IN WHAT SPACES IS EVERY CLOSED NORMAL CONE REGULAR?

by C. W. McARTHUR †

(Received 1st September 1969)

1. Introduction

It is known (**13**, p. 92) that each closed normal cone in a weakly sequentially complete locally convex space is regular and fully regular. Part of the main theorem of this paper shows that a certain amount of weak sequential completeness is necessary in order that each closed normal cone be regular. Specifically, it is shown that each closed normal cone in a Fréchet space is regular if and only if each closed subspace with an unconditional basis is weakly sequentially complete. If E is a strongly separable conjugate of a Banach space it is shown that each closed normal cone in E is fully regular. If E is a Banach space with an unconditional basis it is shown that each closed normal cone in E is fully regular if and only if E is the conjugate of a Banach space.

Throughout this paper all spaces dealt with will be real Fréchet spaces (locally convex, complete, metrizable). A *cone* in a real vector space E is a set K such that $K+K \subset K$, $\lambda K \subset K$ for $\lambda \geqq 0$, and $K \cap -K = \{\theta\}$. If K is a cone in E, then its *dual wedge* K' is defined to be the set of those continuous linear functionals on E which are non-negative on K. Corresponding to the cone K in E is the partial ordering " $\leqq$ " defined on E by $x \leqq y$ for $x, y \in E$ means $y-x \in K$. A cone K in E is *normal* if there is a family of seminorms $P = \{p\}$ which generates the topology of E and which has the property that if $p \in P$ and $x, y \in K$ with $x \leqq y$ then $p(x) \leqq p(y)$. A cone K is *regular* if each non-decreasing sequence in K which is majorised by an element of K converges. A cone is *fully regular* if each non-decreasing bounded sequence in K converges. Krasnosel'skiĭ (**9**) has shown that closed fully regular cones in Banach spaces are regular and closed regular cones are normal. It is in fact true (**11**) that in complete metric linear spaces closed regular cones are normal. On the other hand the natural positive cone in (m) is an example of a closed normal cone which is not regular.

2. The Main Theorem

Let ω be the set of positive integers and $\mathscr{F}(\omega)$ the finite subsets of ω. A series $\sum_{i=1}^{\infty} x_i$ in E is *unordered bounded* if and only if the set $\{\sum_{i \in \sigma} x_i : \sigma \in \mathscr{F}(\omega)\}$

† This research was supported by the National Science Foundation Grant NSF-GP-9632.

is bounded. It is well known that $\sum_{i=1}^{\infty} x_i$ is unordered bounded if and only if $\sum_{i=1}^{\infty} |f(x_i)| < +\infty$ for all $f \in E'$. A sequence $\{z_n\}$ in E is a (topological) *basis* for E if and only if corresponding to each $x \in E$ there is a unique sequence of scalars $\{a_i\}$ such that $x = \sum_{i=1}^{\infty} a_i z_i$. A sequence $\{z_n\}$ in E is *basic* if and only if $\{z_n\}$ is a basis for its closed linear span which we denote by $[z_n]$. The following fundamental lemma is due to Bessaga and Pełczyński (**2**; **12**).

Lemma 1. *For a Fréchet space E the following conditions are equivalent:*

(i) *there exists in E an unordered bounded series which is not unconditionally convergent;*

(ii) *there exists in E an unordered bounded series* $\sum_{i=1}^{\infty} x_i$ *and a neighbourhood U of* θ *such that* $x_i \notin U$, $i = 1, 2, \ldots$;

(iii) *E contains a subspace* E_0 *which is isomorphic to* (c_0), (*i.e. there is a linear homeomorphism of* (c_0) *onto* E_0).

Generalising work of (**6**, **3**, **4**) the following lemma has been shown (**5**, **16**).

Lemma 2. *If E is a Fréchet space with an unconditional basis the following are equivalent:*

(a) *E has no subspace isomorphic to* (c_0);
(b) *E is weakly sequentially complete;*
(c) *the basis is boundedly complete.*

Theorem 1. *If E is a Fréchet space the following are equivalent:*

(i) *each closed normal cone is regular;*
(ii) *each closed normal cone is fully regular;*
(iii) *E has no subspace isomorphic to* (c_0);
(iv) *each closed subspace of E with an unconditional basis is weakly sequentially complete;*
(v) *each unconditionally basic sequence in E is boundedly complete.*

Proof. That (ii) implies (i) is immediate, since order-intervals determined by normal cones are bounded.

(i) implies (iii). Suppose E has a subspace E_0 isomorphic to (c_0). Then (**1**, p. 181) since (c_0) and (c) are isomorphic there exists a linear homeomorphism T from (c) on to E_0. Let $K_c = \{a = \{a_i\} \in (c): a_i \geqq 0, i = 1, 2, \ldots\}$. Let e_i, $i = 1, 2, \ldots$ denote the unit vectors in (c) and let e denote the element of (c) all of whose coordinates are 1. Now $\left\{\sum_{i=1}^{n} e_i\right\}_{n \in \omega}$ is a non-decreasing sequence in K_c majorised by e so K_c is not regular but K_c is closed and normal

https://doi.org/10.1017/S001309150000938X Published online by Cambridge University Press

(since the usual norm for (c) is monotone on K_c). It follows that the cone $T(K_c)$ is closed and normal but non-regular in E.

(iii) implies (ii). Suppose K is a closed, normal non-fully regular cone in E. Thus there exists a sequence $\{x_n\}$ in K which is bounded, nonconvergent and $\theta \leqq x_1 \leqq x_2 \leqq \ldots \leqq x_n \leqq x_{n+1} \leqq \ldots$ where the partial order is defined by K. It follows that there exists a neighbourhood U of θ and a sequence of positive integers $p_1 < q_1 < p_2 < q_2 < \ldots < p_n < q_n < \ldots$ such that if $y_n = x_{q_n} - x_{p_n}$ then $y_n \notin U$. Note that $y_n \in K$. Also for arbitrary $\sigma \in \mathscr{F}(\omega)$ there exists $m \in \omega$, e.g. $m = \max\{q_n : n \in \sigma\}$, such that $\sum_{n \in \sigma} y_n \leqq x_m$. Thus if $f \in K'$,

$$0 \leqq f\Big(\sum_{n \in \sigma} y_n\Big) \leqq \lim_m f(x_m) < +\infty$$

since $\{x_n\}$ is non-decreasing and bounded and $f \in K'$. Since K is normal, $E' = K' - K'$, so if $f \in E'$, $f = f_1 - f_2$ for some $f_1, f_2 \in K'$ and

$$\Big| f\Big(\sum_{n \in \sigma} y_n\Big)\Big| \leqq \lim_m f_1(x_m) + \lim_m f_2(x_m) < +\infty.$$

Hence the set $\{\sum_{n \in \sigma} y_n : \sigma \in \mathscr{F}(\omega)\}$ is weakly bounded and therefore bounded. The series $\sum_{i=1}^{\infty} y_i$ is unordered bounded and there is a neighbourhood U of θ such that $y_i \notin U$, $i = 1, 2, \ldots$. By Lemma 1, E has a subspace isomorphic to (c_0).

We have now shown that (i), (ii) and (iii) are equivalent. Recalling that (c_0) is not weakly sequentially complete and that the unit vector basis for (c_0) is unconditional the equivalence of (iii), (iv) and (v) is clear by Lemma 2.

3. Applications

The following theorem strengthens a theorem of Karlin (**8**; **13**, p. 98).

Theorem 2. *Let E be a Banach space whose topological dual space E' is separable with the strong topology. Then with respect to the strong topology each closed normal cone in E' is fully regular and hence regular.*

Proof. It has been shown (**2**) that if E' has a subspace isomorphic to (c_0) then it has a subspace isomorphic to (m) so E' could not be separable since (m) is not separable. Thus, since E' is separable, E' has no subspace isomorphic to (c_0) and the conclusion is given by Theorem 1.

Theorem 3. *If E is a Banach space with an unconditional basis the following are equivalent:*

(i) *each closed normal cone in E is fully regular;*
(ii) *E is weakly sequentially complete;*
(iii) *E is isomorphic to the strong topological dual of a Banach space.*

Proof. That (i) implies (ii) follows from Theorem 1 and that (ii) implies (i) (**13**, p. 92) is true for locally convex spaces in general. By Theorem 2, we have (iii) implies (i). Conversely, if (i) holds then by Lemma 2 the basis is boundedly complete so (**7**, Theorem 10) (iii) follows.

A Banach space E has property (u) if and only if for every weak Cauchy sequence $\{x_n\} \subset E$ there is a sequence $\{x'_n\} \subset E$ such that

(i) $\sum_{n=1}^{\infty} |f(x'_n)| < +\infty$ for each $f \in E'$;

(ii) the sequence $x_n - \sum_{i=1}^{n} x'_i$ converges weakly to θ. Pełczyński (**14**) has shown that if E is a Banach space with property (u) then E is weakly sequentially complete if and only if no subspace of E is isomorphic to (c_0). He shows that a Banach space with an unconditional basis has property (u), (**14**). A Banach space E is called a *cyclic space* if E is the closed linear span of the set

$$\{P(x_0): P \in B\}$$

for some $x_0 \in E$ and a σ-complete (not necessarily atomic) Boolean algebra of projections B on E. A Banach space with an unconditional basis is a cyclic space and any cyclic Banach space has property (u) (**15**, Lemma 2). The following theorem is immediate from the above results and Theorem 1.

Theorem 4. *If the Banach space E has the property* (u), *then each closed normal cone in E is fully regular if and only if E is weakly sequentially complete.*

Theorem 5. *If E is a Banach lattice then each closed normal cone in E is fully regular if and only if E is weakly sequentially complete.*

Proof. It has been shown (**10**) that if E is a Banach lattice then E is weakly sequentially complete if and only if E contains no subspace isomorphic to (c_0). The conclusion follows from this and Theorem 1.

REFERENCES

(1) S. Banach, *Théorie des opérations linéaires*, Warszawa, 1933.

(2) C. Bessaga and A. Pełczyński, On bases and unconditional convergence of series in Banach spaces, *Studia Math.* **17** (1958), 151-164.

(3) C. Bessaga and A. Pełczyński, A generalization of results of R. C. James concerning absolute bases in Banach spaces, *Studia Math.* **17** (1958), 165-174.

(4) M. M. Day, *Normed linear spaces* (Springer-Verlag, Berlin-Gottingen-Heidelberg, 1962).

(5) E. Dubinsky and J. R. Retherford, Schauder bases and Köthe sequence spaces, *Trans. Amer. Math. Soc.* **130** (1968), 265-280.

(6) R. C. James, Bases and reflexivity of Banach spaces, *Ann. of Math.* (2) **52** (1950), 518-527.

(7) S. KARLIN, Bases in Banach spaces, *Duke Math. J.* **15** (1948), 971-985.

(8) S. KARLIN, Positive operators, *J. Math. Mech.* **8** (1959), 907-935.

(9) M. A. KRASNOSEL'SKIĬ, *Positive solutions of operator equations* (Noordhoff, Groningen, 1964).

(10) G. JA. LOZANOVSKIĬ, *Funkcional. Anal. i Priložen.* **1** (1967), *no.* 3, 92. MR **36** # 3110.

(11) C. W. MCARTHUR, Convergence of monotone nets in ordered topological vector spaces, *Studia Math.*, **34** (1970), 1-16.

(12) C. W. MCARTHUR and J. R. RETHERFORD, Some applications of an inequality in locally convex spaces, *Trans. Amer. Math. Soc.* **137** (1969), 115-123.

(13) A. L. PERESSINI, *Ordered topological vector spaces* (Harper & Row, New York and London, 1967).

(14) A. PEŁCZYŃSKI, A connection between weakly unconditional convergence and weakly completeness of Banach spaces, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys.* **6** (1958), 251-253.

(15) L. TZAFRIRI, Reflexivity in Banach lattices and their subspaces, to appear.

(16) L. J. WEILL, *Unconditional bases in locally convex spaces* (Dissertation, Florida State University, 1966).

THE FLORIDA STATE UNIVERSITY
TALLAHASSEE, FLORIDA

https://doi.org/10.1017/S001309150000938X Published online by Cambridge University Press