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IN-PLANE NONLINEAR DYNAMICS OF WIND TURBINE BLADES

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ABSTRACT

The partial differential equation that governs the in-plane motion of a wind turbine blade subject to gravitational loading and which accommodates for aerodynamic loading is developed using the extended Hamilton principle. This partial differential equation includes nonlinear terms due to nonlinear curvature and nonlinear foreshortening, as well as parametric and direct excitation at the frequency of rotation. The equation is reduced using an assumed cantilevered beam mode to produce a single second-order ordinary differential equation (ODE) as an approximation for the case of constant rotation rate. Embedded in this ODE are terms of a nonlinear forced Mathieu equation. The forced Mathieu equation is analyzed for resonances by using the method of multiple scales. Superharmonic and subharmonic resonances occur. The effect of various parameters on the response of the system is demonstrated using the amplitude-frequency curve. A superharmonic resonance persists for the linear system as well.

INTRODUCTION

This paper introduces a model of the lead-lag (in-plane) vibration motion of an operating wind turbine subjected to gravitational loading and aerodynamic loading, and provides initial analyses of resonances by using a simplification of the single-mode reduced-order model.

The reliability of wind turbines is a major issue for the industry. Drivetrain and blade failures are common, costly and not fully understood. Designers must thus examine and understand the key parameters that influence reliability. As wind turbines increase in size, the blades are designed to be more lightweight and flexible, increasing the potential for large-displacement oscillations during operation. This necessitates the incorporation of nonlinearity in the formulation of the model to completely understand the dynamics and stability characteristics. Also, oscillations in the blade impart dynamic loading onto the gearbox. Understanding these dynamic loads is essential for the design of reliable gears and bearings, and hence economically viable wind turbines. Traditional studies of wind turbines have focused on the aerodynamic performance of the blades, the reliability of gearbox components and grid failures and improvements in power distribution. The aspect of blade vibration from a dynamics point of view has garnered interest but not been fully developed and understood.

There is a large body of work on modeling of rotating blades under aerodynamic and gravitational loading. Wendell [1] developed the partial differential equations of motion for a rotating wind turbine blade. Additional efforts have been made by to include gravity and pitch action by Kallosóe [2] . Caruntu [3] has developed nonlinear equations that model the flexural potential in non-uniform beam that can be extended to large flexible blades of wind turbines. Chopra et al. [4] have modeled the blades as a hinged structure and constructed equations of motion to study

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its dynamic behavior. Jonkman [5, 6] used the modal information of blade vibration to calculate the tip displacement in blade, the torques and moments experienced at the hub amongst other things. These calculation have been used to develop the code FAST that is used in research and industry as a basis for performance evaluations.

Extensive literature also exists on the analysis the related problem of the vibration of helicopter blades [7–10]. Hodges and Dowell [11] also developed the nonlinear partial differential equations for a twisted helicopter rotor blade. These rotate at much higher rotational velocities than wind turbine blades, and in a horizontal plane, unlike the wind turbine blades. Hence the gravitational influence on the vibration characteristics of the wind turbine blade would be much different. However, many features of helicopter blade dynamics, such as modeling of centrifugal effects and deflected geometry, carry over to the wind turbine blade models.

In this work we will focus on the role of cyclic gravitational loading on the in-plane vibration of an isolated blade. We also include nonlinearity associated with large deflections (nonlinear curvature and nonlinear foreshortening). Accounting for nonlinearity and cyclic gravitational loading exposes the presence of resonances other than one-to-one resonance of the structural modes.

Modeling the in-plane blade motion in its full depth, i.e. including all the nonlinearities and gravitational loading, leads to a nonlinear parametrically excited equation. Our goal is to simplify the system of equations and look for analytical descriptions of representative phenomena. For this we apply reduced-order modeling to obtain analytical expressions of response characteristics as functions of parameters. We make assumed modal reductions using the cantilevered beam modes to obtain a single mode model of blade motion in the lead-lag direction. The resulting second order differential equation is then simplified to a parametrically and directly excited Mathieu/Duffing equation. This simplified equation thus becomes the focus of our analysis. The simplifications can be relaxed in future work where the analysis will be more numerical.

There have been extensive studies on systems with parametric excitation that fit in into a minor variation of the Mathieu equation. Most of the work involving both direct and parametric excitation has the parametric excitation at twice the frequency as the direct excitation. Rhoads and Shaw [12] have studied the influence of parametric resonance in MEMS structures. They have identified systems with direct and parametric excitation and studied their behavior. Experimental results that demonstrate exploitation of parametric amplification [13]. Other work has been on forced and unforced parametrically excited systems with van der Pol, Rayleigh and Duffing nonlinearities. Rand [14–17] in collaboration with others have analyzed the dynamics and bifucations of a forced Mathieu equation and properties of superharmonic resonances at 2:1 and 4:1. Belhaq [18] has studied quasi-periodicity in systems with parametric and external excitation. Veerman [19] has done analysis on the dynamic response of the van der Pol Mathieu equation. Arrowsmith, Marathe [20, 21] have studied at the stability region for the Mathieu equation. Reference [22] is an extensive compilation of the various fields of study in which the characteristics of Mathieu equation are found and employed.

Using the method of multiple scales (MMS) [23–26], we analyze resonances of the forced nonlinear Mathieu equation, as an approximate representative of blade motion. The analysis reveals the existence of sub-harmonic and super-harmonic resonances. We unfold the super- and sub-harmonic resonance cases present in the system and identify the critical ones which a wind turbines may be close to during normal operation.

1 Equation of Motion

We formulate the equations of motion of in-plane deflection of a beam attached to a hub which rotates at a constant rate. The equation of motion of the rotating beam is obtained by applying the extended Hamilton's principle [23]. The energy formulation for an inextensible beam with large deflections, rotating about a horizontal axis with flexure is discussed in this section.

The gravitational potential energy of the rotating beam is integrated over the beam elements with the help of Figure 1. The height $h(x,t)$ of an element dx, located at a point x of the beam, rotated by an angle ϕ, is given by the angular inclination $x(1-\cos\phi)$ plus the inclined foreshortening $s(y,x,t)$, plus the inclined transverse beam displacement, $y(x,t)$, such that $h(x,t) = x(1-\cos\phi) + s(x,t)\cos\phi + y(x,t)\sin\phi$, where $s(y,x,t) = \int_0^x \left(\frac{y'^2}{2} + \frac{y'^4}{8}\right) dx$ and the primes indicate partial derivatives with respect to x. Then the gravitational potential energy is

$$\begin{aligned} V_g &= \int_0^L m(x) g h(x,t) dx \\ &= \int_0^L m(x) g [x(1-\cos\phi) + s(x,t)\cos\phi + y(x,t)\sin\phi] dx \end{aligned} \quad (1)$$

where L is the length and $m(x)$ is the mass per unit length.

If the beam is rotating at a fixed angular speed Ω, then $\phi = \Omega t$. In a wind turbine, the speed will naturally be a slowly changing dynamic variable, effected by the aerodynamic forces on the blades, and the moment on the hub including such effects as generator dynamics, gear box dynamics, friction and control. In our resonance analyses of this work, we will start with a fixed angular speed.

The flexural potential energy will include geometric nonlinearity, for example following Caruntu [4]. The nonlinear curvature is given as $k = \frac{y''}{(1+y'^2)^{3/2}}$. The square of the curvature is

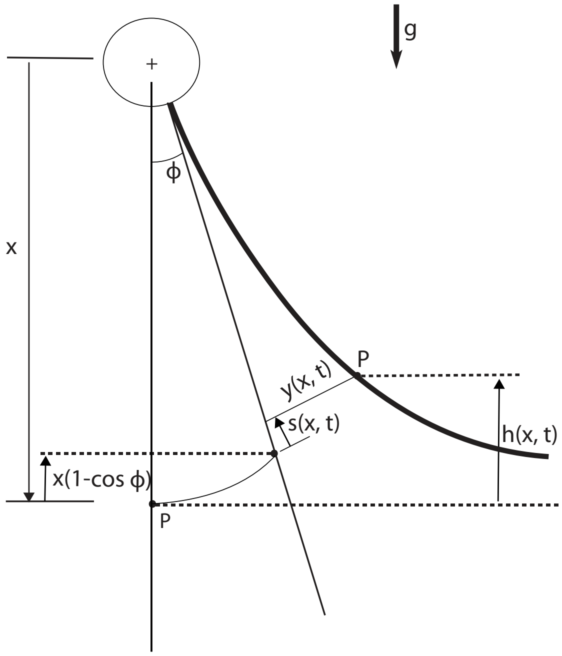

FIGURE 1. THE ARBITRARY CONFIGURATION OF THE POSITION OF A POINT P ON THE ROTATED AND DEFLECTED BEAM. THE POINT P CORRESPONDS TO LOCATION X ON THE UNDEFLECTED BEAM.

expanded into the form $k^2 \approx y''^2(1-3y'^2)$, and inserted into the bending potential energy as

$$V_b = \int_0^L \frac{1}{2}EI(x)k^2 dx \approx \int_0^L \frac{1}{2}EI(x)y''^2(1-3y'^2)dx, \quad (2)$$

where E is the Young's modulus and I is the area moment of inertia of a cross-section of the beam.

The kinetic energy of the beam is formulated from the velocity of each element. The position of an element is $\mathbf{r} = (x-s(x,t))\mathbf{e}_r + y(x,t)\mathbf{e}_\phi$ in terms of the unit vectors in the direction of the undeformed, rotated beam ($\mathbf{e}_r$) and transverse in the ϕ direction ($\mathbf{e}_\phi$). Using $\dot{\mathbf{e}}_r = \dot{\phi}\mathbf{e}_\phi$ and $\dot{\mathbf{e}}_\phi = -\dot{\phi}\mathbf{e}_r$, and considering the case where $\dot{\phi} = \Omega$, the velocity is $\mathbf{v} = (-\dot{s} - \Omega y)\mathbf{e}_r + [\Omega(x-s)+\dot{y}]\mathbf{e}_\phi$. The contribution to the integral of the rotational energy density, $\frac{1}{2}J(x)(y'+\dot{\phi})^2$, where $J(x)$ is the mass moment of inertia per unit length, can also be added. Then

$$T = \int_0^L \frac{1}{2}m(x)\mathbf{v}\cdot\mathbf{v}dx + \int_0^L \frac{1}{2}J(x)(y'+\dot{\phi})^2 dx \quad (3)$$

is the total kinetic energy.

The extended Hamilton's principle is now applied, such that

$$\int_{t_1}^{t_2} (\delta T - \delta V + \delta W)dt = 0 \quad (4)$$

under the constraint that the varied path coincides with the true path at $t = t_1$ and $t = t_2$. Constructing δT, δV, and δW, and integrating by parts to obtain a common δy term in the integrands, yields the partial differential equation and boundary conditions.

We assume the $x = 0$ boundary to be clamped, and thus impose $y(0,t) = y'(0,t) = 0$ as geometric boundary conditions.

To this end, we obtain an integral-partial differential equation (IPDE) of motion and boundary conditions. For the case when $\phi = \Omega t$,

$$\begin{aligned} &-m(a(\ddot{s},s,\dot{y},x,t) + g\cos\Omega t)(y' + y'^3/2) \\ &+\textstyle\int_x^L m(a(\ddot{s},s,\dot{y},z,t) + g\cos\Omega t)dz(y'+y'^3/2)' \\ &+mb(\ddot{y},y,\dot{s},x,t) + (J(x)\ddot{y}')' - (EIy'' - 3EIy''y'^2)'' - \\ &(3EIy''^2y')' + f(y,\dot{y},x,t) = mg\sin\Omega t, \end{aligned} \quad (5)$$

where the dots are partial derivatives with respect to time. with boundary conditions $y(0,t) = y'(0,t) = 0$ at $x = 0$ and

$$-J(L)\ddot{y}' + (EIy'' - 3EIy''y'^2 + 3EIy''^2y') = 0 \quad (6)$$

$$EIy'' - 3EIy''y'^2 = 0 \quad (7)$$

at $x = L$, where $a(\ddot{s},s,\dot{y},x,t) = \ddot{s} + 2\Omega\dot{y} + \Omega^2(x-s)$, $b(\ddot{y},y,\dot{s},x,t) = -\ddot{y} + \Omega^2 y + 2\Omega\dot{s}$, and $f(y,\dot{y},x,t)$ represents distributed aeroelastic loads.

Interpreting these equations of motion, the IPDE has nonlinear inertial terms due to the geometric nonlinear term $m(a(\ddot{s},s,\dot{y},x,t) + g\cos\Omega t)(y'+y'^3/2)$, along with linear inertial terms, including rectilinear, centripetal, and Coriolis effects. The IPDE has parametric excitation, with the $\cos\Omega t$ terms, as well as direct excitation $mg\sin\Omega t$ at the same frequency. These come from the gravitational potential energy, with parametric excitation involving large-deflection nonlinearity. The aerodynamic force f has cyclic components and will contribute to direct, and possibly parametric, excitation terms as well. The EI terms are nonlinear curvature terms from large deflection. The aeroelastic $f(y,\dot{y},x,t)$ terms may have further dependence on s and $\dot{s}$, and may produce additional direct or parametric excitation terms. An aeroelastic formulation may also be accompanied by the appropriate fluid mechanics equation, coupled through boundary conditions.

2 Modal Reduction of the Equation of Motion

To make the partial differential equation amenable to the first level of analysis, we perform a modal reduction to project the PDE onto a finite number of ordinary differential equations (ODEs). We use cantilevered beam modes as assumed modes and apply a Galerkin projection. To this end, we approximate the transverse deflection as

$$y(x,t) \approx \sum_{i=1}^{N} q_i(t)\psi_i(x), \tag{8}$$

where N is the number of retained modes, $\psi_i(x)$ are the assumed modal functions, and $q_i(t)$ are the assumed modal coordinates.

We substitute this expression into equation (5), multiply by $\psi_j(x)$, and integrate over the length of the blade to obtain the j^{th} second-order ODE. To simplify the analysis, we take $N = 1$ and neglect the effect of $J(x)$. Neglecting the contributions of the rotational inertia is a common approximation for the analysis of thin beams [23]. Also, we expect the dominant failure mode to be in the first resonant mode of the beam. Including higher order modes will reveal additional details but our aim is to start simple and analyze the representative system for dynamic instabilities.

Employing the aforementioned simplifications to the model, the differential equation takes the form

$$\begin{aligned}\ddot{q} + b\ddot{q}q^2 + c\dot{q}q + d\dot{q}^2 q + (e_1 + e_2\cos\Omega t)q + (f_1 + f_2\cos\Omega t)q^3 \\ = g\sin\Omega t \int_0^L \psi(x)\,dx - \int_0^L \psi(x) f(y,\dot{y},s,\dot{s},x,t)\,dx\end{aligned} \tag{9}$$

where $b, c, d, e_1, e_2, f_1, f_2$ are constant coefficients and can be evaluated according to

$$b = \int_0^L \psi(x)\left(\psi''(x)\int_x^L\int_0^z \psi'^2(u)\,du\,dz - \psi'(x)\int_0^x \psi'^2(v)\,dv\right)dx$$

$$c = \int_0^L \psi(x)\left(-2\Omega\psi(x)\psi'(x) + 2\psi''(x)\Omega\int_x^L \psi(z)\,dz\right)dx$$

$$d = \int_0^L \psi(x)\left(\psi'(x)\int_0^x \psi'^2(v)\,dv + \psi''(x)\int_x^L\int_z^L \psi'^2(u)\,du\,dz\right)dx$$

$$e_1 = \int_0^L \psi(x)\left(\Omega^2 x\psi'(x) + \Omega^2\psi''(x)\left(\frac{L^2 - x^2}{2}\right)\psi(x) - (EI\psi''(x))''\right)dx$$

$$e_2 = \int_0^L \psi(x)\left(\psi'(x)g + g\psi''(x)(L - x)\right)dx$$

$$\begin{aligned}f_1 = \int_0^L \psi(x)\Big(&\left(-\frac{\psi'^3(x)}{2}\right)\Omega^2 x + \psi'(x)\Omega^2\int_0^x \frac{\psi'^2(v)}{2}\,dv + \\ &\frac{\Omega^2}{2}(L^2 - x^2)\left(\frac{\psi'^3(x)}{2}\right)' - \left[\int_x^L\int_0^z \frac{\psi'^2}{2}\,du\,dz\right]\psi''(x)\Omega^2 + \\ &EI\psi''(x)(3\psi'^2(x))' - (3EI\psi''^2(x)\psi'(x))'\Big)\,dx\end{aligned}$$

$$f_2 = \int_0^L \psi(x)\left(-\frac{\psi'^3(x)}{2}g + \frac{(L-x)g}{2}(\psi'^3(x))'\right)dx$$

These coefficients are dependent on the assumed modal function $\psi(x)$ and the distributed parameters $m(x)$ and $EI(x)$. The form of $f(y,\dot{y},s,\dot{s},x,t)$ influences the last integral in equation (9), and the q terms born from it.

For example, if we approximate the system as a uniform beam, such that $m(x)$ and $EI(x)$ are constants, and use the first modal function of a uniform cantilevered Euler-Bernoulli beam [27], given by

$$\psi = [(\cosh(\lambda x/L) - \cos(\lambda x/L)) - \delta(\sinh(\lambda x/L) + \sin(\lambda x/L)], \tag{10}$$

where $\lambda = 1.87510407$ and $\delta = 0.7341$, then we obtain parameter values that produce

$$\begin{aligned}&\ddot{q} - 0.94\ddot{q}q^2 - 0.918\dot{q}q + 1.37\dot{q}^2 q + (-2.66EIL^{-2} + 0.363L^2\Omega^2 \\ &+5.12L\cos\Omega t)q + (5.352EIL^{-3} + 13.05EIL^{-4} - 0.226\Omega^2 \\ &-8.534L^{-1}\cos\Omega t)q^3 = 1.65L\sin\Omega t - \int_0^L \psi(x) f(y,\dot{y},s,\dot{s},x,t)\,dx\end{aligned} \tag{11}$$

Including a model for aeroelastic loading via f will contribute some terms already present in (11) and also some new terms. Wind shear and tower passing will introduce cyclic terms in f which will contribute to the direct forcing term, and possibly other parametrically excited terms. Aeroelastic forces will also be significant on models of flap-wise (out-of-plane) deflection. It is apparent, from inspection, that this resulting ODE in q has linear and cubic stiffness effects, with parametric excitation on both the linear and cubic terms, along with direct excitation.

In addition we include a damping term in our model to account for operational damping in the system due to resistance in blade motion in the plane of motion. Also the cross coupled terms in $\ddot{q}, \dot{q}, q$ give rise to lot more interplay between the terms in a purely analytical case study. Since the mode shapes are fixed, the coefficients are only dependent on the geometry of the blade (beam), its material properties and the rotational frequency.

Our intent is to analyze this system, rebuild the approximate $y(x,t)$ under resonant cases, and determine the resulting loads applied to the low speed shaft and hub via $y(x,t)$ and its derivatives. However, equation (9) has elements of a fundamental vibration equation, namely a forced nonlinear Mathieu equation, which is of dynamical interest in itself and warrants its own study. Therefore, we will analyze the forced nonlinear Mathieu equation, and use it to reveal phenomena that we expect to be significant to wind turbine blades. In this paper we will focus on the superharmonic resonances of the system.

3 Resonances of the Forced Nonlinear Mathieu Equation

Embedded in the single mode ODE (11) are terms of a Mathieu-Duffing equation: since this is a fundamental equation of dynamic systems, equation (12) takes our attention and becomes the focus of this initial study.

$$\ddot{q}+\varepsilon\mu\dot{q}+(\delta+\varepsilon\gamma\cos\Omega t)q+\varepsilon\alpha q^3 = F\sin\Omega t \tag{12}$$

The analysis of the full model is bound to yield a different set of results due to the various cross coupled and higher order terms. Also the analysis in this section is based on relative magnitudes of parameters which in a strict sense may not hold true for a wind turbine blade model. However, we expect the fundamental characteristics of the Mathieu-Duffing equation to exist in the full-scale model and its analysis reveals some interesting dynamical phenomenon.

When the forcing in equation (12) is of order ε, the analysis will indicate a primary resonance. The forcing in the above expression is of order one—also known as hard forcing. This will help us unfold secondary resonances. Using MMS, we allow our system to have fast and slow time scales (T_0,T_1) and also variations in amplitude. This allows for a dominant solution q_0 and a variation of that solution q_1, i.e.

$$q = q_0(T_0,T_1)+\varepsilon q_1(T_0,T_1),$$

where $T_i = \varepsilon^i T_0$. Then $\frac{d}{dt} = D_0+\varepsilon D_1$ and $D_i = \frac{d}{dT_i}$.

We substitute this formulation into equation (12) and then simplify and extract coefficients of $\varepsilon^0, \varepsilon^1$. The expression for the coefficient of ε^0 is

$${D_0}^2 q_0+\omega^2 q_0 = F\sin\Omega T_0,$$

where $\delta=\omega^2$ is the linearized natural frequency. The solution for this is

$$q_0 = Ae^{i\omega T_0}+\Lambda e^{i\Omega T_0}+c.c. \tag{13}$$

where $\Lambda = \dfrac{F}{2(\Omega^2-\omega^2)}$, $\omega=\sqrt{\delta}$ and $A=\dfrac{1}{2}ae^{i\beta}$.

The expression for the coefficient of ε^1 is

$${D_0}^2 q_1+\delta q_1 = -\mu D_0 q_0 - 2D_0 D_1 q_0 - \gamma q_0\cos\Omega T_0 - \alpha {q_0}^3 \tag{14}$$

Substituting the solution for q_0 from equation (13), we expand the terms on the right hand side of (14). We need to eliminate coefficients of $e^{i\omega T_0}$ that constitute the secular terms and would make the solutions unbounded. The solvability condition is thus set by equating the coefficients of $e^{i\omega T_0}$ terms to zero.

3.1 Non-Resonant Case

If there is no specific relation between Ω and the natural frequency (ω) of the system then, the solvability condition is

$$-2A'i\omega-\mu Ai\omega-\alpha(3A^2\bar{A}+6\Lambda^2 A)=0, \tag{15}$$

where the bar indicates the complex conjugate. (Note $\Lambda=\bar{\Lambda}$.) Letting $A=\dfrac{1}{2}ae^{i\beta}$ and $A'=\dfrac{1}{2}(a'+ai\beta')e^{i\beta}$ we get

$$i(a'+ai\beta')\omega+\frac{\mu}{2}ai\omega+\alpha(3\Lambda^2 a+\frac{3}{8}a^3)=0.$$

Splitting the above equation into real and imaginary parts we get

$Re: -a\beta'\omega+\alpha a(3\Lambda^2+\dfrac{3}{8}a^2)+\frac{\delta a}{2}=0$

$Im: a'\omega+\dfrac{\mu a\omega}{2}=0.$

From these we can conclude that $a\to 0$ is the steady-state solution and hence there is no effect of the nonlinear terms in the non-resonant case, i.e. when Ω is some non-specific value.

3.2 Superharmonic Resonances

3.2.1 $3\Omega\approx\omega$ (Nonlinear Case)

If $3\Omega\approx\omega$, the cubic nonlinearity contributes to the secular terms. We detune the frequency of excitation such that $3\Omega=\omega+\varepsilon\sigma$. Then $3\Omega T_0=(\omega+\varepsilon\sigma)T_0=\omega T_0+\sigma T_1$.

The solvability condition will have additional terms that have $3\Omega T_0$ as an exponential argument in equation (14). It takes the form

$$-2A'i\omega-\mu Ai\omega-\alpha(3A^2\bar{A}+6\Lambda^2 A)-\alpha\Lambda^3 e^{i\sigma T_1}=0 \tag{16}$$

Letting $A=\frac{1}{2}ae^{i\beta}$ and $\phi=\sigma T_1-\beta$, the real and imaginary parts of equation (16) lead to

$Re: a\phi'\omega-\sigma\omega a+3\alpha a(\Lambda^2+\dfrac{a^2}{8})+\alpha\Lambda^3\cos(\phi)=0$

$Im: a'\omega+\dfrac{\mu a\omega}{2}+\alpha\Lambda^3\sin(\phi)=0$

in $\phi$and a.

These expressions are similar to the ones we would get in the synthesis of a standard Duffing equation with hard excitation [25]. For the steady state solution $a'=\phi'=0$, which is satisfied if

$$\left[\frac{\mu}{2}^2 + (\sigma - 3\alpha\Lambda^2 - \frac{3}{8}\alpha a^2)^2\right] a^2 = \frac{\alpha^2\Lambda^6}{\omega^2}.$$

This is a quadratic expression in the detuning parameter σ. Solving for σ we get,

$$\sigma = 3\alpha\Lambda^2 + \frac{3}{8}\alpha a^2 \pm (\frac{\alpha^2\Lambda^6}{\omega^2 a^2} - \frac{\mu^2}{2})^{1/2} \quad (17)$$

This suggests that a non-zero $a(\sigma)$ can occur at steady-state. If stable,the leading order homogenous term $a\cos(t_0 + \theta)$ can be sustained. Then with $3\Omega = \omega + \varepsilon\sigma$ we get

$$x_0 = a\cos(3\Omega T_0 - \phi) + 2\Lambda\cos(\Omega t)$$

This shows that the response is periodic with harmonics at Ω and 3Ω.

The peak amplitude can be deduced from the equation (17) as

$$a_p = \frac{\alpha\Lambda^3}{\mu\omega}$$

The corresponding value of σ (after substituting value for a_p and doing a few simplifications) is

$$\sigma_p = \frac{3\alpha\Lambda^2}{\omega}(1 + \frac{\alpha^2\Lambda^4}{8\omega^2\mu^2})$$

If Λ, i.e. F, is small, then a_p is really small and there is little non-linear effect. If on the other hand Λ is O(1), so is a_p. Also the peak frequency is dependent on both Λ, α.

3.2.2 $2\Omega \approx \omega$ (Linear + Nonlinear Case)

If $2\Omega = \omega + \varepsilon\sigma_2$, then the $\gamma q_0 \cos\Omega t$ term from equation (14) contributes to the secular terms. In this case the solvability condition can be written as

$$-2A'i - \mu Ai - \alpha(3A^2\bar{A} + 6\Lambda^2 A) - \frac{\gamma\Lambda}{2}\exp i\sigma_2 T_1 = 0 \quad (18)$$

Following the analysis done in the previous section for the 3Ω superharmonic resonance, we obtain slow flow equations. At steady-state the relationship between the response amplitude and the detuning parameter σ_2

$$\sigma_2 = 3\alpha\Lambda^2 + \frac{3}{8}\alpha a^2 \pm (\frac{\gamma^2\Lambda^2}{4a^2} - \mu^2)^{1/2}$$

The peak amplitude would be

$$a_p = \frac{\gamma\Lambda}{2\mu}.$$

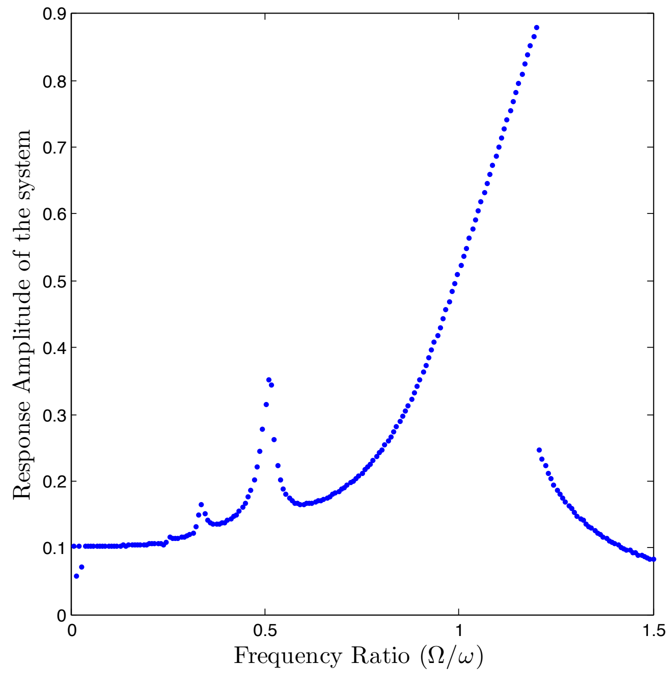

FIGURE 2. RESPONSE OF A SYSTEM SIMILAR TO 9 SHOWING SUPERHARMONIC RESONANCES AT 1/2 AND 1/3; $\mu = 0.01, \varepsilon = 0.1, \alpha = 0.05, F = 0.1$

The corresponding value of σ_2 is

$$\sigma_{2p} = 3\alpha\Lambda^2(1 + \frac{\gamma^2}{32\mu^2})$$

Hence we can conclude the following

- the peak value is independent of α
- α, F and Λ affect the peak location and as they increase, $|\sigma_p|$ increases.
- the sign of α determines the sign of σ_p

A sample response of equation (12) is numerically simulated. The response curve shown in Figure 2 has the primary resonance and two superharmonics shown at $1/3$ and $1/2$ the natural frequency.

The wind turbine blades are generally designed such that the natural frequency of the blade in lead-lag (in-plane) motion is below the rotational frequency. This analysis implies potential existence of superharmonic resonances which would also provide additional critical frequencies where the response of the blade would be dominant. This would imply increased loading on the gearbox and other components and increased bending of the blades.

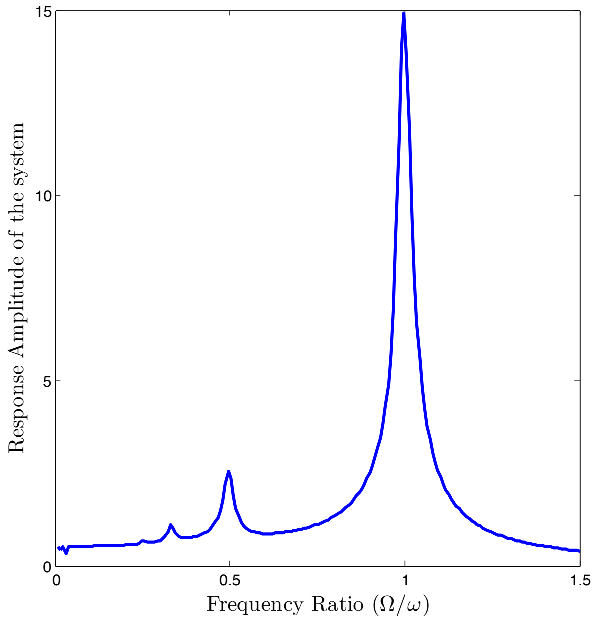

FIGURE 3. RESPONSE OF A SYSTEM SIMILAR TO 9 SHOWING SUPERHARMONIC RESONANCES AT 1/2 AND 1/3; $\mu = 0.01, \varepsilon = 0.1, \alpha = 0, F = 0.1$

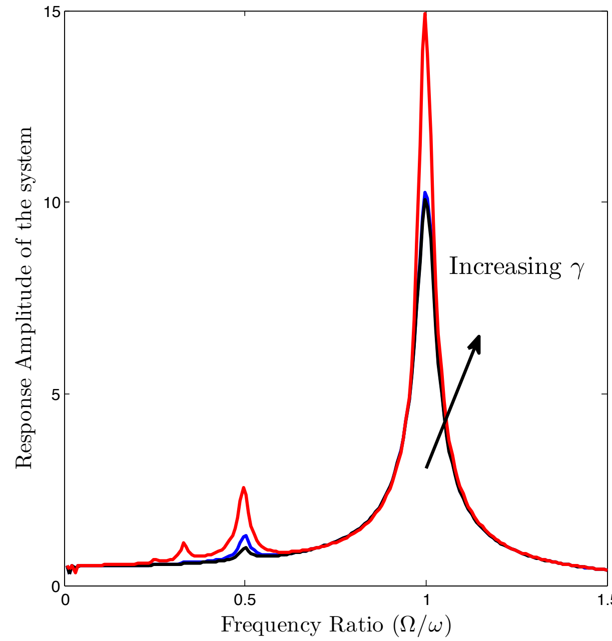

FIGURE 4. EFFECT OF PARAMETRIC FORCING TERM ON SYSTEM RESPONSE

4 Discussion

We have shown the details of a first-order analysis of superharmonic resonances for the system of interest. At first order, the super harmonic resonance of order 1/3 is the same phenomenon as in the Duffing equation. The nonlinear parameter, α, scales the peak response, while both the nonlinear parameter and the direct excitation level affect the frequency value of the peak response.

The superharmonic response of order 1/2 involves interaction between the parametric excitation and both the nonlinear parameter and the direct excitation. In fact, if the nonlinearity is not present, i.e. if $\alpha = 0$, this resonance persists. As such, a *linear system* excited both parametrically and directly at the same frequency can exhibit a superharmonic resonance. This complements linear primary resonance phenomena and linear subharmonic resonances exploited in parametric amplification [12, 13].

There also exists a primary resonance which is the same at first order as that of the Duffing equations. The parametric excitation, when at the same frequency as the direct excitation, does not affect primary resonance amplitude at first order. Subharmonic resonances were not analyzed in detail here, although they exist to first order at orders 2 and 3. Order-two subharmonics involve interactions with the parametric excitation term, while order-three subharmonic resonance is the same as that of the Duffing equation to first order. We did not pursue the details of the subharmonic resonances since the wind turbines, which motivate the story, are design to operate below the natural frequency of the rotating blade.

The variation in system responses for changes in some parameters are shown. Their behavior has been summarized below

1. Effect of parametric forcing term: The system responses increases with increased parametric forcing. (see Figure 4.) Beyond a certain value of $\gamma$the system goes unstable at primary resonance. This could be unfolded doing a higher order perturbation analysis.
2. Effect of direct forcing term: As it would be expected, increase in the direct forcing term F, increases the overall magnitude of response evenly over the entire spectrum. (See Figure 5.)
3. Effect of damping term: Here as well, an increase in the damping, decreases the overall magnitude of response over the entire spectrum. The spectrum resonance peaks scale as $1/\mu$.
4. Effect of nonlinear term: An increase in α causes the response curve to bend over more significantly. This bending can induce jump instabilities as the frequency slowly varies. Our equations are similar to a Duffing equation with hardening for $\alpha > 0$. Also, in the presence of strong enough nonlinearity, the relative magnitudes of the superharmonic at half the frequency is comparable in magnitude to the primary frequency, as seen in Figure 6.

This work focuses on resonances and not stability. The

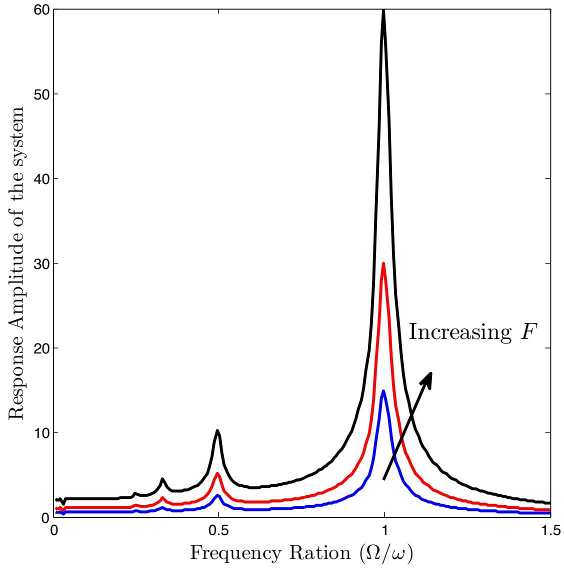

FIGURE 5. EFFECT OF DIRECT FORCING TERM ON SYSTEM RESPONSE

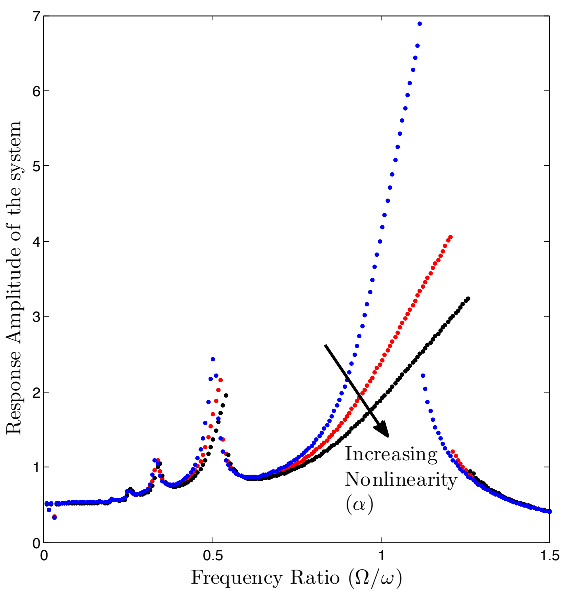

FIGURE 6. EFFECT OF NONLINEAR TERM TERM ON SYSTEM RESPONSE

Mathieu equation is well known to have instabilities in the space of stiffness and parametric excitation parameters. We have observed instabilities at primary resonance in simulations, and an analysis of stability transitions is currently underway.

5 Conclusion

This work has provided an integral-partial differential equation of motion that model the in-plane dynamics of a wind turbine blade. Nonlinear effects and cyclic loading have been considered. In the present work, the model accommodates aerodynamic force although these details are not yet incorporated. Gravitational loading on these structures gives rise to the cyclic parametric and direct excitation of the structure at the same frequency as rotation. Cyclic aerodynamic loading will probably contribute further to both direct forcing and parametric excitation.

The single mode reduction led to the parametrically and directly forced nonlinear ODE of equation (9). This motivated the investigation of a forced nonlinear Mathieu equation as a paradigm in dynamical systems research. Furthermore, we found that when typical wind turbine parameters are included, equation (9) may not have the convenience of a small parameter. So as a preliminary study, we examined the small parameter nonlinear forced Mathieu equation, with the expectation that the phenomena uncovered may carry over to wind turbines. To this end, we performed a perturbation analysis of the nonlinear forced Mathieu equation (12), which revealed multiple super- and subharmonic resonances. With nonlinearity, as the frequency slowly increases and decreases, there can be hysteresis and jump phenomena in the occurrence of large responses. Jump phenomena are dangerous, as a small change in frequency near a jump point can induce a suddenly large steady state. (Transients may be even larger.) Furthermore, presence of superharmonic resonance for a linear Mathieu equation with direct forcing was unfolded.

Some of these resonance phenomena are expected to be relevant to wind turbine blade dynamics. Further analytical study is required to completely understand the dynamics of the system. Resonances can be amplified beyond linear predictions if there is simultaneous superharmonic and primary excitation as well. More detailed modeling is underway to include out-of-plane motion, geometry and aerodynamic effects.

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REFERENCES

[1] Wendell, J., 1982. Simplified aeroelastic modeling of horizontal-axis wind turbines. Technical Report DOE/NASA/3303-3; NASA-CR-168109, Massachusetts Inst. of Tech., Cambridge, USA, September.

[2] Kallesøe, B. S., 2007. “Equations of motion for a rotor

blade, including gravity and pitch action". *Wind Energy,* ***10***(3), February, pp. 209–230.
[3] Caruntu, D., 2008. "On nonlinear forced response of nonuniform blades". In Proceeding of the 2008 ASME Dynamic Systems and Control Conference, no. DSCC2008-2157, ASME.
[4] Chopra, I., and Dugundji, J., 1979. "Nonlinear dynamic response of a wind turbine blade". *Journal of Sound and Vibration,* ***63***(2), pp. 265–286.
[5] Jonkman, J., 2010. NWTC design codes (FAST), Novermber.
[6] Jonkman, J., 2003. Modeling of the UAE wind turbine for refinement of FAST AD. Technical Report TP-500-34755, NREL, Golden, CO, December.
[7] Wang, Y. R., and Peters, D. A., 1996. "The lifting rotor inflow mode shapes and blade flapping vibration system eigen-analysis". *Computer Methods in Applied Mechanics and Engineering,* ***134***(1-2), pp. 91–105.
[8] Bhat, S. R., and Ganguli, R., 2004. "Validation of comprehensive helicopter aeroelastic analysis with experimental data". *Defence Science Journal,* ***54***(4), pp. 419–427.
[9] Rand, O., and Barkai, S. M., 1997. "A refined nonlinear analysis of pre-twisted composite blades". *Composite structures,* ***39***(1-2), pp. 39–54.
[10] Smith, E. C., and Chopra, I., 1993. "Aeroelastic response, loads and stability of a composite rotor in flight forward". *AIAA Journal,* ***31***(7), pp. 1265–1273.
[11] Hodges, D., and Dowell, E., 1974. Nonlinear equations of motion for elastic bending and torsion of twisted nonuniform rotor blades. Technical Note D-7818, NASA.
[12] Rhoads, J., and Shaw, S., 2010. "The impact of nonlinearity on degenerate parametric amplifiers". *Applied Physics Letters,* ***96***(23), June.
[13] Rhoads, J., Miller, N., Shaw, S., and Feeny, B., 2008. "Mechanical domain parametric amplification". *Journal of Vibration and Acoustics,* ***130***(6), December, p. 061006(7 pages).
[14] Pandey, M., Rand, R., and Zehnder, A. T., 2007. "Frequency locking in a forced Mathieu-van der Pol-Duffing system". *Nonlinear Dynamics,* ***54***(1-2), February, pp. 3–12.
[15] Month, L., and Rand, R., 1982. "Bifurcation of 4-1 subharmonics in the non-linear Mathieu equation". *Mechanics Research Communication,* ***9***(4), pp. 233–240.
[16] Newman, W. I., Rand, R., and Newman, A. L., 1999. "Dynamics of a nonlinear parametrically excited partial differential equation". *Chaos,* ***9***(1), March, pp. 242–253.
[17] Ng, L., and Rand, R., 2002. "Bifurcations in a Mathieu equation with cubic nonlinearities". *Chaos Solitions and Fractals,* ***14***(2), August, pp. 173–181.
[18] Belhaq, M., and Houssni, M., 1999. "Quasi-periodic oscillations, chaos and suppression of chaos in a nonlinear oscillator driven by parametric and external excitations". *Nonlinear Dynamics,* ***18***(1), June, pp. 1–24.
[19] Veerman, F., and Verhulst, F., 2009. "Quasiperiodic phenomena in the van der Pol-Mathieu equation". *Journal of Sound and Vibration,* ***326***(1-2), September, pp. 314–320.
[20] Arrowsmith, D., and Mondragón, R., 1999. "Stability region control for a parametrically forced Mathieu equation". *Mecannica,* ***34***, December, pp. 401–410.
[21] Marathe, A., and Chatterjee, A., 2006. "Asymmetric Mathieu equations". *Proceedings of The Royal Society A*(462), February, pp. 1643–1659.
[22] McLachlan, N., 1964. *Theory and Application of Mathieu Functions*. Dover Publications, New York.
[23] Meirovitch, L., 1997. *Principles and Techniques of Vibration*. Upper Saddle River, New Jersey.
[24] Stoker, J., 1967. *Nonlinear Vibrations*. Macmillan, New York.
[25] Nayfeh, A. H., and Mook, D. T., 1979. *Nonlinear Oscillations*. Wiley Interscience Publications. John Wiley and Sons, New York.
[26] Rand, R., 2005. Lecture notes on nonlinear vibration; available at http://audiophile.tam.cornell.edu/randdocs/nlvibe52.pdf. online, Ithaca, NY.
[27] Blevins, R. D., 1979. *Formulas for natural frequency and mode shape*. Krieger Pub Co.