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INCLUSION AND NEIGHBORHOOD PROPERTIES OF A CERTAIN SUBCLASSES OF P-VALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract

By means of Ruscheweyh derivative operator , we introduced and investigated two new subclasses of p-valent analytic functions.The various results obtained here for each of these function class include coefficient bounds and distortion inequalities, associated inclusion relations for the (n, θ)-neighborhoods of subclasses of analytic and multivalent functions with negative coefficients, which are defined by means of non-homogenous differential equation.

1 Introductin

Let $T_p(n)$ denote the class of functions of the form:

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k \quad (a_k \geq 0; p, n \in N = \{1, 2, ...\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disc $U = \{z : |z| < 1\}$. The modified Hadamard product (or convolution) of the function $f(z)$ given by (1.1) and the function $g(z) \in T_p(n)$ given by

$$g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k \quad (b_k \geq 0; p, n \in N) \tag{1.2}$$

is defined by

$$(f * g)(z) = z^p - \sum_{k=n+p}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.3}$$

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We introduce here an extended linear derivative operator of Ruscheweyh type (see [14]):

$$D^{\mu,p} : T_p \to T_p \qquad (T_p = T_p(1)),$$

which is defined by the following convolution:

$$D^{\mu,p} f(z) = \frac{z^p}{(1-z)^{\mu+p}} * f(z) \qquad (\mu > -p; f(z) \in T_p), \tag{1.4}$$

which in view of (1.1) (with $n = 1$) becomes

$$D^{\mu,p} f(z) = z^p - \sum_{k=p+1}^{\infty} \binom{k+\mu-1}{k-p} a_k z^k$$

$$= z^p - \sum_{k=p+1}^{\infty} \frac{\Gamma(k+\mu)}{(k-p)!\Gamma(p+\mu)} a_k z^k \qquad (\mu > -p; f(z) \in T_p). \tag{1.5}$$

In particular, when $\mu = n$ ($n \in N_0 = N \cup \{0\}$), it is easy observed from (1.4) and (1.5) that

$$D^{n,p} f(z) = \frac{z^p (z^{n-p} f(z))^{(n)}}{n!} \qquad (p \in N; n \in N_0), \tag{1.6}$$

so that

$$D^{1-p,p} f(z) = f(z) \quad \text{and} \quad D^{1,p} f(z) = (1-p) f(z) + z f^{'}(z). \tag{1.7}$$

For a function $f(z) \in T_p(n)$, we have (see [9])

$$(D^{\mu,p} f(z))^{(q)} = \delta(p,q) z^{p-q} - \sum_{k=n+p}^{\infty} \binom{k+\mu-1}{k-p} \delta(k,q) a_k z^{k-q},$$

$$= \delta(p,q) z^{p-q} - \sum_{k=n+p}^{\infty} \frac{\Gamma(k+\mu)}{(k-p)!\Gamma(p+\mu)} \delta(k,q) a_k z^{k-q}$$

$$(p \in N; q \in N_0; p > q), \tag{1.8}$$

where

$$\delta(p,q) = \begin{cases} 1 & (q = 0) \\ p(p-1) \,..........(p-q+1) & (q \neq 0). \end{cases} \tag{1.9}$$

Now, making use of the operator $D^{\mu,p} f(z) (\mu > -p, p \in N)$ given by (1.5), we now introduce a new subclass $T_\mu^q(n,p,\lambda,\beta)$ of the p-valent analytic function class $T_p(n)$ which consist of functions $f(z) \in T_p(n)$ satisfying the inequality:

$$\left| \left\{ \frac{\lambda z (D^{\mu,p} f(z))^{(q+1)} + (1-\lambda) z (D^{1+\mu,p} f(z))^{(q+1)}}{\lambda (D^{\mu,p} f(z))^{(q)} + (1-\lambda)(D^{1+\mu,p} f(z))^{(q)}} - (p-q) \right\} \right| < \beta$$

$$(p \in N; q \in N_0; 0 \leq \lambda \leq 1; p > \max(q, -\mu); 0 < \beta \leq 1). \tag{1.10}$$

We note that:

(i) $T_{\mu}^{0}(n, 1, \lambda, \beta) = T_{\mu}(n, \lambda, \beta)$ (Irmak et al. [10]);

(ii) $T_{0}^{0}(n, p, \lambda, \beta\,|b|) = S_n(b, \lambda, \beta)(b \in C\backslash\{0\})$(Altintas et al. [5]);

(iii) $T_{\mu}^{0}(n, p, 1, \beta\,|b|) = S(b, \mu, \beta)\,(b \in C\backslash\{0\})$(Murugusundaramoorthy and Srivastava [13]).

Also in this paper we shall derive several results for functions in the subclass $H_{\mu}^{q}(n, p, \lambda, \beta; \gamma)$ of the function class $T_p(n)$, which is defined as follows:

A function $f(z) \in T_p(n)$ is said to belong to the class $H_{\mu}^{q}(n, p, \lambda, \beta; \gamma)$ if $w = f(z)$ satisfies the following non-homogenous Cauchy-Euler differential equation :

$$z^2 \frac{d^{2+q}w}{dz^{2+q}} + 2(1+\gamma)z\frac{d^{1+q}w}{dz^{1+q}} + \gamma(1+\gamma)\frac{d^q w}{dz^q} = (p-q+\gamma)(p-q+\gamma+1)\frac{d^q g(z)}{dz^q}, \tag{1.11}$$

where $g(z) \in T_{\mu}^{q}(n, p, \lambda, \beta)$ and $\gamma > q - p, \gamma \in R$.

Several other interesting subclasses of the class $T_p(n)$ were investigated recently, for example, by Chen et al. [8], Chen [7], Srivastava and Aouf [16], Murugusundarmoorthy et al. [12], Altinatas [1], and Altinatas et al. ([3] and [4]), (see also Srivastava and Owa [17]).

In this paper we investigate the geometric characteristics of the classes $T_{\mu}^{q}(n, p, \lambda, \beta)$ and $H_{\mu}^{q}(n, p, \lambda, \beta; \gamma)$ also we investigate some (n,θ)-neighborhood properties.

2 Basic properties of the class $T_{\mu}^{q}(n, p, \lambda, \beta)$

We begin by proving a necessery and sufficient condition for a function belonging to the class $T_p(n)$ to be in the class $T_{\mu}^{q}(n, p, \lambda, \beta)$.

Theorem 1. *Let the function $f(z)$ be defined by* (1.1). *Then $f(z)$ is in the class $T_{\mu}^{q}(n, p, \lambda, \beta)$ if and only if*

$$\sum_{k=n+p}^{\infty} \frac{(k+\beta-p)\left[(k+\mu)-\lambda(k-p)\right]\Gamma(k+\mu)\delta(k,q)}{(k-p)!} a_k \leq \beta\Gamma(p+1+\mu)\delta(p,q). \tag{2.1}$$

Proof. If the condition (2.1) holds true, we find from (1.1) and (2.1) that

$$\left|\lambda z(D^{\mu,p}f(z))^{(q+1)} + (1-\lambda)z(D^{1+\mu,p}f(z))^{(q+1)} - (p-q)\left[\lambda(D^{\mu,p}f(z))^{(q)} - (1-\lambda)(D^{1+\mu,p}f(z))^{(q)}\right]\right| - \beta\left|\lambda(D^{\mu}f(z))^{(q)} + (1-\lambda)(D^{1+\mu,p}f(z))^{(q)}\right|$$

$$= \left|\sum_{k=n+p}^{\infty} \frac{(k-p)\left[(k+\mu)-\lambda(k-p)\right]\Gamma(k+\mu)\delta(k,q)}{(k-p)!\Gamma(p+1+\mu)} a_k z^{k-p}\right|$$

$$-\beta\left|\delta(p,q) - \sum_{k=n+p}^{\infty} \frac{\left[(k+\mu)-\lambda(k-p)\right]\Gamma(k+\mu)\delta(k,q)}{(k-p)!\Gamma(p+1+\mu)} a_k z^{k-p}\right|$$

$$\leq \sum_{k=n+p}^{\infty} \frac{(k+\beta-p)\left[(k+\mu)-\lambda(k-p)\right]\Gamma(k+\mu)\delta(k,q)}{(k-p)!\Gamma(p+1+\mu)} a_k - \beta\delta(p,q)$$
$$\leq 0 \qquad (z \in \partial U = \{z : z \in C \quad \text{and} \quad |z| = 1\}).$$

Hence , by the maximum modulus theorem, $f(z) \in T_{\mu}^{q}(n,p,\lambda,\beta)$.

Conversely, let $f(z) \in T_{\mu}^{q}(n,p,\lambda,\beta)$ be given by (1.1). Then, from (1.8) and (1.10), we have

$$\left| \frac{\lambda z(D^{\mu,p}f(z))^{(q+1)} + (1-\lambda)z(D^{1+\mu,p}f(z))^{(q+1)}}{\lambda(D^{\mu,p}f(z))^{(q)} + (1-\lambda)(D^{1+\mu,p}f(z))^{(q)}} - (p-q) \right|$$

$$= \left| \frac{-\sum\limits_{k=n+p}^{\infty} \frac{(k-p)\left[(k+\mu)-\lambda(k-p)\right]\Gamma(k+\mu)\delta(k,q)}{(k-p)!\Gamma(p+1+\mu)} a_k z^{k-p}}{\delta(p,q) - \sum\limits_{k=n+p}^{\infty} \frac{\left[(k+\mu)-\lambda(k-p)\right]\Gamma(k+\mu)\delta(k,q)}{(k-p)!\Gamma(p+1+\mu)} a_k z^{k-p}} \right| < \beta. \qquad (2.2)$$

Putting $z = r(0 \leq r < 1)$ on the right-hand side of (2.2), and noting the fact that for $r = 0$, the resulting expression in the denominator is positive, and remains so for all $r \in (0,1)$, the desrired inequality (2.1) follows upon letting $r \to 1^{-}$.

Corollary 1. *Let the function $f(z) \in T_p(n)$ be given by* (1.1). *If $f(z) \in T_{\mu}^{q}(n,p,\lambda,\beta)$, then*

$$a_k \leq \frac{(k-p)!\beta\Gamma(p+1+\mu)\delta(p,q)}{(k+\beta-p)[(k+\mu)-\lambda(k-p)]\Gamma(k+\mu)\delta(k,q)} \ (k \geq n+p; p,n \in N). \quad (2.3)$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{(k-p)!\beta\Gamma(p+1+\mu)\delta(p,q)}{(k+\beta-p)[(k+\mu)-\lambda(k-p)]\Gamma(k+\mu)\delta(k,q)} z^k$$
$$(k \geq n+p; p,n \in N). \qquad (2.4)$$

We next prove the following growth and distortion property for the functions of the form (1.1) belonging to the class $T_{\mu}^{q}(n,p,\lambda,\beta)$.

Theorem 2. *If a function $f(z)$ defined by* (1.1) *is in the class $T_{\mu}^{q}(n,p,\lambda,\beta)$. Then*

$$||f(z)| - |z|^p| \leq$$

$$\frac{n!\beta\Gamma(p+1+\mu)\delta(p,q)}{(n+\beta)[(p+\mu)+n(1-\lambda)]\Gamma(n+p+\mu)\delta(n+p,q)} |z|^{n+p}, \qquad (2.5)$$

and (in general),

$$\left| \left| f^{(m)}(z) \right| - \delta(p,m) |z|^{p-m} \right| \leq$$

$$\frac{n!\beta\Gamma(p+1+\mu)\delta(p,q)(n+p-q)!}{(n+\beta)[(p+\mu)+n(1-\lambda)]\Gamma(n+p+\mu)(n+p-m)!} |z|^{n+p-m} \qquad (2.6)$$

$$(z \in U; p,n \in N; m,q \in N_0; m \leq q < p; p > \max(m,q,-\mu)).$$

The results are sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{n!\beta\Gamma(p+1+\mu)\delta(p,q)}{(n+\beta)[(p+\mu)+n(1-\lambda)]\Gamma(n+p+\mu)\delta(n+p,q)} z^{n+p}. \tag{2.7}$$

Proof. In view of Theorem 1, we have

$$\frac{(n+\beta)[(p+\mu)+n(1-\lambda)]\Gamma(n+p+\mu)\delta(n+p,q)}{n!} \sum_{k=n+p}^{\infty} a_k$$

$$\leq \sum_{k=n+p}^{\infty} \frac{(k+\beta-p)[(k+\mu)-\lambda(k-p)]\Gamma(k+\mu)\delta(k,q)}{(k-p)!} a_k$$

$$\leq \beta\Gamma(p+1+\mu)\delta(p,q),$$

which readily yields

$$\sum_{k=n+p}^{\infty} a_k \leq \frac{n!\beta\Gamma(p+1+\mu)\delta(p,q)}{(n+\beta)[(p+\mu)+n(1-\lambda)]\Gamma(n+p+\mu)\delta(n+p,q)}. \tag{2.8}$$

Also, (2.1) yields

$$\sum_{k=n+p}^{\infty} k!a_k \leq \frac{n!(n+p-q)!\beta\Gamma(p+1+\mu)\delta(p,q)}{(n+\beta)[(p+\mu)+n(1-\lambda)]\Gamma(n+p+\mu)}. \tag{2.9}$$

Now, by differentiating both sides of (1.1) m-times, we have

$$f^{(m)}(z) = \delta(p,m)z^{p-m} - \sum_{k=n+p}^{\infty} \delta(k,m)a_k z^{k-m}$$

$$(p, n \in N; m \in N_0; p > m). \tag{2.10}$$

Theorem 2 follows from (2.8), (2.9) and (2.10)

Finally, it is easy to see that the bounds in Theorem 2 are attained for the function $f(z)$ given by (2.7).

Theorem 3. *Let the function $f(z)$ defined by* (1.1) *be in the class $T^q_\mu(n,p,\lambda,\beta)$, then*

(i) *$f(z)$ is p-valently close-to- convex of order α $(0 \leq \alpha < p)$ in $|z| < r_1$, where*

$$r_1 = \inf_k \left\{ \left(\frac{p-\alpha}{k} \right) \theta(p,q,\lambda,\mu,\beta;k) \right\}^{\frac{1}{k-p}}, \tag{2.11}$$

(ii) *$f(z)$ is p-valently starlike of order α $(0 \leq \alpha < p)$ in $|z| < r_2$, where*

$$r_2 = \inf_k \left\{ \left(\frac{p-\alpha}{k-\alpha} \right) \theta(p,q,\lambda,\mu,\beta;k) \right\}^{\frac{1}{k-p}}, \tag{2.12}$$

(iii) $f(z)$ *is p-valently convex of order* $\alpha(0 \leq \alpha < p)$ *in* $|z| < r_3$, *where*

$$r_3 = \inf_k \left\{ \frac{p(p-\alpha)}{k(k-\alpha)} \theta(p,q,\lambda,\mu,\beta;k) \right\}^{\frac{1}{k-p}}, \tag{2.13}$$

where

$$\theta(p,q,\lambda,\mu,\beta;k) = \frac{(k+\beta-p)[(k+\mu)-\lambda(k-p)]\Gamma(k+\mu)\delta(k,q)}{\beta(k-p)!\Gamma(p+1+\mu)\delta(p,q)} \tag{2.14}$$

$$(k \geq n+p; p,n \in N; q \in N_0; p > q; \mu > -p; 0 \leq \lambda \leq 1; 0 \leq \alpha < p; 0 < \beta \leq 1).$$

Each of these results is sharp for the function $f(z)$ *given by* (2.4).

Proof. It is sufficient to show that

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p - \alpha \quad (|z| < r_1; 0 \leq \alpha < p; p \in N), \tag{2.15}$$

$$\left| \frac{zf'(z)}{f(z)} - p \right| \leq p - \alpha \quad (|z| < r_2; 0 \leq \alpha < p; p \in N), \tag{2.16}$$

and that

$$\left| 1 + \frac{zf''(z)}{f'(z)} - p \right| \leq p - \alpha \quad (|z| < r_3; 0 \leq \alpha < p; p \in N), \tag{2.17}$$

for a function $f(z) \in T_\mu^q(n,p,\lambda,\beta)$, where r_1, r_2 and r_3 are defined by (2.11), (2.12) and (2.13), repectively. The details involved are fairly straightforward and may be omitted.

3 Properties of the class $H_\mu^q(n,p,\lambda,\beta;\gamma)$

Applying the results of Section 2, which were obtained for the function $f(z)$ of the form (1.1) belonging to the class $T_\mu^q(n,p,\lambda,\beta)$, we now derive the corresponding results for the function $f(z)$ belonging to the class $H_\mu^q(n,p,\lambda,\beta;\gamma)$.

Theorem 4. *If a function* $f(z)$ *defined by* (1.1) *is in the class* $H_\mu^q(n,p,\lambda,\beta;\gamma)$, *then*

$$||f(z)| - |z|^p| \leq$$

$$\frac{n!\beta\Gamma(p+1+\mu)\delta(p,q)(p-q+\gamma)(p-q+\gamma+1)}{(n+\beta)[(p+\mu)+n(1-\lambda)]\Gamma(n+p+\mu)\delta(n+p,q)(n+p-q+\gamma)} |z|^{n+p} \tag{3.1}$$

and (in general),

$$\left| \left| f^{(m)}(z) \right| - \delta(p,m) \left| z \right|^{p-m} \right| \le$$

$$\frac{n!\beta\Gamma(p+1+\mu)\delta(p,q)(p-q+\gamma)(p-q+\gamma+1)(n+p-q)! \ \left| z \right|^{n+p-m}}{(n+\beta)[(p+\mu)+n(1-\lambda)]\Gamma(n+p+\mu)\delta(n+p,q)(n+p-q+\gamma)(n+p-m)!} \tag{3.2}$$

$$(z \in U; p,n \in N; m,q \in N_0; m \le q < p).$$

The results in (3.1) *and* (3.2) *are sharp for the function* $f(z)$ *given by*

$$f(z) = z^p -$$

$$\frac{n!\beta\Gamma(p+1+\mu)\delta(p,q)(p-q+\gamma)(p-q+\gamma+1)}{(n+\beta)[(p+\mu)+n(1-\lambda)]\Gamma(n+p+\mu)\delta(n+p,q)(n+p-q+\gamma)} z^{n+p}. \tag{3.3}$$

Proof. Assume that $f(z) \in T_p(n)$ is given by (1.1). Also, let function $g(z) \in H^q_\mu(n,p,\lambda,\beta)$, occuring in the non-homogenous differential equation(1.11) be of the form:

$$g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k \ \ (b_k \ge 0; p,n \in N). \tag{3.4}$$

Then, we readily find from (1.11) that

$$a_k = \frac{(p-q+\gamma)(p-q+\gamma+1)}{(k-q+\gamma)(k-q+\gamma+1)} b_k \ \ (k \ge n+p; p,n \in N), \tag{3.5}$$

so that

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k = z^p - \sum_{k=n+p}^{\infty} \frac{(p-q+\gamma)(p-q+\gamma+1)}{(k-q+\gamma)(k-q+\gamma+1)} b_k z^k, \tag{3.6}$$

and

$$\left| \left| f(z) \right| - \left| z \right|^p \right| \le \left| z \right|^{n+p} \sum_{k=n+p}^{\infty} \frac{(p-q+\gamma)(p-q+\gamma+1)}{(k-q+\gamma)(k-q+\gamma+1)} b_k \ \ (z \in U). \tag{3.7}$$

Next, since $g(z) \in T^q_\mu(n,p,\lambda,\beta)$, therefore, on using the assertion (2.8) of Theorem 2, we get the following coefficient ineqality :

$$b_k \le \frac{n!\beta\Gamma(p+1+\mu)\delta(p,q)}{(n+\beta)[(p+\mu)+n(1-\lambda)]\Gamma(n+p+\mu)\delta(n+p,q)} \ \ (k \ge n+p; p,n \in N), \tag{3.8}$$

which in conjunction with (3.6) and (3.7) yield

$$\left| \left| f(z) \right| - \left| z \right|^p \right| \le$$

$$\frac{n!\beta\Gamma(p+1+\mu)\delta(p,q)}{(n+\beta)[(p+\mu)+n(1-\lambda)]\Gamma(n+p+\mu)\delta(n+p,q)} \left| z \right|^{n+p}.$$

$$\cdot \sum_{k=n+p}^{\infty} \frac{(p-q+\gamma)(p-q+\gamma+1)}{(k-q+\gamma)(k-q+\gamma+1)} \quad (z \in U). \tag{3.9}$$

By noting the following summation result:

$$\sum_{k=n+p}^{\infty} \frac{(p-q+\gamma)(p-q+\gamma+1)}{(k-q+\gamma)(k-q+\gamma+1)} = \frac{(p-q+\gamma)(p-q+\gamma+1)}{(n+p-q+\gamma)}, \tag{3.10}$$

where $\gamma \in R^* = R \backslash \{-n-p, -n-p-1, ...\}$. The assertion (3.1) of Theorem 4 follows from (3.9) and (3.10). The assertion (3.2) of Theorem 4 can be established by similarly applying (2.9), (3.5) and (3.10).

Theorem 5. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $H_\mu^q(n,p,\lambda,\beta;\gamma)$, *then* $f(z)$ *is p-valently close-to-convex of order* $\delta(0 \le \delta < p)$ *in* $|z| < r_4$, *where*

$$r_4 = \inf_k \left\{ \theta(p,q,\lambda,\mu,\beta;k) \frac{(p-\delta)(k-q+\gamma)(k-q+\gamma+1)}{k(p-q+\gamma)(p-q+\gamma+1)} \right\}^{\dfrac{1}{k-p}}$$

$$(k \ge n+p; p,n \in N; q \in N_0; p > q; \mu > -p; 0 \le \lambda \le 1; 0 \le \delta < p; 0 < \beta \le 1; \gamma \in R^*), \tag{3.11}$$

where $\theta(p,q,\lambda,\mu,\beta;k)$ *is given by* (2.14). *The result is sharp for the function* $f(z)$ *given by* (3.3).

Proof. Assume that $f(z) \in T_p(n)$ is given by (1.1). Also, let the function $g(z) \in T_\mu^q(n,p,\lambda,\beta)$, occuring in the non-homogenous differential equation (1.11), be given by (3.4). Then, it sufficient to show that

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| < p - \delta \quad (|z| < r_4; 0 \le \delta < p; p \in N).$$

Indeed, we have

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \le \sum_{k=n+p}^{\infty} k a_k \left| z \right|^{k-p},$$

and by using the coefficient relation (3.5) between the functions $f(z)$ and $g(z)$, we get

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \le \sum_{k=n+p}^{\infty} \frac{(p-q+\gamma)(p-q+\gamma+1)}{(k-q+\gamma)(k-q+\gamma+1)} k b_k \left| z \right|^{k-p} \le p - \delta. \tag{3.12}$$

Since $g(z) \in T_\mu^q(n,p,\lambda,\beta)$, and we know from the assertion (2.1) of Theorem 1 that

$$\sum_{k=n+p}^{\infty} \frac{(k+\beta-p)[(k+\mu)-\lambda(k-p)]\Gamma(k+\mu)\delta(k,q)}{(k-p)!} b_k \le \beta \Gamma(p+1+\mu)\delta(p,q),$$

hence, (3.11) is true if

$$\left(\frac{k}{p-\delta}\right)\frac{(p-q+\gamma)(p-q+\gamma+1)}{(k-q+\gamma)(k-q+\gamma+1)}\left|z\right|^{k-p} \le \theta(p,q,\lambda,\mu,\beta;k) \ \ (k \ge n{+}p; p,n \in N), \tag{3.13}$$

where $\theta(p,q,\lambda,\mu,\beta;k)$ is given by (2.14). Solving (3.12) for $|z|$, we obtain

$$|z| \le \left\{\theta(p,q,\lambda,\mu,\beta;k).\frac{(p-\delta)(k-q+\gamma)(k-q+\gamma+1)}{k(p-q+\gamma)(p-q+\gamma+1)}\right\}^{\frac{1}{k-p}} \ (k \ge n{+}p; p,n \in N)$$

which obviously proves Theorem 5.

Remark 1. *We note that the result obtained by Irmak et al. [10 , Theorem 2.3] is not correct. The correct result is given by* (3.11) *with* $p = 1$ *and* $q = 0$.

Theorem 6. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $H_{\mu}^{q}(n,p,\lambda,\beta;\gamma)$, *then* $f(z)$ *is p-valently starlike of order* $\delta (0 \le \delta < p)$ *in* $|z| < r_5$, *where*

$$r_5 = \inf_k \left\{\theta(p,q,\lambda,\mu,\beta;k).\frac{(p-\delta)(k-q+\gamma)(k-q+\gamma+1)}{(k-\delta)(p-q+\gamma)(p-q+\gamma+1)}\right\}^{\frac{1}{k-p}}$$

$$(k \ge n{+}p; p,n \in N; q \in N_0; p > q; \mu > -p; 0 \le \lambda \le 1; 0 \le \delta < p; 0 < \beta \le 1; \gamma \in R^*), \tag{3.14}$$

where $\theta(p,q,\lambda,\mu,\beta;k)$ *is given by* (2.14). *The result is sharp for the function* $f(z)$ *given by* (3.3).

Theorem 7. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $H_{\mu}^{q}(n,p,\lambda,\beta;\gamma)$, *then* $f(z)$ *is p-valently convex of order* $\delta (0 \le \delta < p)$ *in* $|z| < r_6$, *where*

$$r_6 = \inf_k \left\{\theta(p,q,\lambda,\mu,\beta;k).\frac{p(p-\delta)(k-q+\gamma)(k-q+\gamma+1)}{k(k-\delta)(p-q+\gamma)(p-q+\gamma+1)}\right\}^{\frac{1}{k-p}}$$

$$(k \ge n{+}p; p,n \in N; q \in N_0; p > q; \mu > -p; 0 \le \lambda \le 1; 0 \le \delta < p; 0 < \beta \le 1; \gamma \in R^*), \tag{3.15}$$

where $\theta(p,q,\lambda,\mu,\beta;k)$ *is given by* (2.14). *The result is sharp for the function* $f(z)$ *given by* (3.3).

Remark 2. *We note that the results obtained by Irmak et al. [10, Theorems 3.3 and 3.4] are not correct. The correct results are given by* (3.14) *and* (3.15), *respectively, with* $p = 1$ *and* $q = 0$.

4 Inclusion relations involving $(n,\theta)-$neighborhood for the class $T_{\mu}^{q}(n,p,\lambda,\beta)$

Following the works of Goodman[11], Ruscheweyh [15] and Altintas [2] (see also [5], [6], [9], and [13]) we define the $(n,\theta)-$neighborhood of a function $f^{(q)}(z)$ when $f \in T_p(n)$ by

$$N_{n,p}^{\theta}(f^{(q)},g^{(q)}) =$$

$$\left\{ g \in T_p(n) : g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k \text{ and } \sum_{k=n+p}^{\infty} \delta(k,q) k \left| a_k - b_k \right| \leq \theta \right\}. \quad (4.1)$$

It follows from (4.1) that , if

$$h(z) = z^p \qquad (p \in N), \quad (4.2)$$

then

$$N^{\theta}_{n,p}(h) = \left\{ g \in T_p(n) : g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k \text{ and } \sum_{k=n+p}^{\infty} \delta(k,q) k \left| b_k \right| \leq \theta \right\}. \quad (4.3)$$

Next; we establish inclusion relationships for the function class $T^q_\mu(n,p,\lambda,\beta)$ involving the (n,θ)−neighborhood $N^{\theta}_{n,p}(h)$ defined by (4.3).

Theorem 8. *If*

$$\theta = \frac{\beta\Gamma(p+1+\mu)\delta(p,q)n!}{[p+\mu+n(1-\lambda)]\Gamma(n+p+\mu)} \left(\frac{n+p}{n+\beta} \right), \quad (4.4)$$

then

$$T^q_\mu(n,p,\lambda,\beta) \subset N^{\theta}_{n,p}(h). \quad (4.5)$$

Proof. Let $f \in T^q_\mu(n,p,\lambda,\beta)$.Then , in view of the assertion (2.1) of Theorem 1, we have

$$\frac{(n+\beta)[p+\mu+n(1-\lambda)]\Gamma(n+p+\mu)}{n!} \sum_{k=n+p}^{\infty} \delta(k,q) a_k$$
$$\leq \beta\Gamma(p+1+\mu)\delta(p,q) \quad (4.6)$$

so that

$$\sum_{k=n+p}^{\infty} \delta(k,q) a_k \leq \frac{\beta\Gamma(p+1+\mu)\delta(p,q)n!}{(n+\beta)[p+\mu+n(1-\lambda)]\Gamma(n+p+\mu)}. \quad (4.7)$$

On the other hand, we also find from (2.1) and (4.7) that

$$\frac{[p+\mu+n(1-\lambda)]\Gamma(n+p+\mu)}{n!} \sum_{k=n+p}^{\infty} \delta(k,q) k a_k \leq \beta\Gamma(p+1+\mu)\delta(p,q)+$$

$$\frac{(p-\beta)[(p+\mu+n)]\Gamma(n+p+\mu)}{n!} \sum_{k=n+p}^{\infty} \delta(k,q) a_k \leq \beta\Gamma(p+1+\mu)\delta(p,q)+$$

$$(p-\beta)\frac{[p+\mu+n(1-\lambda)]\Gamma(n+p+\mu)}{n!} \frac{\beta\Gamma(p+1+\mu)\delta(p,q)n!}{(n+\beta)[p+\mu+n(1-\lambda)]\Gamma(n+p+\mu)}$$
$$= \beta\Gamma(p+1+\mu)\delta(p,q)\left(\frac{n+p}{n+\beta} \right),$$

that is

$$\sum_{k=n+p}^{\infty} \delta(k,q) k a_k \leq \beta \frac{\Gamma(p+1+\mu)\delta(p,q)n!}{[p+\mu+n(1-\lambda)]\Gamma(n+p+\mu)} \left(\frac{n+p}{n+\beta} \right) = \theta. \tag{4.8}$$

Remark 3. *Putting $q = 0$ and $p = 1$ in Theorem 8, we obtain the following corollary.*

Corollary 2. *If*

$$\theta = \frac{\beta\Gamma(2+\mu)n!}{[1+\mu+n(1-\lambda)]\Gamma(n+1+\mu)} \left(\frac{n+1}{n+\beta} \right), \tag{4.9}$$

then

$$T_\mu(n,\lambda,\beta) \subset N_n^\theta(h). \tag{4.10}$$

5 Neighborhood for the class $T_\mu^{q,\alpha}(n,p,\lambda,\beta)$

In this section we determine the neighborhood for the class $T_\mu^{q,\alpha}(n,p,\lambda,\beta)$ which we define as follows. A function $f \in T_p(n)$ is said to be in the class $T_\mu^{q,\alpha}(n,p,\lambda,\beta)$ if there exist a function $g \in T_\mu^q(n,p,\lambda,\beta)$ such that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < p - \alpha \quad (z \in U;\ 0 \leq \alpha < p). \tag{5.1}$$

Theorem 9. *If $g \in T_\mu^q(n,p,\lambda,\beta)$ and*

$$\alpha = p - \frac{\theta(n+\beta)[p+\mu+n(1-\lambda)]\Gamma(n+p+\mu)}{(n+p)\{[(n+\beta)[p+\mu+n(1-\lambda)]\Gamma(n+p+\mu)\delta(n+p,q) - \beta\Gamma(p+1+\mu)\delta(p,q)n!\}}, \tag{5.2}$$

where

$$\theta \leq p(n+p) \times \\ \times \left\{ \delta(n+p,q) - \beta\Gamma(p+1+\mu)\delta(p,q)n!)[(n+\beta)[p+\mu+n(1-\lambda)]\Gamma(n+p+\mu)]^{-1} \right\},$$

then

$$N_{n,p}^\theta(g) \subset T_\mu^{q,\alpha}(n,p,\lambda,\beta). \tag{5.4}$$

Proof. Suppose that $f \in N_{n,p}^\theta(g)$, then we find from the definition (4.1) that

$$\sum_{k=n+p}^{\infty} \delta(k,q) k \left| a_k - b_k \right| \leq \theta, \tag{5.5}$$

which implies the coefficient inequality

$$\sum_{k=n+p}^{\infty} |a_k - b_k| \leq \frac{\theta}{(n+p)\delta(n+p,q)} \qquad (p > q, n, p \in N, q \in N_0). \tag{5.6}$$

Next, since $g \in T_{\mu}^{q}(n,p,\lambda,\beta)$, we have

$$\sum_{k=n+p}^{\infty} b_k \leq \frac{\beta\Gamma(p+1+\mu)\delta(p,q)n!}{(n+\beta)[p+\mu+n(1-\lambda)]\Gamma(n+p+\mu)\delta(n+p,q)}, \tag{5.7}$$

so that

$$\left|\frac{f(z)}{g(z)}-1\right| \leq \frac{\sum\limits_{k=n+p}^{\infty}|a_k-b_k|}{1-\sum\limits_{k=n+p}^{\infty}|b_k|}$$

$$\leq \frac{\dfrac{\theta}{(n+p)\delta(n+p,q)}}{1-\dfrac{\beta\Gamma(p+1+\mu)\delta(p,q)n!}{(n+\beta)[p+\mu+n(1-\lambda)]\Gamma(n+p+\mu)\delta(n+p,q)}}$$

$$= \frac{\theta(n+\beta)[p+\mu+n(1-\lambda)]\Gamma(n+p+\mu)}{(n+p)\{[(n+\beta)[p+\mu+n(1-\lambda)]\Gamma(n+p+\mu)\delta(n+p,q)-\beta\Gamma(p+1+\mu)\delta(p,q)n!\}} \leq p-\alpha,$$

where α given by (5.2). This implies that $f \in T_{\mu}^{q,\alpha}(n,p,\lambda,\beta)$.

Remark 4. *Putting $q=0$ and $p=1$ in Theorem 9, we obtain the following corollary.*

Corollary 3. *If $g \in T_{\mu}(n,\lambda,\beta)$, and*

$$\alpha = 1-\frac{\theta(n+\beta)[1+\mu+n(1-\lambda)]\Gamma(n+1+\mu)}{(n+1)\{(n+\beta)[1+\mu+n(1-\lambda)]\Gamma(n+1+\mu)-\beta\Gamma(2+\mu)n!)\}},$$

where

$$\theta \leq (n+1)\{1-\beta\Gamma(2+\mu)n![(n+\beta)[1+\mu+A(1-\lambda)]\Gamma(n+1+\mu)]^{-1}\}$$

then

$$N_n^{\theta}(g) \subset T_{\mu}^{(\alpha)}(n,\lambda,\beta).$$

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References

[1] O. Altinatas, *On a subclass of certain starlike functions with negative coefficients*, Math. Japon., 36 (1991), no.3, 489-495.

[2] O. Altinatas, *Neighborhoods of certian subclasses of p-valently analytic functions with negative coefficients*, Appl. Math. Comput. 187 (2007) no. 1, 47-53.

[3] O. Altinatas, H. Irmak and H. M. Srivastava, *A subclass of analytic functions defined by using certain operators of fractinal calculus*, Comput. Math. Appl. 30 (1995), no.1, 1-9.

[4] O. Altinatas, H. Irmak and H. M. Srivastava, *Fractinal calulus and certain starlike functions with negative coefficients*, Comput. Math. Appl. 30(1995), no.2, 9-15.

[5] O. Altinatas, H. Irmak and H. M. Srivastava, *Neighborhoods for certian subclasses of multivalently analytic functions defined by differential operator*, Comput. Math. Appl. 55 (2008), no.9, 331-338.

[6] O. Altinatas, Ö. Özkan and H. M. Srivastava, *Neighborhoods of certain family of multivalent functions with negative coefficients*, Comput. Math. Appl. 47(2004), 1167-1672.

[7] M. -P. Chen, *Multivalent functions with negative coefficients in the unit disc*, Tamkang J.Math.17(1986), 127-137.

[8] M. -P. Chen, H. Irmak and H. M. Srivastava, *A certain subclass of analytic functions involving operator of fractional calculus*, Comput. Math. Appl. 35 (1998), no.5, 83-91.

[9] B. A. Frasin, *Neighborhoods of certian multivalent analytic functions with negative coefficients*, Appl. Math.Comput. 193 (2007), no.1, 1-6.

[10] H. Irmak, S. B. Joshi and R. K. Raina, *On certain novel subclasses of analytic functions*, Kyungpook Math. J. 46(2006), 543-552.

[11] A. W. Goodman, *Univalent functions and non-analytic curves*, Proc. Amer. Math. Soc. 8 (1957), 598-601.

[12] G. Murugusundarmoorthy, P. Balasubramanyam and K. G. Suramanian, *On a geralization of a class of analytic functions with negative coefficient*, Chinese J. Math. 22(1994), 11-19.

[13] G. Murugusundarmoorthy and H. M. Srivastava, *Neighborhoods of certain classes of analytic functions of complex order*, J. Inequal. Pure Appl. Math., 5(2)(2004), Art. 24; 1-8.

[14] St. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc. 49(1957), 109-115.

[15] St. Ruscheweyh, *Neighborhoods of univalent functions*, Proc. Amer. Math. Soc. 81 (1981), 521-527.

[16] H. M. Srivastava and M. K. Aouf, *A certain fractional derivative operator and its applicatins to new class of analytic multivalent functions with negative coefficient. I and II.*, J. Math. Anal. Appl. 171(1992), 1-13; ibid. 19(1995), 673-688.

[17] H. M. Srivastava and S. Owa (Editors), *Current Topics in Analytic Function Theory*, World Scientific Publishing Company, Singapore, New Jersey, London, and Hong Kong, 1992.

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