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INCENTIVE COMPATIBILITY OF LARGE CENTRALIZED MATCHING MARKETS

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Abstract

This paper discusses the strategic manipulation of stable matching mechanisms. We provide a model of a two-sided matching market, in which a firm hires a worker, and each of them receives non-transferable utility. Assuming that the utilities are randomly drawn from underlying distributions, we measure the likelihood of differences in utilities from different stable matchings. Our key finding is that in large markets, most agents are close to being indifferent among partners in different stable matchings. Specifically, as the number of firms and workers becomes large, the expected proportion of firms and workers whose utilities from all stable matchings are within an arbitrarily small difference of one another converges to one. It is known that the utility gain by manipulating a stable matching mechanism is limited by the difference between utilities from the most and the least preferred stable matchings. Thus, the finding also implies that the expected proportion of agents who may obtain a significant utility gain from manipulation vanishes in large markets. This result reconciles successful stable mechanisms in practice with the theoretical concerns about strategic manipulation.

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1 Introduction

1.1 Overview

Two-sided matching markets are a class of markets where one kind of agent is matched with another such as firms and workers in professional labor markets, schools and students in school choice problems, men and women who use online dating services, and birth mothers and potential adoptive parents in child adoption. Some markets, such as public school choice programs and entry-level labor markets for doctors and lawyers, are centralized. In a labor market, for instance, firms submit their preferences for workers and workers submit their preferences for firms. A centralized mechanism then produces a matching between firms and workers. Market designers seeking to achieve desirable outcomes to these matching markets have introduced centralized clearinghouses. The best-known example is the National Resident Matching Program (NRMP). Each year, approximately $16,000$ U.S. medical school students and $4,000$ residency programs participate in the NRMP. In addition, another $20,000$ independent applicants compete for the approximately $25,000$ available residency positions (Roth and Peranson, 1999).[1]

In market design, the concept of "stability" is of central importance. A matching is regarded as stable if no agent is matched with an unacceptable partner, and there is no pair of agents on opposite sides of the market who prefer each other to their current partners. In practice, successful mechanisms often implement a stable matching with respect to the submitted preferences (Roth and Xing, 1994; Roth, 2002). The NRMP also uses a particular stable matching mechanism, called the doctor-proposing Gale-Shapley algorithm (Roth and Peranson, 1999). Table 1 below lists whether each clearinghouse produces a stable matching with respect to submitted preferences, and whether these clearinghouses are still in use or no longer operating. With few exceptions, the table suggests that stable matching mechanisms have been successful for the most part whereas unstable mechanisms have mostly failed.[2]

From a theoretical perspective, however, stable matching mechanisms have a significant shortcoming. While the mechanisms produce stable matchings by assuming that

[1]Independent applicants include former graduates of U.S. medical schools, U.S. osteopathic students and graduates, Canadian students and graduates, and students and graduates of international medical schools. For a recent report on the NRMP, see *http://www.nrmp.org/data/resultsanddata2011.pdf.*

[2]Table 1 is reorganized from tables in Roth (2002) and McKinney, Niederle, and Roth (2003). The clearinghouse for the gastroenterology fellowship market is a rare case in which a stable matching mechanism started to fail in 1996, was abandoned in 2000, and then was reinstated in 2006 (Niederle and Roth, 2005; Roth, 2008).

	Still in use	No longer in use
Stable	**The NRMP**: over 40 specialty markets and submarkets for first year postgraduate positions, and 15 for second year positions **Specialty matching services**: over 30 subspecialty markets for advanced medical residencies and fellowships **British regional medical markets**: Edinburgh ($\geq$‘69), Cardiff **Dental residencies**: 3 specialties **Other healthcare markets**: Osteopaths ($\geq$‘94), Pharmacists, Clinical psychologists ($\geq$‘99) **Canadian lawyers:** multiple regions	**Dental residencies**: Periodontists($<$‘97), Prosthodontists ($<$‘00) **Canadian lawyers:** British Columbia($<$‘96)
Unstable	**British regional medical markets**: Cambridge, London Hospital	**British regional medical markets**: Birmingham, Edinburgh ($<$‘67), Newcastle, Sheffield **Other healthcare markets**: Osteopaths ($<$‘94)

Table 1: Stable and unstable (centralized) mechanisms.

all participants reveal their true preferences, in fact *no stable matching mechanism is strategy-proof* (Roth, 1982). Participants may achieve a more preferred matching by misrepresenting their preferences, either by changing the order of the preference lists or by announcing that some acceptable agents are unacceptable. Even the current NRMP, while widely acknowledged as a model of a successful matching program, cannot rule out such incentives of strategic misrepresentation. Unfortunately, the possibility of such manipulation is mostly unavoidable. Whenever there is more than one stable matching, at least one agent can profitably misrepresent her preferences (Roth and Sotomayor, 1990), and the conditions under which preference profiles contain a unique stable matching seem to be quite restrictive (Eeckhout, 2000; Clark, 2006).[3] Thus, markets are most likely to have agents with an incentive to manipulate a stable matching mechanism. In addition, Pittel (1989) shows that the number of stable matchings tends to increase as the market becomes large. Accordingly, concerns regarding strategic manipulation are heightened when market designers deal with markets containing a large number of participants.

This paper analyzes this discrepancy between the success of matching programs in practice and their inherent manipulability in theory. We consider a theoretical matching market in which each firm hires one worker, a model which is known as a one-to-one matching or marriage market. We measure incentives to manipulate a stable matching mechanism by assuming that each firm-worker pair receives utilities, one for the firm and the other for the worker, which in turn determine ordinal preferences. In order to study

[3]The necessary and sufficient conditions of a preference profile containing a unique stable matching is an open question.

the likelihood of an agent having a significant incentive to manipulate, we assume that utilities are randomly drawn from some underlying distributions. Moreover, in light of the large number of participants in applications, we evaluate the measure of utility gain when the number of participants becomes large. The key finding of this paper is that the proportion of participants who can potentially achieve a significant utility gain vanishes as the market becomes large. This result holds both when each agent knows the preferences of all other agents (complete information), and when an agent may not know about preferences of other agents (incomplete information). Given the tangible and intangible costs of strategic behavior in real life, we believe that this result reconciles successful stable matching mechanisms with the theoretical concerns about manipulability.

1.2 A Motivating Example

To understand the logic behind strategic manipulation, consider a simple labor market with three firms and three workers. We illustrate that in such a situation an agent can achieve a better partner by misrepresenting preferences, and that the best achievable partner must be a partner in a stable matching under true preferences. The following example is a simplest example illustrating both of these features of strategic manipulations.

$$\begin{array}{ccccccc} \mathbf{f_1} & : & w_3 & \succ & \langle w_1 \rangle & \succ & [w_2] \\ \mathbf{f_2} & : & \langle w_2 \rangle & \succ & [w_1] & \succ & w_3 \\ \mathbf{f_3} & : & \langle [w_3] \rangle & \succ & w_1 & \succ & w_2 \end{array} \quad , \quad \begin{array}{ccccccc} \mathbf{w_1} & : & [f_2] & \succ & f_3 & \succ & \langle f_1 \rangle \\ \mathbf{w_2} & : & [f_1] & \succ & \langle f_2 \rangle & \succ & f_3 \\ \mathbf{w_3} & : & f_2 & \succ & \langle [f_3] \rangle & \succ & f_1 \end{array}$$

Table 2: An example of a two-sided matching market with 3 firms and 3 workers.

Table 2 lists preferences of firms for workers, and of workers for firms which are known to all participants: For instance, firm 1 prefers worker 3 most, followed by worker 1 and worker 2. Similarly, worker 1 prefers firm 2 most, followed by firm 3 and firm 1. Under these preferences, there are two stable matchings. In one stable matching (marked by $\langle \cdot \rangle$), f_1, f_2, and f_3 are matched with w_1, w_2, and w_3, respectively; whereas in the second stable matching (marked by $[\cdot]$), f_1, f_2, and f_3 are matched with w_2, w_1, and w_3, respectively.

Suppose that all agents submit their true preferences, and a stable matching mechanism produces the second stable matching marked by $[\cdot]$. In that case, if firm 1 submits preferences such that worker 3 and 1 are acceptable, but not worker 2, then only the first stable matching remains stable for those announced preferences. The stable matching mechanism, which produces a stable matching for any submitted preference profile, will produce the matching marked by $\langle \cdot \rangle$. Ultimately, firm 1 is matched with worker 1 rather than worker 2, and is therefore better off.

However, whichever preference list firm 1 submits, the firm will not be matched with worker 3. The pair (f_3, w_3) would otherwise block the matching. For instance, if f_1 declares that only w_3 is acceptable, then the only stable matching matches f_2 with w_2, and f_3 with w_3, and firm 1 will remain unmatched.[4] In fact, more broadly, whenever a stable matching mechanism is applied, participants cannot be matched with a partner who is strictly preferred to all stable matching partners with respect to true preferences (Demange, Gale, and Sotomayor, 1987). Since participants are guaranteed to be matched with one of their stable matching partners, the gain from strategic manipulation is bounded above by the difference between the most and the least preferred stable matching partners.

Based on the above observation, we mainly focus on the difference in utilities from the firm-optimal stable matching and the worker-optimal stable matching. As we show that this difference vanishes when the market becomes large, we can derive the vanishing incentive to manipulate a stable matching mechanism.

1.3 Description of Main Results

We consider a sequence of one-to-one matching markets, each of which has n firms and an equal number of workers. Preferences of each firm for workers or of each worker for firms are generated by utilities, which are randomly drawn from given distributions on $\mathbb{R}_+$.[5] We formulate utilities as a convex combination of a *common value component* and an *independent private value component*. That is, when a firm f is matched with a worker

[4] We call such a strategy a *truncation strategy*. An agent does not change the relative ranks, but only misrepresents the number of acceptable partners. In one-to-one matching, truncation strategies are known to be *exhaustive* (Roth and Vande Vate, 1991): For any submitted preferences of other agents, each agent always has a truncation strategy as a best response. In the example, we prove that no truncation strategy allows f_1 to be matched with w_3, implying that f_1 cannot be matched with w_3 by misrepresenting her preferences. In a many-to-one matching, Kojima and Pathak (2009) show that a broader class of strategies, referred to as *dropping strategies*, is exhaustive.

[5] The only restrictions on distributions are bounded supports and some conditions for continuity.

w, the firm receives

$$U_{f,w} = \lambda\, U^o_w + (1-\lambda)\,\zeta_{f,w} \qquad (0 \leq \lambda \leq 1),$$

where U^o_w is the intrinsic value of w, which is common to all firms, and $\zeta_{f,w}$ is w's value as independently evaluated by firm f. In other words, U^o_w is received by any firm that is matched with worker w, and $\zeta_{f,w}$ is received only by firm f. We similarly define the utilities of the workers. The common value component introduces a commonality of preferences, which is prevalent in real applications. In the entry-level labor market for doctors, for instance, US News and World Report's annual rankings are often referred to as a guideline to the best hospitals. We also consider the pure private value model ($\lambda = 0$) for theoretical interest. In matching theory, commonality has been considered a driving force of a preference profile having a unique stable matching, where no agent has an incentive to manipulate a stable matching mechanism (Eeckhout, 2000; Clark, 2006; Samet, 2011). The pure private value model would be the worst case.[6]

The main finding of the paper is that the expected proportion of agents whose utilities vary only slightly from one another for all stable matchings converges to one as the market becomes large. That is, while agents typically have multiple stable partners, most of the agents are close to being indifferent among the stable partners. We observed in the previous example that when a stable matching mechanism is applied, the best a firm (by misrepresenting its preferences) can achieves is matching with the firm-optimal stable matching partner with regard to the true preferences; similarly, the best a worker can achieves is matching with the worker-optimal stable matching partner with regard to the true preferences (Demange, Gale, and Sotomayor, 1987). As such, my main finding implies that when a stable matching mechanism is applied, the expected proportion of agents who have an incentive to manipulate the mechanism vanishes as the market becomes large.

The proof is based on a *random bipartite graph model*, which, to the best of our knowledge, has not been used before in the matching literature. In a random bipartite graph model, we are given a set of nodes, which is partitioned into two disjoint sets. Each pair of nodes, one from each partition, is joined by an edge independently with a

[6] Worst-case scenarios vary depending on the concern of the failure of stable matching mechanisms. Halaburda (2010) considers a model where firms and workers may unravel a centralized mechanism by contracting on their own, and not participating in the clearinghouse. In terms of unravelling instead of strategic preference misrepresentation, a stable matching mechanism may have a higher chance to fail when preferences have a strong commonality.

fixed probability. For each realized graph, we consider subsets of nodes, which are also partitioned into two disjoint sets, and every pairs of nodes, one from each partition, is joined by an edge. It is known that, as the set of nodes becomes large, the possibility of having a large such subset of nodes ultimately becomes infinitesimal (Dawande, Keskinocak, Swaminathan, and Tayur, 2001). In our matching model, we consider a subset of firms and workers whose common values are above certain levels. We show that the firms and the workers are most likely to achieve relatively high levels of utility in every stable matching. Taking the subset of firms and workers as a set of partitioned nodes, we join each firm-worker pair by an edge if one of their independent private values is significantly lower than the upper bound of the support. It turns out that every firm-worker pair where both the firm and worker fail to achieve high levels of utility in a stable matching must be joined by an edge. Their private values would otherwise both be so high that they would prefer each other to their current partners, thereby blocking the stable matching. Referring to the result of the random bipartite graph model, we can conclude that the set of firms and workers who fail to achieve high levels of utility will remain relatively small as the market becomes large.

This paper mainly focuses on the case of complete information, in which all participants are aware of the preferences of all other agents. However, we will exploit its findings to a market with incomplete information, in which each agent is partially informed about a preference profile. Various setups are conceivable: An agent may know (i) only her own utilities from agents on the other side; (ii) her own utilities and common values of agents on the other side; (iii) her own utilities, common values of agents on the other side, and her own common value to agents on the other side; or (iv) her own utilities and all agents' common values. Regardless of the information structure, most agents are ex-ante close to being indifferent among the different stable matchings as the market becomes large.[7] This is because there is a high probability that agents are close to being indifferent among the realized partners from all stable matchings, which is this study's key finding in the context of complete information.

[7] We study the incentive of misrepresenting preferences given that other agents reveal their true preferences. The expected differences in utility from stable matchings are conditioned on an agent's private information about the preference profile, but not on the other agents' strategies.

1.4 Related Literature

As strategic manipulability has been a major concern of market design applications, a number of studies have addressed the incentives for misrepresenting preferences in a stable matching mechanism (Roth and Peranson, 1999; Immorlica and Mahdian, 2005; Kojima and Pathak, 2009). These studies consider a particular stable matching mechanism, which is called the worker-proposing Gale-Shapley algorithm, which implements a stable matching favorable to workers. As truthfully revealing preferences is a dominant strategy for workers in this mechanism (Roth, 1982; Dubins and Freedman, 1981), the papers focus on firms' incentives to manipulate the mechanism. Unlike the current paper, these studies assume that firms will manipulate a mechanism regardless of how much benefit the firms can obtain. Accordingly, the primary goal is to find conditions of a preference profile in which most firms have a unique stable matching partner, since a firm has no incentive to misrepresent its preferences if and only if it has a unique stable matching partner (Roth and Sotomayor, 1990). A crucial assumption is that agents on one side (either firms or workers) consider only up to a fixed number of agents on the other side acceptable, even when the market size has become large. Operating from this assumption, Roth and Peranson, based on a computational experiment, show that the proportion of firms who have more than one stable matching partner converges to zero. Convergence is theoretically proved by Immorlica and Mahdian and extended to many-to-one matchings by Kojima and Pathak.

As Roth and Peranson (1999) also point out, the key feature driving the non-manipulability of a stable matching mechanism is the assumption that each worker considers only up to a fixed number of firms acceptable. However, this approach does not seem to well represent real markets. First, the assumption itself is questionable, especially in the case of large markets where workers may consider a great number of firms acceptable. Furthermore, even with a weak commonality of preferences, the proportion of firms who are accepted at least by some workers becomes small as the market becomes large. In this case, most firms do have a unique stable matching partner, but quite often the unique stable matching partner is only the firm itself: i.e. *most agents remain unmatched*.

Figure 1 provides an example of this phenomenon by illustrating the result of simulations where each worker considers only up to 30 most preferred firms acceptable. Each graph represents the proportion of firms (or workers) unmatched in stable matchings averaged over 10 repetitions.[8] The utility of a firm is defined as $U_{f,w} = \lambda\, U^o_w + (1-\lambda)\, \zeta_{f,w}$,

[8]The set of unmatched agents is the same for all stable matchings (McVitie and Wilson, 1970).

the utility of a worker is similarly defined, and all values are drawn from a uniform distribution over $[0, 1]$. Even with modest levels of commonality of preferences, the proportion of agents who remain unmatched increases when the market becomes large.

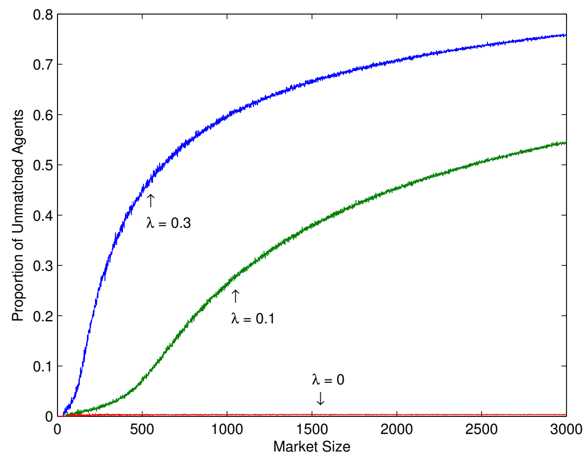

Figure 1: Proportion of agents unmatched in stable matchings.

Another strand of literature on large matching markets considers a market where a finite number of firms are matched with a continuum of workers (Azevedo and Leshno, 2011). It is shown that generically each market has a unique stable matching, to which the set of stable matchings in markets with large discrete workers converges. Based on this model, Azevedo (2010) studies firms' incentives to manipulate capacities to hire workers. The paper also compares welfare effects between situations where each firm pays its employees equally (uniform wage) and those where each firm may pay different wages to different workers (personalized wage). While previous studies with fixed capacities suggest that a uniform wage may induce inefficient matching and compress workers' wages (Bulow and Levin, 2006; Crawford, 2008), if firms can manipulate their capacities, the uniform wage may produce higher welfare as they cause less capacity reduction.

The large market approach is not limited to the standard matching model. In particular, Ashlagi, Braverman, and Hassidim (2011) and Kojima, Pathak, and Roth (2010) develop models of large matching markets with couples. When couples are present, notwithstanding the concerns about strategic manipulation, a market does not necessarily have a stable matching (Roth, 1984). These studies show that the probability that a market with couples contains a stable matching converges to one as the market becomes large. Moreover, when a mechanism produces a stable matching with high probability, it is an approximate equilibrium for all participants to submit their true preferences.

The results are based on the condition that the number of couples grows slower than the market size, with some additional regularity conditions. (The details are different between the two models.)

In the assignment problem, allocating a set of indivisible objects to agents, Kojima and Manea (2010) study incentives in the probabilistic serial mechanism (Bogomolnaia and Moulin, 2001). The probabilistic serial mechanism is proposed as a mechanism improving ex-ante efficiency on the random priority mechanism: all agents have higher chances of obtaining more preferred objects by using the probabilistic serial mechanism. However, while the random priority mechanism is strategy-proof, the probabilistic serial mechanism is not. Kojima and Manea show that for a fixed set of object types and an agent with a given utility function, if there is a sufficiently large number of copies of each object type, then reporting truthful preferences is a weakly dominant strategy for the agent.[9]

The rest of this paper is organized as follows: In Section 2, we introduce our model – a sequence of matching markets with random utilities. In Sections 3 and 4, we state the main theorem informally and then formally, and illustrate the intuition of the proof using a random bipartite graph model. In Section 5, we study a market with incomplete information, and show that the main results hold when an agent does not fully observe preferences of all other agents. The conclusion of the paper is provided in Section 6. All detailed proofs are relegated to the Appendix, which also includes definitions and related theorems of asymptotic statistics.

2 Model

2.1 Two-sided Matching with Random Utilities

The model is based on the standard one-to-one matching model (see, e.g., Roth and Sotomayor (1990)). We add a utility structure, which in turn generates ordinal preferences.

There are n firms and an equal number of workers. We denote the set of firms by F and the set of workers by W. $U = [U_{f,w}]$ and $V = [V_{f,w}]$ are $n \times n$ random matrices with distributions commonly known to all agents. When a firm f and a worker w match

[9]Che and Kojima (2010) show that the random assignments in the two distinct mechanisms converge to each other as the number of copies of each object type goes to infinity. This can be also observed in Azevedo and Leshno (2011) by assuming that all firms have an identical preference list for workers.

with one another, f receives utility $U_{f,w}$ and w receives utility $V_{f,w}$. We use u and v to denote the realized matrices of U and V. A (random) market is defined as a tuple $\langle F, W, U, V\rangle$. We use $\langle F, W, u, v\rangle$ to denote a market instance. Given $\langle F, W, u, v\rangle$, if each firm f receives distinct utilities from different workers, we can define a strict preference list $\succ_f$ as

$$\succ_f = w_1, w_2, f, w_3 \ldots, w_4$$

if and only if

$$u_{f,w_1} > u_{f,w_2} > 0 > u_{f,w_3} \cdots > u_{f,w_4}.$$

This preference list indicates that w_1 is firm f's first choice, w_2 is the second choice, and that w_3 is the least preferred worker that the firm still wants to hire. We also write $w \succ_f w'$ to mean that f prefers w to w'. We call a worker w **acceptable** to f if $w \succ_f f$, otherwise we call the worker **unacceptable**. We define $\succ_w$ similarly for each $w \in W$, and call $\succ := ((\succ_f)_{f\in F}, (\succ_w)_{w\in W})$ **a preference profile** induced by (u, v). We shall assume that utilities are randomly drawn from some underlying distributions, ensuring that realized utility values are all distinct with probability 1, so (u, v) has a strict preference profile with probability 1.

A **matching** μ is a function from the set $F \cup W$ onto itself such that (i) $\mu^2(x) = x$, (ii) if $\mu(f) \neq f$ then $\mu(f) \in W$, and (iii) if $\mu(w) \neq w$ then $\mu(w) \in F$. We say a matching μ is **individually rational** if each firm or worker is matched to an acceptable partner, or otherwise remains unmatched. For a given matching μ, a pair (f, w) is called a **blocking pair** if $w \succ_f \mu(f)$ and $f \succ_w \mu(w)$. We say a matching is μ **stable** if it is individually rational and has no blocking pair.

For two stable matchings μ and μ', we write $\mu \succeq_i \mu'$ if an agent i weakly prefers μ to μ': i.e. $\mu(i) \succ_i \mu'(i)$ or $\mu(i) = \mu'(i)$. We also write $\mu \succeq_F \mu'$ if every firm weakly prefers μ to μ': i.e $\mu(f) \succeq_f \mu'(f)$ for every $f \in F$. Similarly, we write $\mu \succeq_W \mu'$ if every worker weakly prefers μ to μ': i.e. $\mu(w) \succeq_w \mu'(w)$ for every $w \in W$. A stable matching μ_F is **firm-optimal** if every firm weakly prefers it to any other stable matching μ: i.e. $\mu_F \succeq_F \mu$. Similarly, a stable matching μ_W is **worker-optimal** if every worker weakly prefers it to any other stable matching μ: i.e. $\mu_W \succeq_W \mu$. It is known that every market instance has a firm-optimal stable matching μ_F and a worker-optimal stable matching μ_W (Gale and Shapley, 1962): i.e. for any stable matching μ, we have $\mu_F \succeq_F \mu$ and $\mu_W \succeq_W \mu$. Moreover if μ and μ' are both stable matchings, then $\mu \succeq_F \mu'$ if and only if $\mu' \succeq_W \mu$ (Knuth, 1976). Thus for any stable matching μ, it must be the case that

$\mu \succeq_F \mu_W$ and $\mu \succeq_W \mu_F$.

We abuse notation and use μ to denote a function $\succ \longmapsto \mu(\succ)$ so that its domain is the set of all preference profiles and its image is the set of all matchings. We call the function μ a **matching mechanism**, and say that a mechanism μ is **stable** if $\mu(\succ)$ is a stable matching with respect to the preference profile $\succ$.[10] We also use μ_F and μ_W to denote firm-optimal and worker-optimal stable matching mechanisms. A matching mechanism induces a game in which each agent $i \in F \cup W$ states her preference list $\succ_i$. If for all $\succ_i$ and $\succ_{-i}$,

$$\mu(\succ_i^*, \succ_{-i}) \succeq_i \mu(\succ_i, \succ_{-i}),$$

then we call $\succ_i^*$ a **dominant strategy** for the agent i. A mechanism μ is called **strategy-proof** if it is a dominant strategy for every agent to state her true preference list.

We study the asymptotic properties of stable matchings in a sequence of random markets $\langle F_n, W_n, U_n, V_n \rangle_{n=1}^{\infty}$. The index n will be omitted whenever it does not lead to confusion.

2.2 Utility Specification

For each pair (f, w),

$$\begin{aligned} U_{f,w} &= \lambda\, U_w^o + (1-\lambda)\, \zeta_{f,w} \qquad \text{and} \\ V_{f,w} &= \lambda\, V_f^o + (1-\lambda)\, \eta_{f,w} \qquad (0 \leq \lambda \leq 1). \end{aligned}$$

We call U_w^o and V_f^o *common value components*, and $\zeta_{f,w}$ and $\eta_{f,w}$ *private value components.* Common values are defined as random vectors

$$U_n^o := \langle U_w^o \rangle_{w \in W_n} \quad \text{and} \quad V_n^o := \langle V_f^o \rangle_{f \in F_n}.$$

Each U_w^o and V_f^o are drawn from continuous distributions with strictly positive density functions and with bounded supports in $\mathbb{R}_+$. Private values are defined as $n \times n$ random

[10]We can define stable matching mechanisms more generally so that the mechanisms may use utilities as well as preference profiles. We may consider even random mechanisms, randomly selecting a stable matching with respect to submitted utilities. However, firm-optimal and worker-optimal stable matchings are uniquely determined by ordinal preferences, and thus the firm-optimal and the worker-optimal stable matching mechanisms are intact in such a general definition. Since these two mechanisms are the main focus of this paper, we, without loss of generality, continue with the standard stable matching mechanisms.

matrices

$$\zeta = [\zeta_{f,w}] \quad \text{and} \quad \eta = [\eta_{f,w}].$$

Each $\zeta_{f,w}$ and $\eta_{f,w}$ are randomly drawn from continuous distributions with bounded supports in $\mathbb{R}_+$.

The model includes both cases of commonality of preferences ($\lambda > 0$) and pure private values ($\lambda = 0$). The common value component introduces commonality of preferences among firms over workers, or among workers over firms. When $\lambda > 0$, firms with high level of common values tend to be ranked higher by workers, and vice versa. If $\lambda = 0$, all utilities are i.i.d, so firms' orderings of workers are equally likely to be any permutation from the set of all permutations of n workers. Similarly, workers' orderings of firms are equally likely to be any permutation from the set of all permutations of n firms.

In practice, commonality of preferences is prevalent. In the NRMP, some hospitals are considered prestigious and some doctors are considered very well-qualified. For example, US News and World Report's annual rankings are frequently referred to as a guideline to the best hospitals. The common value component provides a way of taking into account such commonality of preferences, while retaining the tractability of the model. In the pure private value model ($\lambda = 0$), agents have no commonality of preferences. Although the case hardly represents any real application, it is theoretically valuable to include it in the model. In matching theory, commonality has been considered a driving force of unique stable matching (Eeckhout, 2000; Clark, 2006). In fact, if preferences have an extreme commonality ($\lambda = 1$), there is a unique stable matching.[11] When there exists a unique stable matching, no agent has an incentive to misrepresent her preferences in a stable matching mechanism (Roth and Sotomayor, 1990). Additionally, Samet (2011) proposes commonality as a source establishing a small core: the small differences in utilities between stable matchings favorable to firms, and to workers. Thus, commonality of preferences may contribute to non-manipulability of stable matching mechanisms. In this regard, we consider the pure private value model the worst-case scenario in terms of incentives to manipulate a stable matching mechanism.

[11] When $\lambda = 1$, a stable matching mechanism sorts firms and workers, so a firm and a worker in the same rank will be matched with one another: i.e. an assortative matching. To see the intuition, fix a market instance and consider the firm-worker pair with the highest common values. The pair must be matched in a stable matching. If it were otherwise, the firm would prefer the worker to his partner and the worker would prefer the firm to her partner, and thus they would form a blocking pair. By sequentially applying the same argument to pairs with the next highest common values, we find that assortative matching is the unique stable matching.

3 Main Results

We first state the main theorem informally, and then later restate it with formal expressions.

3.1 Informal Statement

Theorem 3.1. *For every $\delta > 0$, the expected proportion of firms (and workers) whose utilities from all stable matchings are within δ of one another converges to one as the market becomes large.*

In other words, while agents typically have multiple stable partners, most of the agents are close to being indifferent among the stable partners as the market becomes large. It has been known that no stable matching mechanism is strategy-proof (Roth, 1982). For instance, when the worker-optimal matching mechanism (e.g. worker-proposing Gale-Shapley algorithm) is applied, thereby yielding a worker-optimal stable matching for each submitted preference profile, there might be a firm which can become better off by misrepresenting its preference list.[12] Noting that a matching mechanism is defined over all possible preference profiles, we may expect that a stable matching mechanism is not manipulable in most cases of preference profiles. Unfortunately, though, it turns out that whenever there is more than one stable matching, at least one agent can profitably misrepresent her preferences (Roth and Sotomayor, 1990), and the condition of a preference profile containing a unique stable matching seems to be quite restrictive (Eeckhout, 2000; Clark, 2006).

However, the gain by misrepresenting preferences is limited even when agents form a coalition and coordinate the members' strategic behavior. Not all firms will prefer the new matching outcome to the firm-optimal stable matching with respect to the true preferences, and not all workers will prefer the new matching outcome to the worker-optimal stable matching with respect to the true preferences. Formally, let $\succ$ be the true preference profile, and let $\succ'$ differ from $\succ$ in that some coalition S of firms and workers misstate their preferences. Then, there is no matching, stable for $\succ'$, which is preferred to every stable matching under $\succ$ by all members of S (Demange, Gale, and Sotomayor, 1987). If a coalition consists of a single firm, then the best the firm (by misrepresenting its preferences) can achieves is matching with the firm-optimal stable matching partner

[12] When the worker-optimal stable matching mechanism is applied, it is a dominant strategy for every worker to state his true preference list (Roth, 1982; Dubins and Freedman, 1981).

with the true preferences. Likewise, the best a worker can achieve is matching with the worker-optimal stable matching partner. Since every firm and worker is guaranteed to be matched with a stable matching partner without any strategic manipulation, the gain by misrepresenting preferences is limited to the difference between utilities from the firm-optimal stable matching partner and the worker-optimal stable matching partner.

As such, we can reinterpret Theorem 3.1 in that agents are mostly likely to have only a slight utility gain by misrepresenting their preferences. Whenever there is any cost of manipulating a mechanism, participants are most likely to find no incentive to misrepresent their preferences. In addition, we show with the pure private value model ($\lambda = 0$) that a commonality of preferences may establish, but is not necessary for, a small core.

Corollary 3.2. *For any given cost of misrepresenting preferences, if other agents truthfully reveal their preferences, then the expected proportion of agents who have no incentive to manipulate a stable matching mechanism converges to one as the market becomes large.*

3.2 Formal Statement

Given a market instance $\langle F, W, u, v \rangle$ and a matching μ, we use $u_\mu(\cdot)$ and $v_\mu(\cdot)$ to denote utilities from the matching outcome: i.e. $u_\mu(f) := u_{f,\mu(f)}$, and $v_\mu(w) := v_{\mu(w),w}$. For each $f \in F$, we define $\Delta(f; u, v)$ as the range of utilities from all stable matching outcomes: i.e.

$$\Delta(f; u, v) := u_{\mu_F}(f) - u_{\mu_W}(f).$$

Then, for every $\delta > 0$, we have the set of firms whose utilities are not within δ of one another for all stable matchings, which we denote by

$$A^F(\delta; u, v) := \{ f \in F \mid \Delta(f; u, v) > \delta \}.$$

The following theorem is a formal statement of Theorem 3.1, using the notation defined thus far. We have similar notation and a theorem for workers, which are omitted here.

Theorem 3.1* *For every $\delta > 0$,*

$$E\left[\frac{\left|F \backslash A^F(\delta; U, V)\right|}{n}\right] \to 1, \quad as \quad n \to \infty.$$

For the proof of Theorem 3.1*, we take distinct approaches depending on the value of λ. When $\lambda = 0$, Theorem 3.1* is relatively easily derived from Pittel (1989). Pittel considers a model that is essentially the same as our pure private value model ($\lambda = 0$), and analyzes the sum of each firm's partner's rank number in the worker-optimal stable matching as the market becomes large.[13] When each firm ranks workers in order of preferences (i.e. the most preferred worker is ranked 1, the next worker is ranked 2, and so on), Pittel shows that the sum of the rank numbers of firms' partners in the worker-optimal stable matching is asymptotically equal to $n^2 \log^{-1} n$. Then, the rank number of each firm is roughly $n \log^{-1} n$ on average. In turn, as we normalize the rank number by the market size n, the normalized average rank number is roughly equal to $\log^{-1} n$, converging to 0. As the utility values are randomly drawn from distributions with bounded supports, even the worst stable matching (i.e worker-optimal stable matching) gives utility values asymptotically close to the upper bound. That is, all stable matchings yield only slightly different utility values.

Once we introduce common values ($0 < \lambda < 1$), however, the probability distribution over preference profiles becomes complicated and intractable. As such, we directly analyze the asymptotic utility values rather than referring to the corresponding rank numbers. In doing so, we use a random bipartite graph model. Since a random bipartite graph model has not been used before in the two-sided matching literature, we describe this technique in greater depth in the following section. We relegate detailed proofs of $\lambda = 0$ and $0 < \lambda < 1$ to Appendix B and Appendix C, respectively. We omit the proof for the case of $\lambda = 1$.[14]

4 Intuition of the Proof

4.1 A Random Bipartite Graph Model

A **graph** G is a pair (V, E), where V is a set called **nodes** and E is a set of unordered pairs (i, j) or (j, i) of $i, j \in V$ called **edges**. The nodes i and j are called the **endpoints** of (i, j). We say that a graph $G = (V, E)$ is **bipartite** if its node set V can be partitioned

[13]Pittel does not consider utilities, but a model with random preference profiles. As all preference profiles are equally likely to occur, though, the model is essentially the same as our pure private value model ($\lambda = 0$).

[14]For intuition of the proof, see footnote 11.

into two disjoint subsets V_1 and V_2 such that each of its edges has one endpoint in V_1 and the other in V_2. A **biclique** of a bipartite graph $G = (V_1 \cup V_2, E)$ is a set of nodes $U_1 \cup U_2$ such that $U_1 \subseteq V_1$, $U_2 \subseteq V_2$, and for all $u_1 \in U_1$ and $u_2 \in U_2$, $(u_1, u_2) \in E$. In other words, a biclique is a complete bipartite subgraph of G. We say that a biclique is **balanced** if $|U_1| = |U_2|$, and refer to a balanced biclique with the maximum number of nodes as **a maximum balanced biclique**.

Given a partitioned set $V_1 \cup V_2$, we consider a random bipartite graph model $G(V_1 \cup V_2, p)$. A bipartite graph $G = (V_1 \cup V_2, E)$ is constructed so that each pair of nodes, one in V_1 and the other in V_2, is included in E independently with probability p. We use the following theorem in the proof of the main theorem.

Theorem 4.1 (Dawande, Keskinocak, Swaminathan, and Tayur (2001))**.** *Consider a random bipartite graph $G(V_1 \cup V_2, p)$, where $0 < p < 1$ is a constant, $|V_1| = |V_2| = n$, and $\alpha(n) = \log n / \log \frac{1}{p}$. If the maximum balanced biclique of this graph has size $\alpha \times \alpha$, then*

$$P\left(\alpha(n) \leq \alpha \leq 2\alpha(n)\right) \to 1, \quad \textit{as} \quad n \to \infty.$$

4.2 Intuition of the Proof ($0 < \lambda < 1$)

We use a random bipartite graph model to find an asymptotic lower bound on the utilities from stable matchings. In order to illustrate the technique more easily, we first apply the random bipartite graph model to a matching market with tiers, where firms and workers are partitioned into three tiers.[15] That is, F is partitioned into F_1, F_2, and F_3; and W is partitioned into W_1, W_2, and W_3. For simplicity, we assume that all tiers are of equal size: i.e.

$$|F_t| = |W_t| = n \qquad (t = 1, 2, 3).$$

If $f \in F_t$ and $w \in W_s$ are matched with one another, then they receive utilities

$$U_{f,w} = u_s^o + \zeta_{f,w} \quad \text{and} \quad V_{f,w} = v_t^o + \eta_{f,w}.$$

Common values are uniquely defined by tiers such that

$$u_1^o > u_2^o > u_3^o, \quad \text{and} \quad v_1^o > v_2^o > v_3^o,$$

[15] Although we call each group of firms and workers tiers, the tier structure is not a decisive factor in the preferences. Depending on the relative magnitudes of tier-specific utilities and private values, the tier structure may be diluted by the private value components.

and private values $\zeta_{f,w}$ and $\eta_{f,w}$ are randomly drawn from uniform distributions over $[0, \bar{u}]$ and $[0, \bar{v}]$, respectively. In other words, the firm receives a tier-specific value corresponding to the worker's tier added to private value, and the worker receives tier-specific value corresponding to the firm's tier added to private value. We, without loss of generality, ignore λ and $(1-\lambda)$ by incorporating the weights into the tier-specific utilities and the distributions of private values.

We first find an asymptotic lower bound on utilities that tier-1 firms receive in a stable matching mechanism. The lower bound is defined as the level arbitrarily close to the maximum utility that a firm achieves by matching with tier-2 workers: i.e. $u_2^o + \bar{u} - \varepsilon$. That is, firms in tier-1 are achieving high levels of utility by leveraging on the existence of tier-2 workers. Although not necessarily being matched with tier-2 workers, firms in tier-1 would otherwise make blocking pairs with workers in tier-2. In order to show that the level is an asymptotic lower bound, we define the set of firms that fail to achieve the utility level in the worker-optimal stable matching as

$$\bar{F} := \{f \in F_1 \mid u_{\mu_W}(f) \leq u_2^o + \bar{u} - \varepsilon\},$$

and show that

$$E\left[|\bar{F}|/n\right] \to 0, \quad \text{as} \quad n \to \infty.$$

For each realized private value, we construct a bipartite graph with the set of firms in tier-1, and workers in tiers up to 2 as a partitioned set of nodes (see the left figure in Figure 2). Each pair of $f \in F_1$ and $w \in W_1 \cup W_2$ is joined by an edge if and only if

$$\zeta_{f,w} \leq \bar{u} - \varepsilon \quad \text{or} \quad \eta_{f,w} \leq \bar{v} - (v_1^o - v_2^o).$$

We define the set of workers in tiers up to 2 matched with non tier-1 firms as

$$\bar{W} := \{w \in W_1 \cup W_2 \mid \mu_W(w) \notin F_1\}.$$

Then, $\bar{F} \cup \bar{W}$ is a biclique: i.e. every firm-worker pair from $\bar{F}$ and $\bar{W}$ is joined by an edge (as illustrated by the right figure in Figure 2).

To see why $\bar{F} \cup \bar{W}$ is a biclique, suppose that $f \in \bar{F}$ and $w \in \bar{W}$ are not joined. Since $f \in \bar{F}$,

$$u_{\mu_W}(f) \leq u_2^o + \bar{u} - \varepsilon.$$

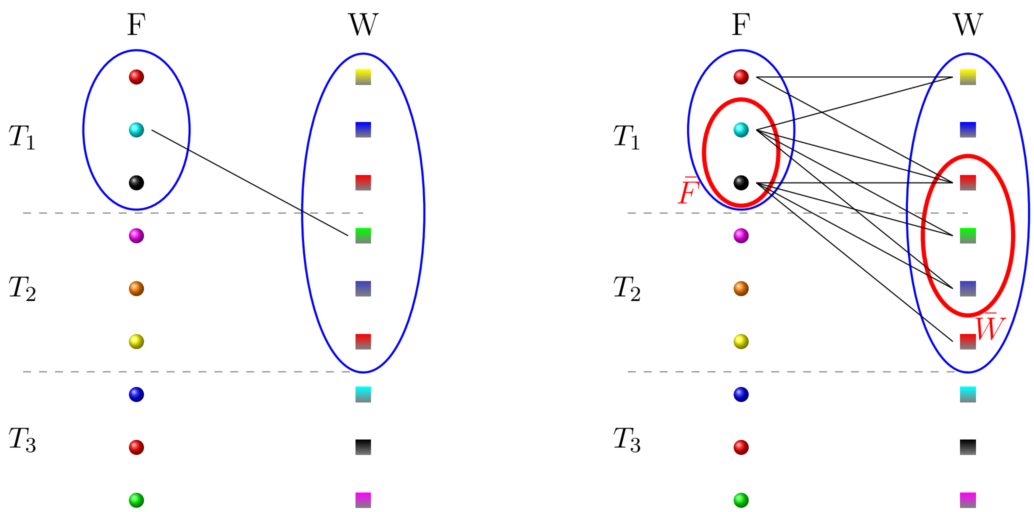

Figure 2: For each realized utility, we draw a bipartite graph with firms in tier-1 and workers in tiers up to 2 as the partitioned set of nodes (left). Firms in tier-1 receiving low utilities ($\bar{F}$) and workers in tiers up to 2 matched with non tier-1 firms ($\bar{W}$) form a biclique (right).

Since $w \in \bar{W}$, the worker is not matched with a tier-1 firm, and thus

$$v_{\mu_W}(w) \leq u_2^o + \bar{v}.$$

That is, f and w mutually fail to achieve high levels of utility.

On the other hand, since they are not joined by an edge,

$$\zeta_{f,w} > \bar{u} - \varepsilon \quad \text{and} \quad \eta_{f,w} > \bar{v} - (v_1^o - v_2^o),$$

and therefore

$$u_{f,w} > u_2^o + \bar{u} - \varepsilon \quad \text{and} \quad v_{f,w} > v_1^o + \bar{v} - (v_1^o - v_2^o) = v_2^o + \bar{v}.$$

In other words, the pair could have achieved high utilities by making a blocking pair, which contradicts that μ_W is a stable matching.

This construction of a bipartite graph fits into a random bipartite graph model. Given that the tier-structure specifies the set of nodes, a bipartite graph is constructed from each profile of realized private values. Since the private values are i.i.d, each firm-worker pair is joined by an edge independently and with an identical probability. Theorem 4.1 shows that, if the partitioned set of nodes has a size on the order of n, and each pair of nodes is joined by an edge independently with a fixed probability, then the maximum balanced biclique has a size on the order of $\log(n)$ with a sequence of probabilities converging to 1 as n gets large. In addition, $\bar{W}$ contains at least n workers, since there are $2n$ workers in tiers up to 2, but only n firms in tier-1: i.e. $\bar{W}$ has a size on the order of n. Therefore, $\bar{F}$ must have a size that is, at most, on the order of $\log(n)$ with a sequence of probabilities converging to 1. The biclique $\bar{F} \cup \bar{W}$ would otherwise contain a balanced biclique with a size bigger than on the order of $\log(n)$, violating the Theorem 4.1. Lastly, $E\left[|\bar{F}|/n\right] \to 0$ is immediately from $\log(n)/n \to 0$.

For the main theorem (without tier structure), we begin the proof by partitioning the supports of distributions for common values. Suppose the common values are drawn from uniform distribution over $[0, 1]$. We partition the unit interval into T subintervals with equal lengths. Workers and firms are, in turn, grouped into tiers where firms or workers in the same tier have common values in the same subinterval. Basically, we continue the proof as if we have a model with a finite number T of tiers. The tiers, though, need to be carefully handled. This time, because common values are random, the tier structure is random. Moreover, agents in adjacent tiers may have arbitrarily close common values.

As we increase the number of tiers T, the asymptotic lower bound on the utility of firms in tier-t becomes close to the maximum utility achievable by matching with a worker in tier-t. An asymptotic upper bound on the utility of firms in tier-t is identified by referring to the asymptotic lower bounds on the utility of workers in tiers higher than t. As workers in tiers higher than t achieve even higher utility values, the workers are not matched with firms in tier-t, which naturally gives an asymptotic upper bound on utility of firms in tier-t. As we finely partition the supports, the common values of workers in tiers higher than t, but close to it, have a little higher common values than the common values of workers in tier-t. That is, the asymptotic upper bound on the utility of firms in tier-t also becomes close to the maximum utility achievable by matching with a worker in tier-t.

5 Market with Incomplete Information

We have so far implicitly assumed complete information. Agents are assumed to be able to assess the exact gain by misrepresenting preferences, or at least they can compute firm-optimal and worker-optimal stable matchings. It is a strong assumption, especially when we consider large markets. More realistically, we may want to consider a market with incomplete information, where each agent is only partially informed about the preferences of other participants. Moreover, we assume that agents have no incentive to manipulate a mechanism unless they see a significant gain by misrepresenting their preferences. In this respect, a model with incomplete information may be more realistic, since agents are required to investigate others' preferences prior to making strategic manipulations, which costs time and effort.

Nevertheless, we have mainly focused on the case of complete information, and exploit its findings to show that the incentive to misrepresent preferences vanishes under incomplete information. The intuition is clear. The expected utility gain from manipulation under incomplete information is simply a convex combination of the utility gains in all realized market instances. Previously, we showed that the utility gain is most likely to be insignificant, and thus the expected gain is most likely to be negligible as well.

There are two advantages of showing the result in the context of complete information first, and then deriving the same result in the context of incomplete information. First, the results are robust to the information structure. In relaxing the complete information assumption, we may consider various information structures. Each agent may know only the probability distributions in addition to either (i) her own utilities; (ii) her own utilities and common values of the other side; (iii) her own utilities, common values of the other side, and her own common value evaluated by the other side; or (iv) her own utilities and all agents' common values. The intuition of showing the main result by using convex combinations remains valid regardless of the details of the information structure. Secondly, we can stress that non-manipulability of stable matching mechanisms is a property of the two-sided matching market itself, rather than stemming from insufficient information to manipulate the mechanism. Even when an agent can obtain complete knowledge of a preference profile at a small cost, it is not worth incurring that cost since the gain from manipulation will be small.

The following theorem in the context of incomplete information corresponds to Theorem 3.1 and Corollary 3.2 for the model with complete information.

Theorem 5.1. *With any information structure from (i) to (iv), Corollary 3.2 still holds when agents have incomplete information.*

That is, for any given cost of misrepresenting preferences, if other agents truthfully reveal their preferences, then the expected proportion of agents who have no incentive to manipulate a stable matching mechanism converges to one as the market becomes large.

In order to restate Theorem 5.1 formally, we use K_f to denote what f knows about a preference profile. We use k_f to denote its realization. Then the various incomplete information structures are denoted by (i) $K_f = \langle U_{f,w} \rangle_{w \in W}$; (ii) $K_f = \langle U_{f,w}, U^o_w \rangle_{w \in W}$; (iii) $K_f = \langle U_{f,w}, U^o_w \rangle_{w \in W} \cup \{V^o_f\}$; and (iv) $K_f = \langle U_{f,w}, U^o_w \rangle_{w \in W} \cup \langle V^o_{f'} \rangle_{f' \in F}$. Given a market instance $\langle F, W, u, v \rangle$, we define $\Delta^E(f; u, v)$ as the range of the expected utilities from all stable matchings conditioned on k_f. That is,

$$\Delta^E(f; u, v) := E_{U,V} \left[u_{\mu_F}(f) - u_{\mu_W}(f) \mid k_f \right],$$

where the expectations are applied to firm-optimal and worker-optimal stable matchings. For every $\delta > 0$, we correspondingly have the set of firms, whose expected utilities are *not* within δ of one another for all stable matchings, which we denote by

$$B^F(\delta; u, v) := \left\{ f \in F \mid \Delta^E(f; u, v) > \delta \right\}.$$

We obtaine Theorem 5.1 by interpreting the following result.

Theorem 5.1* *Given any information structure from (i) to (iv) and for every $\delta > 0$,*

$$E \left[\frac{\left| F \backslash B^F(\delta; U, V) \right|}{n} \right] \to 1 \quad as \quad n \to \infty.$$

The main intuition of the proof is that an expectation is a convex combination of all realizations. The realized differences between utilities from the firm-optimal and the worker-optimal stable matchings are mostly less than δ (Theorem 3.1). Therefore, their convex combination is most likely to be less than δ as well. We relegate the detailed proofs to Appendix D.

6 Conclusions

This paper demonstrates an asymptotic similarity of stable matchings as the number of participants becomes large. Our measure of similarity is based on utilities, by which ordinal preferences are determined. As the utilities are drawn from some underlying probability distributions, one can analyze the likely differences in utilities from all stable matchings. We take into account the commonality of preferences using a common value structure, and also consider an absence of commonality of preferences as a worst-case scenario in terms of strategic manipulation of a stable matching mechanism.

We show that the expected proportion of firms and workers who are close to being indifferent among all stable partners converges to one as the market becomes large. By applying the fact that the gain from manipulation of a stable matching mechanism is bounded above by the difference between utilities from the firm-optimal and the worker-optimal stable matchings, the result also implies that the expected proportion of agents who have a significant incentive to manipulate the mechanism vanishes in large markets. We prove our results using a random bipartite graph model. As this approach is new in the matching literature, we exemplify the technique by applying it to a simplified model, a matching market with tiers.

This paper is one of many recent studies exploring how the popularly used matching mechanisms really work in practice. It is essential to have a better understanding of stable matching mechanisms as market design applications expand from the NRMP to many other markets, including school choice programs, dental residencies, various medical specialty matching programs, and labor markets for law clerks. Of particular relevance here is the fact that market designers are encouraging economists to adopt a centralized matching program in the market for new Economics PhDs (Coles, Cawley, Levine, Niederle, Roth, and Siegfried, 2010). As such, understanding the stable matching mechanisms in real applications is not only a market designers' question in theory, but also of concrete interest for economists in general.

Appendix A Asymptotic Statistics (Serfling, 1980)

Let $X_1, X_2, \ldots$ and X be random variables on a probability space $(\Omega, \mathcal{A}, P)$. We say that X_n **converges in probability** to X if

$$\lim_{n\to\infty} P\left(|X_n - X| < \varepsilon\right) = 1, \quad \text{every } \varepsilon > 0.$$

This is written $X_n \xrightarrow{p} X$. For two sequences of random variables $\langle X_n \rangle$ and $\langle Y_n \rangle$, the notation $X_n = o_p(Y_n)$ denotes that $\frac{X_n}{Y_n} \xrightarrow{p} 0$.

For $r > 0$, we say that X_n converges in the r^{th} mean (or in the L^r-norm) to X if

$$\lim_{n\to\infty} E\left(|X_n - X|^r\right) = 0.$$

This is written $X_n \xrightarrow{L^r} X$.

Theorem A.1. *If $X_n \xrightarrow{L^r} X$, then $X_n \xrightarrow{p} X$.*

Theorem A.2. *Suppose that $X_n \xrightarrow{p} X$, $|X_n| \leq |Y|$ with probability 1 (for all n), and $E\left(|Y|^r\right) < \infty$. Then, $X_n \xrightarrow{L^r} X$.*

Remark. *In this paper, most random variables represent proportions, which are bounded above by 1 with probability 1. As such, convergence in probability and convergence in the r^{th} mean are equivalent.*

Theorem A.3. *Let $\mathbf{X_1}, \mathbf{X_2}, \ldots,$ and $\mathbf{X}$ be random k-vectors defined on a probability space, and let g be a vector-valued Borel function defined on $\mathbf{R}^k$. If g is continuous with $P_{\mathbf{X}}$-probability 1, then*

$$\mathbf{X_n} \xrightarrow{p} \mathbf{X} \implies g(\mathbf{X_n}) \xrightarrow{p} g(\mathbf{X}).$$

In particular, if $X_n \xrightarrow{p} X$ and $Y_n \xrightarrow{p} Y$, then $X_n + Y_n \xrightarrow{p} X + Y$ and $X_n Y_n \xrightarrow{p} XY$.

Given a univariate distribution function F and $0 < q < 1$, we define q^{th} **quantile** ξ_q as

$$\xi_q := \inf\{x : F(x) \geq q\}.$$

Consider an i.i.d sequence $\langle X_i \rangle$ with distribution function F. For each sample of size n, $\{X_1, X_2, \ldots, X_n\}$, a corresponding **empirical distribution function** F_n is constructed as

$$F_n(x) := \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}\left\{X_i \leq x\right\}, \quad -\infty < x < \infty.$$

The **empirical q^{th} quantile** $\hat{\xi}_{q:n}$ is defined as the q^{th} quantile of the empirical distribution function. That is

$$\hat{\xi}_{q:n} := \inf\{x : F_n(x) \geq q\}.$$

For each x, $F_n(x)$ is a random variable, and therefore, $\hat{\xi}_{q:n}$ is also a random variable.

Theorem A.4. *Suppose that q^{th} quantile ξ_q is the unique solution x of $F(x-) \leq q \leq F(x)$. Then, for every $0 < q < 1$ and $\varepsilon > 0$,*

$$P\left(\left|\hat{\xi}_{q:n} - \xi_q\right| > \varepsilon\right) \leq 2e^{-2n\lambda_\varepsilon^2}$$

for all n, where $\lambda_{1,\varepsilon} = F(\xi_q + \varepsilon) - q$, $\lambda_{2,\varepsilon} = q - F(\xi_q - \varepsilon)$, and $\lambda_\varepsilon = \min\{\lambda_{1,\varepsilon},\ \lambda_{2,\varepsilon}\}$.

For each sample of size n, $\{X_1, X_2, \ldots, X_n\}$, the ordered sample values

$$X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$$

are called the **order statistics**.

In view of

$$X_{k:n} = \hat{\xi}_{k/n:n}, \quad 1 \leq k \leq n, \tag{1}$$

we will carry out proofs in terms of empirical quantiles, even when variables are defined as order statistics.

Appendix B Proof of Theorem 3.1 ($\lambda = 0$)

Let $\zeta = [\zeta_{f,w}]$ be an i.i.d sample from a continuous distribution Γ^W with support $[0, \bar{u}]$, and $\eta = [\eta_{f,w}]$ be an i.i.d sample from a continuous distribution Γ^F with support $[0, \bar{v}]$.[16]

For $\delta > 0$, we define the set of firms whose utility from the worst stable matching is significantly below the upper bound $\bar{u}$, which we shall write as

$$\bar{A}(\delta; u, v) := \{f \in F | u_{\mu_W}(f) < \bar{u} - \delta\}.$$

[16]We use Γ^W instead of Γ^M to represent the distribution of utilities of firms, and interpret it as the distribution of private values of workers. This notation will be consistent with the additional notation G^W representing the distribution of workers' common values. By the same reason, we use Γ^M to denote the distribution of private values of firms.

Note from $u_{\mu_F}(f) \leq \bar{u}$ that

$$u_{\mu_F}(f) - u_{\mu_W}(f) \leq \bar{u} - u_{\mu_W}(f),$$

and thus

$$A^F(\delta; u, v) \subseteq \bar{A}(\delta; u, v).$$

Therefore, Theorem 3.1* follows immediately from the following proposition.

Proposition B.1. *For every $\delta > 0$,*

$$E\left[\frac{|\bar{A}(\delta; U, V)|}{n}\right] \to 0 \quad \textit{as} \quad n \to \infty.$$

We divide the proof into two lemmas. For every market instance $\langle F, W, u, v\rangle$, we let $R_{\mu_W}(f)$ be the rank number of firm f's worker-optimal stable matching partner: e.g. $R_{\mu_W}(f) = 1$ if f matches with its most preferred worker. We first observe that for most firms, the rank numbers of worker-optimal matching partners normalized by n converge to 0. The second lemma shows that the corresponding utility levels must be close to the upper bound $\bar{u}$.

Lemma B.2. *For $\gamma > 0$ let*

$$\bar{A}_q(\gamma; u, v) := \left\{ f \in F \mid \frac{R_{\mu_W}(f)}{n} > \gamma \right\} = \left\{ f \in F \mid 1 - \frac{R_{\mu_W}(f)}{n} < 1 - \gamma \right\}.$$

Then, for every sequence $\langle \gamma_n \rangle$ such that $\gamma_n \to 0$ and $(\log n) \cdot \gamma_n \to \infty$,

$$E\left[\frac{|\bar{A}_q(\gamma_n; U, V)|}{n}\right] \to 0 \quad \textit{as} \quad n \to \infty.$$

Proof. For every instance $\langle F, W, u, v\rangle$ and for every sequence $\langle \gamma_n \rangle$ satisfying the conditions,

$$\begin{aligned} \frac{1}{n}\gamma_n \left|\bar{A}_q(\gamma_n; u, v)\right| &< \frac{1}{n} \sum_{f \in \bar{A}_q(\gamma_n; u, v)} \frac{R_{\mu_W}(f)}{n} \\ &\leq \frac{1}{n} \sum_{f \in F_n} \frac{R_{\mu_W}(f)}{n}. \end{aligned}$$

We use Theorem 2 in Pittel (1989) showing that

$$\frac{\sum_{f\in F_n} R_{\mu_W}(f)}{n^2\log^{-1} n} \xrightarrow{p} 1. \tag{2}$$

Applying (2), we shall write

$$\begin{aligned}\frac{\left|\bar{A}_q(\gamma_n;U,V)\right|}{n} &\le \frac{\sum_{f\in F_n} R_{\mu_W}(f)}{n^2}\frac{1}{\gamma_n} \\ &= \frac{\sum_{f\in F_n} R_{\mu_W}(f)}{n^2\log^{-1} n}\frac{1}{\log n\cdot\gamma_n} \xrightarrow{p} 0 \quad \text{as} \quad n\to\infty.\end{aligned}$$

We obtain Lemma B.2 since $\frac{\left|\bar{A}_q(\gamma_n;U,V)\right|}{n}$ is bounded above by 1 for all n so that convergence in probability implies convergence in mean (Theorem A.2). $\square$

Lemma B.3. *For every $\gamma > 0$ let*

$$\bar{A}'(\delta, 1-\gamma; u, v) := \left\{ f\in F \mid \hat{\xi}^f_{1-\gamma;n} < \bar{u}-\delta \right\},$$

where $\hat{\xi}^f_{1-\gamma;n}$ is the realized value of the empirical $(1-\gamma)^{th}$ quantile of $U_f = \langle U_{f,w}\rangle_{w\in W_n}$.

Then, for every $\delta > 0$ and sequence $\langle\gamma_n\rangle$ such that $\gamma_n \to 0$ and $(\log n)\cdot\gamma_n \to \infty$,

$$E\left[\frac{\left|\bar{A}'(\delta,1-\gamma_n;U,V)\right|}{n}\right] \to 0 \quad \textit{as} \quad n\to\infty.$$

Proof. For each n, let $f_n \in F_n$ and consider the resulting sequence $\langle f_n\rangle_{n=1}^{\infty}$.

Note that

$$\xi_{1-\gamma_n;n} - \left|\xi_{1-\gamma_n;n} - \hat{\xi}^{f_n}_{1-\gamma_n;n}\right| \le \hat{\xi}^{f_n}_{1-\gamma_n;n},$$

where $\xi_{1-\gamma_n;n}$ is the $(1-\gamma_n)^{th}$ quantile of Γ^W. We shall write

$$\begin{aligned} P\left(\hat{\xi}^{f_n}_{1-\gamma_n;n} < \bar{u}-\delta\right) &\le P\left(\xi_{1-\gamma_n;n} - \left|\xi_{1-\gamma_n;n} - \hat{\xi}^{f_n}_{1-\gamma_n;n}\right| < \bar{u}-\delta\right) \\ &= P\left(\left|\xi_{1-\gamma_n;n} - \hat{\xi}^{f_n}_{1-\gamma_n;n}\right| > \bar{u} - \xi_{1-\gamma_n;n} + \delta\right) \\ &\le P\left(\left|\xi_{1-\gamma_n;n} - \hat{\xi}^{f_n}_{1-\gamma_n;n}\right| > \delta\right). \end{aligned} \tag{3}$$

Regarding to Theorem A.4,

$$\lambda_{1,\delta} = \Gamma^W(\xi_{1-\gamma_n;n} + \delta) - (1 - \gamma_n) = \gamma_n, \quad \text{and}$$
$$\lambda_{2,\delta} = (1 - \gamma_n) - \Gamma^W(\xi_{1-\gamma_n;n} - \delta), \quad \text{with large } n.$$

Since $\lambda_{1,\delta} \to 0$ and $\lambda_{2,\delta} \to 1 - \Gamma^W(\bar{u} - \delta) > 0$, we have

$$\lambda_\delta = \min\{\lambda_{1,\delta}, \lambda_{2,\delta}\} = \lambda_{1,\delta} = \gamma_n, \quad \text{with large } n.$$

Thus, the last term in (3) is bounded above by $2e^{-2n\gamma_n^2}$ which converges to 0, and therefore

$$P\left(\hat{\xi}^{f_n}_{1-\gamma_n;n} < \bar{u} - \delta\right) \to 0.$$

Note that

$$\begin{aligned} E\left[\frac{|\bar{A}'(\delta, 1-\gamma_n; U, V)|}{n}\right] &= \frac{1}{n}\sum_{f\in F_n} E\left[\mathbf{1}\left\{\hat{\xi}^f_{1-\gamma_n;n} < \bar{u} - \delta\right\}\right] \\ &= E\left[\mathbf{1}\left\{\hat{\xi}^{f_n}_{1-\gamma_n;n} < \bar{u} - \delta\right\}\right] \\ &= P\left(\hat{\xi}^{f_n}_{1-\gamma_n;n} < \bar{u} - \delta\right) \to 0. \end{aligned}$$

□

We complete the proof of Proposition B.1 using the following observation. For each $\langle F, W, u, v\rangle$ and for every sequence $\langle \gamma_n \rangle$ such that $\gamma_n \to 0$ and $(\log n) \cdot \gamma_n \to \infty$,

$$\begin{aligned} \bar{A}(\delta; u, v) &= \left(\bar{A}(\delta; u, v) \cap \bar{A}_q(\gamma_n; u, v)\right) \cup \left(\bar{A}(\delta; u, v) \cap (F\backslash \bar{A}_q(\gamma_n; u, v))\right) \\ &\subseteq \bar{A}_q(\gamma_n; u, v) \cup \left(\bar{A}(\delta; u, v) \cap (F\backslash \bar{A}_q(\gamma_n; u, v)\right). \end{aligned}$$

Each f in $F\backslash \bar{A}_q(\gamma_n; u, v)$ matches in μ_W with a worker of a normalized rank less than γ_n. Nevertheless if f obtains utility less than $\bar{u} - \delta$ in μ_W (i.e. $f \in \bar{A}(\delta; u, v)$), then the (realized) empirical $(1-\gamma_n)^{th}$ quantile of his utilities is below $\bar{u} - \delta$.

That is,

$$\bar{A}(\delta; u, v) \cap F\backslash \bar{A}_q(\gamma_n; u, v) \subseteq \bar{A}'(\delta, 1 - \gamma_n; u, v),$$

and therefore

$$\bar{A}(\delta; u, v) \subseteq \bar{A}_q(\gamma_n; u, v) \cup \bar{A}'(\delta, 1-\gamma_n; u, v).$$

We proved in Lemma B.2 and B.3 that both $\frac{|\bar{A}_q(\gamma_n; U,V)|}{n}$ and $\frac{|\bar{A}'(\delta, 1-\gamma_n; U,V)|}{n}$ converge to 0 in mean, which completes the proof.

Appendix C Proof of Theorem 3.1 ($0 < \lambda < 1$).

To simplify notations, we compress λ and $1-\lambda$, and consider utilities defined as

$$U_{f,w} = U^o_w + \zeta_{f,w} \quad \text{and} \quad V_{f,w} = V^o_f + \eta_{f,w}.$$

We do not lose generality, since we can regard common values and private values as the ones already incorporated λ and $1-\lambda$ in their distributions.

Let U^o_n and V^o_n be i.i.d samples of size n from distributions G^W and G^F, respectively. G^W has a strictly positive density function on the support $[0, \bar{u}^o]$ in $\mathbb{R}_+$, and G^F has a strictly positive density function on the support $[0, \bar{v}^o]$ in $\mathbb{R}_+$. $\zeta = [\zeta_{f,w}]$ is an i.i.d sample from a continuous distribution Γ^W with support $[0, \bar{u}]$, and $\eta = [\eta_{f,w}]$ is an i.i.d sample from a continuous distribution Γ^F with support $[0, \bar{v}]$.

We prove that $\frac{|A^F(\delta; U,V)|}{n}$ converges to 0 in probability, which is equivalent to proving convergence in mean (Theorem A.2). That is, we fix $\delta > 0$ and $T \in \mathbb{N}$ and prove that

$$P\left(\frac{|A^F(\delta; U, V)|}{n} > \frac{9}{T}\right) \to 0, \quad \text{as} \quad n \to \infty.$$

First, we partition the supports of the common value distributions into T intervals. Then for each market instance, in particular for each realized profile of common values, we group firms and workers into *two* versions of finite number of tiers, where agents in the same tier have similar common values. In Proposition C.1, we find that tier-t firms are most likely to achieve a utility level higher than an arbitrary ε less than the maximum achievable utility from a worker in tier-$(t+3)$.[17] For the proof, we use a random bipartite

[17]In Section 4, we showed with a market with tiers that firms in tier-t are most likely to achieve a utility level higher than an arbitrary ε less than the maximum achievable utility from a worker in tier-$(t+1)$. In the model with tiers, each tier has a distinct tier-specific common value, so there is a clear-cut distinction between tier-t and tier-$(t+1)$ specific values. In the general model (without tiers),

graph model explained in Section 4. Once we find an asymptotic lower bound on utilities of firms in each tier, we find an asymptotic upper bound on utilities of firms in a tier, say t, with referencing to the asymptotic lower bounds on utilities of workers in tiers higher than t (Proposition C.2). As workers in high tiers achieve high utilities, they are most likely to match with firms in high tiers, rather than firms in tier-t. Accordingly, utility of tier-t firms is asymptotically bounded above by the maximum that they can achieve by matching with workers in tiers of near-t.

As we finely partition the supports of the common value distributions, the differences in common values between adjacent tiers become small. Then, the asymptotic lower bound will get close to the sum of the lowest common value of tier-t workers and $\bar{u}$. In addition, the asymptotic upper bound also becomes close to the same level, since the common values of workers in tiers of near-t will be close to the lowest common value of workers in tier-t.

We divide the proof into three subsections. First in subsection C.1, we construct *two* tier-structures from realized common values. Then in subsection C.2, we define three events related to the tier-structures, and show that the all three events occur with probability converging to 1 as the market becomes large. We often write the probability that the events do not occur as a remainder term converging to 0, and continue the proof under the condition that the events all occur. The real proof begins in subsection C.3. During the proof, we shall focus on the market instances where realized firms' or workers' common values are all distinct. G^F and G^W are continuous, ensuring that realized common values are all distinct with probability 1.

C.1 Tier-Grouping

We use the following notations.

1. ξ_q^F and ξ_q^W : q^{th} quantile of G^F and G^W.
2. $\hat{\xi}_{q;n}^F$ and $\hat{\xi}_{q;n}^W$: empirical q^{th} quantile of n-size samples from distributions G^F and G^W, respectively. We also use $\hat{\xi}_{q;n}^F$ and $\hat{\xi}_{q;n}^W$ to denote their realizations.

however, there is no such distinction in common values between the adjacent tiers. The highest common value of workers in tier-$(t+1)$ can be arbitrarily close to the lowest common value of workers in tier-t. This leads us to set the maximum achievable utility from a worker in tier-$(t+3)$ rather than tier-$(t+1)$ as an asymptotic lower bound on utilities of tier-t firms.

3. $U^o_{i;n}$ and $V^o_{i;n}$: i^{th} highest values of n order statistics from G^W and G^F. Note that $U^o_{i;n} = \hat{\xi}^W_{(1-i/n);n}$ by the relationship between order statistics and empirical quantiles (see Equation (1)).

We partition the support of G^W into

$$
\begin{aligned}
I_1^W &:= (\xi^W_{1-1/T}, \infty] \\
I_2^W &:= (\xi^W_{1-2/T}, \xi^W_{1-1/T}] \\
&\vdots \\
I_t^W &:= (\xi^W_{1-t/T}, \xi^W_{1-(t-1)/T}] \\
&\vdots \\
I_T^W &:= [0, \xi^W_{1/T}].
\end{aligned}
$$

We define

$$W_t(u) := \left\{ w \mid u^o_w \in I^W_t \right\} \quad \text{for} \quad t = 1, 2, \ldots, T,$$

and call this set **the set of workers in tier-t** (with respect to *workers'* common values).

For an instance $\langle F_n, W_n, u, v \rangle$, if the corresponding realized common values $u^o_n = \langle u^o_w \rangle_{w \in W_n}$ and $v^o_n = \langle v^o_f \rangle_{f \in F_n}$ are all distinct, we index firms and workers from $i = 1$ to n in the order of their common values: i.e.

$$v^o_{f_i} > v^o_{f_j} \quad \text{and} \quad u^o_{w_i} > u^o_{w_j}, \quad \text{if} \quad i < j.$$[18]

Then, **the set of firms in tier-t** (with respect to *workers'* common values) is defined as

$$F_t(u) := \{ f_i \in F_n \mid w_i \in W_t(u) \}.$$

We will use the following notations.

1. $l_t(u) := |F_t(u)| = |W_t(u)|$: The size of tier-$t$ (with respect to *workers'* common values).

2. $u^o_t := \xi^W_{1-\frac{t}{T}}$: The threshold level of tier-t and tier-$t+1$ workers' common values. Note, $w \in W_t(u)$ if and only if $u^o_t < u^o_w \leq u^o_{t-1}$.

[18] G^F and G^W have positive density functions, ensuring that realized common values are all distinct with probability 1.

Remark. *The set of tier-t workers is defined with respect to workers' common values, which is a random sample. Therefore, $W_t(U)$ is random, and so is $F_t(U)$. In particular, the size of tier-t, $l_t(U)$, is random; whereas, u_t^o is a constant.*

In parallel, we partition the support of the firms' common value distribution function into

$$\begin{aligned} I_1^F &:= (\xi_{1-1/T}^F, \infty] \\ I_2^F &:= (\xi_{1-2/T}^F, \xi_{1-1/T}^F] \\ &\vdots \\ I_t^F &:= (\xi_{1-t/T}^F, \xi_{1-(t-1)/T}^F] \\ &\vdots \\ I_T^F &:= [0, \xi_{1/T}^F]. \end{aligned}$$

We define **the set of firms in tier-t** (with respect to *firms'* common values) as

$$F_t(v) := \left\{ f \mid v_f^o \in I_t^F \right\} \quad \text{for} \quad t = 1, 2, \ldots, T,$$

and define **the set of workers in tier-t** (with respect to *firms'* common values) as

$$W_t(v) := \{ w_i \in W_n \mid f_i \in F_t(v) \}.$$

Accordingly, we use the following notations.

1. $l_t(v) := |F_t(v)| = |W_t(v)|$: The size of tier-$t$ (with respect to *firms'* common values).
2. $v_t^o := \xi_{1-\frac{t}{T}}^F$: The threshold level of tier-t and tier-$t+1$ firms' common values. Note, $f \in F_t(u)$ if and only if $v_t^o < v_f^o \leq v_{t-1}^o$.

Remark. *Tiers with respect to workers' common values are in general not the same as tiers with respect to firms' common values. In particular, we are most likely to have $l_t(u) \neq l_t(v)$.*

Throughout the proof, we mainly use tiers defined with respect to workers' common values. However, we need both definitions of tier-structures in the last part of the proof. We simply write "tier-t" to denote tier-t with respect to workers' common values, and use "(w.r.t firm) tier-t" to denote tier-t with respect to firms' common values.

C.2 High-Probability Events

We introduce three events and show that the events occur with probability converging to 1 as the market becomes large. In the next section, we will leave the probability that the following events do not occur as a remainder term converging to zero, and focus on the probabilities conditioned that the following events all occur.

C.2.1 No vanishing tiers

Event $(\mathcal{E}_1)$**.** *Let $\bar{T} > T$. For all $t = 1, 2, \ldots, T$,*

$$\frac{l_t(U)}{n} > \frac{1}{\bar{T}}.$$

Proof. By definition,

$$\frac{l_t(U)}{n} := \frac{1}{n} \sum_{w \in W_n} \mathbf{1}\{U_w^o \in I_t^W\},$$

which converges to $\frac{1}{T}$ in probability by the (weak) law of large numbers. □

C.2.2 Distinct common values of the firms in non-adjacent tiers.

Let $\delta^v > 0$ such that for any $v, v' \in [0, \xi^F_{1-1/T}]$ and $|v - v'| \leq \delta^v$,

$$|G^F(v) - G^F(v')| < \frac{1}{3T}.$$

G^F is uniformly continuous on the interval, so there exists such a δ^v.

Event $(\mathcal{E}_2)$**.** *For every $t = 1, 2, \ldots, T-2$,*

$$\min_{\substack{f \in F_t(U) \\ f' \in F_{t+2}(U)}} |V_f^o - V_{f'}^o| > \delta^v.$$

Proof. Fix $t \in 1, 2, \ldots, T-2$ and realized u. For every $w_i \in W_t(u)$ and $w_j \in W_{t+2}(u)$,

$$u_{w_i}^o > u_t^o = \xi^W_{1-t/T}, \quad \text{and} \quad u_{w_j}^o \leq u_{t+1}^o = \xi^W_{1-(t+1)/T}. \tag{4}$$

For any $q \in (0,1)$, $\hat{\xi}^W_{q;n} \xrightarrow{p} \xi^W_{q;n}$ (Theorem A.4), from which the following inequalities hold with probability converging to 1 as $n \to \infty$.

$$\xi^W_{1-t/T} > \hat{\xi}^W_{1-\frac{t}{T}-\frac{1}{4T}} \quad \text{and} \quad \xi^W_{1-(t+1)/T} < \hat{\xi}^W_{1-\frac{t+1}{T}+\frac{1}{4T}}. \tag{5}$$

Considering (4), if (5) holds, then we have

$$1 - \frac{t}{T} - \frac{1}{4T} < \min_{w_i \in W_t(u)} \left(1 - \frac{i}{n}\right) = \min_{f_i \in F_t(u)} \left(1 - \frac{i}{n}\right)$$

and

$$1 - \frac{t+1}{T} + \frac{1}{4T} > \max_{w_j \in W_{t+2}(u)} \left(1 - \frac{j}{n}\right) = \max_{f_j \in F_{t+2}(u)} \left(1 - \frac{j}{n}\right).$$

Then for every $f_i \in F_t(u)$ and $f_j \in F_{t+2}(u)$,

$$v^o_{f_i} > \hat{\xi}^F_{1-\frac{t}{T}-\frac{1}{4T}} \quad \text{and} \quad v^o_{f_j} < \hat{\xi}^F_{1-\frac{t+1}{T}+\frac{1}{4T}}.$$

Therefore,

$$\begin{aligned} P\Big(\inf_{\substack{f_i \in F_t(U) \\ f_j \in F_{t+2}(U)}} \left|V^o_{f_i} - V^o_{f_j}\right| \le \delta^v \Big) &\le P\left(\left| \hat{\xi}^F_{1-\frac{t}{T}-\frac{1}{4T}} - \hat{\xi}^F_{1-\frac{t+1}{T}+\frac{1}{4T}} \right| \le \delta^v \right) + R_n \\ &\le P\left(\left| G^F(\hat{\xi}^F_{1-\frac{t}{T}-\frac{1}{4T}}) - G^F(\hat{\xi}^F_{1-\frac{t+1}{T}+\frac{1}{4T}}) \right| < \frac{1}{3T} \right) + R_n \quad (6) \end{aligned}$$

where R_n corresponds to the probability that (5) is violated: i.e. $R_n \to 0$. The last inequality is from the definition of δ^v.

Note that

$$G^F(\hat{\xi}^F_{1-\frac{t}{T}-\frac{1}{4T}}) - G^F(\hat{\xi}^F_{1-\frac{t+1}{T}+\frac{1}{4T}}) \xrightarrow{p} \frac{1}{2T}$$

by Theorem A.4 and continuity of G^F(Theorem A.3). As a result, the right hand side of (6) converges to 0.

□

C.2.3 Similarity between tiers defined with workers' common values and tiers defined with firms' common values

The following event is the case that firms in tier-t with respect to workers' common values are in a tier near t with respect to firms' common values, and vice versa.

Event $(\mathcal{E}_3)$**.** *For every* $t = 2, 3, \ldots, T-2$,

$$F_t(U) \subseteq \bigcup_{t'=t-1}^{t+1} F_{t'}(V) \quad \textit{and} \quad W_t(V) \subseteq \bigcup_{t'=t-1}^{t+1} W_{t'}(U).$$

Proof. We prove the first part and omit the proof of the second part.

For each realized (u, v), we have

$$\{u_w^o | w \in W_t(u)\} \subseteq \left(u_t^o, u_{t-1}^o\right] = \left(\xi^W_{1-\frac{t}{T}}, \xi^W_{1-\frac{t-1}{T}}\right]. \tag{7}$$

Suppose

$$\left(\xi^W_{1-\frac{t}{T}}, \xi^W_{1-\frac{t-1}{T}}\right] \subseteq \left(\hat{\xi}^W_{1-\frac{t}{T}-\frac{1}{2T}}, \hat{\xi}^W_{1-\frac{t-1}{T}+\frac{1}{2T}}\right], \tag{8}$$

and

$$\left(\hat{\xi}^F_{1-\frac{t}{T}-\frac{1}{2T}}, \hat{\xi}^F_{1-\frac{t-1}{T}+\frac{1}{2T}}\right] \subseteq \left(\xi^F_{1-\frac{t+1}{T}}, \xi^F_{1-\frac{t-2}{T}}\right]. \tag{9}$$

If (8) hold, then (7) implies that for every tier-t worker w_i, we have

$$u^o_{w_i} \in \left(\hat{\xi}^W_{1-\frac{t}{T}-\frac{1}{2T}}, \hat{\xi}^W_{1-\frac{t-1}{T}+\frac{1}{2T}}\right],$$

and thus,

$$1 - \frac{i}{n} \in \left(1 - \frac{t}{T} - \frac{1}{2T}, 1 - \frac{t-1}{T} + \frac{1}{2T}\right].$$

Then for any tier-t firm f_i, we have

$$v^o_{f_i} \in \left(\hat{\xi}^F_{1-\frac{t}{T}-\frac{1}{2T}}, \hat{\xi}^F_{1-\frac{t-1}{T}+\frac{1}{2T}}\right],$$

which implies that

$$\left\{v_f^o \mid f \in F_t(u)\right\} \subseteq \left(\hat{\xi}^F_{1-\frac{t}{T}-\frac{1}{2T}}, \hat{\xi}^F_{1-\frac{t-1}{T}+\frac{1}{2T}}\right].$$

Consequently if both (8) and (9) hold, then

$$\begin{aligned}\left\{v_f^o \mid f \in F_t(u)\right\} &\subseteq \left(\hat{\xi}^F_{1-\frac{t}{T}-\frac{1}{2T}}, \hat{\xi}^F_{1-\frac{t-1}{T}+\frac{1}{2T}}\right] \\ &\subseteq \left(\xi^F_{1-\frac{t+1}{T}}, \xi^F_{1-\frac{t-2}{T}}\right] \\ &= \bigcup_{t'=t-1}^{t+1} I^F_{t'}.\end{aligned}$$

In other words,

$$F_t(u) \subseteq \bigcup_{t'=t-1}^{t+1} F_{t'}(v).$$

Since (8) and (9) occur with probability converging to 1 (Theorem A.4), the event $\mathcal{E}_3$ also occurs with probability converging to 1. □

C.3 Proof of the Theorem 3.1

We choose T large enough that

$$\max_{1\le t\le T-1}\left|u_t^o - u_{t+1}^o\right| \equiv \max_{1\le t\le T-1}\left|\xi^W_{1-\frac{t}{T}} - \xi^W_{1-\frac{t+1}{T}}\right| < \frac{\delta}{9}. \quad (10)$$

[19]

We divide the proof into two propositions. The first proposition finds an asymptotic lower bound on utilities of firms in each tier, using a random bipartite graph model. The second proposition derives an asymptotic upper bound on utilities of firms in each tier, by referencing the lower bounds on utilities of workers in higher tiers.

Proposition C.1. *For each instance $\langle F_n, W_n, u, v\rangle$ and for each $\bar{t} = 1, 2, \ldots, T-2$, define*

$$\hat{A}^F_{\bar{t}}(\varepsilon; u, v) := \left\{f \in F_{\bar{t}}(u) : u_{\mu_W}(f) \le u^o_{\bar{t}+2} + \bar{u} - \varepsilon\right\}.$$

[20]

Then for $\varepsilon > 0$,

$$\frac{|\hat{A}^F_{\bar{t}}(\varepsilon; U, V)|}{n} \xrightarrow{p} 0 \quad as \quad n \to \infty.$$

Proof. For each instance $\langle F_n, W_n, u, v\rangle$ and for each $t = 1, 2, \ldots, T$, let $F_{\le t}(u) := \bigcup_{t'\le t} F_{t'}(u)$ and $W_{<t}(u) := \bigcup_{t'<t} W_{t'}(u)$.

[19] We can always satisfy the condition since G^W has a strictly positive density function.

[20] Note that $u^o_{\bar{t}+2} + \bar{u}$ is the highest utility level a firm can achieve by matching with a worker in tier-($\bar{t}+3$).

Let $\bar{t} = 1, 2, \ldots, T-2$. We construct a bipartite graph with $F_{\bar{t}}(u) \cup W_{\leq \bar{t}+2}(u)$ as a partitioned set of nodes. (see Section 4.1 for the related definitions.) Two vertices $f \in F_{\bar{t}}(u)$ and $w \in W_{\leq \bar{t}+2}(u)$ are joined by an edge if and only if

$$\zeta_{f,w} \leq \bar{u} - \varepsilon \quad \text{or} \quad \eta_{f,w} \leq \bar{v} - \delta^v,$$

where δ^v is the value taken before, while defining $\mathcal{E}_2$.

Let $\bar{W}(u, v)$ be the set of workers in tiers up to $\bar{t}+2$ who are not matched with firms in tiers up to $\bar{t}+1$. That is,

$$\bar{W}_{\leq \bar{t}+2}(u, v) := \left\{ w \in W_{\leq \bar{t}+2}(u) \,|\, \mu_W(w) \notin F_{\leq \bar{t}+1}(u) \right\}.$$

We now prove that if $\mathcal{E}_2$ holds, then

$$\hat{A}^F_{\bar{t}}(\varepsilon; u, v) \cup \bar{W}_{\leq \bar{t}+2}(u, v)$$

is a biclique.

Suppose, towards a contradiction, that a pair of $f \in \hat{A}^F_{\bar{t}}(\varepsilon; u, v)$ and $w \in \bar{W}_{\leq \bar{t}+2}(u, v)$ is *not* joined by an edge: i.e.

$$\zeta_{f,w} > \bar{u} - \varepsilon \quad \text{and} \quad \eta_{f,w} > \bar{v} - \delta^v.$$

Then, we first have

$$u_{f,w} = u^o_w + \zeta_{f,w} > u^o_{\bar{t}+2} + \zeta_{f,w} > u^o_{\bar{t}+2} + \bar{u} - \varepsilon, \tag{11}$$

and also have

$$v_{f,w} = v^o_f + \eta_{f,w} \geq \min_{f' \in F_{\bar{t}}(u)} v^o_{f'} + \eta_{f,w} > \min_{f' \in F_{\bar{t}}(u)} v^o_{f'} + \bar{v} - \delta^v.\text{[21]}$$

[21] We should not replace $\min_{f' \in F_{\bar{t}}(u)} v^o_{f'}$ with $v^o_{\bar{t}}$, since $F_{\bar{t}}(u)$ is defined with respect to workers' common values, rather than firms' common values.

Since $\mathcal{E}_2$ holds, we can proceed further and obtain

$$\begin{aligned} v_{f,w} &> \min_{f' \in F_{\bar{t}}(u)} v^o_{f'} + \bar{v} - \left(\min_{f' \in F_{\bar{t}}(u)} v^o_{f'} - \max_{f'' \in F_{\bar{t}+2}(u)} v^o_{f''} \right) \\ &= \max_{f'' \in F_{\bar{t}+2}(u)} v^o_{f''} + \bar{v}. \end{aligned} \tag{12}$$

On the other hand, $f \in \hat{A}^F_{\bar{t}}(\varepsilon; u, v)$ implies that

$$u_{\mu_W}(f) \leq u^o_{\bar{t}+2} + \bar{u} - \varepsilon,$$

and $w \in \bar{W}_{\leq \bar{t}+2}(u, v)$ implies that

$$v_{\mu_W}(w) \leq \max_{f'' \in F_{\bar{t}+2}(u)} v^o_{f''} + \bar{v},$$

since a worker can obtain utility higher than $\max_{f'' \in F_{\bar{t}+2}(u)} v^o_{f''} + \bar{v}$ only by matching with a firm in $F_{\leq \bar{t}+1}(u)$.

Then, (11) and (12) implies that (f, w) must be a blocking pair of μ_W, contradicting that μ_W is stable. Therefore,

$$\hat{A}^F_{\bar{t}}(\varepsilon; u, v) \cup \bar{W}_{\leq \bar{t}+2}(u, v).$$

is a biclique (though not necessarily be a balanced biclique).

We now control the size of $\hat{A}^F_{\bar{t}}(\varepsilon; U, V)$ by referencing Theorem 4.1. Let u^o and v^o be realized common values such that events $\mathcal{E}_1$ and $\mathcal{E}_2$ hold. Then, the remaining randomness of U and V is from ζ and η. Consider a random bipartite graph with $F_{\bar{t}}(U) \cup W_{\leq \bar{t}+2}(U)$ as a partitioned set of nodes, where each pair of $f \in F_{\bar{t}}(U)$ and $w \in W_{\leq \bar{t}+2}(U)$ is joined by an edge if and only if

$$\zeta_{f,w} \leq \bar{u} - \varepsilon \quad \text{or} \quad \eta_{f,w} \leq \bar{v} - \delta^v.$$

In other words, every pair is joined by an edge independently with probability

$$p(\varepsilon) = 1 - \left(1 - \Gamma^W(\bar{u} - \varepsilon)\right) \cdot \left(1 - \Gamma^F(\bar{v} - \delta^v)\right).$$

We write $\beta(n) := 2 \cdot \log(l_{\leq \bar{t}+2}(U))/\log \frac{1}{p(\varepsilon)}$, and show that

$$P\left(|\hat{A}^F_{\bar{t}}(\varepsilon; U, V)| \leq \beta(n)\right) \to 1 \quad \text{as} \quad n \to \infty.^{22}$$

First observe that $\bar{W}_{\leq \bar{t}+2}(U, V)$ is the size of at least $l_{\bar{t}+2}(U)$, since amongst $l_{\leq \bar{t}+2}(U)$ workers in tiers up to $\bar{t}+2$ at most $l_{\leq \bar{t}+1}(U)$ are matched with firms in tiers up to $\bar{t}+1$. In addition, $l_{\bar{t}+2}(U) > \beta(n)$ with large n, since $\mathcal{E}_1$ holds. Therefore, with large n, we shall write

$$P\left(|\hat{A}^F_{\bar{t}}(\varepsilon; U, V)| \leq \beta(n)\right) = P\left(\min\left\{|\hat{A}^F_{\bar{t}}(\varepsilon; U, V)|, |\bar{W}_{\leq \bar{t}+2}(U, V)|\right\} \leq \beta(n)\right). \tag{13}$$

Let $\alpha(U, V) \times \alpha(U, V)$ be the size of maximum balance biclique of the random graph

$$G\left(F_{\bar{t}}(U) \cup W_{\leq \bar{t}+2}(U)\ ,\ p(\varepsilon)\right).$$

Since every realized $\hat{A}^F_{\bar{t}}(\varepsilon; u, v) \cup \bar{W}(u, v)$ is a biclique, it contains a balanced biclique of the size equals to

$$\min\left\{|\hat{A}^F_{\bar{t}}(\varepsilon; u, v)|\ ,\ |\bar{W}(u, v)|\right\}.$$

Therefore,

$$P\left(\min\left\{|\hat{A}^F_{\bar{t}}(\varepsilon; U, V)|, |\bar{W}(U, V)|\right\} \leq \beta(n)\right) \geq P\left(\alpha(U, V) \leq \beta(n)\right). \tag{14}$$

Applying Theorem 4.1 to (14) and using (13),

$$P\left(|\hat{A}^F_{\bar{t}}(\varepsilon; U, V)| \leq \beta(n)\right) \geq P\left(\alpha(U, V) \leq \beta(n)\right) \to 1. \tag{15}$$

Lastly, we consider random utilities U and V, in which even common values are yet

[22] Note that we fixed common values as a realization u^o and v^o for each n such that the events $\mathcal{E}_1$ and $\mathcal{E}_2$ occur. Thus for now, the tier-structure is deterministic, and the sequence $\beta(n)$ is, in turn, a deterministic sequence.

realized. For every $\varepsilon' > 0$,

$$P\left(\frac{|\hat{A}^F_{\bar{t}}(\varepsilon;U,V)|}{n} > \varepsilon'\right) = P\left(|\hat{A}^F_{\bar{t}}(\varepsilon;U,V)| > \varepsilon' \cdot n\right)$$
$$\leq P\left(|\hat{A}^F_{\bar{t}}(\varepsilon;U,V)| > \beta(n) \mid \mathcal{E}_1, \mathcal{E}_2\right) + R_n, \quad \text{with large } n,$$

where R_n is the probability that either $\mathcal{E}_1$ or $\mathcal{E}_2$ does not hold: i.e. $R_n \to 0$. The inequality is from $\varepsilon' \cdot n > \beta(n)$ with large n. We complete the proof by applying (15).

□

We also obtain the counterpart proposition of Proposition C.1 in terms of tiers defined with respect to firms' common values.

Proposition C.1* *For each $\bar{t} = 1, 2, \ldots, T-2$, define*

$$\hat{A}^W_{\bar{t}}(\varepsilon;u,v) := \left\{w \in W_{\bar{t}}(v) | v_{\mu_F}(w) \leq v^o_{\bar{t}+2} + \bar{v} - \varepsilon\right\}.$$

Then for $\varepsilon > 0$,

$$\frac{|\hat{A}^W_{\bar{t}}(\varepsilon;U,V)|}{n} \xrightarrow{p} 0 \quad as \quad n \to \infty.$$

Proof. We omit the proof since it is analogous to the proof of Proposition C.1. □

For each instance $\langle F_n, W_n, u, v\rangle$ and for each $\bar{t} = 1, 2, \ldots, T$, we define

$$A^F_{\bar{t}}(\delta;u,v) := \{f \in F_{\bar{t}}(u) | \Delta(f;u,v) > \delta\}.$$

Proposition C.2. *If $\bar{t} = 7, 8, \ldots, T-2$, then*

$$\frac{|A^F_{\bar{t}}(\delta;U,V)|}{n} \xrightarrow{p} 0 \quad as \quad n \to \infty.$$

Proof. In Proposition C.1* with $t = 1, 2, \ldots, T-3$, we replace ε with

$$\varepsilon_t := v^o_{t+2} - v^o_{t+3},$$

and write

$$\hat{A}_t^W(\varepsilon_t; u, v) = \left\{ w \in W_t(v) | v_{\mu_F}(w) \le v_{t+3}^o + \bar{v} \right\}.^{23}$$

Then,

$$\frac{|\hat{A}_t^W(\varepsilon_t; U, V)|}{n} \xrightarrow{p} 0 \quad \text{as} \quad n \to \infty. \tag{16}$$

Note that a worker receives utility higher than $v_{t+3}^o + \bar{v}$ only by matching with a firm in (w.r.t firm) tiers up to $t+3$.[24] Thus for $t = 5, 6, \ldots, T$,

$$\{w \in W_{\le t-4}(V) : \mu(w) \in F_t(V)\} \subseteq \bigcup_{t'=1}^{t-4} \hat{A}_{t'}^W(\varepsilon_{t'}; U, V). \tag{17}$$

If event $\mathcal{E}_3$ holds, we can translate (17) into tiers with respect to workers' common values. That is, for $t = 7, 8, \ldots, T$,

$$\begin{aligned} \{w \in W_{\le t-6}(U) : \mu_F(w) \in F_t(U)\} &\subseteq \bigcup_{t'=t-1}^{t+1} \{w \in W_{\le t-6}(U) : \mu_F(w) \in F_{t'}(V)\} \\ &\subseteq \bigcup_{t'=t-1}^{t+1} \{w \in W_{\le t-5}(V) : \mu_F(w) \in F_{t'}(V)\} \\ &\subseteq \bigcup_{t'=t-1}^{t+1} \{w \in W_{\le t'-4}(V) : \mu_F(w) \in F_{t'}(V)\} \end{aligned}$$

where the first and second inequalities are from $\mathcal{E}_3$.

Applying (17), we obtain

$$\{w \in W_{\le t-6}(U) : \mu_F(w) \in F_t(U)\} \subseteq \bigcup_{t'=1}^{t-3} \hat{A}_{t'}^W(\varepsilon_{t'}; U, V).$$

It follows that

$$\frac{|\{f \in F_t(U) : \mu_F(f) \in W_{\le t-6}(U)\}|}{n} \xrightarrow{p} 0, \tag{18}$$

[23]Recall that v_t^o is a constant, defined as $v_t^o := \xi_{1-\frac{t}{T}}^F$.

[24]Recall that $f \in F_t(v)$ if and only if $v_t^o < v_f^o \le v_{t-1}^o$. Thus, if $f \in F_{>t+3}(v)$ then $v_f^o \le v_{t+3}^o$.

because for every $\varepsilon > 0$,

$$P\left(\frac{|\{f \in F_t(U) : \mu_F(f) \in W_{\leq t-6}(U)\}|}{n} > \varepsilon\right) \leq P\left(\sum_{t'=1}^{t-3} \frac{|\hat{A}^W_{t'}(\varepsilon_{t'}; U, V)|}{n} > \varepsilon\right) + R_n,$$

where R_n is the probability that $\mathcal{E}_3$ does not hold: i.e. $R_n \to 0$. The right hand side converges to 0 by (16).

We complete the proof of Proposition C.2 by proving the following claim. Proposition C.1 and (18) show that the normalized sizes of two sets on the right hand side of (19) converge to 0 in probability.

Claim C.1. *For $\bar{t} = 7, 8, \ldots, T-2$ and each instance $\langle F, W, u, v\rangle$,*

$$A^F_{\bar{t}}(\delta; u, v) \subseteq \hat{A}^F_{\bar{t}}(\delta/9; u, v) \cup \{f \in F_{\bar{t}}(u) | \mu_F(f) \in W_{\leq \bar{t}-6}(u)\}. \tag{19}$$

Proof of Claim C.1. If a firm $f \in F_{\bar{t}}(u)$ is *not* in $\hat{A}^F_{\bar{t}}(\delta/9; u, v)$, then

$$u_{\mu_W}(f) \geq u^o_{\bar{t}+2} + \bar{u} - \delta/9,$$

and if the firm f is *not* in $\{f \in F_{\bar{t}}(u) | \mu_F(f) \in W_{\leq \bar{t}-6}(u)\}$, then

$$u_{\mu_F}(f) \leq u^o_{\bar{t}-6} + \bar{u}.$$

Therefore, using (10) we obtain

$$u_{\mu_F}(f) - u_{\mu_W}(f) \leq u^o_{\bar{t}-6} - u^o_{\bar{t}+2} + \delta/9 < \delta,$$

and thus f is *not* in $A^F_{\bar{t}}(\delta; u, v)$.

□

□

Lastly, we complete the proof of Theorem 3.1 by the following inequalities.

$$
\begin{aligned}
P\left(\frac{|A^F(\delta;U,V)|}{n} > \frac{9}{T}\right) &= P\left(\sum_{1\le t\le T} \frac{|A_t^F(\delta;U,V)|}{n} > \frac{9}{T}\right) \\
&< P\left(\sum_{7\le t\le T-2} \frac{|A_t^F(\delta;U,V)|}{n} + \sum_{t=1,\ldots,6,T-1,T} \frac{l_t(U)}{n} > \frac{9}{T}\right).
\end{aligned}
$$

The last probability converges to 0. For each $t = 7, \ldots, T-2$, the proportion $\frac{|A_t^F(\delta;U,V)|}{n}$ converges to 0 in probability (Proposition C.2). For each $t = 1, \ldots, 6, T-1, T$, the proportion $\frac{l_t(U)}{n}$ converges to $\frac{1}{T}$ in probability by the (weak) law of large numbers.

Appendix D Proof of Theorem 5.1

For each n, let $f_n \in F_n$ and consider the resulting sequence $\langle f_n \rangle_{n=1}^{\infty}$. For any $\delta > 0$,

$$
\begin{aligned}
E\left[\frac{|B^F(\delta;U,V)|}{n}\right] &= E\left[\mathbf{1}\{f_n \in B^F(\delta;U,V)\}\right] \\
&= P\left(\Delta^E(f_n;U,V) > \delta\right).
\end{aligned}
$$

Thus if $\Delta^E(f_n;U,V) \xrightarrow{p} 0$, then for every δ, $\frac{|B^F(\delta,U,V)|}{n}$ converges to zero in mean, thereby completing the proof.

Claim D.1.

$$
\Delta^E(f_n;U,V) \xrightarrow{p} 0, \quad as \quad n \to \infty.
$$

Proof. For every $\delta > 0$,

$$
\begin{aligned}
P\left(\Delta(f_n;U,V) > \delta\right) &= E\left[\mathbf{1}\{\Delta(f_n;U,V) > \delta\}\right] \\
&= E\left[\frac{|A^F(\delta;U,V)|}{n}\right].
\end{aligned}
$$

The last term converges to 0 by Theorem 3.1*, and thus $\Delta(f_n;U,V) \xrightarrow{p} 0$.

Since $\Delta(f_n;U,V)$ is bounded above by $\lambda\,\bar{u}^o + (1-\lambda)\bar{u}$ with probability 1, we obtain by Theorem A.2 that

$$
\lim_{n\to\infty} E[\Delta^E(f_n;U,V)] := \lim_{n\to\infty} E\left[\Delta(f_n;U,V)|K_{f_n}\right] = \lim_{n\to\infty} E\left[\Delta(f_n;U,V)\right] = 0.
$$

The Claim D.1 follows by Theorem A.1. □

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