Incentive-Based Pricing for Network Games with Complete and Incomplete Information

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Outline

- Previous Work
- A Significant New Direction — Dynamic Pricing
- Complete Information
- Incomplete Information
- Multiple Users: Incentive-Design Problem Formulation
- Conclusions and Extensions

Previous Work

- ➢ Başar and Srikant (2002a) — Hierarchical Stackelberg Game
- ➢ Başar and Srikant (2002b) — Linear Network
- ➢ Shen and Başar (2004a) — Differentiated Pricing
- ➢ Shen and Başar (2004b) — Incomplete Information

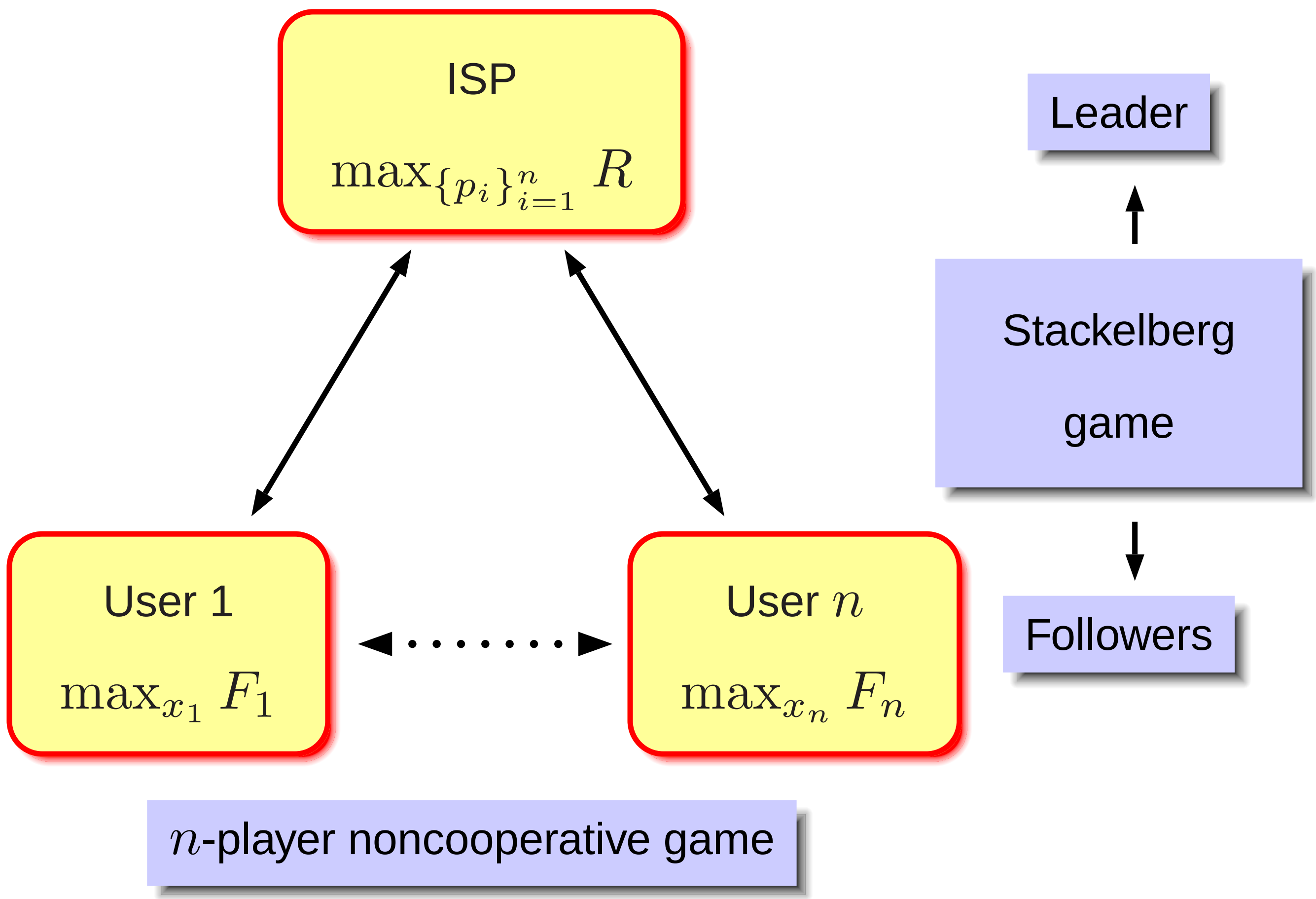
T. Başar and R. Srikant (2002a), "Revenue-maximizing pricing and capacity expansion in a many-users regime," *Proc. IEEE INFOCOM 2002,* pp. 1556–1563.

T. Başar and R. Srikant (2002b), "A Stackelberg network game with a large number of followers," *J. Optimization Theory and Applications,* 115(3): 479–490.

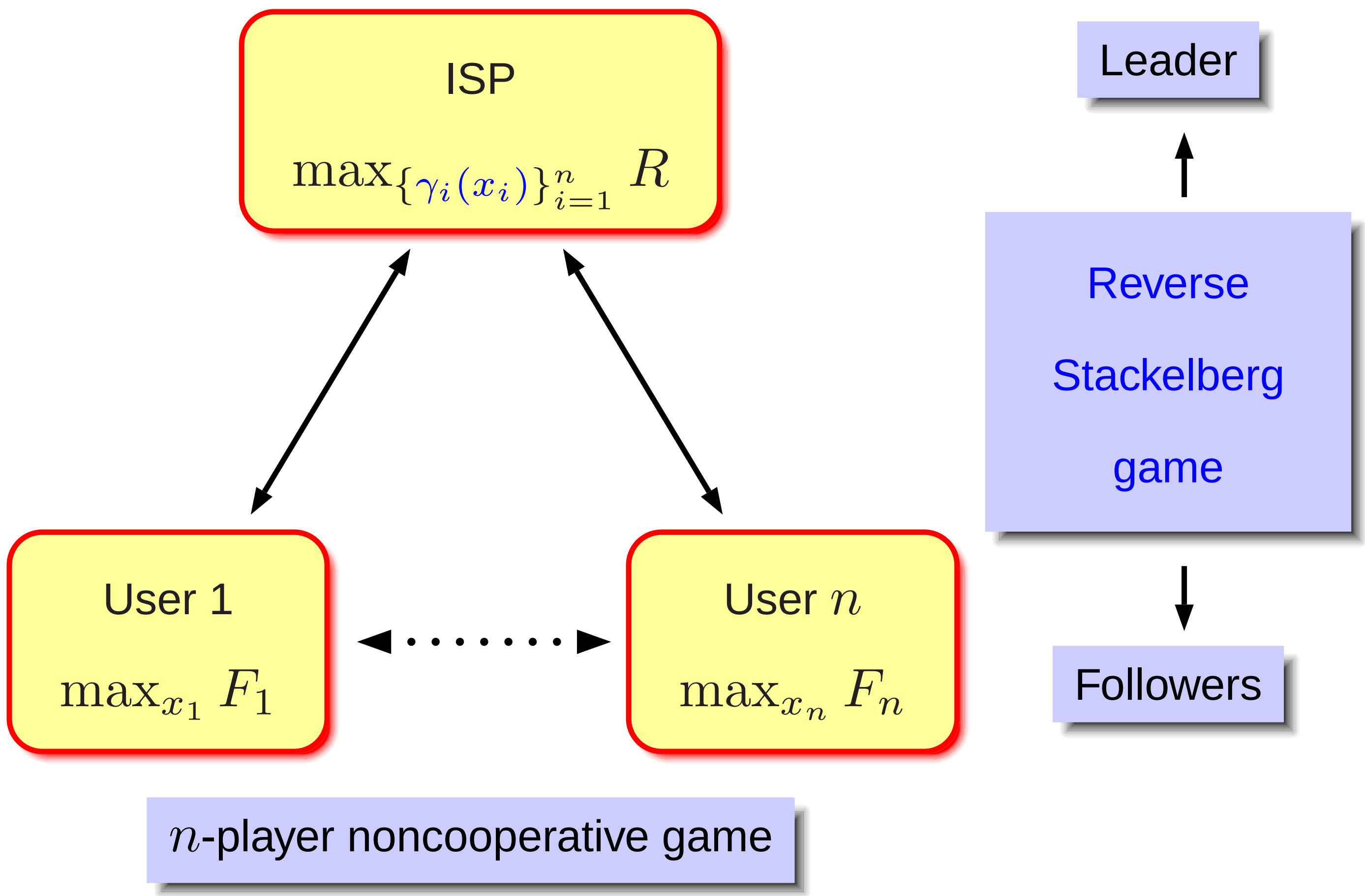
H.-X. Shen and T. Başar (2004a), "Differentiated Internet pricing using a hierarchical network game model," *Proc. IEEE ACC 2004,* pp. 2322–2327.

H.-X. Shen and T. Başar (2004b), "Network game with a probabilistic Description of User Types," to appear in *Proc. IEEE CDC 2004.*

Two-Level Hierarchical Stackelberg Game



A Significant New Direction — Dynamic Pricing



Incentive-Design Problem

- **Incentive Policy** — usage-based policy
- **Team Solution**
  — Pareto-optimal solution (action outcome desired by the leader)
- **Solution of the Incentive-Design Problem**
  — incentive policy to achieve the team solution
- **Incentive Controllability**
  — existence of the solution of the incentive-design problem
- **Complete Information** — the ISP knows the user types
  Incomplete Information — the ISP knows only the probability distribution

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T. Başar and G. J. Olsder (1999), *Dynamic Noncooperative Game Theory,* pp. 392–396.

Complete Information

- Incentive-Design Problem Formulation
- Team Solution
- Incentive-Design Problem Solution
- Incentive Controllability
- Team Solution vs Stackelberg Game Solution

Incentive-Design Problem Formulation

- User's net utility: $F_w(x;r) := w\log(1+x) - \frac{1}{1-x} - r, \quad 0 < x < 1$
- Restriction: for $x = 0, \; r \equiv 0$ and $F_w(0;r) \equiv -1$
- Team solution:

$$\begin{aligned} (x^t, r^t) = & \quad \arg \max_{0 \le x < 1, r \ge 0} r, \\ \text{s. t.} & \quad F_w(x;r) \ge -1 \end{aligned}$$

- Incentive-Design Problem Solution: $\gamma : [0,1) \to \mathcal{R}, \;\; \gamma(0) \equiv 0,$

$$\arg \max_{0 \le x < 1} F_w(x;\gamma(x)) = x^t,$$

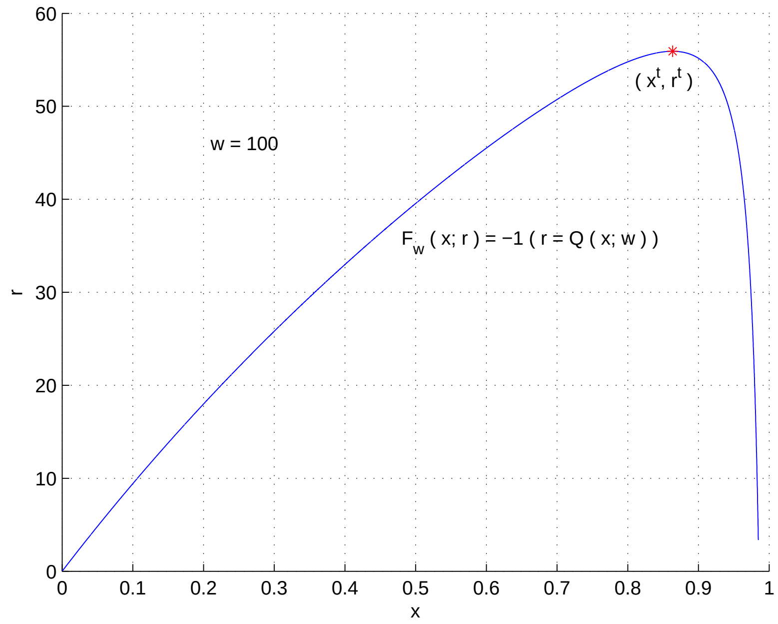
$$\gamma(x^t) = r^t$$

Team Solution

- Define: $Q(x;w) := w\log(1+x) - \frac{1}{1-x} + 1$
- Solve:

$$(x^t, r^t) = \arg\max_{0\le x<1, r\ge 0} r,$$

$$\text{s. t.} \quad F_w(x;r) = Q(x;w) - 1 - r \ge -1$$

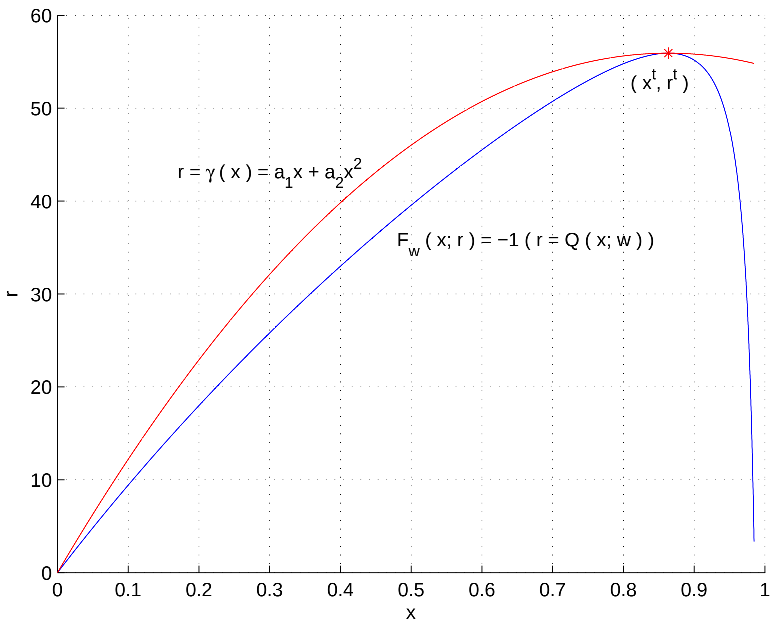


- Team solution: $x^t = \alpha(w) := \frac{1+2w-\sqrt{1+8w}}{2w}, \; w > 1; \;\; r^t = Q(\alpha(w);w)$

Incentive-Design Problem Solution

$$\gamma : [0,1) \to \mathcal{R} : \ \ \gamma(0) \equiv 0; \ \ \arg \max_{0 \le x < 1} F_w(x; \gamma(x)) = x^t, \quad \gamma(x^t) = r^t$$

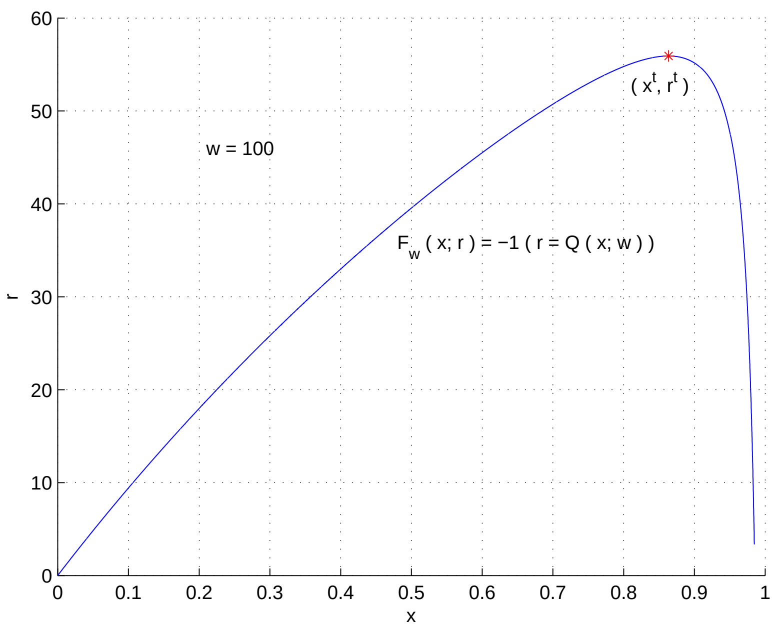
- No linear incentive
- Unique quadratic incentive: $\gamma(x) = a_1 x + a_2 x^2, \ \ a_1 = \frac{2r^t}{x^t}, \ \ a_2 = -\frac{r^t}{(x^t)^2}$



- General incentive
- Incentive controllable?

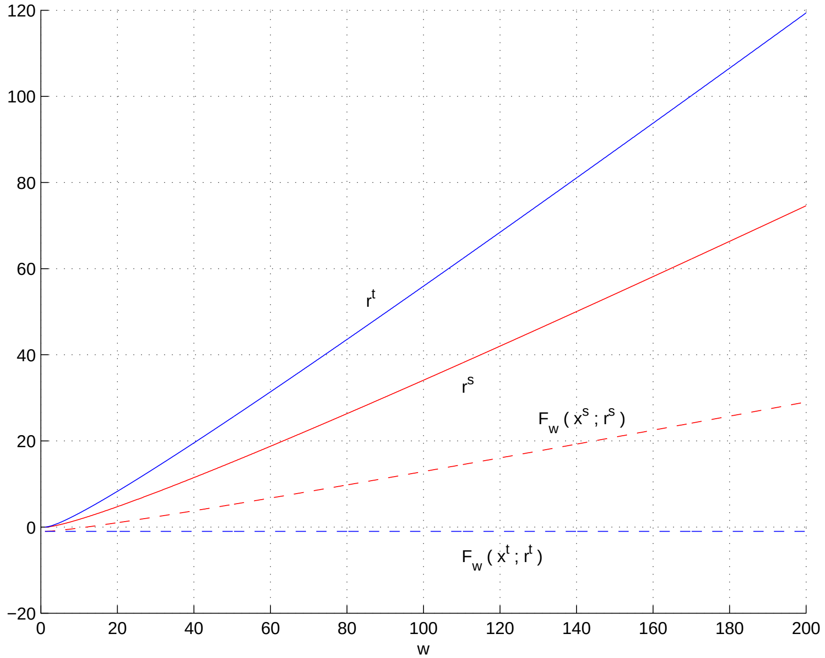
ϵ-Incentive Controllability

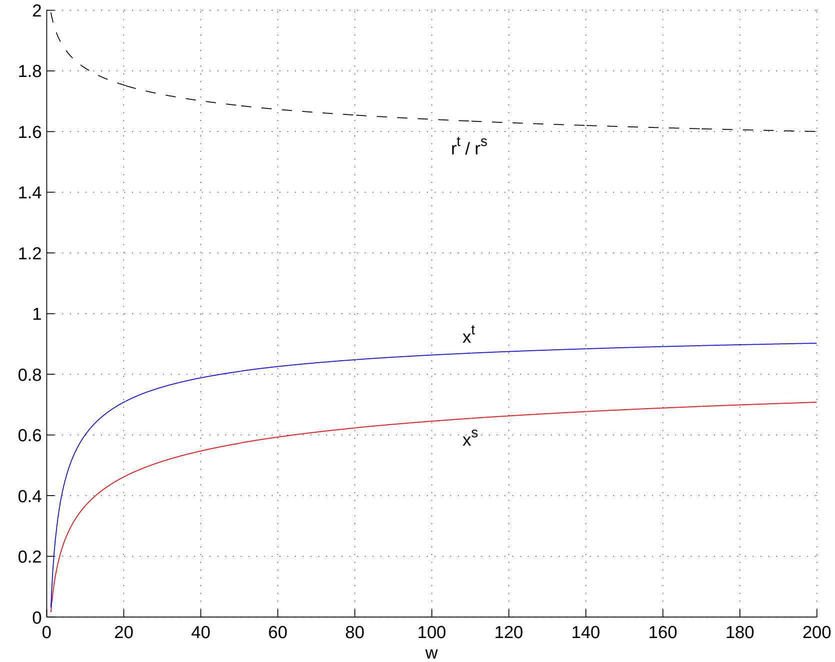
- Make a small "dip": $(x^t, r^t) \to (x^t, r^t - \epsilon)$
- Unique maximizing point for the user: $F_w(x^t; r^t - \epsilon) = -1 + \epsilon$
- ISP's profit: $r^t - \epsilon$



T. Başar and G. J. Olsder (1999), *Dynamic Noncooperative Game Theory,* pp. 392–396.

Team Solution vs Stackelberg Game Solution





Incomplete Information

- Incentive-Design Problem Formulation
- Team Solution
- Incentive-Design Problem Solution
- Incentive Controllability
- Numerical Examples

Incentive-Design Problem Formulation

- User's type: $w = w^i$ w.p. $q_i \in (0,1),\ i = 1, \cdots, m;\ \ \sum_{j=1}^{m} q_j = 1$
- Assumption: $w^1 > \cdots > w^m > 1$
- Team solution:

$$\{(x^{it}, r^{it})\}_{i=1}^{m} = \arg \max_{\{0 \le x^i < 1, r^i \ge 0\}_{i=1}^{m}} \{E[r] = \sum_{j=1}^{m} q_j r^j\},$$

$$\text{s. t.} \quad F_{w^i}(x^i; r^i) \ge -1,\ 1 \le i \le m,$$

$$F_{w^i}(x^i; r^i) \ge F_{w^i}(x^j; r^j),\ 1 \le i \ne j \le m$$

- Incentive-Design Problem Solution: $\gamma : [0,1) \to \mathcal{R},\ \ \gamma(0) \equiv 0,$

$$\arg \max_{0 \le x < 1} F_{w^i}(x; \gamma(x)) = x^{it},\ 1 \le i \le m,$$

$$\gamma(x^{it}) = r^{it},\ 1 \le i \le m$$

Constraint Reduction

- Team solution constraints:

  $F_{w^i}(x^i; r^i) \geq -1; \;\; F_{w^i}(x^i; r^i) \geq F_{w^i}(x^j; r^j), \; 1 \leq i \neq j \leq m$

- Lemma: for $w^i > w^j > 1$,

  $F_{w^i}(x^i; r^i) \geq F_{w^i}(x^j; r^j) \geq F_{w^j}(x^j; r^j) \geq F_{w^j}(x^i; r^i)$ and $x^i \geq x^j$

- Constraint reduction:

  ❶ $F_{w^m}(x^m; r^m) = -1$

  ❷ $F_{w^{m-1}}(x^{m-1}; r^{m-1}) = F_{w^{m-1}}(x^m; r^m); \;\; x^{m-1} \geq x^m$

  ❸ $F_{w^{k-1}}(x^{k-1}; r^{k-1}) = F_{w^{k-1}}(x^k; r^k); \;\; x^{k-1} \geq x^k$

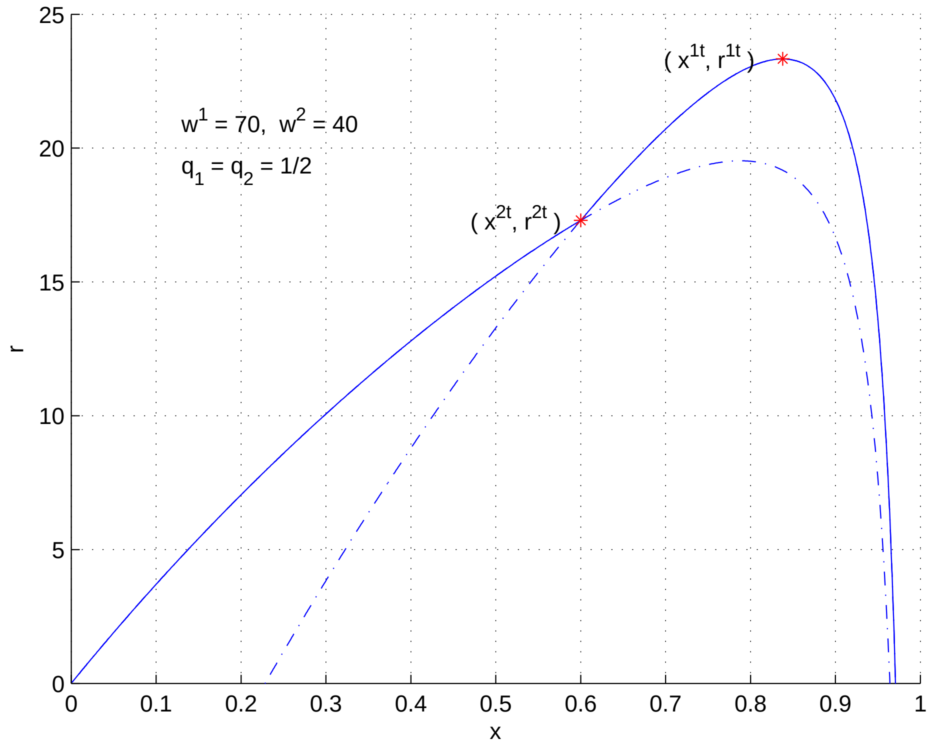
- Reduced constraints: $x^1 \geq \cdots \geq x^m \geq 0;$

  $F_{w^m}(x^m; r^m) = -1$ and $F_{w^i}(x^i; r^i) = F_{w^i}(x^{i+1}; r^{i+1}), \; 1 \leq i \leq m-1$

- Equivalently, $r^m = Q(x^m; w^m); \;\; r^i = r^{i+1} + Q(x^i; w^i) - Q(x^{i+1}; w^i)$

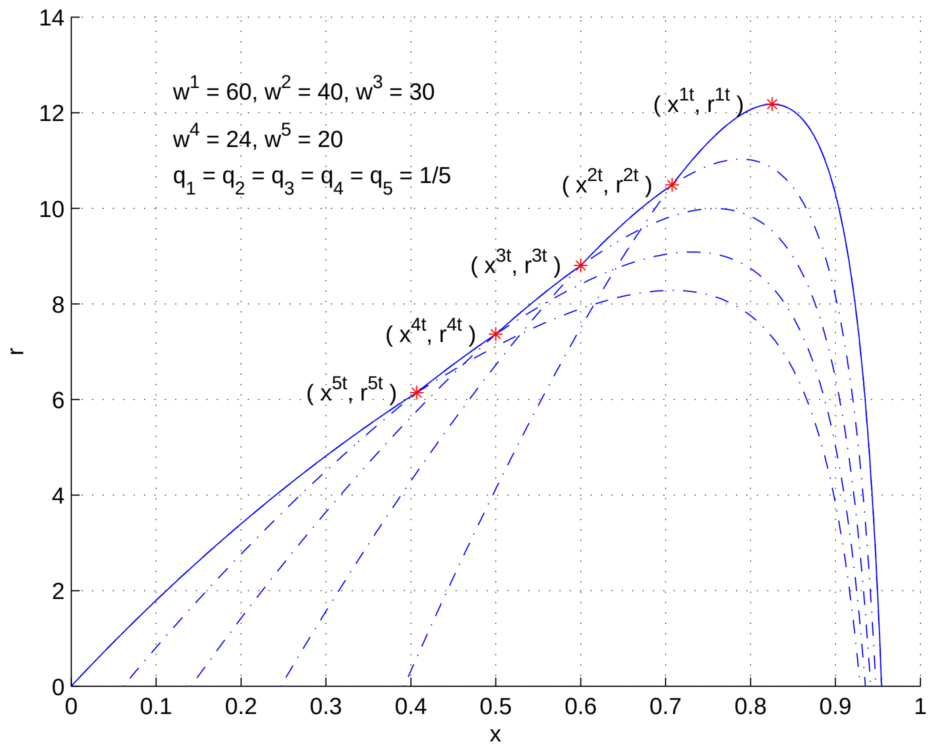
Team Solution — Two Types

- $E[r] = q_1 Q(x^1; w^1) + q_2 Q(x^2; v^{2/2} := \frac{w^2 - q_1 w^1}{q_2})$
- Team solution: $x^{1t} = \alpha(w^1)$ and $x^{2t} = \alpha(v^{2/2})$



Team Solution — Multiple Types

- $E[r] = q_1 Q(x^1; w^1) + \sum_{k=2}^{m} q_k Q(x^k; v^{k/m})$
- If $w^1 > v^{2/m} > \cdots > v^{m/m} > 1$:
  $x^{1t} = \alpha(w^1); \quad x^{it} = \alpha(v^{i/m})$ for $2 \leq i < m$



- Otherwise?

Team Solution — Induction

➢ Reduce m types to $m-1$ or $m-2$ types:

❶ if $v^{m-1/m} \leq v^{m/m}$, $x^{(m-1)t} = x^{mt}$ and $E[r] = q_1 Q(x^1; w^1) + \sum_{k=2}^{m-2} q_k Q(x^k; v^{k/m}) + (q_{m-1} + q_m) Q(x^{m-1}; v^{m-1,m/m})$

❷ if $1 \geq v^{m-1/m} > v^{m/m}$,
$x^{(m-1)t} = x^{mt} = 0$ and $E[r] = q_1 Q(x^1; w^1) + \sum_{k=2}^{m-2} q_k Q(x^k; v^{k/m})$

❸ if $v^{m-1/m} > 1 \geq v^{m/m}$,
$x^{mt} = 0$ and $E[r] = q_1 Q(x^1; w^1) + \sum_{k=2}^{m-1} q_k Q(x^k; v^{k/m})$

❹ if $v^{m-1/m} > v^{m/m} > 1$ and $v^{m-2/m} \leq v^{m-1/m}$,
$x^{(m-2)t} = x^{(m-1)t}$

❺ if $v^{m-2/m} > v^{m-1/m} \geq v^{m/m} > 1$,
at some point $v^{i-1/m} \leq v^{i/m}$ and $x^{(i-1)t} = x^{it}$

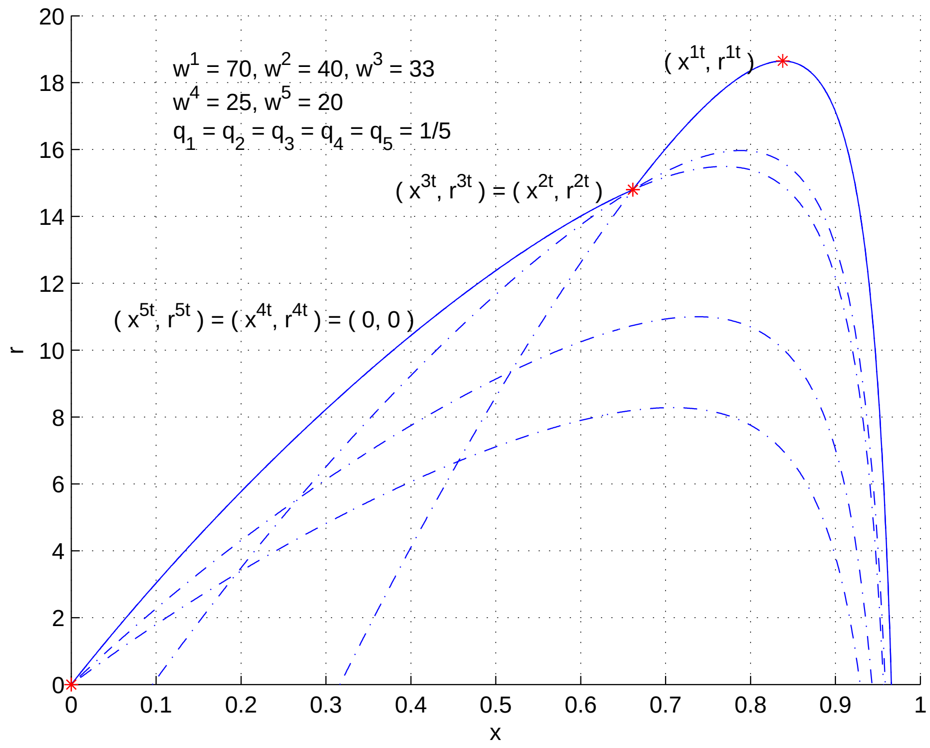
Induction — Example

➢ $v^{1/5} = 70,\ v^{2/5} = 10,\ v^{3/5} = 19,\ v^{4/5} = 1,\ v^{5/5} = 0$

❶ $1 \geq v^{4/5} > v^{5/5}$: $x^{4t} = x^{5t} = 0$

❷ $v^{2/5} \leq v^{3/5}$: $x^{2t} = x^{3t}$ and $v^{2,3/5} = 14.5$

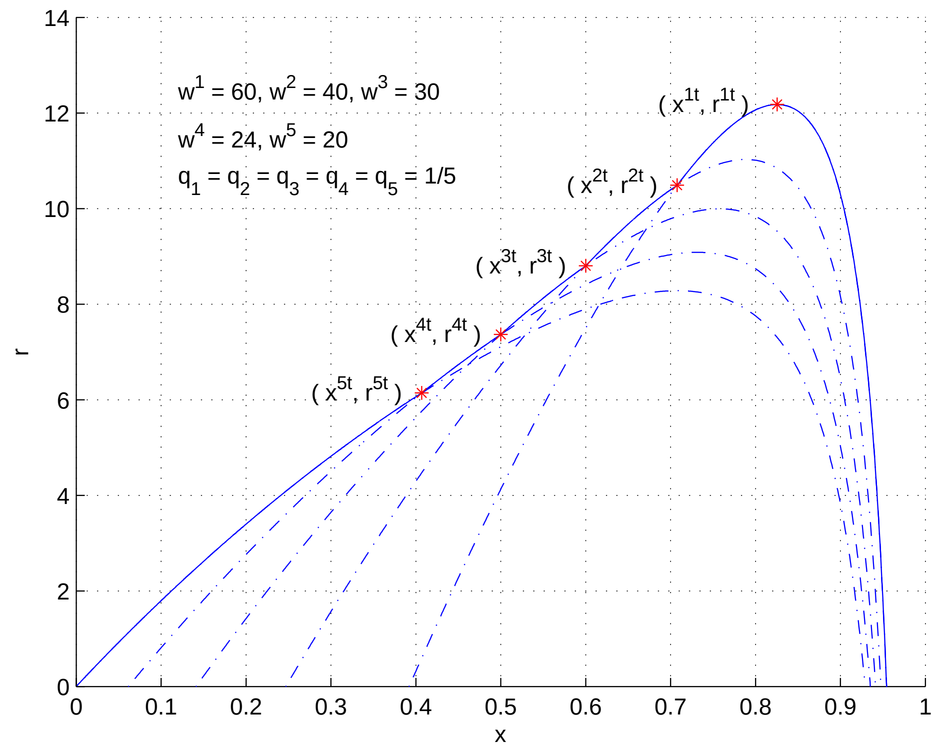
❸ $v^{1/5} > v^{2,3/5} > 1$: $x^{1t} = \alpha(v^{1/5})$ and $x^{2t} = x^{3t} = \alpha(v^{2,3/5})$



Incentive-Design Problem Solution

➢ $\gamma : [0,1) \to \mathcal{R}$: $\gamma(0) \equiv 0$;

$$\arg\max_{0\le x<1} F_{w^i}(x;\gamma(x)) = x^{it}, \quad \gamma(x^{it}) = r^{it}, \quad 1 \le i \le m$$



ϵ-Incentive Controllability

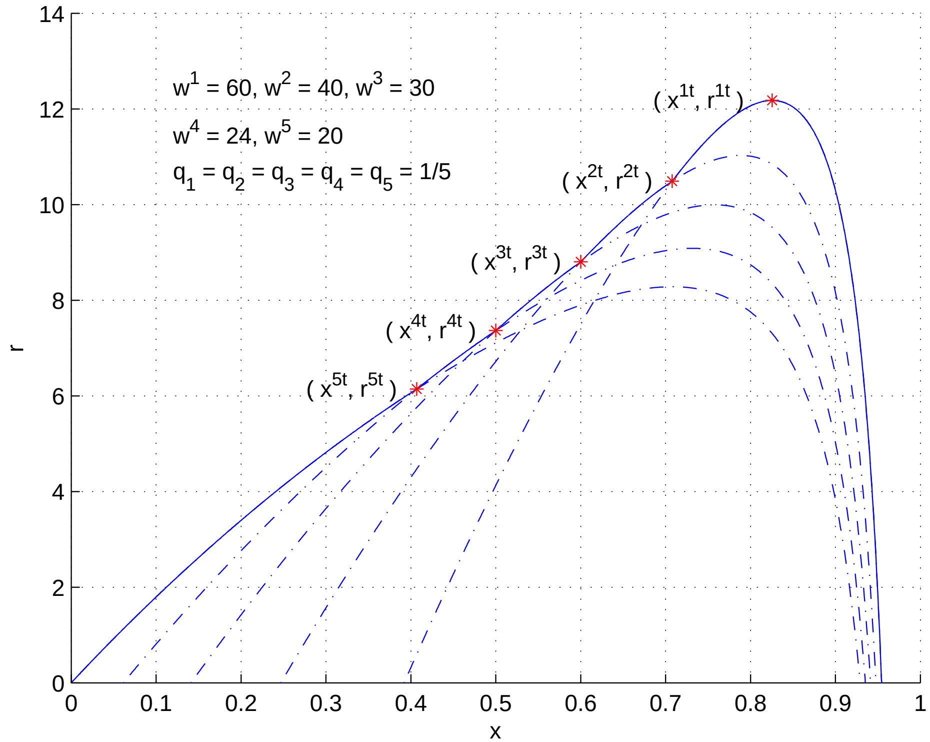
➢ Make a small "dip": $(x^{it}, r^{it}) \rightarrow (x^{it}, r^{it} - \epsilon^i)$

❶ $\epsilon^1 > \epsilon^2 > \epsilon^3 > \epsilon^4 > \epsilon^5 > 0$

❷ ϵ^i's are small enough

➢ Unique maximizing point for type i: $F_w(x^{it}; r^{it} - \epsilon^i) = F_w(x^{it}; r^{it}) + \epsilon^i$

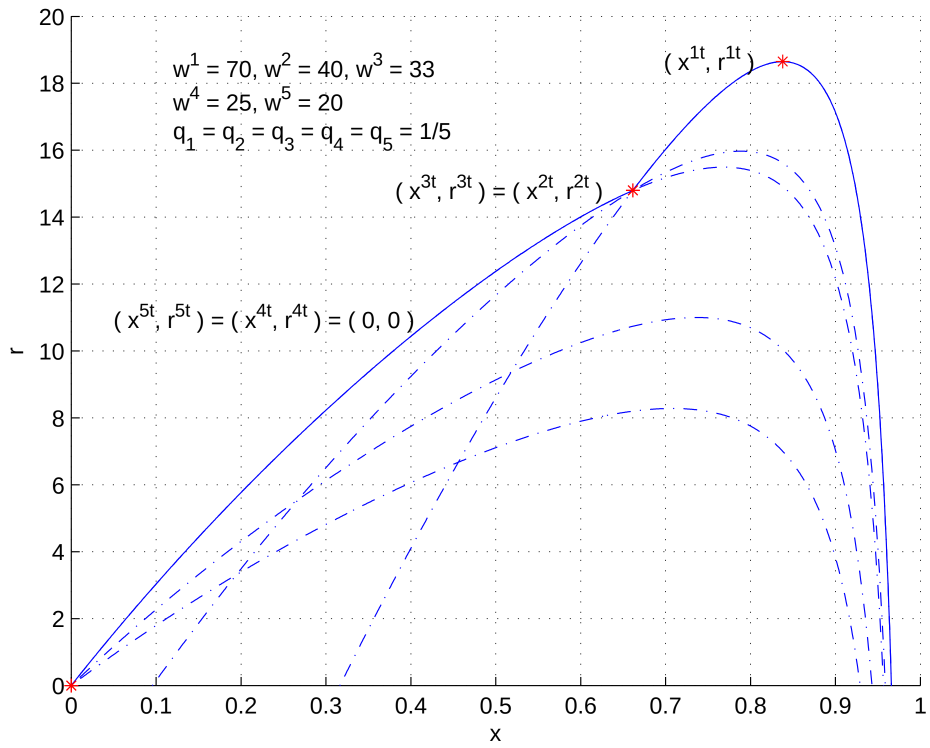
➢ ISP's expected profit: $E[r^t] - \sum_{j=1}^m \epsilon^j$



Another Example

- $\gamma : [0,1) \to \mathcal{R}$: $\gamma(0) \equiv 0$;

$$\arg\max_{0 \le x < 1} F_{w^i}(x;\gamma(x)) = x^{it}, \quad \gamma(x^{it}) = r^{it}, \quad 1 \le i \le m$$



ϵ-Incentive Controllability

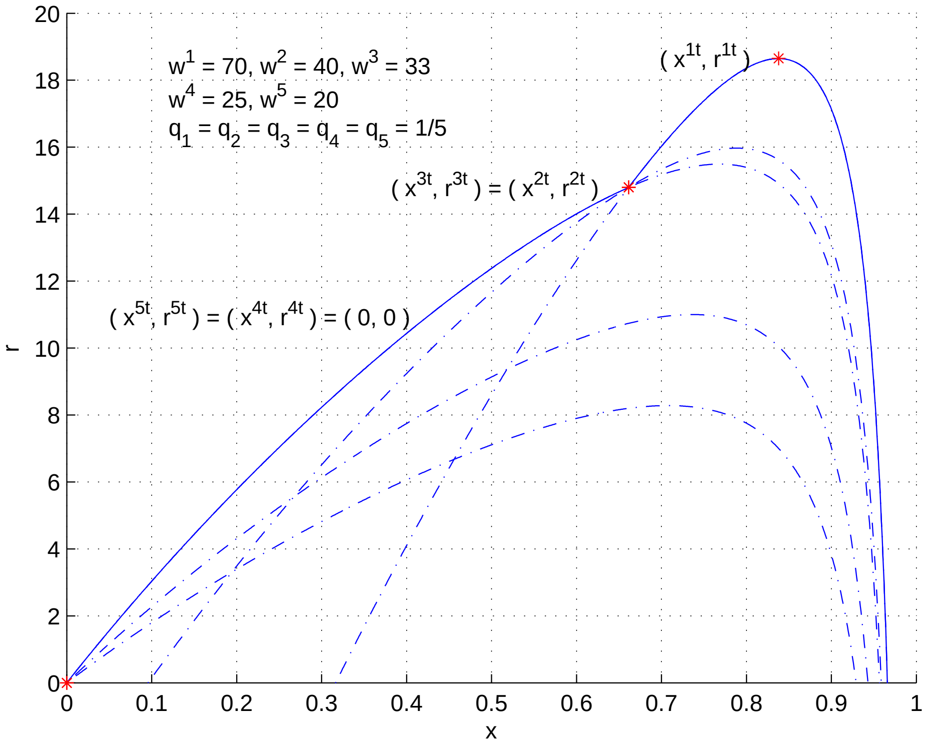
➢ Make a small "dip": $(x^{it}, r^{it}) \longrightarrow (x^{it}, r^{it} - \epsilon^i)$

❶ $\epsilon^1 > \epsilon^2 = \epsilon^3 > \epsilon^4 = \epsilon^5 = 0$

❷ ϵ^i's are small enough

➢ Unique maximizing point for type i: $F_w(x^{it}; r^{it} - \epsilon^i) = F_w(x^{it}; r^{it}) + \epsilon^i$

➢ ISP's expected profit: $E[r^t] - \sum_{j=1}^{m} \epsilon^j$



Multiple Users — Complete Information

- User i's net utility: $F_{w_i}(x_i, x_{-i}; r_i) := w_i \log(1+x_i) - \frac{1}{n-x_i-x_{-i}} - r_i$
- Team solution:

$$\{(x_i^t, r_i^t)\}_{i=1}^n = \arg \max_{\{x_i \geq 0, \sum_{j=1}^n x^j < n, r_i \geq 0\}_{i=1}^n} \sum_{j=1}^n r_j,$$

$$\text{s. t.} \quad F_{w_i}(x_i, x_{-i}; r_i) \geq -\frac{1}{n - x_{-i}}, \ 1 \leq i \leq n$$

- Incentive-Design Problem Solution: $\{\gamma_i\}_{i=1}^n, \ \gamma_i(0) \equiv 0,$

$$\arg \max_{0 \leq x_i < n - x_{-i}^t} F_{w_i}(x_i, x_{-i}^t; \gamma_i(x_i)) = x_i^t,$$

$$\gamma_i(x_i^t) = r_i^t$$

Multiple Users — Incomplete Information

- Independent Users: $w_i = w_i^{j_i}$ w.p. $q_i^{j_i}$; $\vec{J} := (j_1, \cdots, j_n)^T$
- Team solution:

$$\left\{\{(x_i^{\vec{J}t}, r_i^{\vec{J}t})\}_{i=1}^n\right\}_{\vec{J}=\vec{J}_f}^{\vec{J}_l} = \arg\max \sum_{\vec{J}=\vec{J}_f}^{\vec{J}_l} q_1^{j_1} \times \cdots \times q_n^{j_n} \sum_{j=1}^n r_j^{\vec{J}},$$

s. t. $F_{w_i^{j_i}}(x_i^{\vec{J}}, x_{-i}^{\vec{J}}; r_i^{\vec{J}}) \geq -\dfrac{1}{n - x_{-i}^{\vec{J}}}, \;\; 1 \leq i \leq n,$

$$F_{w_i^{j_i}}(x_i^{\vec{J}}, x_{-i}^{\vec{J}}; r_i^{\vec{J}}) \geq F_{w_i^{j_i}}(x_i^{\vec{J}'}, x_{-i}^{\vec{J}}; r_i^{\vec{J}'}), \;\; 1 \leq i \leq n, \;\; \vec{J} \neq \vec{J}'$$

- Incentive-Design Problem Solution: $\{\gamma_i\}_{i=1}^n, \;\; \gamma_i(0) \equiv 0,$

$$\arg \max_{0 \leq x_i < n - x_{-i}^{\vec{J}t}} F_{w_i^{j_i}}(x_i, x_{-i}^{\vec{J}t}; \gamma_i(x_i)) = x_i^{\vec{J}t},$$

$$\gamma_i(x_i^{\vec{J}t}) = r_i^{\vec{J}t}$$

Conclusions

- Dynamic pricing
- Incentive-design problem formulation
- Single ISP, single user
  - ϵ-incentive controllability
  - ϵ-optimal incentive policy

Extensions

- Multiple users, multiple ISPs, continuously distributed user types, . . .

End of the Talk