varphi(\vartheta) = \vartheta$ and $0 < r \leq 1$, we have

$$\left({}^{CF}\mathfrak{D}^r \varphi\right)(\vartheta) = \frac{M(r)}{r}\left(1 - \exp\left(-\frac{r}{1-r}\vartheta\right)\right).$$

(2) For $\varphi(t) = e^{\ell\vartheta}$, $\ell \geq 0$ and $0 < r \leq 1$, we have

$$\left({}^{CF}\mathfrak{D}^r \varphi\right)(\vartheta) = \frac{\ell M(r)}{\ell + (1-\ell)r} e^{\ell\vartheta}\left(1 - \exp\left(-\ell - \frac{r}{1-r}\vartheta\right)\right).$$

Lemma 1.3. *For $\psi \in L^1(\mathbf{I})$, the given linear problem*

$$\begin{cases} \left({}^{CF}\mathfrak{D}_0^r \omega\right)(\vartheta) = \psi(\vartheta), & \vartheta \in \mathbf{I}, \\ \omega(0) = \omega_0, \end{cases} \tag{2}$$

admits the following solution:

$$\omega(\vartheta) = \omega_0 - a_r\psi(0) + a_r\psi(\vartheta) + b_r \int_0^\vartheta \psi(s)\mathrm{d}s, \tag{3}$$

where

$$a_r = \frac{2(1-r)}{(2-r)M(r)}, \quad b_r = \frac{2r}{(2-r)M(r)}.$$

Proof. Let ω satisfy (2). On account of Proposition 1 in [38]; the equation

$$\left({}^{CK}\mathfrak{D}_0^r \omega\right)(\vartheta) = \psi(\vartheta)$$

implies that

$$\omega(t) - \omega(0) = a_r(\psi(\vartheta) - \psi(0)) + b_r \int_0^\vartheta \psi(s)\mathrm{d}s.$$

Taking the initial condition $\omega(0) = \omega_0$ into account, we find that

$$\omega(\vartheta) = \omega_0 - a_r\psi(0) + a_r\psi(\vartheta) + b_r\int_0^\vartheta \psi(s)\mathrm{d}s.$$

Hence, we get (3). □

We set $\mathbb{R}_0^+ := [0, \infty)$.

Definition 1.4. [22,23] For a non empty set M, and $c \geq 1$, a distance $\eth : M \times M \to \mathbb{R}_0^+$ is called b-metric if
($bM1$) $\eth(\mu, \nu) = 0$ if and only if $\mu = \nu$;
($bM2$) $\eth(\mu, \nu) = \eth(\nu, \mu)$;
($bM3$) $\eth(\mu, \xi) \leq c[\eth(\mu, \nu) + \eth(\nu, \xi)]$;
for all $\mu, \nu, \xi \in M$. The tripled $(M, \eth, c)$ is called a b-metric space.

Example 1.5. [22,23] Let $\eth : C(\mathbf{I}) \times C(\mathbf{I}) \to \mathbb{R}_0^+$ be defined by

$$\eth(\omega, \varpi) = \|(\omega - \varpi)^2\|_\infty := \sup_{\vartheta\in\mathbf{I}} \|\omega(\vartheta) - \varpi(\vartheta)\|^2, \quad \text{for all } \omega, \varpi \in C(\mathbf{I}).$$

It is clear that $\eth$ is a b-metric with $c = 2$.

Example 1.6. [22,23] Let $\mathbb{X} = [0, 1]$ and $\eth : \mathbb{X} \times \mathbb{X} \to \mathbb{R}_0^+$ be defined by

$$\eth(\omega, \varpi) = |\omega^2 - \varpi^2|, \quad \text{for all } \omega, \varpi \in \mathbb{X}.$$

Clearly, $\eth$ is not a metric, but is a b-metric space with $r \geq 2$.

We use Φ to indicate the set of all continuous and increasing function $\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that: $\phi(c\mu) \leq c\phi(\mu) \leq c\mu$, for $c > 1$ and $\phi(0) = 0$. We denote by $\mathcal{F}$ the family of all nondecreasing functions $\lambda : \mathbb{R}_+^0 \to \left[0, \frac{1}{c^2}\right)$ for some $c \geq 1$.

Definition 1.7. [22,23] For a b-metric space $(M, \eth, c)$, an operator $F : M \to M$ is called a generalized α-ϕ-Geraghty contraction-type mapping whenever there exist $\alpha : M \times M \to \mathbb{R}_0^+$ and some $L \geq 0$ such that

$$\alpha(\mu, \nu)\phi(c^3\eth(F(\mu), F(\nu)) \leq \lambda(\phi(D(\mu, \nu))\phi(D(\mu, \nu)) + L\psi(N(\mu, \nu)), \tag{4}$$

for all $\mu, \nu \in M$, where $\lambda \in \mathcal{F}$, $\phi\psi \in \Phi$, where

$$N(x, y) = \min\{\eth(x, y), \eth(x, F(x)), \eth(y, F(y))\},$$

and

$$D(x, y) = \max\left\{\eth(x, y), \eth(x, F(x)), \eth(y, F(y)), \frac{\eth(x, F(y)) + \eth(y, F(x))}{2s}\right\}.$$

Remark 1.8. In the case when $L = 0$ in Definition 1.7, and the fact that

$$\eth(x, y) \leq D(x, y),$$

for all $x, y \in M$, inequality (4) becomes

$$\alpha(\mu, \nu)\phi(c^3\eth(F(\mu), F(\nu)) \leq \lambda(\phi(\eth(\mu, \nu))\phi(\eth(\mu, \nu)). \tag{5}$$

Definition 1.9. [22,23,25] Let M be a non-empty set, $F : M \to M$ and $\alpha : M \times M \to \mathbb{R}_0^+$ be given mappings. We say that F is α-admissible if for all $\mu, \nu \in M$, we have

$$\alpha(\mu, F\mu) \geq 1 \;\Rightarrow\; \alpha(F(\mu), F^2(\mu)) \geq 1.$$

Definition 1.10. [22,23] Let $\alpha : M \times M \to \mathbb{R}_0^+$, where $(M, \mathfrak{d}, c)$ is a b-metric space. We say that M is an α-regular if for each sequence $\{x_n\}_{n\in\mathbb{N}}$ in M with $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x$ as $n \to \infty$, there exists a subsequence $\{x_{n(k)}\}_{k\in\mathbb{N}}$ of $\{x_n\}_n$ with $\alpha(x_{n(k)}, x) \geq 1$ for all k.

Theorem 1.11. [22,23,25] *Suppose a self-mapping F on complete b-metric space $(M, \mathfrak{d}, c)$ forms a generalized α-ϕ-Geraghty contraction-type mapping with the following additional assumptions*:
(*i*) *F is α-admissible*;
(*ii*) *there exists $\mu_0 \in M$ such that $\alpha(\mu_0, F(\mu_0)) \geq 1$*;
(*iii*) *either M is α-regular or F is continuous.*
Then T has a fixed point. In addition, if
(*iv*) *for all fixed points μ, ν of F, either $\alpha(\mu, \nu) \geq 1$ or $\alpha(\nu, \mu) \geq 1$,*
then F has a unique fixed point.

2 Main results

Consider the Banach space

$$\mathfrak{PC} = \{\omega : \mathbf{I} \to \mathbb{R} : \omega \in C(\mathbf{I}_k), \quad k = 0,\dots,m, \text{ and there exist } \omega(\vartheta_k^-) \text{ and } \omega(\vartheta_k^+), \quad k = 1,\dots,m, \text{ with } \omega(\vartheta_k^-) = \omega(\vartheta_k)\},$$

normed by

$$\|\omega\|_{\mathfrak{PC}} = \sup_{\vartheta\in I}|\omega(\vartheta)|.$$

Let $(\mathfrak{PC}, \mathfrak{d}, 2)$ be the complete b-metric space with $c = 2$, such that $\mathfrak{d} : \mathfrak{PC} \times \mathfrak{PC} \to \mathbb{R}_0^+$ is given by:

$$\mathfrak{d}(\omega, \varpi) = \|(\omega - \varpi)^2\|_\infty := \sup_{\vartheta\in\mathbf{I}}|\omega(\vartheta) - \varpi(\vartheta)|^2.$$

Then $(\mathfrak{PC}, \mathfrak{d}, 2)$ is a b-metric space in the sense of Definition 1.4.

Definition 2.1. By a solution of problem (1) we mean a function $\omega \in \mathfrak{PC}$ that satisfies $\omega(\vartheta_k^+) = \omega(\vartheta_k^-) + L_k(\omega(\vartheta_k^-))$, $k = 1,\dots, m$, the equation $\left({}^{CF}\mathfrak{D}_{\vartheta_k}^r\omega\right)(\vartheta) = f(\vartheta, \omega(\vartheta))$, $\vartheta \in \mathbf{I}_k$, $k = 0,\dots, m$, on $\mathbf{I}$, and the condition $\omega(0) = \omega_0$.

Lemma 2.2. *Let $h : \mathbf{I} \to \mathbb{R}$ be a continuous function. A function $\omega \in \mathfrak{PC}$ is a solution of the fractional integral equation*:

$$\begin{cases} \omega(\vartheta) = \omega_0 - a_r h(0) + a_r h(\vartheta) + b_r \int_0^\vartheta h(s)\mathrm{d}s, & \text{if } \vartheta \in \mathbf{I}_0, \\ \omega(\vartheta) = \omega_0 - a_r h(0) + \sum_{i=1}^k L_i(\omega(\vartheta_i^-)) + a_r h(\vartheta) + b_r \int_0^\vartheta h(s)\mathrm{d}s, & \text{if } \vartheta \in \mathbf{I}_k, \ k = 1,\dots, m, \end{cases} \tag{6}$$

if and only if ω is a solution of the problem

$$\begin{cases} \left({}^{CF}\mathfrak{D}_{\vartheta_k}^r\omega\right)(\vartheta) = h(\vartheta), & \vartheta \in \mathbf{I}_k, \ k = 0,\dots, m, \\ \omega(\vartheta_k^+) = \omega(\vartheta_k^-) + L_k(\omega(\vartheta_k^-)), & k = 1,\dots, m, \\ \omega(0) = \omega_0. \end{cases} \tag{7}$$

Proof. Assume u satisfies (7). If $\vartheta \in \mathbf{I}_0$, then

$$\left({}^{CF}\mathfrak{D}_0^r \omega\right)(\vartheta) = h(\vartheta).$$

Lemma 1.3 implies that

$$\omega(\vartheta) = \omega_0 - a_r h(0) + a_r h(\vartheta) + b_r \int_0^\vartheta h(s)\,\mathrm{d}s.$$

If $\vartheta \in I_1$, then

$$\left({}^{CF}\mathfrak{D}_{\vartheta_1}^r \omega\right)(\vartheta) = h(\vartheta).$$

Lemma 1.3 implies that

$$\omega(\vartheta) = \omega(\vartheta_1) - a_r h(\vartheta_1) + a_r h(\vartheta) + b_r \int_{\vartheta_1}^\vartheta h(s)\,\mathrm{d}s.$$

Thus,

$$\begin{aligned}\omega(\vartheta) &= L_1(\omega(\vartheta_1^-)) + \omega(\vartheta_1^-) - a_r h(\vartheta_1) + a_r h(\vartheta) + b_r \int_{\vartheta_1}^\vartheta h(s)\,\mathrm{d}s \\ &= L_1(\omega(\vartheta_1^-)) + \omega_0 - a_r h(0) + a_r h(\vartheta_1^-) + b_r \int_0^{\vartheta_1^-} h(s)\,\mathrm{d}s - a_r h(\vartheta_1) + a_r h(\vartheta) + b_r \int_{\vartheta_1}^\vartheta h(s)\,\mathrm{d}s \\ &= L_1(\omega(\vartheta_1^-)) + \omega_0 - a_r h(0) + a_r h(\vartheta) + b_r \int_0^\vartheta h(s)\,\mathrm{d}s.\end{aligned}$$

If $\vartheta \in \mathbf{I}_2$, then

$$\left({}^{CF}\mathfrak{D}_{\vartheta_2}^r \omega\right)(\vartheta) = h(\vartheta).$$

Then,

$$\begin{aligned}\omega(\vartheta) &= \omega(\vartheta_2) - a_r h(\vartheta_2) + a_r h(\vartheta) + b_r \int_{\vartheta_2}^\vartheta h(s)\,\mathrm{d}s \\ &= L_2(\omega(\vartheta_2^-)) + \omega(\vartheta_2^-) - a_r h(\vartheta_2) + a_r h(\vartheta) + b_r \int_{\vartheta_2}^\vartheta h(s)\,\mathrm{d}s \\ &= L_2(\omega(\vartheta_2^-)) + L_1(\omega(\vartheta_1^-)) + \omega_0 - a_r h(0) + a_r h(\vartheta_2^-) + b_r \int_0^{\vartheta_2^-} h(s)\,\mathrm{d}s - a_r h(\vartheta_2) + a_r h(\vartheta) + b_r \int_{\vartheta_2}^\vartheta h(s)\,\mathrm{d}s \\ &= L_2(\omega(\vartheta_2^-)) + L_1(\omega(\vartheta_1^-)) + \omega_0 - a_r h(0) + a_r h(\vartheta) + b_r \int_0^\vartheta h(s)\,\mathrm{d}s.\end{aligned}$$

If $\vartheta \in I_k$, we get (6).

Conversely, assume that ω satisfies (6). If $\vartheta \in \mathbf{I}_0$, then

$$\omega(\vartheta) = \omega_0 - a_r h(0) + a_r h(\vartheta) + b_r \int_0^\vartheta h(s)\,\mathrm{d}s.$$

Thus, $\omega(0) = \omega_0$ and since ${}^{CF}\mathfrak{D}_{t_k}^r$ is the left inverse of ${}^{CF}I_0^r$ we get $\left({}^{CF}\mathfrak{D}_0^r \omega\right)(\vartheta) = h(\vartheta)$.

Now, if $\vartheta \in \mathbf{I}_k$, $k = 1, \ldots, m$, we get $\left({}^{\mathcal{CF}}\mathfrak{D}^r_{\vartheta_k}\omega\right)(\vartheta) = h(\vartheta)$. Also,

$$\omega(\vartheta_k^+) = \omega(\vartheta_k^-) + L_k(\omega(\vartheta_k^-)).$$

Hence, if ω satisfies (6) then we get (7). □

If $h(\vartheta) = f(\vartheta, \omega(\vartheta))$ in Lemma 2.2, then we can conclude:

Lemma 2.3. *A function ω is a solution of problem* (1), *if and only if ω satisfies the following integral equation*:

$$\begin{cases} \omega(\vartheta) = c + a_r f(\vartheta, \omega(\vartheta)) + b_r \int_0^{\vartheta} f(s, \omega(s))\mathrm{d}s, & \text{if } \vartheta \in \mathbf{I}_0, \\ \omega(\vartheta) = c + \sum_{i=1}^{k} L_i(\omega(\vartheta_i^-)) + a_r f(\vartheta, \omega(\vartheta)) + b_r \int_0^{\vartheta} f(s, \omega(s))\mathrm{d}s, & \text{if } \vartheta \in \mathbf{I}_k,\ k = 1, \ldots, m, \end{cases} \tag{8}$$

where $c = \omega_0 - a_r f(0, \omega_0)$.

Assumptions: Here, we list the necessary assumptions to state our main theorem in a proper form.

($Ax1$) There exist $\phi \in \Phi$ and $p, q_k : \mathfrak{PC} \times \mathfrak{PC} \to \mathbb{R}_0^+$ such that for each $\omega, \varpi \in \mathfrak{PC}$;

$$|f(\vartheta, \omega) - f(\vartheta, \varpi)| \le p(\omega, \varpi)\|\omega - \varpi\|_{\mathfrak{PC}}$$

and

$$|L_k(\omega) - L_k(\varpi)| \le q(\omega, \varpi)\|\omega - \varpi\|_{\mathfrak{PC}},$$

with

$$\sum_{i=1}^{k} q_k(\omega, \varpi) + a_r p(\omega, \varpi) + b_r \left\| \int_0^{\vartheta} p(\omega, \varpi)\mathrm{d}s \right\|_{\infty}^2 \le \phi(\|(\omega - \varpi)^2\|_{\mathfrak{PC}}).$$

($Ax2$) There exist $\mu_0 \in \mathfrak{PC}$, a function $\delta : \mathfrak{PC} \times \mathfrak{PC} \to \mathbb{R}$ and $\phi \in \Phi$, such that

$$\delta\left(\mu_0(\vartheta), \sum_{i=1}^{k} L_i(\mu_0(\vartheta_i^-)) + a_r f(\vartheta, \mu_0(\vartheta)) + b_r \int_0^{\vartheta} f(s, \mu_0(s))\mathrm{d}s\right) \ge 0.$$

($Ax3$) For each $\vartheta \in \mathbf{I}$, and $\mu, \nu \in \mathfrak{PC}$, we have: $\delta(\mu(\vartheta), \nu(\vartheta)) \ge 0$ implies

$$\delta\left(a_r f(\vartheta, \mu(\vartheta)) + b_r \int_0^{\vartheta} f(s, \mu(s))\mathrm{d}s,\ a_r f(\vartheta, \nu(\vartheta)) + b_r \int_0^{\vartheta} f(s, \nu(s))\mathrm{d}s\right) \ge 0$$

and

$$\delta\left(\sum_{i=1}^{k} L_i(\mu(\vartheta_i^-)) + a_r f(\vartheta, \mu(\vartheta)) + b_r \int_0^{\vartheta} f(s, \mu(s))\mathrm{d}s,\ \sum_{i=1}^{k} L_i(\nu(\vartheta_i^-)) + a_r f(\vartheta, \nu(\vartheta)) + b_r \int_0^{\vartheta} f(s, \nu(s))\mathrm{d}s\right) \ge 0.$$

($Ax4$) $\mathfrak{PC}$ is δ-regular. That is, for every sequence $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathfrak{PC}$ with $\mu_n \to \mu$ as $n \to \infty$, there exists a subsequence $\{\mu_{n(k)}\}_{k\in\mathbb{N}}$ of $\{\mu_n\}_n$ with $\delta(\mu_{n(k)}, \mu) \ge 1$ for all k.

($Ax5$) For all fixed solutions μ, ν of problem (1), either $\delta(\mu, \nu) \ge 0$ or $\delta(\nu, \mu) \ge 0$.

Theorem 2.4. *Under assumptions* (Ax_1)–(Ax_4), *problem* (1) *has at least one solution defined on* $\mathbf{I}$. *Moreover, if* (Ax_5) *holds, then* (1) *has a unique solution.*

Proof. Let the operator $\Upsilon : \mathfrak{PC} \to \mathfrak{PC}$ be defined by

$$\begin{cases} (\Upsilon\omega)(\vartheta) = c + a_r f(\vartheta, \omega(\vartheta)) + b_r \int_0^\vartheta f(s, \omega(s))\mathrm{d}s, & \text{if } \vartheta \in \mathbf{I}_0, \\ (\Upsilon\omega)(\vartheta) = c + \sum_{i=1}^{k} L_i(\omega(\vartheta_i^-)) + a_r f(\vartheta, \omega(\vartheta)) + b_r \int_0^\vartheta f(s, \omega(s))\mathrm{d}s, & \text{if } \vartheta \in \mathbf{I}_k,\ k = 1, \dots, m. \end{cases} \tag{9}$$

Let $\alpha : \mathfrak{PC} \times \mathfrak{PC} \to \mathbb{R}_0^+$ be the function defined by:

$$\begin{cases} \alpha(\omega, \varpi) = 1, & \text{if } \delta(\omega(\vartheta), \varpi(\vartheta)) \geq 0, \quad \vartheta \in \mathbf{I}, \\ \alpha(\omega, \varpi) = 0, & \text{else.} \end{cases}$$

Our following result is based on Theorem 1.11.

First, we prove that Υ *is a generalized* α-ϕ-*Geraghty contraction operator*:
Let $\omega, \varpi \in \mathfrak{PC}$. For each $\vartheta \in \mathbf{I}_0$, we have

$$\begin{aligned} |(\Upsilon\omega)(\vartheta) - (\Upsilon\varpi)(\vartheta)| &\leq a_r |f(\vartheta, \omega(\vartheta)) - f(\vartheta, \varpi(\vartheta))| + b_r \int_0^\vartheta |f(s, \omega(s)) - f(s, \varpi(s))|\mathrm{d}s \\ &\leq a_r p(\omega, \varpi)(\|\omega(\vartheta) - \varpi(\vartheta)\|^2)^{\frac{1}{2}} + b_r \int_0^\vartheta p(\omega, \varpi)(\|\omega(s) - \varpi(s)\|^2)^{\frac{1}{2}}\mathrm{d}s \\ &\leq a_r p(\omega, \varpi)(\|(\omega - \varpi)^2\|_{\mathfrak{PC}})^{\frac{1}{2}} + b_r \int_0^t p(\omega, \varpi)(\|(\omega - \varpi)^2\|_{\mathfrak{PC}})^{\frac{1}{2}}\mathrm{d}s. \end{aligned}$$

Thus, from (Ax_1) we get

$$\begin{aligned} \alpha(\omega, \varpi)|(\Upsilon\omega)(\vartheta) - (\Upsilon\varpi)(\vartheta)|^2 &\leq a_r \alpha(\omega, \varpi)\|(\omega - \varpi)^2\|_{\mathfrak{PC}} p(\omega, \varpi) + b_r \alpha(\omega, \varpi)\|(\omega - \varpi)^2\|_\infty \left\| \int_0^\vartheta p(\omega, \varpi) d_q s \right\|_{\mathfrak{PC}}^2 \\ &\leq \|(\omega - \varpi)^2\|_{\mathfrak{PC}} \phi(\|(\omega - \varpi)^2\|_{\mathfrak{PC}}). \end{aligned}$$

This gives

$$\alpha(\omega, \varpi)\phi(2^3\eth(\Upsilon(\omega), \Upsilon(\varpi)) \leq \lambda(\phi(\eth(\omega, \varpi))\phi(\eth(\omega, \varpi)) \leq \lambda(\phi(D(\omega, \varpi))\phi(D(\omega, \varpi)) + L\psi(\Upsilon(\omega, \varpi)), \tag{10}$$

where $\lambda \in \mathcal{F}$, $\phi, \psi \in \Phi$ with $\lambda(\vartheta) = \frac{1}{8}\vartheta$, $L = 0$ and $\phi(\vartheta) = \psi(\vartheta) = \vartheta$.

On the other hand, for each $\vartheta \in \mathbf{I}_k :\ k = 1, \dots, m$, we have

$$\begin{aligned} |(\Upsilon\omega)(\vartheta) - (\Upsilon\varpi)(\vartheta)| &\leq \sum_{i=1}^{k} |L_i(\omega(\vartheta_i^-)) - L_i(\varpi(\vartheta_i^-))| + a_r |f(\vartheta, \omega(\vartheta)) - f(\vartheta, \varpi(\vartheta))| + b_r \int_0^\vartheta |f(s, \omega(s)) - f(s, \varpi(s))|\mathrm{d}s \\ &\leq \sum_{i=1}^{k} q_i(\omega, \varpi)(\|\omega(\vartheta) - \varpi(\vartheta)\|^2)^{\frac{1}{2}} + a_r p(\omega, \varpi)(\|\omega(\vartheta) - \varpi(\vartheta)\|^2)^{\frac{1}{2}} + b_r \int_0^\vartheta p(\omega, \varpi)(\|\omega(s) - \varpi(s)\|^2)^{\frac{1}{2}}\mathrm{d}s \\ &\leq \sum_{i=1}^{k} q_i(\omega, \varpi)(\|(\omega - \varpi)^2\|_{\mathfrak{PC}})^{\frac{1}{2}} + a_r p(\omega, \varpi)(\|(\omega - \varpi)^2\|_{\mathfrak{PC}})^{\frac{1}{2}} + b_r \int_0^t p(\omega, \varpi)(\|(\omega - \varpi)^2\|_{\mathfrak{PC}})^{\frac{1}{2}}\mathrm{d}s. \end{aligned}$$

Hence, we obtain (10). So, Υ is generalized α-ϕ-Geraghty contraction.

Next, we verify that Υ *is* α*-admissible*:
Let $\omega, \varpi \in \mathfrak{PC}$ such that $\alpha(\omega, \varpi) \geq 1$. For each $\vartheta \in \mathbf{I}_0$, we have

$$\delta(\omega(\vartheta), \varpi(\vartheta)) \geq 0.$$

This implies from (Ax_3) that $\delta(\Upsilon\omega(\vartheta), \Upsilon\varpi(\vartheta)) \geq 0$, which gives $\alpha(\Upsilon(\omega), \Upsilon(\varpi)) \geq 1$. Hence, Υ is α-admissible.

Now, from (Ax_2), there exists $\mu_0 \in C(\mathbf{I})$ such that $\alpha(\mu_0, \Upsilon(\mu_0)) \geq 1$.

Finally, from (Ax_4), $\mathfrak{PC}$ is α-regular. Indeed, for a given sequence $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathfrak{PC}$ with $\mu_n \to \mu$ as $n \to \infty$, there exists a subsequence $\{\mu_{n(k)}\}_{k\in\mathbb{N}}$ of $\{\mu_n\}_n$ with $\delta(\mu_{n(k)}, \mu) \geq 1$ for all k. This gives $\alpha(\mu_{n(k)}, \mu) \geq 1$ for all k.

Applying now Theorem 1.11, we conclude that Υ has at least one fixed point, which is a solution of problem (1). Moreover, (Ax_5), implies that if ω and ϖ are fixed points of Υ, then either $\delta(\omega, \varpi) \geq 0$ or $\delta(\varpi, \omega) \geq 0$. Thus, we obtain that either $\alpha(\omega, \varpi) \geq 1$ or $\alpha(\varpi, \omega) \geq 1$. Hence, problem (1) has a unique solution. □

3 An example

Let the impulsive Caputo-Fabrizio fractional differential equation

$$\begin{cases} ({}^{CF}\mathfrak{D}^r_{\vartheta_k}\omega)(\vartheta) = f(\vartheta, \omega(\vartheta)), & \vartheta \in \mathbf{I}_k, \ k = 0, \ldots, m, \\ \omega(\vartheta_k^+) = \omega(\vartheta_k^-) + L_k(\omega(\vartheta_k^-)), & k = 1, \ldots, m, \\ \omega(0) = 0, \end{cases} \tag{11}$$

where $\mathbf{I} = [0, 1]$, $r \in (0, 1)$,

$$f(\vartheta, \omega(\vartheta)) = \frac{1 + \sin(|\omega(\vartheta)|)}{4(1 + |\omega(\vartheta)|)}, \quad \vartheta \in (0, 1],$$

and

$$L_k(\omega(t_k^-)) = \frac{1 + |\omega(\vartheta_k^-)|}{3e^5}, \quad k = 1, \ldots, m.$$

Let $(\mathfrak{PC}([0, 1]), \mathfrak{d}, 2)$ be the complete b-metric space, such that $\mathfrak{d} : \mathfrak{PC}([0, 1]) \times \mathfrak{PC}([0, 1]) \to \mathbb{R}_0^+$ is given by:

$$\mathfrak{d}(\omega, \varpi) = \|(\omega - \varpi)^2\|_\infty := \sup_{\vartheta\in[0,1]} |\omega(\vartheta) - \varpi(\vartheta)|^2.$$

For each $\omega, \varpi \in \mathfrak{PC}([0, 1])$, we have

$$|L_k(\omega(\vartheta_k^-)) - L_k(\varpi(\vartheta_k^-))| = \frac{|\omega(\vartheta_k^-) - \varpi(\vartheta_k^-)|}{3e^5}, \quad k = 1, \ldots, m.$$

Let $\vartheta \in (0, 1]$, and $\omega, \varpi \in \mathfrak{PC}([0, 1])$. If $|\omega(\vartheta)| \leq |\varpi(\vartheta)|$, then

$$\begin{aligned} |f(\vartheta, \omega(\vartheta)) - f(\vartheta, \varpi(\vartheta))| &= \left| \frac{1 + \sin(|\omega(\vartheta)|)}{4(1 + |\omega(\vartheta)|)} - \frac{1 + \sin(|\varpi(\vartheta)|)}{4(1 + |\varpi(\vartheta)|)} \right| \\ &\leq \frac{1}{4}||\omega(\vartheta)| - |\varpi(\vartheta)|| + \frac{1}{4}|\sin(|\omega(\vartheta)|) - \sin(|\varpi(\vartheta)|)| \\ &\quad + ||\omega(\vartheta)|\sin(|\varpi(\vartheta)|) - |\varpi(\vartheta)|\sin(|\omega(\vartheta)|)| \\ &\leq |\omega(\vartheta) - \varpi(\vartheta)| + \frac{1}{4}|\sin(|\omega(\vartheta)|) - \sin(|\varpi(\vartheta)|)| \\ &\quad + ||\varpi(\vartheta)|\sin(|\varpi(\vartheta)|) - |\varpi(\vartheta)|\sin(|\omega(\vartheta)|)| \end{aligned}$$

$$
\begin{aligned}
&= |\omega(\vartheta) - \varpi(\vartheta)| + (1 + |\varpi(\vartheta)|)|\sin(|\omega(\vartheta)|) - \sin(|\varpi(\vartheta)|)| \\
&\leq |\omega(\vartheta) - \varpi(\vartheta)| + \frac{1}{2}(1 + |\varpi(\vartheta)|) \\
&\quad \times \left|\sin\left(\frac{||\omega(\vartheta)|-|\varpi(\vartheta)||}{2}\right)\right|\left|\cos\left(\frac{|\omega(\vartheta)|+|\varpi(\vartheta)||}{2}\right)\right| \\
&\leq (2 + \|\varpi\|_{\mathfrak{PC}})\|\omega - \varpi\|_{\mathfrak{PC}}.
\end{aligned}
$$

The case when $|\varpi(\vartheta)| \leq |\omega(\vartheta)|$, we get

$$|f(\vartheta, \omega(\vartheta)) - f(\vartheta, \varpi(\vartheta))| \leq (2 + \|\omega\|_{\mathfrak{PC}})\|\omega - \varpi\|_{\mathfrak{PC}}.$$

Hence,

$$|f(\vartheta, \omega(\vartheta)) - f(\vartheta, \varpi(\vartheta))| \leq \min\{2 + \|\omega\|_{\mathfrak{PC}}, 2 + \|\varpi\|_{\mathfrak{PC}}\}\|\omega - \varpi\|_{\mathfrak{PC}}.$$

Thus, hypothesis (Ax_1) is satisfied with

$$p(\omega, \varpi) = \min\{2 + \|\omega\|_{\mathfrak{PC}}, 2 + \|\varpi\|_{\mathfrak{PC}}\}$$

and

$$q_k(\omega, \varpi) = \frac{1}{3e^5}.$$

Define the functions $\lambda(\vartheta) = \frac{1}{8}\vartheta$, $\phi(\vartheta) = \vartheta$, $\alpha : \mathfrak{PC}([0,1]) \times \mathfrak{PC}([0,1]) \to \mathbb{R}_0^+$ with

$$
\begin{cases}
\alpha(\omega, \varpi) = 1, & \text{if } \delta(\omega(\vartheta), \varpi(\vartheta)) \geq 0, \ \vartheta \in I, \\
\alpha(\omega, \varpi) = 0, & \text{else,}
\end{cases}
$$

and $\delta : \mathfrak{PC}([0,1]) \times \mathfrak{PC}([0,1]) \to \mathbb{R}$ with $\delta(\omega, \varpi) = \|\omega\|_{\mathfrak{PC}} - \|\varpi\|_{\mathfrak{PC}}$.

Hypothesis (Ax_2) is satisfied with $\mu_0(\vartheta) = 0$. Also, (Ax_3) holds from the definition of the function δ. Hence, there exists at least one solution of (11).

Moreover, (Ax_5) is satisfied. Indeed, if ω and ϖ are solutions of (11), then either $\delta(\omega, \varpi) \geq 0$ or $\delta(\varpi, \omega) \geq 0$. This implies that either $\alpha(\omega, \varpi) \geq 1$ or $\alpha(\varpi, \omega) \geq 1$. Consequently, problem (11) has a unique solution.

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