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In the insurance business risky investments are dangerous

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Dedicated to the memory of Vladimir Kalashnikov.

Abstract We find an exact asymptotics of the ruin probability $\Psi(u)$ when the capital of insurance company is invested in a risky asset whose price follows a geometric Brownian motion with mean return a and volatility $\sigma > 0$. In contrast to the classical case of non-risky investments where the ruin probability decays exponentially as the initial endowment u tends to infinity, in this model we have, if $\rho := 2a/\sigma^2 > 1$, that $\Psi(u) \sim Ku^{1-\rho}$ for some $K > 0$. If $\rho < 1$, then $\Psi(u) = 1$.

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1 Introduction

It is well-known that the prosperity of an insurance company is due not only to earnings in its principal business but also to intelligent investments of the money at its disposal. This is the reason why the modern trend in actuarial mathematics is toward incorporating an economic environment into models, see, e.g., [9], [11], [6], and many others. Apparently, risky investment can be dangerous: disasters may arrive at the period when the market value of assets is low and the company will not be able to cover losses by selling these assets just because of price fluctuations. Regulators are rather attentive to this issue and impose stringent constraints on company portfolios. Typically, junk bonds are prohibited, a prescribed (large) part of the portfolio should contain non-risky assets (e.g., Treasury bonds) while in the remaining part only risky assets with good ratings are allowed.

The common idea that investments in an asset with stochastic interest rate may be too risky for an insurance company can be justified mathematically. In [4] it is noticed that in the classical Lundberg–Cramér model the ruin probability may decrease not as an exponential but a power function if the wealth is invested in the stock whose price follows a geometric Brownian motion. In the setting of [4] the risk process is Markov; the equation for the exit (ruin) probability can be reduced to a differential equation which belongs to a well-studied class.

In May 1999 the second author had the pleasure of visiting the University of Copenhagen and discussing with Vladimir Kalashnikov the topic of interest. Vladimir kindly provided him with the manuscript [5] containing upper and lower bounds allowing us to complete the study initiated in [4].

In the present work we consider in detail the model of [4] and find an exact asymptotics for the ruin probability. The conclusion: independently of the safety loading, the investments in an asset with large volatility lead to the bankruptcy with probability one while for the small volatility the ruin probability decreases as a power function. Kalashnikov's bounds (developed further in his joint work with Ragnar Norberg [6]) play an important role in our study. Our techniques is elementary. More profound and general results can be found in [10], [7], and [8].

2 The model

We are given a stochastic basis with a Wiener process w independent of the integer-valued random measure $p(dt,dx)$ with the compensator $\tilde{p}(dt,dx)$.

Let us consider a process $X = X^u$ of the form

$$X_t = u + a\int_0^t X_s ds + \sigma \int_0^t X_s dw_s + ct - \int_0^t \int xp(ds,dx), \qquad (1)$$

where a and σ are arbitrary constants and $c \geq 0$.

We shall assume that $\tilde{p}(dt,dx) = \alpha dt F(dx)$ where $F(dx)$ is a probability distribution on $]0,\infty[$. In this case the integral with respect to the jump measure is simply a compound Poisson process. It can be written as $\sum_{i=1}^{N_t} \xi_i$ where N is a Poisson process with intensity α and ξ_i are random variables with common distribution F; w, N, ξ_i, $i \in \mathbf{N}$, are independent.

In our main result (Theorem 1) we assume that F is an exponential distribution.

Let $\tau^u := \inf\{t : X_t^u \leq 0\}$ (the date of ruin), $\Psi(u) := P(\tau^u < \infty)$ (the ruin probability), and $\Phi(u) := 1 - \Psi(u)$ (the non-ruin probability).

The parameter values $a = 0$, $\sigma = 0$, correspond to the Lundberg–Cramér model for which the risk process is usually written as $X_t = u + ct - \sum_{i=1}^{N_t} \xi_i$. In the considered version (of non-life insurance) the capital evolves due to continuously incoming cash flow with rate c and outgoing random payoffs ξ_i at times forming an independent Poisson process N with intensity α. For the model with positive safety loading and F having a "non-heavy" tail, the Lundberg inequality provides an encouraging information: the ruin probability decreases exponentially as the initial capital u tends to infinity. Moreover, for the exponentially distributed claims the ruin probability admits an explicit expression, see [1] or [2].

The more realistic case $a > 0$, $\sigma = 0$, corresponding to non-risky investments, does not pose any problem.

We study here the case $\sigma > 0$. Now the equation (1) describes the evolution of the capital of an insurance company which is continuously reinvested into an asset with the price following a geometric Brownian motion (i.e. the relative price increments are $adt + \sigma dw_t$)

It is well-known (see, e.g., the discussion in [9] for more general insurance models) that for the Markov process given by (1) the non-exit probability $\Phi(u)$ satisfies the following equation:

$$\frac{1}{2}\sigma^2 u^2 \Phi''(u) + (au + c)\Phi'(u) - \alpha\Phi(u) + \alpha \int_0^u \Phi(u-y)dF(y) = 0. \quad (2)$$

With $\sigma > 0$, this equation is of the second order and, hence, requires two boundary conditions in contrast to the classical case ($a = 0$, $\sigma = 0$) where it degenerates to an equation of the first order requiring a single boundary condition, see [2].

Theorem 1 *Let $F(x) = 1 - e^{-x/\mu}$, $x > 0$. Assume that $\sigma > 0$.*

(i) If $\rho := 2a/\sigma^2 > 1$, then for some $K > 0$

$$\Psi(u) = Ku^{1-\rho}(1 + o(1)), \quad u \to \infty. \quad (3)$$

(ii) If $\rho < 1$, then $\Psi(u) = 1$ for all u.

The same model serves well in the situation where only a fixed part $\gamma \in]0, 1]$ of the capital is invested in the risky asset (one should only replace the parameters a and σ in (1) by $a\gamma$ and $\sigma\gamma$).

The proofs will be given in Sections 5 and 4, respectively. Section 3 contains, in a certain sense, preliminary results which happen to be useful to accomplish an analysis of solutions to the differential equation for ruin probability and obtain its exact asymptotics in the model of interest. For this reason we do not try to look here for more delicate formulations and penetrate, e.g., into a specific structure of coefficients to get rid of the logarithm in Proposition 1. In Section 4 we provide simple arguments revealing the fact that for $\rho < 1$ the imbedded process is ergodic with the invariant measure charging the negative axes and, hence, leaves the positive half-axes with probability one.

3 Kalashnikov's bounds

Here we establish a result for generally distributed claims.

Proposition 1 *Let $\rho := 2a/\sigma^2 > 1$.*

(i) If $E\xi_1^{\rho-1} < \infty$, then there exists a constant $C \geq 0$ such that

$$\Psi(u) \leq Cu^{1-\rho}(\ln u)^{1\vee(\rho-1)}, \quad \forall\, u \geq 2. \quad (4)$$

$ii)$ If $P(\xi_1 > x) > 0$ for all x, then there are constants $b, B, u_0 > 0$ such that

$$\Psi(u) \geq bu^{-B}, \quad \forall\, u \geq u_0. \quad (5)$$

Let τ_n be the instant of the n-th jump of N and let $\theta_n := \tau_n - \tau_{n-1}$ with $\tau_0 := 0$. We define the discrete-time process $S = S^u$ with $S_n := X_{\tau_n}$. Since the ruin may occur only when X jumps downwards, $\Psi(u) = P(T^u < \infty)$ where $T^u := \inf\{n \geq 1 : \ S_n^u \leq 0\}$.

Put $\kappa := a - \sigma^2/2$ and $w_t^n := w_{t+\tau_{n-1}} - w_{\tau_{n-1}}$. Let us introduce the notations

$$\lambda_n := \exp\{\sigma w_{\theta_n}^n + \kappa\theta_n\},$$
$$\eta_n := c\int_0^{\theta_n} \exp\{\sigma(w_{\theta_n}^n - w_u^n) + \kappa(\theta_n - u)\}\, du.$$

Solving the linear stochastic equation we get that

$$S_n = \lambda_n S_{n-1} + \eta_n - \xi_n. \tag{6}$$

Putting $\mathcal{E}_n := \Pi_{k=1}^n \lambda_k$, we may use also the representation

$$S_n = \mathcal{E}_n u + \mathcal{E}_n \sum_{k=1}^n \mathcal{E}_k^{-1}(\eta_k - \xi_k). \tag{7}$$

Notice that λ_n are i.i.d. random variables and

$$E\lambda_1^\nu = \frac{\alpha}{\alpha + (1-\rho-\nu)\nu\sigma^2/2}. \tag{8}$$

We deduce Proposition 1 from results on the general discrete-time process given by (6) where (λ_n, η_n) is a sequence of (two-dimensional) i.i.d. random variables, $\lambda_n > 0$, and each ξ_n is independent from the σ-algebra generated by the family $\{\lambda_k, \eta_k,\ \xi_m,\ k \in \mathbf{N},\ m \in \mathbf{N} \setminus \{n\}\}$. In particular, the assertion (i) follows immediately from (8) and

Proposition 2 *Let $\eta_n \geq 0$. Assume that $E\lambda_1^{-\beta} = q_\beta$ where $q_\beta < 1$ if $\beta \in]0, \beta_0[$ and $q_{\beta_0} = 1$. If $E\xi_1^{\beta_0} < \infty$, then there is a constant C such that*

$$P(T^u < \infty) \leq Cu^{-\beta_0}(\ln u)^{1\vee\beta_0}, \quad u \geq 2. \tag{9}$$

Proof. It is easily seen from the formula (7) that $P(T^u < \infty) \leq P(\zeta_\infty > u)$ where $\zeta_n := \sum_{k=1}^n \mathcal{E}_k^{-1}\xi_k$. Applying Lemma 1 below we get the result. □

Lemma 1 *Let $\zeta_n := \sum_{k=1}^n \chi_k$ where $\chi_k \geq 0$ and $E\chi_k^\beta \leq l_\beta q_\beta^k$ with $q_\beta < 1$ if $\beta \in]0, \beta_0[$ and $q_{\beta_0} = 1$. Then there is a constant C such that*

$$P(\zeta_\infty > u) \leq Cu^{-\beta_0}(\ln u)^{1\vee\beta_0}, \quad u \geq 2.$$

Proof. Let M be a positive integer. Let $\beta_0 \leq 1$. Take arbitrary $\beta \in]0, \beta_0[$. Using the Chebyshev inequality and taking into account that $|x+y|^r \leq |x|^r + |y|^r$, $r \leq 1$, we infer that

$$P(\zeta_M > u/2) \leq \left(\frac{2}{u}\right)^{\beta_0} \sum_{k=1}^M E\chi_k^{\beta_0} \leq \left(\frac{2}{u}\right)^{\beta_0} l_{\beta_0} M$$

and, similarly,

$$P(\zeta_\infty - \zeta_M > u/2) \le \left(\frac{2}{u}\right)^\beta \sum_{k=M+1}^{\infty} E\chi_k^\beta \le \left(\frac{2}{u}\right)^\beta l_\beta \frac{q_\beta^M}{1-q_\beta}.$$

Choosing $M = M_\beta$ as the integer part of $(\ln q_\beta)^{-1} \ln u^{\beta-\beta_0}$, we get the result.

Let $\beta_0 > 1$. The first line above can be modified as follows:

$$\left(\frac{2}{u}\right)^{\beta_0} E\left(\sum_{k=1}^{M} \chi_k\right)^{\beta_0} \le \left(\frac{2}{u}\right)^{\beta_0} M^{\beta_0-1} \sum_{k=1}^{M} E\chi_k^{\beta_0} \le \left(\frac{2}{u}\right)^{\beta_0} l_{\beta_0} M^{\beta_0}.$$

Using the bound for the tail of the series with $\beta = 1$ and putting $M = M_1$, we obtain the desired inequality. □

The assertion (ii) is a corollary of the following general result.

Proposition 3 *Assume that the following conditions hold:*
(a) *there exists a constant* $l < 1$ *such that* $P(\lambda_1 \le l) > 0$*;*
(b) $P(\xi_1 > x) > 0$ *for any* x*.*
Then there are $b, B > 0$ *such that for all sufficiently large* u

$$P(T^u < \infty) \ge b u^{-B}. \tag{10}$$

Proof. The assumption (a) implies that for some constants $K > 0$ and $p_1 > 0$

$$P(\lambda_1 \le l,\ \eta_1 - \xi_1 \le K) = p_1.$$

The assumption (b) and the independence of ξ_1 and (λ_1, η_1) imply that there are constants $L > 0$ and $p_2 > 0$ for which

$$P(\lambda_1 \le L,\ \eta_1 - \xi_1 \le -2LK/(1-l)) = p_2.$$

Let $M := 1 + [(\ln K(1-l) - \ln u)/\ln l]$ where $[.]$ denotes the integer part. Obviously, $l^M u \le K/(1-l)$. Define the sets

$$A_M := \cap_{k=1}^M \{\lambda_k \le l,\ \eta_k - \xi_k \le K\},$$

and

$$D_{M+1} := \{\lambda_{M+1} \le L,\ \eta_{M+1} - \xi_{M+1} \le -2LK/(1-l)\}.$$

On the set A_M

$$S_M = \mathcal{E}_M u + \mathcal{E}_M \sum_{k=1}^{M} \mathcal{E}_k^{-1}(\eta_k - \xi_k) \le l^M u + \sum_{k=1}^{M} l^{M-k} K \le l^M u + \frac{K}{1-l}.$$

This implies that on the set $A_M \cap D_{M+1}$

$$S_{M+1} \le L S_M - 2LK/(1-l) \le L(l^M u - K/(1-l)) \le 0.$$

Thus,

$$P(T^u < \infty) \ge P(A_M \cap D_{M+1}) \ge p_2 p_1^M \ge p_2 p_1^{1+(\ln K(1-l) - \ln u)/\ln l}$$

and we get the desired result with $b = -(\ln p_1)/\ln l$. □

Remark. The exit probability for the solution of the difference equation (6) with random coefficients was studied in [5] and [6]. The results of this section, although slightly different in formulations and proofs, are strongly inspired by these works.

4 Large volatility: the ruin is imminent

We show that the investments in a stock with large volatility, namely, when $\rho < 1$, lead to a ruin with probability one whatever is the initial capital. Clearly, it is sufficient to consider the case where $a > 0$. Inspecting the formula (8) we infer that $E\lambda_1^\nu < 1$ for certain $\nu \in]0,1[$ and the required assertion follows from the general result below on the exit probability for the linear equation (6).

Proposition 4 *Assume that the following conditions hold:*
(a) *there is a constant* $\nu \in]0,1[$ *such that* $E\lambda_1^\nu = q < 1$ *and* $E|\eta_n - \xi_n|^\nu < \infty$*;*
(b) $P(\xi_1 > x) > 0$ *for any* x.
Then $P(T^u < \infty) = 1$ *for every* u.

Proof. Put $\mathcal{E}_k^n := \mathcal{E}_n/\mathcal{E}_k$,

$$S_n(p) := \sum_{k=n-p+1}^{n} \mathcal{E}_k^n(\eta_k - \xi_k),$$

and $\Delta_n(p) := S_n - S_n(p)$, $p \in \mathbf{N}$, $p \le n$. Then

$$\Delta_n(p) = \mathcal{E}_{n-p}^n \left(\mathcal{E}_{n-p} u + \sum_{k=1}^{n-p} \mathcal{E}_k^{n-p}(\eta_k - \xi_k) \right) = \mathcal{E}_{n-p}^n S_{n-p}.$$

Since

$$|S_n|^\nu \le \lambda_n^\nu |S_{n-1}|^\nu + |\eta_n - \xi_n|^\nu$$

and λ_n and S_{n-1} are independent, we obtain from (a) that $E|S_n|^\nu < C$ and $E|\Delta_n(p)|^\nu < Cq^p$ for some constant C.

Let $A_n := \{S_n > 0\}$. For any $\varepsilon > 0$ the set $\cap_{i\le m} A_{pi}$ is a subset of

$$\bigcap_{i\le m} (\{S_{pi}(p) > -\varepsilon\} \cup \{\Delta_{pi}(p) > \varepsilon\}) \subseteq \bigcup_{i\le m} \{\Delta_{pi}(p) > \varepsilon\} \cup \bigcap_{i \le m} \{S_{pi}(p) > -\varepsilon\}.$$

Since $S_{pi}(p)$, $i = 1, 2, ...$, is a sequence of i.i.d. random variables, it follows that

$$P(T^u = \infty) \le P\left(\cap_{i\le m} A_{pi}\right) \le mC\varepsilon^{-\nu} q^p + (P\{S_p(p) > -\varepsilon\})^m. \qquad (11)$$

Notice that the distribution of $S_p(p)$ coincides with the distribution of

$$\vartheta_p := \sum_{k=1}^{p} \mathcal{E}_{k-1}(\eta_k - \xi_k).$$

As ϑ_p is a partial sum of a series absolutely convergent in L^ν, the sequence ϑ_p converges a.s. to a finite random variable ϑ_∞ which takes negative values with positive probability (because ξ_1 is independent of all other random variables and satisfies (b)). Thus, taking the limit in p, we get that

$$P(T^u = \infty) \le P(\vartheta_\infty > -\varepsilon)^m.$$

Choosing ε small enough to ensure that $P(\vartheta_\infty > -\varepsilon) < 1$ and letting m tend to infinity, we obtain the result. □

Remark. One can extend the above arguments and show that S is a Harris-recurrent, hence, ergodic process. The distribution of ϑ is its invariant measure.

5 Small volatility: decay of the ruin probability

Assume that the claims are exponentially distributed, i.e. $F(x) = 1 - e^{-x/\mu}$. Similarly to the classical case, this assumption allows us to obtain for the ruin probability an ordinary differential equation (but of a higher order). Indeed, now the equation (2) is

$$\frac{1}{2}\sigma^2 u^2 \Phi''(u) + (au + c)\Phi'(u) - \alpha\Phi(u) + \frac{\alpha}{\mu}\int_0^u \Phi(u-y)e^{-y/\mu}dy = 0. \quad (12)$$

Notice that

$$\frac{d}{du}\int_0^u \Phi(u-y)e^{-y/\mu}dy = \Phi(u) - \frac{1}{\mu}\int_0^u \Phi(u-y)e^{-y/\mu}dy.$$

Differentiating (12) and excluding the integral term we arrive to a third order differential equation. The good news is that it does not contain the function itself. In other words, we obtain a second order differential equation for $G = \Phi'$ which can be written as

$$G'' + p(u)G' + q(u)G = 0, \quad (13)$$

where

$$p(u) := \frac{1}{\mu} + 2\left(1 + \frac{a}{\sigma^2}\right)\frac{1}{u} + \frac{2c}{\sigma^2}\frac{1}{u^2},$$
$$q(u) := \frac{2a}{\mu\sigma^2}\frac{1}{u} + \left(a - \alpha + \frac{c}{\mu}\right)\frac{2}{\sigma^2}\frac{1}{u^2}.$$

The substitution $G(u) = R(u)Z(u/(2\mu))$ with

$$R(u) := \exp\left\{-\frac{1}{2}\int_1^u p(s)ds\right\}$$

eliminates the first derivative and yields the equation

$$Z'' - (1 + Q_u)Z = 0$$

where

$$Q_u := 2\left(1 - \frac{a}{\sigma^2}\right)\frac{1}{u} + \sum_{i=2}^{4} A_i \frac{1}{u^i}$$

with certain constants A_i which are of no importance.

Notice that Q^2 is integrable at infinity and hence, according to [3], pp. 54-55, the equation has a fundamental solution

$$Z_{\pm}(u) = \exp\left\{\pm\left(u + \frac{1}{2}\int_1^u \tilde{Q}_r dr\right)\right\}(1 + o(1)) = e^{\pm u}u^{\pm(1-a/\sigma^2)}(1 + o(1))$$

as $u \to \infty$. Since $R(u) = e^{-\frac{1}{2\mu}u} u^{-(1+a/\sigma^2)} f(u)$, where f is a decreasing function on $[1, \infty[$ bounded away from zero, $f(1) = e^{\frac{1}{2\mu}}$, we obtain that (13) admits, as solutions, functions with the following asymptotics:

$$G_+(u) = u^{-2a/\sigma^2}(1 + o(1)),$$
$$G_-(u) = u^{-2} e^{-\frac{1}{\mu}u}(1 + o(1)), \quad u \to \infty.$$

The differential equation of the third order for Φ has the solutions $\Phi_0(u) = 1$ and

$$\Phi_+(u) = \int_u^\infty r^{-2a/\sigma^2}(1 + \beta_1(r))\, dr,$$
$$\Phi_-(u) = \int_u^\infty r^{-2} e^{-\frac{1}{\mu}r}(1 + \beta_2(r))\, dr,$$

where $\beta_i(r) \to 0$ as $r \to \infty$. The ruin probability $\Psi := 1 - \Phi$ is the linear combination of these functions, i.e.

$$\Psi(u) = C_0 + C_1\Phi_+(u) + C_2\Phi_-(u).$$

For the case $\rho > 1$ we know from Proposition 1 (i) that $\Psi(\infty) = 0$. Thus,

$$\Psi(u) = C_1 \int_u^\infty r^{-\rho}(1 + \beta_1(r))\, dr + C_2 \int_u^\infty r^{-2} e^{-\frac{1}{\mu}r}(1 + \beta_2(r))\, dr.$$

The first integral decreases at infinity as the power function $u^{1-\rho}/(1-\rho)$ while the second is exponentially decreasing. But Proposition 1 (ii) asserts that Ψ behaves at infinity as a power function. This implies that $C_1 \neq 0$ and we obtain the assertion (i) of Theorem 1. $\Box$

References

1. Asmussen S. *Ruin Probabilities.* World Scientific, Singapore, 2000.
2. Grandell I. *Aspects of Risk theory.* Springer, Berlin, 1990.
3. Fedoryuk M.V. *Asymptotic analysis: linear ordinary differential equations.* Springer, Berlin, 1993.
4. Frolova A.G. Some mathematical models of risk theory. All-Russian School-Colloquium on Stoch. Methods in Geometry and Analysis. Abstracts, 1994, 117-118.
5. Kalashnikov V. Ruin probability under random interest rate. Manuscript, 1999.
6. Kalashnikov V., Norberg R. Power tailed ruin probabilities in the presence of risky investments. Preprint. Laboratory of Actuarial Math., Univ. of Copenhagen, 2000.
7. Nyrhinen H. On the ruin probabilities in a general economic environment. *Stoch. Proc. Appl.*, **83** (1999), 319-330.
8. Nyrhinen H. Finite and infinite time ruin probabilities in a stochastic economic environment. *Stoch. Proc. Appl.*, **92** (2001), 265-285.
9. Paulsen J. *Stochastic Calculus with Applications to Risk Theory.* Lecture Notes, Univ. of Bergen and Univ. of Copenhagen, 1996.
10. Paulsen J. Sharp conditions for certain ruin in a risk process with stochastic return on investments. *Stoch. Proc. Appl.*, **75** (1998), 135-148.
11. Paulsen J., Gjessing H. K. Ruin theory with stochastic return on investments. *Adv. Appl. Probab.*, 29 (1997), 4, 965-985.
12. Shiryaev A.N. *Essentials of Stochastic Finance.* World Scientific, Singapore, 1999.