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INCLUSION WEIERSTRASS-LIKE ROOT-FINDERS WITH CORRECTIONS

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Abstract. **In this paper we present iterative methods of Weierstrass's type for the simultaneous inclusion of all multiple zeros of a polynomial. The order of convergence of the proposed interval method is $1+\sqrt{2}\approx 2.414$ or 3, depending on the type of the applied disk inversion. The criterion for the choice of a proper circular root-set is given. This criterion uses the already calculated entries which increases the computational efficiency of the presented algorithms. Numerical results are given to demonstrate the convergence behavior.**

1. Preliminaries and basic concepts

The problem of finding complex zeros of a polynomial is one of the most important problems involved in mathematical models of many branches of engineering sciences, physics and applied mathematics. The implementation of some of zero-finding iterative procedures requires solving certain practical problems as computationally verifiable initial conditions that guarantee convergence of applied algorithm, the construction of algorithms which possess a fast convergence in the presence of multiplicity of a requested zero, the control of rounding errors, information about error bounds of a complex approximation to the sought zero, and so on.

Some of the above requirements can be fulfilled by applying interval arithmetic. In particular, the problem of determination of complex zeros with automatic error bounds needs complex interval arithmetic. Iterative methods for the simultaneous determination of complex zeros of a given polynomial, realized in complex interval arithmetic, are a new and very efficient device to error estimates for a given set of approximate zeros. More details about iterative inclusion methods, including studies on convergence

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properties, computational efficiency, numerical experiments and parallel implementation, may be found in the books [1], [7], [11] and references cited there. In general, inclusion methods, realized in complex interval arithmetic, produce resulting disks or rectangles containing complex zeros. In this manner, the upper error bounds, given by the radii of disks or the semidiagonals of rectangles, are obtained automatically. This very useful property of self-validated results, together with the ability to incorporate rounding errors without altering the fundamental structure of the iterative formula, led to frequent application of inclusion methods for solving many problems which appear not only in applied mathematics but also in mathematical models of physics and engineering sciences.

A significant improvement of computational efficiency of simultaneous inclusion methods can be achieved by using suitable correction terms. Such an approach, based on Nourein's idea [5] for the simultaneous methods in ordinary complex arithmetic, was applied for the first time in [9] to the Börsch-Supan-like method. This idea was later applied to the Ehrlich-Aberth method [2] (the case of multiple zeros) and the Halley method [8]. The aim of this paper is to present iterative methods of Weierstrass' type for the simultaneous inclusion of multiple zeros of a polynomial where the improved convergence order is attained by using suitable corrections.

Let us consider a monic polynomial of degree $n \geq 3$

(1.1)
$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = (z-\zeta_1)^{\mu_1} \cdots (z-\zeta_k)^{\mu_\nu}$$

with simple or multiple zeros $\zeta_1, ..., \zeta_\nu$ with the known multiplicities $\mu_1, ..., \mu_\nu$, $\mu_1 + \cdots + \mu_\nu = n$. From the factorization (1.1) we obtain the fixed point relations

$$\zeta_k = z - \left[\frac{P(z)}{\prod_{j\neq k}(z-\zeta_j)^{\mu_j}}\right]_*^{1/\mu_k} \quad (k \in I_\nu := \{1, ..., \nu\}). \tag{1.2}$$

The right-hand side in (1.2) will reduce to the zero ζ_k only for one particular value of the μ_k-th root ($\mu_k > 1$). The symbol $*$ indicates that only one (appropriate) of μ_k values of the μ_k-th root of a complex number has to be chosen.

Let $z_1, \ldots, z_\nu$ be distinct approximations to the zeros $\zeta_1, ..., \zeta_\nu$. Farmer and Loizou [4] constructed the following iterative formula for multiple zeros:

$$\hat{z}_k = z_k - \left[\frac{P(z_k)}{\prod_{j\neq k}(z_k-z_j)^{\mu_j}}\right]_*^{1/\mu_k} \quad (k \in I_\nu). \tag{1.3}$$

The order of convergence of this method is *two*. The increase of the convergence rate of the method (1.3) has been obtained by using Newton's correction in the following way (see [12]):

$$\hat{z}_k = z_k - \left[\frac{P(z_k)}{\prod_{j\neq k}(z_k - z_j + N_j)^{\mu_j}}\right]_*^{1/\mu_k} \quad (k \in I_\nu). \tag{1.4}$$

Newton's correction N_j appearing in (1.4), often known as Schröder's correction, is given by $N_j = \mu_j P(z_j)/P'(z_j)$. The order of convergence of the iterative method (4) is *three.* The proper value of the μ_k−th root (which is stressed by $*$) should select according to the criterion described in [12]. In this paper we will extend the iterative methods (1.3) and (1.4) to circular complex arithmetic.

The development and convergence analysis of the algorithm which will be considered in this paper require the basic properties of the so-called circular complex arithmetic introduced by Gargantini and Henrici [3]. More details about circular arithmetic can be found in the books [1, Ch. 5] and [11, Ch. 1]. Throughout this paper, disks in the complex plane will be denoted by capital letters. A circular closed region (disk) $Z := \{z : |z - c| \leq r\}$ with center $c := \operatorname{mid} Z$ and radius $r := \operatorname{rad} Z$ we will denote by parametric notation $Z := \{c; r\}$.

Addition, subtraction and multiplication in circular arithmetic are defined as follows:

$$\{c_1; r_1\} \pm \{c_2; r_2\} = \{c_1 \pm c_2; r_1 + r_2\},$$
$$\{c_1; r_1\} \cdot \{c_2; r_2\} := \{c_1c_2; |c_1|r_2 + |c_2|r_1 + r_1r_2\}.$$

We will consider two types of inversion of a disk Z: the *exact inversion*

$$Z^{-1} = \{c; r\}^{-1} = \frac{\{\bar{c}; r\}}{|c|^2 - r^2} \quad (|c| > r, \text{ i.e. } 0 \notin Z), \tag{1.5}$$

obtained by the Möbius transformation, and the inversion in the *centered form*

$$Z^I := \{c; r\}^I = \left\{\frac{1}{c}; \frac{r}{|c|(|c| - r)}\right\} \quad (|c| > r, \text{ i.e. } 0 \notin Z). \tag{1.6}$$

Let us note that

$$\{c; r\}^{-1} \subset \{c; r\}^I.$$

In the sequel, $INV(Z)$ will denote one of the two inversions Z^{-1} or Z^I.

Using (1.5) and (1.6) the division is defined as

$$Z_1 : Z_2 := Z_1 \cdot Z_2^{-1} \;\text{ or }\; Z_1 : Z_2 := Z_1 \cdot Z_2^I \qquad (0 \notin Z_2).$$

In this paper we will use the following obvious implications:

$$z \in \{c; r\} \Leftrightarrow |z - c| \le r. \tag{1.7}$$

$$z \notin \{c; r\} \Leftrightarrow |z - c| > r. \tag{1.8}$$

For a disk $Z = \{c; r\}$ which does not contain the origin (that is, $|c| > r$), the k-th root of Z is defined by the union (see [11, Ch. 2])

(1.9)

$$Z^{1/k} := \bigcup_{\lambda=0}^{k-1} \Big\{ |c|^{1/k} \exp\Big(\frac{\arg c + 2\lambda\pi}{k} i\Big); |c|^{1/k} - (|c| - r)^{1/k} \Big\}.$$

2. Weierstrass-like algorithm for multiple zeros

Let $Z_1, \dots, Z_n$ be disjoint disks which contain the zeros $\zeta_1, \dots, \zeta_n$ respectively, and let $z_k = \text{mid}\, Z_k$ $(k = 1, \dots, n)$. In [10] the following inclusion methods of Weierstrass' type for simple zeros with Newton's and Weierstrass' corrections were constructed:

$$\hat{Z}_k = z_k - P(z_k) \cdot INV\left(\prod_{\substack{j=1 \\ j \ne k}}^{n} (z_k - Z_j + N_j) \right) \quad (k \in I_\nu), \tag{2.1}$$

$$\hat{Z}_k = z_k - P(z_k) \cdot INV\left(\prod_{\substack{j=1 \\ j \ne k}}^{n} (z_k - Z_j + W_j) \right) \quad (k \in I_\nu), \tag{2.2}$$

where $N_j = P(z_j)/P'(z_j)$ and $W_j = P(z_j)/\prod_{j \ne i}(z_i - z_j)$ are Newton's and Weierstrass' corrections, respectively. INV in (2.1) and (2.2) denotes either the exact inversion (1.5) or the centered inversion (1.6), that is, $INV \in \{()^{-1}, ()^I\}$.

Now, we will extend the iterative inclusion formula (2.1) to the case of multiple zeros. Considering computational costs, Schröder's correction

$$N_k = \mu_j \frac{P(z_k)}{P'(z_k)}$$

is obviously more convenient compared to Weierstrass' correction

$$W_k = \left[\frac{P(z_k)}{\prod_{j \ne k} (z_k - z_j)} \right]^{1/\mu_k}$$

since (already calculated) Schröder's correction enables us to establish an efficient <u>**c**</u>riterion for the <u>**c**</u>hoice of the *proper* <u>**r**</u>oot-disk, referred to as **CCR**, among μ_k (> 1) disks. Aside from this advantage, we note that Schröder's correction N_k is rather simpler for calculation than Weierstrass' correction

W_k. For this reason, we will not study the iterative formula with Weierstrass' correction. However, if all zeros are simple, then both corrections give algorithms (2.1) and (2.2) of the same computational costs.

Let $Z_1, \dots, Z_\nu$ be disjoint disks containing the zeros $\zeta_1, \dots, \zeta_\nu$ respectively, and $z_k = \text{mid } Z_k$. From the fixed point relation (1.2) we obtain

$$\zeta_k \in z_k - \frac{1}{\left[\dfrac{1}{P(z_k)} \prod_{j \neq k} (z_k - Z_j)^{\mu_j}\right]_*^{1/\mu_k}} \quad (k \in I_\nu). \tag{2.3}$$

According to (2.3) we are able to construct the following interval method of Weierstrass' type for the simultaneous inclusion of multiple zeros of a polynomial P:

(2.4)

$$\hat{Z}_k = z_k - INV\left(\frac{1}{P(z_k)} \prod_{j \neq k} (z_k - Z_j)^{\mu_j}\right]_*^{1/\mu_k}\right) \qquad (k \in I_\nu).$$

Let us introduce disks

$$Q_k = \frac{1}{P(z_k)} \prod_{j \neq k} (z_k - Z_j + N_j)^{\mu_k} \quad (k \in I_\nu).$$

Assuming that $\zeta_j \in Z_j - N_j$ (under some suitable conditions), from the fixed point relation (1.2) we obtain

(2.5)

$$\zeta_k \in z_k - \frac{1}{\left[\dfrac{1}{P(z_k)} \prod_{j \neq k} (z_k - Z_j + N_j)^{\mu_j}\right]_*^{1/\mu_k}} = z_k - \frac{1}{[Q_k]_*^{1/\mu_k}} \quad (k \in I_\nu).$$

Let again $Z_1, \dots, Z_\nu$ be disjoint disks containing the zeros $\zeta_1, \dots, \zeta_\nu$ respectively, and $z_k = \text{rad } Z_k$. According to (2.5) we can construct the following interval method of Weierstrass' type with Schröder's correction for the simultaneous inclusion of multiple zeros of a polynomial P:

$$\hat{Z}_k = z_k - INV\left(\frac{1}{P(z_k)} \prod_{j \neq k} (z_k - Z_j + N_j)^{\mu_j}\right]_*^{1/\mu_k}\right) = z_k - INV\left([Q_k]_*^{1/\mu_k}\right)$$
$$(k \in I_\nu).$$

3. Criterion for the root selection

First, we are concerned with the selection of the appropriate disk in (2.6). The main idea has been already presented in [12] in ordinary complex arithmetic so that we give only the outline of **CCR** in complex circular arithmetic in the same spirit as in [6] (see, also, [7, Ch. 3]). Let $\mu_k > 1$ and assume that the disk Q_k^{1/μ_k} does not contain the origin. Then Q_k^{1/μ_k} is the union of μ_k disjoint disks (see formula (1.9)), one of which contains $(z_k-\zeta_j)^{-1}$. Let this disk be denoted by $\big[Q_k\big]_*^{1/\mu_k} = \{c_k^*; d_k^*\}$. Denote the remaining $\mu_k - 1$ disks $Q_{k,\lambda}^{1/\mu_k}$, which do not contain $(z_k-\zeta_j)^{-1}$, with $\{c_{k,\lambda}; d_k\},\ \lambda = 1, ..., \mu_k - 1$. Let us note that $d_k^* = d_k$ and $c_{k,\lambda} = c_k^* \exp\big(i\frac{2\lambda\pi}{\mu_k}\big),\ \lambda = 1, ..., \mu_k - 1$ adopting $c_{k,0} = c_k^*$.

Let us introduce

(3.1)
$$\rho = \min_{\substack{1\le i,j\le \nu\\ i\ne j}} \{|z_i - z_j| - r_j\}, \quad r = \operatorname*{rad}_{1\le j\le\nu} r_j, \quad \mu = \min_{1\le k\le\nu} \mu_k,$$

where $z_j = \operatorname{mid} Z_j,\ r_j = \operatorname{rad} Z_j$. The criterion for the choice of the appropriate disk $\big[Q_k\big]_*^{1/\mu_k}$ is based on the following assertion:

Lemma 3.1. *If $\rho > (n-1)r$ and*

$$d_k < \frac{\rho - (n-\mu_k)r}{2\mu_k \rho r},$$

then for each $k = 1, \dots, \nu$ it follows

(i) $|N_k^{-1} - c_k^*| \le \dfrac{n-\mu_k}{\rho\mu_k} + d_k;$

(ii) $|N_k^{-1} - c_k| \ge \dfrac{n-\mu_k}{\rho\mu_k} + 3d_k.$

Proof. Using the logarithmic derivative we find

(3.2)
$$N_k^{-1} = \frac{1}{\mu_k}\frac{P'(z_j)}{P(z_j)} = \frac{1}{\mu_k}\Big[\frac{d}{dz}\ln\prod_{j=1}^{\nu}(z-\zeta_j)^{\mu_j}\Big]_{z=z_k} = \frac{1}{\mu_k}\sum_{j=1}^{\nu}\frac{\mu_j}{z_k-\zeta_j}.$$

We will prove Lemma 3.1 using a geometric construction displayed in Fig. 1 (the case $\mu_k = 3$, where the boundaries of the disks $\big[Q_k\big]_*^{1/3} = Q_{k,0}^{1/3},\ Q_{k,1}^{1/3}$ and $Q_{k,2}^{1/3}$ are denoted by $\gamma_k^{(0)}, \gamma_k^{(1)}, \gamma_k^{(2)}$, respectively).

Since $N_k^{-1} - (z_k - \zeta_k)^{-1} \in \{N_k^{-1} - c_k^*; d_k^*\}$, according to (3.2) we have

$$|N_k^{-1} - c_k^*| \le \Big|\frac{1}{\mu_k}\sum_{j=1}^{\nu}\frac{\mu_j}{z_k-\zeta_j} - \frac{1}{z_k-\zeta_k}\Big| + d_k^* < \frac{n-\mu_k}{\rho\mu_k} + d_k,$$

which proves (i).

To prove (ii) we use (1.8) and the elementary inequality $m \sin \frac{\pi}{m} > 1$ $(m \geq 2)$. We estimate for $\lambda = 1, ..., \mu_k - 1$

$$\begin{aligned} |N_k^{-1} - c_{k,\lambda}| &\geq \Big| \frac{1}{\mu_k} \sum_{j=1}^{\nu} \frac{\mu_j}{z_k - \zeta_j} - \frac{1}{z_k - \zeta_k} \exp\Big(i \frac{2\lambda\pi}{\mu_k} \Big) \Big| - d_k \\ &\geq \frac{1}{|z_k - \zeta_k|} \Big| 1 - \exp\Big(i \frac{2\lambda\pi}{\mu_k} \Big) \Big| - \frac{1}{\mu_k} \sum_{j \neq k} \frac{\mu_j}{|z_k - \zeta_j|} - d_k \\ &\geq \frac{2}{r_k} \sin \frac{\lambda\pi}{\mu_k} - \frac{n - \mu_k}{\rho\mu_k} - d_k \geq \frac{2}{r\mu_k} - \frac{n - \mu_k}{\rho\mu_k} - d_k \\ &= \frac{2\rho - (n - \mu_k) r}{r\rho\mu_k} - 4d_k + 3d_k > \frac{n - \mu_k}{\rho\mu_k} + 3d_k. \end{aligned}$$

According to (1.7) and the relation (induced by (i))

$$|N_k^{-1} - c_k^*| \leq \Big(\frac{n - \mu_k}{\rho\mu_k} + 2d_k \Big) - d_k,$$

we conclude that the disk $[Q_k]_*^{1/\mu_k} = \{c_k^*; d_k^*\}$, containing $(z_k - \zeta_k)^{-1}$, lies in the interior of the disk

$$\Gamma_k = \Big\{ N_k^{-1}; \frac{n - \mu_k}{\rho\mu_k} + 2d_k \Big\}.$$

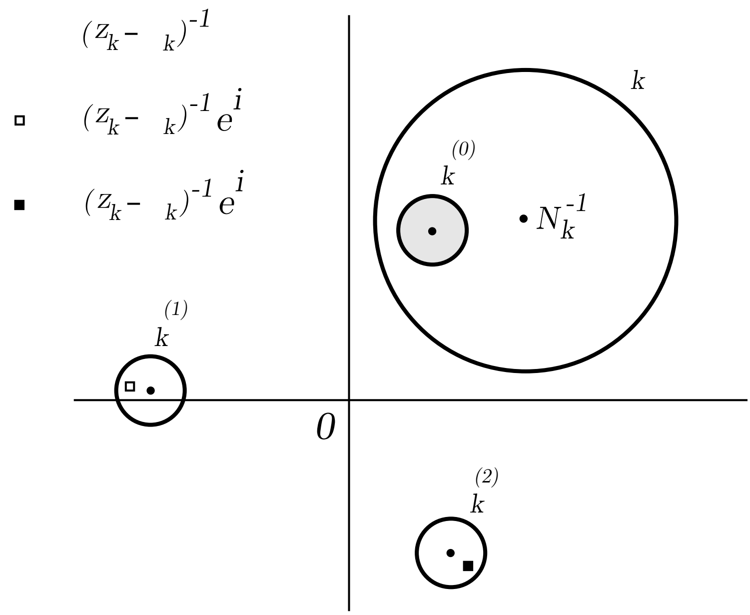

Fig. 1 Selection of a proper disk root

Besides, in regard to (1.8) and the relation (induced by (ii))

$$|N_k^{-1} - c_{k,\lambda}| \geq \Big(\frac{n-\mu_k}{\rho\mu_k} + 2d_k\Big) + d_k,$$

it follows that all remaining $\mu_k - 1$ disks $Q_{k,\lambda}^{1/\mu_k} = \{c_{k,\lambda}; d_k\}$ $(\lambda = 1, \dots, \mu_k - 1)$ (that is, the disks with the boundaries $\gamma_k^{(1)}$ and $\gamma_k^{(2)}$ in Fig. 1), which do not contain $(z_k - \zeta_k)^{-1}$, lie outside of the disk Γ_k. □

Regarding the assertions (i) and (ii) of Lemma 3.1, as well as Fig. 1, there follows that we have to choose the disk *whose center minimizes* $|N_k^{-1} - c_{k,\lambda}|$, $\lambda = 0, 1, ..., \mu_k - 1$. Of course, in finding this minimum we use the already computed value N_k which is necessary in the iterative formula (2.6).

4. Convergence theorems

Let ρ, r and μ be the abbreviations given by (3.1). Assume that the initial condition

$$\rho > \frac{7}{2}(n-\mu)r \tag{4.1}$$

holds. Under this condition, using the same technique presented in [10] for the inclusion method (2.4) we are able to prove the following assertions for the method (2.6) taking into account **CCR** defined by Lemma 3.1.

Lemma 4.1. *Let* $Z_1, ..., Z_\nu$ *be inclusion disks for the zeros* $\zeta_1, ..., \zeta_\nu$, $\zeta_k \in Z_k$, *and let* $z_k = \operatorname{mid} Z_k$, $r_k = \operatorname{rad} Z_k$, $\varepsilon_k = z_k - \zeta_k$. *If the inclusion disks* $Z_1, ..., Z_\nu$ *are chosen so that the inequality* (4.1) *is satisfied, then we have for* $k = 1, \dots, \nu$:

(i) $\zeta_k \in Z_k \Rightarrow \zeta_k \in Z_k - N_k$;
(ii) $0 \notin H_{N,k} := \prod_{j \neq k}(z_k - Z_j + N_j)^{\mu_j}$.

Furthermore, introducing for any $m = 0, 1, ...$ the quantities

$$\varepsilon_k^{(m)} := z_k^{(m)} - \zeta_k, \quad \varepsilon_m := \max_{1 \leq k \leq \nu} |\varepsilon_k^{(m)}|, \quad r_m := \max_{1 \leq k \leq \nu} r_k^{(m)} = r^{(m)},$$

we can prove

Lemma 4.2. *Let* β *be equal 1 if* $INV = ()^{-1}$ *and 0 if* $INV = ()^I$. *Then for the inclusion method* (2.6) *we have*

(i) $r_{m+1} = \mathcal{O}(\varepsilon_m r_m)$;
(ii) $\varepsilon_{m+1} = \mathcal{O}(\varepsilon_m^3) + \beta\mathcal{O}(\varepsilon_m r_m^2)$,

where $\mathcal{O}$ *is the Landau symbol.*

The proofs of Lemmas 4.1 and 4.2 are tedious and rather extensive but elementary and we omit them.

Lemma 4.1 gives conditions under which the inclusion method is defined, while Lemma 4.2 describes the behavior of the sequences of centers and radii of disks produced by (2.6). Using these lemmas we establish the convergence theorem for the inclusion method (2.6):

Theorem 4.3. *Let* $(Z_1, ..., Z_\nu) := (Z_1^{(0)}, ..., Z_\nu^{(0)})$ *be initial disks such that* $\zeta_k \in Z_k$ $(k = 1, ..., \nu)$ *and let* $\{Z_k^{(m)}\}$ $(k = 1, ..., \nu)$ *denote the sequences of disks produced by* (2.6)*, where* $m = 0, 1, ...$ *is the iteration index. If the initial condition*

$$\rho^{(0)} > \frac{7}{2}(n - \mu)r^{(0)}$$

holds, then for any $k \in \{1, ..., \nu\}$ *and* $m = 0, 1, ...$ *there holds* $\zeta_k \in Z_k^{(m)}$ *and the sequences of radii* $\{r_k^{(m)}\}$ $(k = 1, ..., \nu)$ *tend monotonically towards 0.*

The convergence rate of the inclusion method (2.6) for multiple zeros is the subject of the following theorem:

Theorem 4.4. *Let* $O_R(2.6)$ *denote the R-order of the iterative interval method* (2.6)*, where* $INV \in \{()^{-1}, ()^I\}$*. Then*

$$O_R(2.6) \geq \begin{cases} 1 + \sqrt{2} \cong 2.414 & \text{if} \quad INV = ()^{-1}, \\ 3 & \text{if} \quad INV = ()^I. \end{cases}$$

Theorems 4.3 and 4.4 are proved in a similar way as in [10] for simple zeros.

The convergence speed of the second order interval method (2.4) can be accelerated by applying the Gauss-Seidel approach (single step mode). The single step version of (2.4) reads

$$\hat{Z}_k = z_k - INV\left(\frac{1}{P(z_k)}\left[\prod_{j<k}(z_k - \hat{Z}_j)^{\mu_j} \prod_{j>k}(z_k - Z_j)^{\mu_j}\right]^{1/\mu_k}\right)$$
$$(k = 1, ..., \nu).$$

The lower bound of the R-order of (4.2) is given by

$$O_R((4.2)) \geq 1 + x_n,$$

where $x_n > 1$ is the unique positive root of the polynomial equation $x^n - x - 1 = 0$ (see Alefeld and Herzberger [1, Ch. 8]).

In the similar way, the presented inclusion method (2.6) of Weierstrass' type with Schröder's correction can be further accelerated using the Gauss-Seidel procedure. Starting from (2.6) we can state the following single step inclusion method

$$\hat{Z}_k = z_k - INV\left(\frac{1}{P(z_k)}\left[\prod_{j<k}(z_k-\hat{Z}_j)^{\mu_j}\prod_{j>k}(z_k-Z_j+N_j)^{\mu_j}\right]^{1/\mu_k}\right)$$

$$(k \in I_\nu).$$

Using the technique presented in [8] (see, also, [2], [9]), it can be proved that the R-order of convergence of the single step method (4.3) decreases as the polynomial degree grows, and lies in the interval $(2.414, 3.214)$ if the exact inversion is applied, and in the interval $(3, 3.732)$ in the case of the centered inversion.

5. Numerical example

In this section we illustrate numerically the presented inclusion methods with/without corrections. The proposed algorithms were implemented on PC PENTIUM IV using the programming package *Mathematica* 5 with multiple precision arithmetic. The type of inversion is stressed by the subscript indices "E" (*exact*) and "C" (*centered*); for instance, $(2.6)_E$ and $(2.6)_C$ denote two versions of the inclusion method (2.6) where the exact inversion $()^{-1}$ and the centered inversion $()^I$ are applied, respectively. For demonstration, we present the following example.

Example 1. We have considered the polynomial

$$\begin{aligned} P(z) &= z^7-(6+4i)z^6+(6+20i)z^5+(20-20i)z^4-(27+36i)z^3 \\ &\quad -(30-56i)z^2 \quad +(28+16i)z+24-32i \end{aligned}$$

with the zeros $\zeta_1 = -1$, $\zeta_2 = 2$, and $\zeta_3 = 1+2i$, with the respective multiplicities $\mu_1 = 2$, $\mu_2 = 3$, $\mu_3 = 2$. As the initial inclusion disks we have taken the circular regions

$$Z_1^{(0)} = \{-1.1+0.1i; 0.3\}, \quad Z_2^{(0)} = \{1.9+0.1i; 0.3\}, \quad Z_3^{(0)} = \{1.1+2.1i; 0.3\}.$$

We have tested the total step methods (2.4) and (2.6) using the inversions $()^{-1}$ and $()^I$. The radii of the inclusion disks obtained after the third iteration are displayed in Table 1.

methods	$(2.4)_E$	$(2.4)_C$	$(2.6)_E$	$(2.6)_C$
$r_1^{(3)}$	1.19(−6)	2.23(−8))	7.16(−8)	2.08(−14)
$r_2^{(3)}$	4.79(−7)	2.90(−9)	2.73(−8)	1.66(−14)
$r_3^{(3)}$	1.18(−6)	9.07(−8)	1.03(−7)	3.45(−14)

Table 1 The results of the third iteration of the total step methods (2.4) and (2.6).

The corresponding single step methods have also been tested in the same example. After the third iteration we obtained considerably better results shown in Table 2.

methods	$(4.2)_E$	$(4.2)_C$	$(4.3)_E$	$(4.3)_C$
$r_1^{(3)}$	$2.92(-10)$	$1.49(-10)$	$7.88(-12)$	$1.14(-16)$
$r_2^{(3)}$	$1.19(-13)$	$4.06(-15)$	$4.37(-17)$	$3.94(-31)$
$r_3^{(3)}$	$4.44(-18)$	$1.70(-19)$	$5.22(-24)$	$2.55(-44)$

Table 2 The results of the third iteration of the single step methods (4.2) and (4.3).

The results obtained in this example, as well as many other examples, coincide very well with the theoretical results given in Theorems 4.3 and 4.4. In particular, from Tables 1 and 2 we observe that the centered inversion (1.6) gives smaller disks compared to the exact inversion (1.5), which could seem surprising regarding the inclusion $Z^{-1} \subset Z^I$ for arbitrary Z. However, this is caused by the better convergence of the midpoints which behave as the third order method (1.4) when the inversion (1.6) is applied. This forces the faster convergence of the sequences of radii of disks produced by the employed inclusion method. The convergence of the midpoints of disks in the case of the exact inversion is somewhat slower due to the shifted centers of disks.

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