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Income-Smoothing Behavior Under Selected Stochastic Processes

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Income-Smoothing Behavior Under Selected Stochastic Processes

Abstract

This paper attempts to provide a framework for clarifying and testing a version of the "income-smoothing" hypothesis. This hypothesis has been variously expressed in terms of income levels, rates of change in income, and rates of return, inter alia. In general, each version of the hypothesis is concerned with the extent to which managers may attempt to affect the volatility of a series of reported accounting numbers (or, in the case of rates of return, a series of relationships among accounting numbers) via selections and applications of accounting procedures. The alleged motivation for this behavior is a desire to reduce the extent to which "bad times" and-at the other extreme-"good times" are revealed by reported accounting numbers. It is suggested by some that a "smoothed" series of accounting numbers, particularly income numbers, will enhance the value of a firm. A typical statement of this argument was provided by Hepworth: "Certainly the owners and creditors of an enterprise will feel more confident toward a corporate management which is able to report stable earnings than if considerable fluctuation of reported earnings exists."

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I

This paper attempts to provide a framework for clarifying and testing a version of the “income-smoothing” hypothesis. This hypothesis has been variously expressed in terms of income levels, rates of change in income, and rates of return, *inter alia*.¹ In general, *each version* of the hypothesis is concerned with the extent to which managers may attempt to affect the volatility of a series of reported accounting numbers (or, in the case of rates of return, a series of relationships among accounting numbers) via selections and applications of accounting procedures. The alleged motivation for this behavior is a desire to reduce the extent to which “bad times” and—at the other extreme—“good times” are revealed by reported accounting numbers. It is suggested by some that a “smoothed” series of accounting numbers, particularly income numbers, will enhance the value of a firm. A typical statement of this argument was provided by Hepworth: “Certainly the owners and creditors of an enterprise will feel more confident toward a corporate management which is able to report stable earnings than if considerable fluctuation of reported earnings exists.”²

* University of Chicago. I am indebted to Ray Ball, Philip Brown, Eugene Fama, Robert S. Kaplan, and, in particular, Warren Dent for their comments on earlier drafts of this paper.

1. See, e.g., M. J. Gordon, B. N. Horowitz, and P. T. Meyers, “Accounting Measurements and Normal Growth of the Firm,” in *Research in Accounting Measurement*, ed. R. K. Jaedicke (Menasha, Wis.: American Accounting Association, 1966), pp. 221–31.

2. S. R. Hepworth, “Smoothing Periodic Income,” *Accounting Review* 28 (January 1953): 34. A similar rationale for income-smoothing actions was advanced in M. J. Gordon, “Postulates, Principles, and Research in Accounting,” *Accounting Review* 39 (April 1964): 32–39. Direct and indirect test of versions of the income-smoothing hypothesis are discussed in, e.g., Gordon, Horowitz, and Meyers; R. M. Copeland, “Income-Smoothing,” *Empirical Research in Accounting: Selected Studies, 1966*, in *Journal of Accounting Research* 4 (suppl.): 101–16; R. M. Copeland and R. L. Licastro, “A Note on Income-Smoothing,” *Accounting Review* 43 (July 1968): 540–45; N. Dopuch and D. Drake, “The Effect of Alternative Accounting Rules for Nonsubsidiary Investments,” *Empirical Research in Accounting: Selected Studies, 1966*, in *Journal of Accounting Research* 4 (suppl.): 192–219; J. Gagnon, “Purchase versus Pooling of Interests: The Search for a Predictor,” *Empirical Research in Accounting: Selected Studies, 1967*, in *Journal of Accounting Research* 5 (suppl.): 187–204; T. R. Archibald, “The Return to Straight-Line Depreciation: An Analysis of a Change in Accounting Method,” *Empirical Research in Accounting: Selected Studies, 1967*, in *Journal of Accounting Research* 5 (suppl.): 229–35; B. E. Cushing, “An Empirical Study of Changes in Accounting Policy,” *Journal of Accounting Research* 7 (Autumn 1969): 196–203; P. E. Dascher and R. E. Malcolm, “A Note on Income Smoothing in the Chemical Industry,” *Journal of Accounting Research* 8 (Autumn 1970):

One characteristic of the available studies of the income-smoothing hypothesis is a lack of rigorously derived *explicit* statements about what one should expect if, in fact, income smoothing is practiced.³ Such statements are particularly important for empirical tests of the smoothing hypothesis, since it is difficult to test for the existence of something that cannot even be identified when, in fact, that something does exist. Additionally, with one exception, the available studies do not even *ask* whether income smoothing is optimal, given the stochastic process applicable to the accounting numbers of interest *and* the alleged objectives of income-smoothing actions; the one exception is a study by Ball and Watts.⁴ Finally, the available studies pay little (if any) attention to the multipieriod consequences of income-smoothing actions; the importance of this factor will be made clear in Section III of this paper.

Most of this paper is concerned with the *optimality* of income-smoothing actions. In particular, a few beginning steps will be taken toward establishing the conditions under which income-smoothing actions are optimal (in a sense defined below). *Throughout this paper, attention is restricted to the income-smoothing hypothesis expressed in terms of rates of return* (e.g., rates of return on common equity, rates of return on total assets, etc.). Two classes of stochastic processes will be considered with respect to rates of return (*before* the effects, if any, of smoothing actions on these rates); these classes are: martingales and mean-reverting processes. Note that the techniques used herein are applicable to other versions of the smoothing hypothesis. Furthermore, if any of the stochastic processes assumed for rates of return are applicable to some other version of the income-smoothing hypothesis, then, of course, the conclusions of this paper also hold for that other version. Finally, it is important to note that this paper is only concerned with the effects (if any) of smoothing actions on *reported accounting numbers*. It does *not* deal, for example, with the effects (if any) of smoothing actions on aggregate capital market behavior.⁵

Before proceeding to technical developments, it may be worthwhile to briefly illustrate the importance of determining whether income-

253–59; C. E. White, “Discretionary Accounting Decisions and Income Normalization,” *Journal of Accounting Research* 8 (Autumn 1970): 260–73; and R. Ball and R. Watts, “Some Time Series Properties of Accounting Income” (Chicago: University of Chicago, 1971) (to appear in the *Journal of Finance*).

3. See references in n. 2 above.

4. Ball and Watts; but, as will be indicated (see remark 5), their conclusions regarding the effects of attempted smoothing actions are not accurate, *in general*.

5. In this regard, it is not clear that such effects (if any) could persist, given the observed *efficiency* of capital markets (in particular, the New York Stock Exchange). An extensive recent review of the available theory and evidence regarding efficient capital markets is provided in E. F. Fama, “Efficient Capital Markets: A Review of Theory and Empirical Work,” *Journal of Finance* 25 (May 1970): 383–417. Some implications of capital market efficiency for external accounting are discussed in N. J. Gonedes, “Efficient Capital Markets and External Accounting,” *Accounting Review* 47 (January 1972): 11–21.

smoothing actions are optimal. This is done in Section II. Section III provides a conceptual framework for dealing with the smoothing hypothesis. Section IV provides characterizations of optimal smoothing actions. A summary is provided in Section V; the latter section also outlines some areas for additional research suggested by the results presented in this paper.

II

The existence or nonexistence of income-smoothing actions can have important implications for (1) the observed time-series properties of reported income numbers and rates of return based upon such numbers, and (2) the interpretations of these properties regarding the growth and decline of firms—two topics that have attracted considerable attention.⁶ For example, one might argue that, given monopolistic conditions, one should expect persistence in companies' periodic growth rates. Yet, such a time-series pattern might also be attributable to "smoothing" actions: "There are many . . . monopolistic situations that could induce persistence in earnings progress, but the same result could follow from accounting practices designed at 'managing earnings.' A clear example exists when provisions such as pension fund reserves are varied specifically to achieve a smooth earnings progression. A less blatant case arises when the profit from one effective economic transaction is taken into the income statement over several years."⁷

On the other hand, one can also argue that some attempts to "smooth" accounting number series will have *destabilizing* effects; in this regard, Lintner and Glauber assert the following: "Shifts for whatever reason from one method of reporting certain types of transactions or change to another method . . . may be quite *destabilizing* to the pattern of reported earnings, as may decisions regarding so-called extraordinary or nonrecurring credits and charges."⁸

The contradictory nature of the above citations regarding the effects of attempted "smoothing" actions on the time-series properties of accounting numbers simply reflects the absence of explicit statements about the optimality of smoothing actions relative to the alleged motivation for such actions. For example, if smoothing actions have destabilizing effects on an accounting-number series and if the motivation behind attempted

6. See, e.g., I. M. D. Little, "Higgledy Piggledy Growth," *Institute of Statistics*, vol. 24, no. 4 (November 1962); I. M. D. Little and A. C. Rayner, *Higgledy Piggledy Growth Again* (Oxford: Basil Blackwell, 1966); J. Lintner and R. Glauber, "Higgledy Piggledy Growth in America" (paper presented at the Seminar on the Analysis of Security Prices, University of Chicago, May 1967); J. Lintner and R. Glauber, "Further Observations on Higgledy Piggledy Growth" (paper presented at the Seminar on the Analysis of Security Prices, University of Chicago, May 1969); R. A. Brealey, *An Introduction to Risk and Return from Common Stocks* (Cambridge, Mass.: M.I.T. Press, 1969); and Ball and Watts.

7. Brealey, p. 89.

8. Lintner and Glauber, p. 6.

smoothing actions is stabilization of the number series, then why should one expect managers to pursue the application of such smoothing actions?

It has been suggested that smoothing behavior may also affect empirical results regarding the relationship between firm size and profitability.⁹ For example, it was suggested by Hall and Weiss¹⁰ that the effects of smoothing actions will be such that *observed* profitability differences across firms of different sizes will be “understated.” But this specific conclusion is provided without sufficient justification. It is not obvious that (1) the underlying stochastic processes generating accounting numbers and (2) the smoothing strategies chosen by managements will combine to produce the alleged effects. Moreover, this conclusion begs the question as to whether any kind of smoothing is, in fact, optimal from management’s perspective. That is the question to which we now turn.

III

In order to deal with the smoothing hypothesis, it seems necessary to distinguish between two stochastic processes: (1) the process generating *basic* accounting numbers and (2) that generating *reported* accounting numbers. Consider a firm that uses a given set of accounting procedures in order to generate its accounting numbers. Selection of these procedures may have been based upon “convenience” issues, tax regulations, industry practices, etc. *Given these procedures*, the resultant accounting numbers will reflect the events that affected the firm’s operations. These events include: (1) events that occur within the factor-input markets in which the firm is a transactor and (2) events that occur within the firm’s output markets. Such events may be specific to a particular industry, they may be economy-wide events, or they may be specific to the firm’s operations.¹¹ Let r_t be an accounting rate of return generated at time t under the given accounting procedures. Then, when looking into the future, *basic* rates of return may be defined as the stochastic process $\{\tilde{r}_t; t = 1,$

9. This relationship was examined by, e.g., S. S. Alexander, “The Effect of Size of Manufacturing Corporation on the Distribution of Rate of Return,” *Review of Economics and Statistics* 31 (August 1949): 229–35; H. O. Stekler, *Profitability and Size of Firm* (Berkeley: Institute of Business and Economic Research, University of California, 1963); and Marshall Hall and L. Weiss, “Firm Size and Profitability,” *Review of Economics and Statistics* 49 (August 1967): 319–31.

10. Hall and Weiss, p. 321.

11. In this regard, see P. Brown and R. Ball, “Some Preliminary Findings on the Association between the Earnings of a Firm, Its Industry and the Economy,” *Empirical Research in Accounting: Selected Studies, 1967*, in *Journal of Accounting Research* 6 (suppl.): 55–77; and N. J. Gonedes, “Evidence on the Information Content of Accounting Income Numbers” (report no. 7115, Center for Mathematical Studies in Business and Economics, University of Chicago, March 1971); and idem “Properties of Accounting Numbers: Models and Tests” (working paper no. 64–71–2, Graduate School of Industrial Administration, Carnegie-Mellon University, January 1972).

2, . . . } induced by the multiperiod joint-distribution function of the aforementioned events. For each t , this stochastic process is associated with a conditional distribution function for \tilde{r}_t , say, $F(\tilde{r}_t | r_{t-k}; k = 1, 2, \dots)$, $\forall t$. In the *absence* of smoothing actions, and given the fixed set of accounting procedures, the firm will be reporting realizations from these conditional distribution functions. That is, in this situation (and for each t), the firm's reported rate of return will be equal to the realized *basic* rate of return.

Now suppose that the management of the firm—in view of the *basic* rate of return process that it faces—decides that in each period t , $t = 1, 2, \dots, N$, it would like to report a rate of return that is: (1) “as high as possible” and (2) “sufficiently close” to the previous period's reported rate of return. In other words, the firm's management wants to report “high” rates of return, but it does not want the *reported* rate-of-return series to be “too erratic.” In order to satisfy its objective, the firm's management might have to report a rate of return (for each t) that is equal to the realized *basic* rate of return plus a “smoothing” (or “adjustment”) factor. More generally, in this case, the *reported* rate-of-return process is a *transformed basic* rate-of-return process. The transformations are defined by deliberate “smoothing” actions, or accounting “manipulations.” Of course, for all t , the rate of return reported in period t will be the “previously reported” rate of return for period $t + 1$. Hence, the selection of a smoothing factor for the current period must be based upon the *preceding* period's *reported* rate of return, the *current* period's realized *basic* rate of return, and the joint-distribution function of *future* periods' *basic* rates-of-return.

As before, let $\{\tilde{r}_t; t = 1, 2, \dots, N\}$ be the *basic* rate-of-return process; let $\{\tilde{r}_t^o; t = 1, 2, \dots, N\}$ be the *reported* rate-of-return process where, for each

$$t, \tilde{r}_t^o = \tilde{r}_t + \alpha_t \text{ and } \{\alpha_t \begin{matrix} > \\ < \end{matrix} 0; t = 1, 2, \dots, N\}$$

is a sequence of smoothing factors selected by the firm's management. Until otherwise indicated, I shall assume that any selected smoothing strategy $\{\alpha_t; 1 \leq t \leq N\}$ is such that

$$\sum_{t=1}^N \alpha_t = \Omega = 0;$$

nonzero values of Ω will be considered in remark 6. This assumption implies that the intended effect of the smoothing strategy is simply a redistribution of rates of return (*ex ante*), rather than an alteration that has a “permanent” effect (relative to the finite horizon $1 \leq t \leq N$). For example, the distribution function for \tilde{r}_{t^*} , $\forall t^* > 1$, may be based upon

a pattern of installment-sales-revenue recognition and a particular cost-amortization scheme. If these sales and the expenditures underlying the costs were effected in some \bar{t} such that $\bar{t} < t^*$, then (with few exceptions) the firm will know the associated cost amortizations and revenue recognitions for some \hat{t} , $\hat{t} \geq t^*$. If at the end of some $\bar{t} > 1$, $\bar{t} < \bar{t} < t^*$, the firm concludes that its *realized* basic rate of return for that \bar{t} is “too low” relative to r_{t-1}^e , then it might attempt to recognize some revenue items prematurely (relative to t^*), thus selecting $\alpha_{\bar{t}} > 0$. But, ex ante, this action reduces the revenue recognitions for future periods. According to my formulation, all such adjustments to the basic rate of return must cumulate to zero as of $t = N$. So, it is important to note that any smoothing action in period \bar{t} has implications for reported rates-of-return in some $t > \bar{t}$. In general, if a firm seeks to report a “high,” but not “too erratic,” rate-of-return series over the finite horizon $[1, N]$, then the selection of a smoothing action at the end of a given period must recognize the consequences of current smoothing actions vis-à-vis future periods’ *reported* rates of return.

The preceding remarks may be formalized by expressing the smoothing problem as follows:¹²

$$\max_{\{\alpha_t; 1 \leq t \leq N\}} \mathcal{L} = E \left(\sum_{t=1}^N \tilde{r}_t^e - \lambda (\tilde{r}_t^e - \tilde{r}_{t-1}^e)^2 \right), \quad (1.1)$$

$$\text{subject to } \tilde{r}_t^e = \tilde{r}_t + \alpha_t, \quad (1.2)$$

$$\sum_{t=1}^N \alpha_t = 0, \quad (1.3)$$

where E is the expectation operator, and $\lambda > 0$ is a parameter defining the “disutility” of squared deviations of \tilde{r}_t^e from \tilde{r}_{t-1}^e , $\forall t$. The succeeding analysis is essentially unaffected by changing the exponent of $(\tilde{r}_t^e - \tilde{r}_{t-1}^e)$ in (1.1) from 2 to any integer multiple of 2. Also, as will be indicated below (see remark 6), the constraint (1.3) is less restrictive than may be inferred at first sight.

According to my formulation of the problem, the smoothing action for period \hat{t} , $\alpha_{\hat{t}}$, will be selected *at the end of \hat{t} , after* observing the realization of $\hat{r}_{\hat{t}}$. This selection is to be made so that $r_{\hat{t}}^e$ is “high enough” and not “too far from” $r_{\hat{t}-1}^e$ (which is also *known* at time \hat{t}). In effect, $r_{\hat{t}}$, $r_{\hat{t}-k}^e$, and $\alpha_{\hat{t}-k}$, $k = 1, 2, \dots$, define the state of the system when $\alpha_{\hat{t}}$ is to be selected, and all $t \geq \hat{t}$ define the remaining stages of the N -stage

12. Here, I am using one formulation of the objective guiding income-smoothing behavior, namely, minimization of a function of $[\tilde{r}_t^e - \tilde{r}_{t-1}^e]$. This objective appears to be consistent with the alleged motivation for smoothing: “stabilization” of reported number series. Some statements of the income-smoothing hypothesis would call for minimization of a function of: $[\tilde{r}_t^e - E(\tilde{r}_t^e)]$.

problem (1.1)–(1.3). A solution to this problem, that is, a sequence of optimal smoothing actions $\{\alpha_t^*; 1 \leq t \leq N\}$ should satisfy the principle of optimality: “An optimal policy has the property that, whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.”¹³

With this perspective in mind, I define the following: $f_k(S_k) =$ the k -stage *expected* return associated with the optimal action α_k^* and the state vector $S_k = (r_{k+1}^c, r_k, r_{k+1}, \dots, r_N, \alpha_{k+1}, \alpha_{k+2}, \dots, \alpha_N)$. Note that at stage k the state vector for stage $k-1$ is a random vector because it contains the random variable \tilde{r}_{k-1} .

The sequence $\{f_k(\bullet)\}$ is defined for $k = N, N - 1, \dots, 1$ and $f_0(\bullet) \equiv 0$. Here, $f_N(\bullet)$ applies to the *first* stage of the problem, and $f_1(\bullet)$ applies to the *last* stage of the problem.

At the *last* stage of the problem ($k = 1$),

$$f_1(S_1) = \max_{\alpha_1} \{r_1 + \alpha_1 - \lambda\{r_1^c - r_2^c\}^2\} \tag{2}$$

$$r_1^c = r_1 + \alpha_1.$$

According to the constraint (1.2), the only feasible value of α_1 is

$$\alpha_1 = -\sum_{k=N}^2 \alpha_k.$$

Hence, the optimal decision for $k = 1$ is

$$\alpha_1^* = -\sum_{k=N}^2 \alpha_k^*,$$

where α_k^* for $N \geq k > 1$ are known.

For $k = N, N - 1, \dots, 2$, one has

$$f_k(S_k) = \max_{\alpha_k} E\{r_k^c - \lambda(r_k^c - r_{k+1}^c)^2 + f_{k-1}(\tilde{S}_{k-1})\}, \tag{3}$$

where $\tilde{S}_{k-1} = (r_k^c, \tilde{r}_{k-1}, r_k, \dots, r_N, \alpha_k^*, \alpha_{k+1}^*, \dots, \alpha_N^*)$. Since, at stage k , r_k and r_{k+1}^c are known, (3) becomes

$$f_k(S_k) = \max_{\alpha_k} \{r_k^c - \lambda(r_k^c - r_{k+1}^c)^2 + E f_{k-1}(\tilde{S}_{k-1})\}. \tag{4}$$

Using (2) and (4), one can define the sequence of optimal smoothing actions $\{\alpha_k^*, N \geq k \geq 1\}$ associated with the stochastic process $\{\tilde{r}_k\}$. And, since future rates of return are known only in terms of their distri-

13. R. E. Bellman, *Dynamic Programming* (Princeton, N.J.: Princeton University Press, 1957), p. 83.

bution functions, these optimal actions will be in the form of *decision rules*. With the aid of some tedious algebraic manipulations, it can be shown that the solution to the problem represented by (2) and (4) is (assuming $E(\tilde{r}_k^2) < \infty, \forall k$)

$$\alpha_k^* = -\frac{\omega_k^0}{\gamma_k} r_k + \frac{\omega_k'}{\gamma_k} r_{k+1}^c + \frac{(\omega_k^0 - \omega_k')}{\gamma_k(k-1)} \left\{ \sum_{j=k-1}^1 E(\tilde{r}_j) \right\} - \frac{(\omega_k^0 - \omega_k')}{\gamma_k(k-1)} \sum_{i=N}^{k+1} \alpha_i^* \tag{5.1}$$

for $k = N, N - 1, \dots, 2$; and for $k = 1$,

$$\alpha_1^* = -\sum_{i=N}^2 \alpha_i^*, \tag{5.2}$$

where, for (5.1),

$$\omega_k^0, \omega_k', \gamma_k > 0, \tag{5.3}$$

$$\omega_k^0 > \omega_k', \tag{5.4}$$

$$\omega_k^0 - \omega_k' > (k - 1), \text{ and } \omega_k^0 - \omega_k' > \omega_k'. \tag{5.5}$$

The quantities ω_k^0, ω_k' , and γ_k are parameters of the decision rule for α_k^* ; the values of these parameters emerge from the solution of the problem represented by (2) and (4). *These parameters are functions of k .* In general, $(\omega_k^0 - \omega_k') / [\gamma_k(k - 1)]$ decreases as k increases; ω_k' / γ_k increases and $(-\omega_k^0 / \gamma_k)$ decreases as k increases.

The decision rules in (5.1) and (5.2) may be used (in a straightforward manner) in order to demonstrate several interesting propositions about optimal smoothing behavior. The next section does this for some selected stochastic processes, $\{\tilde{r}_t; N \geq t \geq 1\}$.

IV

This section provides characterizations of optimal smoothing actions (as defined in [5.1] and [5.2]) for selected stochastic processes. The analysis that follows considers the following question for each type of stochastic process: as of the end of stage $k = N$ (i.e., after observing the realized value of r_N^c), what is the optimal value of α_k^* ; *in particular, is the optimal value nonzero?* It turns out that the answers to this question may be had via some straightforward algebraic manipulations of the decision rule in (5.1). With a slight adjustment (see remark 1), the analysis can be extended to any stage of the process (but the conclusions for $k = N$ need not carry over to other stages of the process). *Throughout, it is assumed that $r_N > 0$; $r_N \leq 0$ is easily recognized by appropriately changing the sense of the inequalities that appear in the propositions.*

Proposition A (Strict Martingale)¹⁴

Suppose that the stochastic process for basic rates of return is of the form

$$\tilde{r}_t = b\tilde{r}_{t-1} + \tilde{\epsilon}_t, \quad \text{with } b = 1, \tag{6}$$

where $E(\tilde{\epsilon}_t) = 0$ and $E(\tilde{\epsilon}_t^2) < \infty$. Then

$$\alpha_N^* = \frac{\omega_N'}{\gamma_N} (r_{N+1} - r_N),$$

and, thus, $\text{sgn}\{\alpha_N^*\} = \text{sgn}\{r_{N+1} - r_N\}$.¹⁵

Proof

Under the assumptions, $E(\tilde{r}_{N-s}) = b^{s+1}r_{N+1}$, $s \geq 0$; incorporating this into (5.1) for $k = N$ yields

$$\alpha_N^* = \frac{\omega_N^0}{\gamma_N} \left\{ \frac{1}{(N-1)} \sum_{i=1}^{N-1} b^i - 1 \right\} r_N + \frac{\omega_N'}{\gamma_N} \left\{ r_{N+1} - \frac{r_N}{(N-1)} \sum_{i=1}^{N-1} b^i \right\}, \tag{7}$$

and, since $b = 1$, one has

$$\alpha_N^* = \frac{\omega_N'}{\gamma_N} (r_{N+1} - r_N). \tag{8}$$

Since $\omega_N'/\gamma_N > 0$, the proposition holds.

Remark 1.—The results in (7) and (8) may be easily extended to α_k^* for $k < N$ and for the process in (6). Using (5.1) again, one has

$$\alpha_k^* = \frac{\omega_k'}{\gamma_k} r_{k+1} - \frac{\omega_k^0}{\gamma_k} r_k + \left(\frac{\omega_k^0 - \omega_k'}{\gamma_k(k-1)} \right) (k-1)r_k - \frac{\omega_k^0 - \omega_k'}{\gamma_k(k-1)} \sum_{i=N}^{k+1} \alpha_i^* \\ \alpha_k^* = \frac{\omega_k'}{\gamma_k} (r_{k+1} - r_k) - \left(\frac{\omega_k^0 - \omega_k'}{\gamma_k(k-1)} \right) \sum_{i=N}^{k+1} \alpha_i^*.$$

As indicated in the commentary under expression (5.5), (ω_k'/γ_k) increases as k increases, and $(\omega_k^0 - \omega_k')/[\gamma_k(k-1)]$ decreases as k increases. Hence, if k is “large,” the impact of α_s^* , $s > k$, on α_k^* will be “small” relative to the impact of $(r_{k+1} - r_k)$.

14. Martingale processes are discussed in J. L. Doob, *Stochastic Processes* (New York: John Wiley & Sons, 1953); and L. Brieman, *Probability* (Reading, Mass.: Addison-Wesley Publishing Co., 1968).

15. The *sgn* (read: “signum”) function is defined by $\text{sgn}(x) = 1$ if $x > 0$, 0 if $x = 0$, -1 if $x < 0$.

Proposition B (Multiplicative Linear Semimartingale)

Let the process for basic rates of return be as in (6) but with $b \neq 1$, and let $u_N = r_N - r_{N+1}$. Let $b \equiv (1 + \epsilon)$. Then, for $N \geq 2$,

$$b > 1 \text{ and } u_N < 0 \text{ imply } \alpha_N^* > 0; \tag{a.1}$$

$$b > 1 \text{ and } u_N < \epsilon \frac{N}{2} r_N \text{ imply } \alpha_N^* > 0 \tag{a.2}$$

(if $b > 1, \epsilon > 0$);

$$b < 1 \text{ and } u_N > 0 \text{ imply } \alpha_N^* < 0; \tag{b.1}$$

$$0 < b < 1 \text{ and } u_N > \epsilon r_N \text{ imply } \alpha_N^* < 0 \tag{b.2}$$

(if $0 < b < 1, 0 > \epsilon > -1$).

(Note that [a.2] implies [a.1] and, for $\epsilon > -1$, [b.2] implies [b.1]. Since the proofs for parts [a.1] and [b.1] differ from those for [a.2] and [b.2], respectively, these parts are stated separately.)

Proofs

Part a.—From (5.1) one gets

$$\alpha_N^* = \frac{\omega_N^0}{\gamma_N} \Gamma + \frac{\omega_N'}{\gamma_N} \Phi$$

where

$$\Gamma \equiv \left\{ \frac{r_N}{N-1} \sum_{i=1}^{N-1} b^i - r_N \right\}, \tag{9.1}$$

and

$$\Phi \equiv \left\{ (r_N - u_N) - \frac{r_N}{N-1} \sum_{i=1}^{N-1} b^i \right\} = -\Gamma - u_N. \tag{9.2}$$

If $b > 1, 1 < b < b^2 < \dots < b^{N-1}$ and, thus, $\Gamma > 0$. Since

$$\alpha_N^* = \frac{\omega_N^0 - \omega_N'}{\gamma_N} \Gamma - \frac{\omega_N'}{\gamma_N} u_N \tag{10}$$

and $\omega_N^0 > \omega_N'$ (see [5.1]–[5.5]), part (a.1) of proposition B holds.

In order to prove part (a.2), we note the following: If $b > 1$, that is, $\epsilon > 0$, then $b^i \equiv (1 + \epsilon)^i \geq 1 + i\epsilon$, with equality holding only when $i = 0$ or $i = 1$. Thus, for $N \geq 2$,

$$\sum_{i=1}^{N-1} b^i \geq \sum_{i=1}^{N-1} (1 + i\epsilon) = (N-1) + \epsilon \frac{(N-1)N}{2}. \tag{11}$$

Incorporating (11) into the definition of Γ yields the result $\Gamma > \epsilon(N/2)r_N$ for $N \geq 3$ (and, of course, $\Gamma = \epsilon(N/2)r_N$ for $N = 2$). Incorporating the lower bound on Γ into (10) yields

$$\alpha_N^* \geq \frac{\omega_N^0 - \omega_N'}{\gamma_N} \frac{\epsilon N}{2} r_N - \frac{\omega_N'}{\gamma_N} u_N \quad (12)$$

for $N \geq 2$. A sufficient condition for $\alpha_N^* > 0$ is

$$\frac{\omega_N^0 - \omega_N'}{\omega_N'} \epsilon \frac{N}{2} r_N > u_N. \quad (13)$$

Recall that $[(\omega_N^0 - \omega_N')/\omega_N'] > 1$. Thus, since $\epsilon > 0$,

$$r_N \frac{\omega_N^0 - \omega_N'}{\omega_N'} \epsilon \frac{N}{2} > r_N \epsilon \frac{N}{2}.$$

Consequently, a sufficient condition for (13) to hold (and thus a sufficient condition for $\alpha_N^* > 0$) is

$$u_N < \epsilon \frac{N}{2} r_N. \quad (14)$$

And, of course,

$$\alpha_N^* \geq 0 \quad \text{if} \quad u_N = \epsilon \frac{N}{2} r_N.$$

Part b.—To see this part, use (9.1) and (9.2) and recognize that, if $b < 1$, then $1 > b > b^2 > \dots > b^{N-1}$ and $\Gamma < 0$. Using (10) again, one has $\alpha_N^* < 0$ if $u_N > 0$.

In order to prove part (b.2), note that, if $0 < b < 1$ (i.e., $0 > \epsilon > -1$), then, for $N \geq 2$,

$$\sum_{i=1}^{N-1} b^i \leq (N-1)b = (N-1)(1+\epsilon), \quad (15)$$

where the equality in (15) holds only when $N = 2$. Incorporating (15) into (9.1) yields $\Gamma < \epsilon r_N$ for $N \geq 3$ and $\Gamma = \epsilon r_N$ for $N = 2$. Using the upper bound on Γ in (9.1) yields, for $N \geq 2$,

$$\alpha_N^* \leq \frac{\omega_N^0 - \omega_N'}{\gamma_N} \epsilon r_N - \frac{\omega_N'}{\gamma_N} u_N. \quad (16)$$

A sufficient condition for $\alpha_N^* < 0$ is

$$\frac{\omega_N^0 - \omega_N'}{\omega_N'} \epsilon r_N < u_N.$$

But, since $\epsilon < 0$,

$$\frac{\omega^0 - \omega_N'}{\omega_N'} \epsilon r_N < \epsilon r_N.$$

Hence, a sufficient condition for $\alpha_N^* < 0$ is $u_N > \epsilon r_N$. And, of course, $\alpha_N^* \leq 0$ if $u_N = \epsilon r_N$.

Remark 2.—Note from (14) that $\epsilon(N/2)r_N \rightarrow \infty$ as $N \rightarrow \infty$. Thus, letting $Pr(\cdot)$ denote probability, $Pr(u_N < \epsilon[N/2]r_N) \rightarrow 1$ as $N \rightarrow \infty$. Consequently, $Pr(\alpha_N^* > 0) \rightarrow 1$ as $N \rightarrow \infty$ for $b = (1 + \epsilon) > 1$.

Remark 3.—Expressions for α_k^* , $k < N$ may be obtained by imitating the manipulations used for remark 1. As in the latter remark, α_k^* , $k < N$, for $b \neq 1$, will depend upon α_s^* , $s > k$. And, once again, the effect of α_s^* , $\forall s > k$ on α_k^* will be a decreasing function of the length of the horizon (i.e., N).

Suppose that the generating process for basic rates of return is an additive semimartingale:

$$\tilde{r}_t = \tilde{r}_{t-1} + g + \tilde{\epsilon}_t, \tag{17}$$

where g is a nonzero constant, $E(\tilde{\epsilon}_t) = 0$, and $E(\tilde{\epsilon}_t^2) < \infty$. (If $g = 0$, the process is a strict martingale; see proposition A.) For this generating process, we have for $j \geq 1$

$$E(\tilde{r}_{N-j}) = r_N + jg. \tag{18}$$

Proposition C (Additive Linear Semimartingale)

Let the generating process for basic rates of return be as in (17), and let $u_N = r_N - r_{N+1}$. Then,

$$g > 0 \quad \text{and} \quad u_N < g \frac{N}{2} \quad \text{imply} \quad \alpha_N^* > 0; \tag{c.1}$$

$$g < 0 \quad \text{and} \quad u_N > g \frac{N}{2} \quad \text{imply} \quad \alpha_N^* < 0. \tag{c.2}$$

Proof

Using (5.1) for $k = N$, one has:

$$\alpha_N^* = \frac{-\omega_N^0}{\gamma_N} r_N + \frac{\omega_N'}{\gamma_N} r_{N+1} + \left(\frac{\omega_N^0 - \omega_N'}{\gamma_N(N-1)} \right) \sum_{j=1}^{N-1} E(r_{N-j}). \tag{19}$$

From (18),

$$\begin{aligned} \sum_{j=1}^{N-1} E(r_{N-j}) &= (N-1)r_N + g \sum_{j=1}^{N-1} j \\ &= (N-1)r_N + g \frac{(N-1)N}{2}. \end{aligned} \tag{20}$$

Incorporating (20) and $u_N = r_N - r_{N+1}$ into (19) and rearranging yields

$$\alpha_N^* = \frac{\omega_N^0 - \omega_N'}{\gamma_N} g \frac{N}{2} - \frac{\omega_N'}{\gamma_N} u_N. \quad (21)$$

A sufficient condition for $\alpha_N^* > 0$ is

$$\frac{\omega_N^0 - \omega_N'}{\omega_N'} g \frac{N}{2} > u_N. \quad (22)$$

Recall that $\omega_N^0 - \omega_N' > \omega_N'$. Thus, a sufficient condition for $\alpha_N^* > 0$ when $g > 0$ is $u_N < g(N/2)$. This proves part (c.1).

A sufficient condition for $\alpha_N^* < 0$ is

$$\frac{\omega_N^0 - \omega_N'}{\omega_N'} g \frac{N}{2} < u_N. \quad (23)$$

When $g < 0$, a sufficient condition for (23), and hence a sufficient condition for $\alpha_N^* < 0$, is $u_N > g(N/2)$. This proves part (c.2).

Remark 4.—Note that $g(N/2) \rightarrow \infty$ as $N \rightarrow \infty$. Thus, $Pr[u_N < g(N/2)] \rightarrow 1$ as $N \rightarrow \infty$, and $Pr(\alpha_N^* > 0) \rightarrow 1$ as $N \rightarrow \infty$, if $g > 0$. When $g < 0$, $Pr[u_N > g(N/2)] \rightarrow 1$ as $N \rightarrow \infty$. Hence, for $g < 0$, $Pr(\alpha_N^* < 0) \rightarrow 1$ as $N \rightarrow \infty$.

Proposition D (Mean-reverting Process)

Suppose that $E(\tilde{r}_k) = \mu$, $\forall k$, and let $\tilde{u}_k = \tilde{r}_k - \mu$, where $E(\tilde{u}_k^2) < \infty$. Then

$$\alpha_N^* = 0 \quad \text{if} \quad \frac{u_N}{u_{N+1}} = \frac{\omega_N'}{\omega_N^0} < 1, \quad (a)$$

$$\alpha_N^* < 0 \quad \text{if} \quad u_{N+1} < 0 < u_N, \quad (b)$$

and

$$\alpha_N^* > 0 \quad \text{if} \quad u_N < 0 < u_{N+1}. \quad (c)$$

Proof

Some straightforward algebra is needed to see this. Using (5.1), for $k = N$, $E(\tilde{r}_k) = \mu$, and $\tilde{u}_k + \mu = \tilde{r}_k$, one has

$$\begin{aligned} \alpha_N^* &= \frac{-\omega_N^0}{\gamma_N} (u_N + \mu) + \frac{\omega_N'}{\gamma_N} (u_{N+1} + \mu) + \frac{\omega_N^0 - \omega_N'}{\gamma_N} \mu; \\ \alpha_N^* &= \frac{-\omega_N^0}{\gamma_N} u_N + \frac{\omega_N'}{\gamma_N} u_{N+1}. \end{aligned} \quad (24)$$

Clearly, the proposition holds. (Recall that $\omega_N^0 > \omega_N' > 0$.)

Remark 5.—If $u_k < 0$ is interpreted as indicative of “bad times” and $u_k > 0$ as indicative of “good times,” then (24) suggests that, if “bad times” are followed by “good times,” then the effect of the optimal smoothing action will be a dampening of the magnitude of the “good times.” If “good times” are succeeded by “bad times,” the effect will

be a dampening of the magnitude of “bad times” (or an attempt to return to “good times”). This intuitive characterization seems to be consistent with the general, but loosely worded, statements of the smoothing hypothesis: firms will attempt to reduce the extent to which “bad times” and (at the other extreme) “good times” are revealed. And this behavior is consistent with a mean-reverting process (as described in proposition D) because “bad times” ($u_N < 0$) and “good times” ($u_N > 0$) are, on average, transient phenomena. But note that nonzero smoothing actions may be *optimal* for *non-mean-reverting* processes as well (see propositions A, B, and C). Hence, contrary to the assertions of Ball and Watts,¹⁶ smoothing behavior is *not*, in general, “nonsense” with respect to non-mean-reverting processes; in particular, it is not “nonsense” for the problem formulation given in (1.1)–(1.3).

Remark 6.—Throughout this paper, I have imposed the constraint,

$$\sum_{k=N}^1 \alpha_k = 0. \quad (25)$$

This constraint appears to be appropriate if one is examining a horizon before which and after which no nonzero smoothing actions exist. If this is not the case, then an appropriate constraint is:

$$\sum_{k=N}^1 \alpha_k = \Omega, \quad (26)$$

where Ω is some nonzero constant. Imposing the constraint (26) rather than (25) would have resulted in the inclusion of the constant term, Ω , in the total adjustment to α_k^* for α_s^* , $s > k$. For example, instead of (5.2), that is,

$$\alpha_1^* = - \sum_{i=N}^2 \alpha_i^*,$$

I would have had

$$\alpha_1^* = \Omega - \sum_{i=N}^2 \alpha_i^*.$$

V

This paper considered the “income-smoothing” hypothesis (rate-of-return version) within the context of two kinds of stochastic processes: martingales and mean-reverting processes. A characterization of optimal smoothing action for an N -period horizon was derived via dynamic programming tools. This characterization provided access to several propositions regarding optimal smoothing actions under each kind of

16. Ball and Watts, p. 3.

stochastic process and at various stages within the N -period horizon. These propositions indicate what one should expect if optimal smoothing is pursued by firms' managements. Moreover, they indicate that nonzero smoothing is *not* always optimal. The paper only dealt with the effects (if any) of optimal smoothing actions on reported accounting numbers. It did not consider the effects (if any) of smoothing actions on aggregate capital market behavior.¹⁷

The conclusions of this paper were derived by postulating an optimizing model for smoothing behavior. In principle, this approach may reduce the generality of the conclusions, but, in return, one gets specific theorems about the phenomena of interest. Empirically, of course, we only require that managers behave *as if* they were guided by the postulated model.

Some topics for future research are immediately suggested by the framework employed and the results presented in the above discussion. For example, given a characterization of optimal smoothing actions, it is possible to generate simulated *reported* rates-of-return series for simulated *basic* rates-of-return series. These results should permit identifications of the manner in which smoothing behavior (for various horizon lengths) may alter the statistical properties of basic rates-of-return series. That is, if the *basic* rate of return process is adequately characterized as, say, a *strict* martingale process, is it the case that smoothing actions will affect this process in a manner such that *reported* rates of return will appear to be adequately described by, say, a *semi*-martingale process? Observe that these kinds of simulation results would also yield insights on the descriptive validity of the optimization model *postulated* in (1.1)–(1.3). The available *empirical* evidence suggests that *reported* rates of return are well described by martingale models.¹⁸ And, if model (1.1)–(1.3) has any descriptive validity, one should observe that, for some *basic* rate-of-return processes, the *reported* rate-of-return processes generated via (1.1)–(1.3) are adequately described by martingale models. In other words, if model (1.1)–(1.3) has descriptive validity, the time-series properties of *simulated reported* rates of return generated by this model, for some *simulated basic* rate-of-return processes, should be consistent with the time-series properties of *actual reported* rates of return.

Finally, given the kinds of simulation results described above *and* given *empirical* proxies for *basic* rates-of-return series, it may be possible to conduct more refined empirical tests of the existence of income-smoothing behavior. This assertion is based on the seemingly reasonable assumption that it is futile to test for the existence of something which cannot even be (directly or indirectly) identified when, in fact, that something does exist.

17. But see n. 5 above.

18. See, e.g., Ball and Watts.