Inapproximability of the Perimeter Defense Problem

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Abstract

We model the problem of detecting intruders using a set of infrared beams by the perimeter defense problem: given a polygon P, find a minimum set of edges $\mathcal{S}$ of the polygon such that any straight line segment crossing the polygon intersects at least one of the edges in $\mathcal{S}$. We observe that this problem is equivalent to a new hiding problem, the **M**ax-**H**idden-**E**dge-**S**et problem. We prove the APX-hardness of the **M**ax-**H**idden-**E**dge-**S**et problem for polygons without holes and rectilinear polygons without holes, by providing gap-preserving reductions from the **M**ax-**5**-**O**ccurrence-**2**-**S**at problem.

1 Introduction

Given a region of interest to be defended, we are interested in detecting the presence of an intruder inside the region who originated from outside the region. We model the region of interest by a polygon, and the trajectory of the intruder by a curve intersecting the interior of the polygon. For arbitrary curves, or for line segments that can terminate inside the polygon, there is no choice but to defend the entire perimeter of the polygon. Therefore, we consider the case when the path of the intruder is a straight line segment that *crosses* the polygon (intersects the perimeter of the polygon in at least two distinct edges) and require the intruder to be detected before exiting the polygon.

Infrared beam sensors are an increasingly popular way of achieving intruder detection. Such a device consists of a matched transmitter-receiver pair; the transmitter emits an infrared beam to a receiver module. Usually the beam distance can be adjusted. An intruder going across the beam would interrupt the circuit and be detected. In several applications, it may make sense to place the beams only on the perimeter of the polygon, as allowing beams to intersect either the interior or the exterior of the polygon may lead to false alarms. We are interested in minimizing the number of infrared beams to be placed on the perimeter that are required to ensure that any intruder L whose path crosses the polygon will be detected. This implies that the transmitter and receiver should be placed on adjacent vertices of the polygon, so that the beam is aligned with the edge between them. Our intruder detection problem can therefore be modeled as follows:

Definition 1 ***M****inimum-**E**dge-**P**erimeter-**D**efense (MEPD): Given a polygon P, find a minimum-sized subset $\mathcal{S}$ of edges of P such that any straight line segment L crossing P intersects at least one edge in $\mathcal{S}$.*

It is not difficult to see that this problem can be reduced to a *hiding* problem, i.e. finding a maximum-sized subset of mutually invisible edges of the polygon [1]. Indeed $\mathcal{S}$ is a solution to the perimeter defense problem if and only if all elements in $\overline{\mathcal{S}}$ are mutually invisible. In what follows, we focus on the **M**ax-**H**idden-**E**dge-**S**et problem:

Definition 2 ***M****ax-**H**idden-**E**dge-**S**et (MHES): Given a polygon P, find a maximum-sized subset of mutually invisible edges of the polygon.*

Guarding and hiding problems have been studied extensively in the literature. The *Maximum Hidden Set* (MHS) problem introduced in [2] is to find a maximum-sized set of mutually invisible points in a polygon. In the *Maximum Hidden Vertex Set* ($MHVS$), the points are constrained to be vertices of the polygon. Hiding and guarding problems are combined in the *Minimum Hidden Guard Set* ($MHGS$) and the *Minimum Hidden Vertex Guard Set* ($MHVGS$) and *Hidden Vertex Guard Admissibility* problems. All these problems were shown to be NP-complete and lower and upper bounds for their approximation ratios were given in [2]. The restriction of the problem instance to a terrain was proved to be NP-complete in [4].

In [4, 7] it was shown that for polygons with holes or terrains, the MHS and $MHVS$ problems cannot be approximated by a polynomial time algorithm with an approximation ratio of n^ϵ for some $\epsilon > 0$. For polygons without holes, these problems were shown to be APX-hard. Recently, Eidenbenz [8] presented an inapproximability result for the $MHGS$ problem. He proved

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[1] Two edges e_1 and e_2 are invisible from each other iff for every $p_1 \in e_1$ and $p_2 \in e_2$ such that the line connecting p_1 and p_2 lies entirely within the polygon, at least one of p_1 and p_2 is an endpoint of its edge.

that for input polygons with or without holes or terrains, the $MHGS$ problem is also APX-hard. Notice that the $MHVGS$ problem is a much harder problem. It is NP-hard to even determine whether a feasible solution exists [2].

To our knowledge, the complexity of the MHES problem has not been studied. Since a set of mutually invisible edges in a polygon is an independent set of vertices in the visibility graph of the edges of the polygon, the MHES problem can be reduced to the Maximum Independent Set problem, and therefore is approximable with an $O(n(\log\log n)^2/\log^3 n)$ approximation ratio [3]. Not every graph is a visibility graph of a polygon (for example, $K_{2,3}$), and therefore, the reduction does not go through in the other direction.

Our Results

In this paper, we prove that the MHES problem is APX-hard for polygons without holes. The proof is using a reduction from **M**ax-**5**-**O**ccurrence-**2**-**S**at problem, which was shown to be APX-hard in [5, 6]. In fact, we show that the MHES problem is APX-hard even when restricted to rectilinear polygons without holes. It follows that the MEPD problem is also APX-hard even for rectilinear polygons without holes. Due to space limitations, we omit many of the proofs. The interested reader can find all details in [1].

2 APX-hardness of MHES for an Arbitrary Polygon

In this section, we show that the **M**ax-**5**-**O**ccurrence-**2**-**S**at problem is transformable in polynomial time to MHES by an approximation-preserving (gap-preserving) reduction [9].

Definition 3 *Let Φ be a boolean formula given in conjunctive normal form, with at most two literals in each clause and each variable appearing in at most 5 five clauses. The* ***M****ax-****5****-****O****ccurrence-****2****-****S****at problem consists of finding a truth assignment for the variables of Φ such that the number of satisfied clauses is maximum.*

The goal is to accept an instance of **M**ax-**5**-**O**ccurrence-**2**-**S**at as input and in polynomial time to construct a connected simple polygonal region P such that the difference in the number of hidden edges obtained by the optimal and approximation algorithms preserves the gap between the optimal and approximate results (the number of satisfied clauses) in **M**ax-**5**-**O**ccurrence-**2**-**S**at . The construction is similar to the one proposed in [7]. As shown in Figure 1, the main body is a convex polygon without holes inside; we refer to it as the *center polygon*. For each clause, a *clause pattern* is built on the top right of the center polygon, and for each variable, a *variable pattern* is built on the bottom left of the center polygon. Variable patterns are separated by the *cb*-edges and form a convex curve along the center polygon's bottom. A basic unit in both types of patterns is a *dent*: a set of continuous line segments that form a convex shape. It is clear that at most one edge from any dent can be included in the MHES . Now we show how to construct the clause and variable patterns.

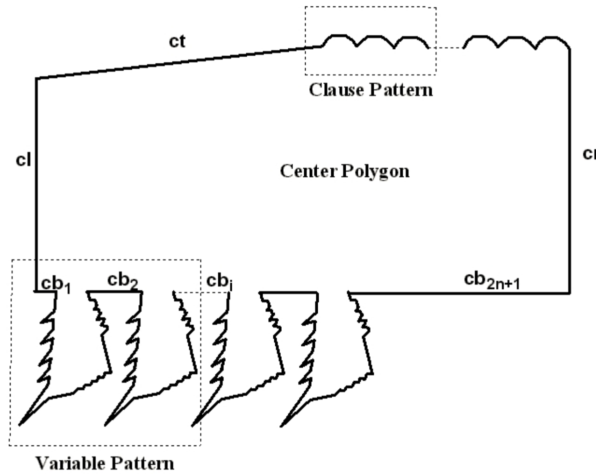

Figure 1: Overview of Construction (Arbitrary Polygon)

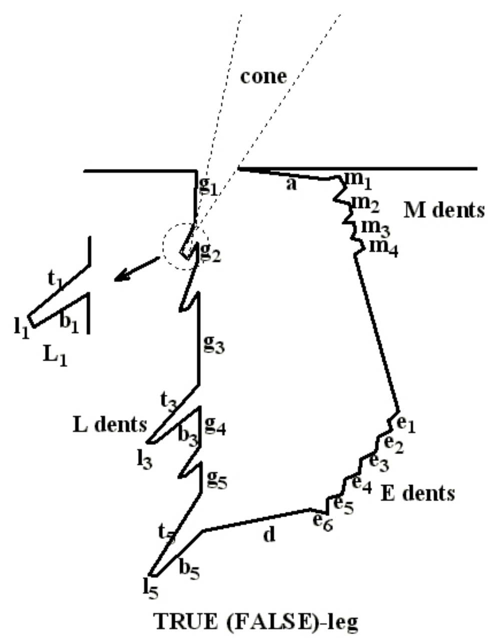

Figure 2: Variable Pattern (Arbitrary Polygon)

Clause Patterns: The clause pattern is shown in Figure 3. Each pattern consists of 15 adjacent edges form-

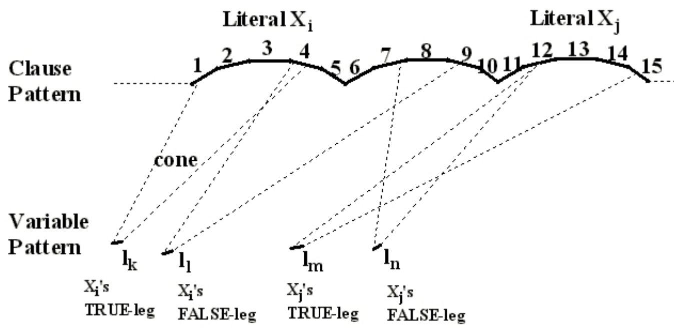

Figure 3: Relationship between Clause and Variable Patterns (Arbitrary Polygon)

ing three dents. Each dent has exactly five edges. Without loss of generality, we assume that all clauses contain two literals. Then for each clause we use the left and right dents to represent the two literals and the middle dent to represent the satisfiability of the clause.

Variable Patterns: The variable pattern is shown in Figure 2. Each variable has a TRUE-leg and a FALSE-leg, each consisting of three components: L-dents, M-dents and E-dents. Observe that no two dents in the same component can see each other. Five L-dents are arranged along a line, separated by g-edges, and each L-dent (L_i) consists of one t-edge (t_i), one b-edge (b_i) and one l-edge (l_i). Each occurrence of a variable matches a pair of L-dents, one from the variable's TRUE-leg and the other from the variable's FALSE-leg. Since each variable appears in at most 5 clauses, we only need five L-dents for each leg. Opposite to the L-dents are four adjacent M-dents and six adjacent E-dents. Each M-dent consists of three edges. We denote the middle edge for a dent M_i by m_i. Finally, six E-dents form a simple zig-zag line under the M-dents. We pick up every alternate edge from this collection of dents and denote the edge chosen from E_i by e_i. By choosing the length and direction of each edge appropriately, each leg of the variable pattern can be constructed with the following properties.

P1 For $1 \leq i \leq 5$, t_i, g_i, a and d can each see all the edges in component M and E, and edge g_i can see d and a.

P2 For $1 \leq i \leq 5$, b_i and l_i see all the edges in component M and edge a, but cannot see any edge in component E or edge d.

P3 For $1 \leq i \leq 4$, m_i can see all the edges in component L and g, but no edge in component E.

P4 For $1 \leq i \leq 5$, if l_i does not match any occurrence of a variable, l_i sees only the M-dent and edge a.

P5 Components M and E are angled so that they cannot be seen by any clause pattern.

The last step of the construction is to establish the relationship between the clause and variable patterns. As shown in Figure 3, we connect them by cones. Each cone starts at an l-edge and ends at a clause pattern and the clause pattern's edges inside the cone are visible to the l-edge. Consequently, if we add an l-edge (cone's bottom) to the hidden edge set, we cannot add any edge at the top of the cone. Further, the cones are overlapped in specific ways, as shown in Figure 3.

Next we show the relationship between a satisfying assignment for a **M**ax-**5**-**O**ccurrence-**2**-**S**at instance and a hidden edge set for the corresponding polygon. Let I be an instance of the **M**ax-**5**-**O**ccurrence-**2**-**S**at problem with n variables and m clauses and I' be the instance of the corresponding MHES problem. We assume without loss of generality that every variable has more than one occurrence in I (if not, then the unique clause containing the variable can definitely be satisfied).

Lemma 1 *If I has an assignment S that satisfies at least $(1-\epsilon)m$ clauses, then I' has a solution S' with at least $21n + 2m + (1-\epsilon)m$ edges.*

Proof. For a variable with a TRUE assignment, we add all the i-edges and e-edges from its TRUE-leg and all the m-edges and e-edges from its FALSE-leg (and vice-versa for a variable with a FALSE assignment). So no matter what the truth value is, we can add 5 l-edges, 4 m-edges and 12 e-edges. Since there are n variable patterns, we add $21n$ edges to S'.

Next we add edges from the clause patterns. We show that we can always add 3 edges for each satisfied clause and 2 edges for each unsatisfied clause. Consider a clause with two literals (x_i, x_j), see Figure 3 (other cases,$(x_i, \overline{x_j})$, $(\overline{x_i}, x_j)$ and $(\overline{x_i}, \overline{x_j})$, are shown in [1]). We examine all the possible assignments of x_i and x_j in S.

Suppose x_i and x_j are both TRUE. Then, all l-edges in the TRUE-legs of variable patterns corresponding to x_i and x_j have already been added to S'. Therefore we cannot add any of the edges 1-4 or 12-15 from the clause pattern to S', because all these edges are visible to the l-edges mentioned above. On the other hand, no l-edge of both variables' FALSE-legs belongs to S', thus any of edges 5-11 can be added to S'. Since for each dent at most one edge belongs to the hidden set, we can add edges 5, 11, and any of the edges 6-10 to S'. A similar analysis can be used for all other truth value combinations for x_i and x_j to show each unsatisfied clause contributes 2 edges and each satisfied clause contributes 3 edges to S'. Since S satisfies at least $(1-\epsilon)m$ clauses, we conclude that S' has at least $21n+2(m-(1-\epsilon)m)+3(1-\epsilon)m = 21n+2m+(1-\epsilon)m$ edges. □

The next two lemmas give us the relationship in the other direction (see proofs in [1]).

Lemma 2 *Given a solution S' to the MHES instance I', without decreasing the number of the hidden edges, we can transform it such that the contribution from each variable pattern leg to S' is all its e-edges and either some subset of its l- or all of its m-edges.*

Lemma 3 *If the MHES instance I' has a solution S' with at least $21n + 3m - (\epsilon + \gamma)m$ edges, then the corresponding **M**ax-**5**-**O**ccurrence-**2**-**S**at instance I has an assignment S which satisfies at least $(1-\epsilon-\gamma)m$ clauses.*

Now we show the APX-hardness of the MHES problem. Let I be an instance of the **M**ax-**5**-**O**ccurrence-**2**-**S**at problem with n variables and m clauses and I' be the instance of the corresponding MHES problem. We denote the optimal solutions for these problems by $OPT(I)$ and $OPT(I')$ respectively. From Lemma 1 and Lemma 3 we have:

1. $|OPT(I)| \geq (1-\epsilon)m \rightarrow |OPT(I')| \geq 21n + 2m + (1-\epsilon)m$
2. $|OPT(I)| < (1-\epsilon-\gamma)m \rightarrow |OPT(I')| < 21n + 3m - (\epsilon+\gamma)m$

It is known that **M**ax-**5**-**O**ccurrence-**2**-**S**at is APX-hard. Therefore, for an instance I such that either $|OPT(I)| \geq (1-\epsilon)m$ or $|OPT(I)| < (1-\epsilon-\gamma)m$ for some constants $\epsilon, \gamma > 0$, it is NP-hard to decide which case is true. We claim that unless P=NP, no polynomial time approximation algorithm for MHES can achieve an approximation ratio better than $\frac{21n+2m+(1-\epsilon)m}{21n+3m-(\epsilon+\gamma)m}$. Suppose to the contrary, that a polynomial time approximation algorithm denoted by APO has a performance ratio $< \frac{21n+2m+(1-\epsilon)m}{21n+3m-(\epsilon+\gamma)m}$. Given an instance of **M**ax-**5**-**O**ccurrence-**2**-**S**at such that either $|OPT(I)| \geq (1-\epsilon)m$ or $|OPT(I)| < (1-\epsilon-\gamma)m$ for some constants $\epsilon, \gamma > 0$, we apply our reduction to obtain an instance I' of MHES . If $|APO(I')| \geq 21n + 3m - (\epsilon+\gamma)m$ then $|OPT(I')| \geq 21n+3m-(\epsilon+\gamma)m$, which further means $|OPT(I)| \geq (1-\epsilon-\gamma)m$ (because of (2) above). Because I can belong only to one of the two categories, we know that $|OPT(I)| \geq (1-\epsilon)m$. If instead $|APO(I')| < 21n + 3m - (\epsilon+\gamma)m$, then $\frac{|OPT(I')|}{|APO(I')|} < \frac{21n+2m+(1-\epsilon)m}{21n+3m-(\epsilon+\gamma)m}$ implies $|OPT(I')| < \frac{21n+2m+(1-\epsilon)m}{21n+3m-(\epsilon+\gamma)m}|APO(I')| < 21n + 2m + (1-\epsilon)m$. Therefore $|OPT(I)| < (1-\epsilon)m$ which implies $|OPT(I)| < (1-\epsilon-\gamma)m$. Consequently, it will be possible to decide in polynomial time which category the instance I belongs to, which contradicts the APX-hardness of the **M**ax-**5**-**O**ccurrence-**2**-**S**at problem. We now calculate the ratio. Using the fact that $n < 2m$, we have

$$\begin{aligned}\frac{21n+2m+(1-\epsilon)m}{21n+3m-(\epsilon+\gamma)m} &= \frac{1}{1-\frac{\gamma m}{21n+3m-\epsilon m}} \\ &\geq \frac{1}{1-\frac{\gamma m}{42m+3m-\epsilon m}} \geq 1+\epsilon'\end{aligned} \tag{1}$$

Theorem 4 *There exists a constant $\epsilon > 0$ such that no polynomial time approximation algorithm for the MHES problem on polygons without holes can have an approximation ratio of $1+\epsilon$, unless $P = NP$.*

The construction above can be modified to use a rectilinear polygon, which gives the following result (see proof in [1]).

Theorem 5 *MHES is APX-hard even when restricted to a rectilinear polygon without holes.*

Corollary 6 *MEPD is APX-hard, even for rectilinear polygons without holes.*

The NP-hardness of MEPD problem follows immediately from Corollary 6.

3 Discussion

We proved the APX-hardness of the MHES problem for rectilinear polygons, which implies the APX-hardness of the MEPD problem. In light of the lower bound of $n^{1-O(1/(\log n)^{\gamma})}$ (where γ is a constant) for the approximation ratio for the Maximum Independent Set problem, it would be interesting to know if a higher lower bound also applies for our problems. The complexity of MHES for monotone rectilinear problems remains an interesting open problem.

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