HIROSHIMA MATH. J.
9 (1979), 451–463

Incomparability in Ring Extensions

Hirohumi UDA

(Received January 16, 1979)

Introduction

Throughout this paper rings will be all commutative rings with units and morphisms will mean unitary ring-homomorphisms.

The purpose of this paper is to study some properties of *an incomparable morphism* (cf. [3]) and to introduce the notion of *universally incomparable morphisms* which will play an important role in this paper.

We shall discuss in § 1 some basic properties of an incomparable morphism. In § 2, we shall define a universally incomparable morphism and shall examine its properties. Let k be a field. For a k-algebra A, we shall prove in Theorem 2.9 that $k \to A$ is a universally incomparable morphism if and only if A is integral over k, and also if and only if $k[X] \to A[X]$ is an incomparable morphism. We shall also give in Theorem 2.11 and in Theorem 2.12 some necessary and sufficient conditions for a morphism f: $A \to B$ to be a universally incomparable one. Moreover, in Proposition 2.17, we shall show that if a morphism f of finite type is incomparable, then f is a universally incomparable morphism.

In § 3, we shall discuss incomparability for some special ring extensions. In Corollary 3.2, we shall give some necessary and sufficient conditions for a morphism: $A \to A[X]/I$ to be an incomparable one, where I is an ideal of $A[X]$. In Corollary 3.6, we shall also give two necessary and sufficient conditions for incomparability to hold for $A \to \bigotimes_{i=1}^{n} A[X]/I_i$, where I_i is an ideal of $A[X]$ for each i. In Proposition 3.11, we shall show that $A \to A[\alpha]$ is an incomparable morphism for each $\alpha \in \Omega$, where A is a Prüfer domain and Ω is the algebraic closure of the quotient field of A.

Acknowledgment

The author wishes to express his hearty thanks to Professor M. Nishi and Professor H. Yanagihara for their kind advices and constant encouragements. He is also indebted to his friends S. Itoh and A. Ōishi for their stimulating and kind comments.

Notation and terminology

Let A be a ring. We let Spec A, Max A and Min A stand for the set of all prime ideals of A, that of all maximal ideals of A and that of all minimal prime ideals of A respectively. For $P \in \operatorname{Spec} A$, we denote by $\kappa(P)$ the quotient field of A/P. Let $f: A \to B$ be a morphism. For an ideal J of B, we understand that $J \cap A$ means $f^{-1}(J)$ and we say that J *lies over* the ideal $J \cap A$ in B and that $J \cap A$ is *the contraction* of J into A. Moreover, we define three properties that a morphism: $A \to B$ might satisfy (cf. [3]).

(LO) For any $P \in \operatorname{Spec} A$ there exists a prime ideal $Q \in \operatorname{Spec} B$ with $Q \cap A = P$.

(GU) Given prime ideals $P \subset P_0$ in A and $Q \in \operatorname{Spec} B$ with $Q \cap A = P$, there exists a prime ideal $Q_0 \in \operatorname{Spec} B$ satisfying $Q \subset Q_0$ and $Q_0 \cap A = P_0$.

(GD) The same with $\subset$ replaced by $\supset$.

For a ring A, we denote the Krull dimension of A by $\dim A$. Moreover, we put $\dim A = 0$ even if $A = 0$.

§1. Basic properties of an incomparable morphism

Let $f: A \to B$ be a morphism. We say that $f: A \to B$ is *an incomparable morphism* if two different prime ideals of B with the same contraction into A can not be comparable. Then it follows easily from the definition that f is an incomparable morphism if and only if $\dim (B \otimes_A \kappa(P)) = 0$ for each $P \in \operatorname{Spec} A$. In this section, we examine basic properties of an incomparable morphism. Although the following Proposition 1.1, 1.2 and Corollary 1.3 can be proved easily, these are very useful.

PROPOSITION 1.1. *Let $f: A \to B$ be a morphism. Then we have the following statements.*

(1) *If f is integral, then f is an incomparable morphism.*

(2) *If f is surjective, then f is an incomparable morphism.*

(3) *If f is a localization, then f is an incomparable morphism.*

PROPOSITION 1.2. *Let $f: A \to B$ and $g: B \to C$ be two morphisms. Then we have the following statements.*

(1) *If both f and g are incomparable morphisms, then so is $g \circ f$.*

(2) *If $g \circ f$ is an incomparable morphism, then so is g.*

(3) *Assume that $g \circ f$ is an incomparable morphism. If g satisfies* GU *or if g satisfies* GD *and* LO, *then f is an incomparable morphism.*

COROLLARY 1.3. *Let $f: A \to B$ be an incomparable morphism. Then we* *have the following statements.*

(1) *If J is an ideal of B with $J \cap A = I$, then $A/I \to B/J$ is an incomparable morphism.*

(2) *If S and T are two multiplicatively closed subsets of A and B respectively with $f(S) \subset T$, then $A_S \to B_T$ is an incomparable morphism.*

We now give characterizations of an incomparable morphism which follow immediately from the above results.

PROPOSITION 1.4. *Let $f: A \to B$ be a morphism. Then the following statements are equivalent.*

(1) *f is an incomparable morphism.*

(2) *For each $M \in \operatorname{Max} A$, $f_M: A_M \to B_M$ is an incomparable morphism.*

(3) *For each $P \in \operatorname{Spec} A$, $f_P: A_P \to B_P$ is an incomparable morphism.*

(4) *For each $Q \in \operatorname{Max} B$ with $Q \cap A = P$, $A_P \to B_Q$ is an incomparable morphism.*

(5) *For each $Q \in \operatorname{Min} B$ with $Q \cap A = P$, $A/P \to B/Q$ is an incomparable morphism.*

As for the change of rings, we have the following

PROPOSITION 1.5. *Let $f: A \to B$ and $g: A \to C$ be two morphisms. Then the following statements hold.*

(1) *If f is an incomparable morphism and the contraction map: $\operatorname{Spec}(B \otimes_A C) \to \operatorname{Spec} B$ is injective, then $C \to B \otimes_A C$ is an incomparable morphism.*

(2) *If f is an incomparable morphism and if g is surjective or a localization, then $C \to B \otimes_A C$ is an incomparable morphism.*

(3) *If $C \to B \otimes_A C$ is an incomparable morphism and g satisfies* LO, *then f is an incomparable morphism.*

PROOF. The assertion (1) is obvious.

(2) If g is surjective (resp. a localization), then $B \to B \otimes_A C$ is surjective (resp. a localization). Therefore, (2) follows immediately from (1).

(3) Let $P \in \operatorname{Spec} A$. Since g satisfies LO, there exists a prime ideal $Q \in \operatorname{Spec} C$ such that $Q \cap A = P$. It follows from Proposition 5 of (1.3.3) in [1] that $B \otimes_A \kappa(P) \to B \otimes_A \kappa(Q)$ is faithfully flat, and hence $B \otimes_A \kappa(P) \to B \otimes_A \kappa(Q)$ satisfies GD and LO. Since $\dim(B \otimes_A \kappa(Q)) = 0$ by the assumption, $\dim(B \otimes_A \kappa(P)) = 0$, which implies that f is an incomparable morphism.

In the next proposition, we give a characterization of incomparability in the category of A-algebras.

PROPOSITION 1.6. *Let A be a ring, and B, C be two A-algebras. Let*

$f: B \to C$ be a morphism of A-algebras. Then f is an incomparable morphism if and only if $B \otimes_A \kappa(P) \to C \otimes_A \kappa(P)$ is an incomparable morphism for each $P \in \operatorname{Spec} A$.

In particular, $A \to B$ is an incomparable morphism if and only if $\kappa(P) \to B \otimes_A \kappa(P)$ is an incomparable morphism for each $P \in \operatorname{Spec} A$.

PROOF. Assume that $B \otimes_A \kappa(P) \to C \otimes_A \kappa(P)$ is an incomparable morphism for each $P \in \operatorname{Spec} A$. Let $Q_1, Q_2 \in \operatorname{Spec} C$ with $Q_1 \subset Q_2$ and $Q_1 \cap B = Q_2 \cap B$. We put $Q_1 \cap A = Q_2 \cap A = P$, and denote $B \otimes_A \kappa(P)$ and $C \otimes_A \kappa(P)$ by $\bar{B}$ and $\bar{C}$ respectively. Since $Q_1\bar{C}, Q_2\bar{C} \in \operatorname{Spec} \bar{C}$ and $Q_1\bar{C} \cap \bar{B} = Q_2\bar{C} \cap \bar{B}$, we have $Q_1\bar{C} = Q_2\bar{C}$ by the assumption. Thus, $Q_1 = Q_2$. This implies that f is an incomparable morphism.

The converse follows immediately from Corollary 1.3.

Here we give some properties of an incomparable morphism of finite type.

PROPOSITION 1.7. *Let A be a finitely generated k-algebra with k a field. Then the following statements are equivalent.*

(1) *$k \to A$ is an incomparable morphism.*

(2) *A is integral over k.*

(3) *$\operatorname{Spec} A$ is a finite set.*

PROOF. (2)⇒(1). It is well known.

(1)⇒(3). We can readily see that $\dim A = 0$, and hence $\operatorname{Spec} A$ is a finite set since A is a Noetherian ring.

(3)⇒(2). By virtue of Theorem 147 in [3], any prime ideal of A is maximal. Let $A = k[\alpha_1, \alpha_2, \dots, \alpha_n]$ and $M \in \operatorname{Spec} A$. Put $\beta_i = \alpha_i$ modulo M. Then $A/M = k[\beta_1, \beta_2, \dots, \beta_n]$ is a field. Therefore, $\beta_1, \beta_2, \dots, \beta_n$ are all integral over k from Theorem 23 in [3]. On the other hand, $\operatorname{Spec} A$ is a finite set. Thus, $\alpha_1, \alpha_2, \dots, \alpha_n$ are all integral over k. This completes the proof.

COROLLARY 1.8 (cf. (6.11.5) in [2]). *Let $f: A \to B$ be a morphism of finite type. Then the following statements are equivalent.*

(1) *f is an incomparable morphism.*

(2) *For each $P \in \operatorname{Spec} A$, $B \otimes_A \kappa(P)$ is a finite dimensional vector space over $\kappa(P)$.*

(3) *For each $P \in \operatorname{Spec} A$, $\operatorname{Spec}(B \otimes_A \kappa(P))$ is a finite set.*

PROOF. This corollary follows easily from Proposition 1.6 and Proposition 1.7.

REMARK 1.9. The condition that $\operatorname{Spec}(B \otimes_A \kappa(M))$ is a finite set for each $M \in \operatorname{Max} A$ does not necessarily imply that $A \to B$ is an incomparable morphism;

in fact the morphism: $\boldsymbol{Z}\to\boldsymbol{Q}[X]$, where $\boldsymbol{Z}$ is *the integers* and $\boldsymbol{Q}$ is *the rational number field*, is such an example.

§2. Universally incomparable morphisms

Let $f: A\to B$ be a morphism. We say that $f: A\to B$ is *a universally incomparable morphism* if for each morphism $A\to C$, $C\to B\otimes_A C$ is an incomparable morphism. If f is a universally incomparable morphism, then f is obviously an incomparable morphism. In this section, we examine some properties of a universally incomparable morphism and give some characterizations.

Throughout this section we shall denote by X an indeterminate. We begin with the following

PROPOSITION 2.1. *Let $f: A\to B$ and $g: B\to C$ be two morphisms. Then we have the following statements.*

(1) *If both f and g are universally incomparable morphisms, then so is $g\circ f$.*

(2) *If $g\circ f$ is a universally incomparable morphism, then so is g.*

PROOF. These assertions follow immediately from definitions and Proposition 1.2.

As for the change of rings, we have the following

PROPOSITION 2.2. *Let $f: A\to B$ and $g: A\to C$ be two morphisms. Then we have the following statements.*

(1) *If f is a universally incomparable morphism and g is an incomparable morphism, then $A\to B\otimes_A C$ is an incomparable morphism.*

(2) *If both f and g are universally incomparable morphisms, then so is $A\to B\otimes_A C$.*

(3) *If $C\to B\otimes_A C$ is a universally incomparable morphism and g satisfies* LO, *then f is a universally incomparable morphism.*

PROOF. The assertion (1) follows immediately from definitions and (1) of Proposition 1.2.

(2) Let $A\to D$ be a morphism. Since g is a universally incomparable morphism, $D\to D\otimes_A C$ is an incomparable morphism. Since f is a universally incomparable morphism, $D\otimes_A C\to(D\otimes_A C)\otimes_A B$ is an incomparable morphism. Therefore, $D\to D\otimes_A(B\otimes_A C)$ is an incomparable morphism by (1) of Proposition 1.2. Thus, $A\to B\otimes_A C$ is a universally incomparable morphism.

(3) Let $A\to D$ be a morphism, and let $C\to C\otimes_A D$ be the change of rings for it. Then our assumption means that $C\otimes_A D\to(C\otimes_A D)\otimes_C(B\otimes_A C)$ is an incomparable morphism. Thus, $C\otimes_A D\to(C\otimes_A D)\otimes_D(B\otimes_A D)$ is an incomparable

morphism. Since g satisfies LO, $D \to C \otimes_A D$ satisfies LO. Therefore, $D \to B \otimes_A D$ is an incomparable morphism from (3) of Proposition 1.5. Thus, f is a universally incomparable morphism.

REMARK 2.3. With the notation of Proposition 2.2, assume that both f and g are incomparable morphisms. In this case, $A \to B \otimes_A C$ is not necessarily an incomparable morphism (cf. (1) in Proposition 2.2). For example, let X and Y be two indeterminates and k be a field. Then both $k \subset k(X)$ and $k \subset k(Y)$ are incomparable morphisms, but $k \to k(X) \otimes_k k(Y)$ is not an incomparable morphism.

For k-algebras with k a field, we give a characterization of a universally incomparable morphism.

PROPOSITION 2.4. *Let A be a k-algebra with k a field. Then $k \to A$ is a universally incomparable morphism if and only if for each field extension L of k, $L \to L \otimes_k A$ is an incomparable morphism.*

PROOF. We have only to prove the 'if' part. Let B be a k-algebra and $P \in \operatorname{Spec} B$. From the assumption, $\kappa(P) \to A \otimes_k \kappa(P)$ is an incomparable morphism, hence $\dim(A \otimes_k \kappa(P)) = 0$. Since $(B \otimes_k A) \otimes_B \kappa(P) = A \otimes_k \kappa(P)$, $B \to B \otimes_k A$ is an incomparable morphism by Proposition 1.6. This implies that $k \to A$ is a universally incomparable morphism.

To characterize universally incomparable morphisms, we need the following lemmas.

LEMMA 2.5. *For a field extension $F \to K$, the following statements are equivalent.*

(1) *$F[X] \subset K[X]$ is an incomparable morphism.*
(2) *K is algebraic over F.*
(3) *$K[X]$ is integral over $F[X]$.*

PROOF. (2)⇒(3)⇒(1). These implications are well known.
(1)⇒(2). If K is not algebraic over F, then there exists an element α of K which is not algebraic over F. Since $(X-\alpha)K[X] \cap F[X] = 0$, $F[X] \subset K[X]$ is not an incomparable morphism. This is a contradiction. Thus, K is algebraic over F.

COROLLARY 2.6 (cf. THEOREM 2 in [4]). *Let $A \subset B$ be integral domains. Then there is no non-zero prime ideal of $B[X]$ lying over 0 in $A[X]$ if and only if the quotient field of B is algebraic over that of A.*

PROOF. Let F and K be the quotient fields of A and B respectively. Assume that there is no non-zero prime ideal of $B[X]$ lying over 0 in $A[X]$. Then this implies that $F[X] \subset K[X]$ is an incomparable morphism. By Lemma 2.5, K is algebraic over F.

Conversely, assume that K is algebraic over F. Let $Q \in \operatorname{Spec} B[X]$. Suppose that $Q \cap A[X]=0$. We put $Q \cap B=P$. Assume that $P \neq 0$. Since $P \cap A=0$, there exists an element α of P such that $\alpha \notin A$. On the other hand, K is algebraic over F. Therefore, there are elements $a_0, a_1, \ldots, a_n$ of A such that $\sum_{i=0}^{n} a_i\alpha^i=0$ and $a_0 a_n \neq 0$. Then $a_0=-\sum_{i=1}^{n} a_i\alpha^i \in \alpha B \cap A \subset P \cap A=0$. This is a contradiction. Thus, $P=0$, and hence we have $QK[X] \in \operatorname{Spec} K[X]$. Since $F[X] \subset K[X]$ is an incomparable morphism by Lemma 2.5, we have $QK[X]=0$. Thus, $Q=0$, which completes the proof.

LEMMA 2.7. *Let A be an integral domain and B be a ring containing A. Then there exists a prime ideal $P \in \operatorname{Min} B$ such that $P \cap A=0$.*

PROOF. Let $S=A-\{0\}$. Then $A_S \subset B_S$. Since $B_S \neq 0$ and A_S is a field, there exists a prime ideal $Q \in \operatorname{Spec} B_S$ with $Q \cap A_S=0$. The assertion follows immediately from the above result.

COROLLARY 2.8. *Let A be a k-algebra with k a field. If A/P is integral over k for each $P \in \operatorname{Min} A$, then A is integral over k.*

PROOF. Assume that there exists an element t of A which is transcendental over k. Since $k[t]$ is an integral domain, there exists a prime ideal $P \in \operatorname{Min} A$ with $P \cap k[t]=0$ from Lemma 2.7, hence $k \subset k[t] \subset A/P$. On the other hand, A/P is integral over k. This is a contradiction, which settles the proof.

With these preparations, we give two more characterizations of a universally incomparable morphism of k-algebras, where k is a field.

THEOREM 2.9. *Let A be a k-algebra with k a field. Then the following statements are equivalent.*

(1) *$k \to A$ is a universally incomparable morphism.*

(2) *A is integral over k.*

(3) *$k[X] \to A[X]$ is an incomparable morphism.*

PROOF. The implications (2)$\Rightarrow$(1)$\Rightarrow$(3) are obvious.
(3)$\Rightarrow$(2). Let $P \in \operatorname{Min} A$. By (5) of Proposition 1.4, $k[X] \to A/P[X]$ is an incomparable morphism, and hence $k[X] \to \kappa(P)[X]$ is an incomparable morphism by (3) of Proposition 1.1 and (1) of Proposition 1.2. Therefore, $k \to A/P$ is algebraic by Lemma 2.5. Thus, A is integral over k by Corollary 2.8.

COROLLARY 2.10. *Let D be an integral domain which contains a field k. Then $k \to D$ is a universally incomparable morphism if and only if D is a field algebraic over k.*

PROOF. The assertion follows easily from Theorem 2.9.

We will now proceed to the general case.

THEOREM 2.11. *Let $A \to B$ be a morphism. Then the following statements are equivalent.*

(1) *$A \to B$ is a universally incomparable morphism.*
(2) *For each morphism $A \to C$, $\dim(B \otimes_A C) \leqq \dim C$.*
(3) *For each morphism $A \to K$ with K a field, $K \to B \otimes_A K$ is an incomparable morphism.*
(4) *For each $P \in \operatorname{Spec} A$, $\kappa(P) \to B \otimes_A \kappa(P)$ is a universally incomparable morphism.*

PROOF. The implications (1)⇒(2)⇒(3) are obvious.
(3)⇒(4). Let $P \in \operatorname{Spec} A$ and L be a field extension of $\kappa(P)$. By the assumption, $L \to L \otimes_A B$ is an incomparable morphism. On the other hand, $L \otimes_A B = L \otimes_{\kappa(P)} (B \otimes_A \kappa(P))$. Therefore, $\kappa(P) \to B \otimes_A \kappa(P)$ is a universally incomparable morphism by Proposition 2.4.
(4)⇒(1). Let $A \to C$ be a morphism. Let $Q \in \operatorname{Spec} C$ and put $Q \cap A = P$. Since $\kappa(P) \to B \otimes_A \kappa(P)$ is a universally incomparable morphism, $\kappa(Q) \to (B \otimes_A \kappa(P)) \otimes_{\kappa(P)} \kappa(Q)$ is an incomparable morphism. That is, $\kappa(Q) \to B \otimes_A \kappa(Q)$ is an incomparable morphism. Therefore, $\dim((B \otimes_A C) \otimes_C \kappa(Q)) = \dim(B \otimes_A \kappa(Q)) = 0$. Thus, $C \to B \otimes_A C$ is an incomparable morphism, and hence $A \to B$ is a universally incomparable morphism.

The following theorem gives two further necessary and sufficient conditions for $A \to B$ to be a universally incomparable morphism.

THEOREM 2.12. *Let $A \to B$ be a morphism. Then the following statements are equivalent.*

(1) *$A \to B$ is a universally incomparable morphism.*
(2) *For each $Q \in \operatorname{Spec} B$ with $Q \cap A = P$, $\kappa(Q)$ is algebraic over $\kappa(P)$.*
(3) *$A[X] \to B[X]$ is an incomparable morphism.*

PROOF. The implication (1)⇒(3) is obvious.
(3)⇒(2). Let $Q \in \operatorname{Spec} B$ and put $Q \cap A = P$. Since $QB[X] \cap A[X] = PA[X]$, $\kappa(P)[X] \to \kappa(Q)[X]$ is an incomparable morphism by Corollary 1.3. Therefore, $\kappa(Q)$ is algebraic over $\kappa(P)$ by Lemma 2.5.
(2)⇒(1). Let $P \in \operatorname{Spec} A$. We shall prove that $\kappa(P) \to B \otimes_A \kappa(P)$ is a universally incomparable morphism. To do this we may assume that $B \otimes_A \kappa(P) \neq 0$. Then there exists a prime ideal $Q \in \operatorname{Spec} B$ such that $Q \cap A = P$. The assumption of (2) means that $(B \otimes_A \kappa(P))/M$ is algebraic over $\kappa(P)$ for each $M \in \operatorname{Spec}(B \otimes_A \kappa(P))$, and hence $B \otimes_A \kappa(P)$ is integral over $\kappa(P)$ by Corollary 2.8. Therefore, $\kappa(P) \to B \otimes_A \kappa(P)$ is a universally incomparable morphism by Theorem 2.9. Thus, $A \to B$

is a universally incomparable morphism by Theorem 2.11.

REMARK 2.13. Let $f: A \to B$ be a morphism. Then it is obvious from Theorem 2.12 that the condition (2) in Theorem 2.12 implies that f is an incomparable morphism. This fact can also be proved directly by Corollary 2.6 in the following way. Assume that $\kappa(Q)$ is algebraic over $\kappa(P)$ for any $Q \in \operatorname{Spec} B$ with $Q \cap A = P$. Then there is no non-zero prime ideal of B/Q lying over 0 in A/P by Corollary 2.6. This implies that f is an incomparable morphism.

On the other hand, it is obvious that an incomparable morphism does not necessarily imply the condition (2) in Theorem 2.12.

COROLLARY 2.14. *Let $f: A \to B$ and $g: B \to C$ be two morphisms. Assume that $g \circ f$ is a universally incomparable morphism. If $B[X] \to C[X]$ satisfies* GU *or if $B[X] \to C[X]$ satisfies* GD *and* LO, *then $A \to B$ is a universally incomparable morphism.*

PROOF. This corollary follows immediately from (3) of Proposition 1.2 and Theorem 2.12.

COROLLARY 2.15. *Let $f: A \to B$ be a morphism. Then f is a universally incomparable morphism if and only if so is $A[X] \to B[X]$.*

PROOF. The assertion follows easily from Theorem 2.12.

COROLLARY 2.16. *Let $X_1, X_2, \ldots, X_n$ be indeterminates. Let $f: A \to B$ be a morphism. Then the following statements are equivalent.*

(1) *$A[X_1] \to B[X_1]$ is an incomparable morphism.*

(2) *$A[X_1, X_2, \ldots, X_n] \to B[X_1, X_2, \ldots, X_n]$ is an incomparable morphism for some $n \geqq 1$.*

(3) *$A[X_1, X_2, \ldots, X_n] \to B[X_1, X_2, \ldots, X_n]$ is an incomparable morphism for all $n \geqq 0$. Here, $A[X_1, X_2, \ldots, X_n] = A$, if $n = 0$.*

PROOF. The assertion follows immediately from Corollary 2.15.

Finally, we prove that the notion of incomparability coincides with that of universal incomparability for any morphism of finite type (cf. [2]).

PROPOSITION 2.17. *Let $f: A \to B$ be a morphism of finite type. Then f is an incomparable morphism if and only if f is a universally incomparable morphism.*

PROOF. It is sufficient to prove the 'only if' part. Assume that f is an incomparable morphism. Let $P \in \operatorname{Spec} A$. Then $B \otimes_A \kappa(P)$ is a finitely generated $\kappa(P)$-algebra, and hence by Proposition 1.6, $\kappa(P) \to B \otimes_A \kappa(P)$ is an incomparable morphism. Therefore, $B \otimes_A \kappa(P)$ is integral over $\kappa(P)$ by Proposition 1.7, and hence $\kappa(P)\to B\otimes_A \kappa(P)$ is a universally incomparable morphism by Theorem 2.9. Thus, $A\to B$ is a universally incomparable morphism by Theorem 2.11.

COROLLARY 2.18. *Let $f: A\to B$ and $g: A\to C$ be two morphisms. If both f and g are incomparable morphisms and if f is of finite type, then $A\to B\otimes_A C$ is an incomparable morphism.*

PROOF. The assertion follows immediately from (1) of Proposition 2.2 and Proposition 2.17.

REMARK 2.19. In general, an incomparable morphism is not necessarily a universally incomparable morphism (cf. Corollary 2.10).

§3. Incomparability for certain ring extensions

In this section, we give some necessary and sufficient conditions for a morphism: $A\to \bigotimes_{i=1}^{n} A[X]/I_i$ to be an incomparable one, where I_i is an ideal of $A[X]$ for each i. We also show a result on incomparability for simple extensions of Prüfer domains.

Throughout this section, A will be a ring and $X, X_1, X_2,\dots, X_n$ will be indeterminates. Let I be an ideal of $A[X_1, X_2,\dots, X_n]$. We denote by $c(I)$ the ideal generated by all coefficients of all polynomials in I and we call it *the content* of I. In particular, if $I=(f)$, then $c(I)$ will be denoted by $c(f)$.

THEOREM 3.1. *Let I be an ideal of $A[X]$ and put $B=A[X]/I$. Let $P\in \operatorname{Spec} A$. Then $\operatorname{Spec}(B\otimes_A \kappa(P))$ is a finite set if and only if $c(I)\not\subset P$.*

PROOF. There is a one-to-one correspondence between prime ideals of B lying over P and prime ideals of $\kappa(P)[X]$ containing $\bar{I}$, where $\bar{I}$ is the ideal generated by the homomorphic image of I in $\kappa(P)[X]$. Assume that $c(I)\subset P$. Then $I\subset PA[X]$, hence $\bar{I}=0$. Thus, $\operatorname{Spec}(B\otimes_A \kappa(P))$ is an infinite set.

Conversely, assume that $c(I)\not\subset P$. Then $\bar{I}\neq 0$. Therefore, since $\kappa(P)[X]$ is a 1-dimensional Noetherian domain, $\operatorname{Spec}(B\otimes_A \kappa(P))$ is a finite set.

COROLLARY 3.2. *With the notation of Theorem 3.1, the following statements are equivalent.*

(1) *$A\to B$ is an incomparable morphism.*
(2) $c(I)=A$.
(3) *For each $M\in \operatorname{Max} A$, $\operatorname{Spec}(B\otimes_A \kappa(M))$ is a finite set.*

PROOF. The assertion follows easily from Theorem 3.1 and Corollary 1.8.

COROLLARY 3.3. *Let $f(X)\in A[X]$. Then $A\to A[X]/(f(X))$ is an incomparable morphism if and only if $c(f)=A$.*

REMARK 3.4. Let I be an ideal of $A[X_1, X_2,\dots, X_n]$ and put $B=A[X_1, X_2,\dots, X_n]/I$. If $A\to B$ is an incomparable morphism, we have obviously $c(I)=A$. Again, if $A\to B$ is an incomparable morphism, then by Corollary 1.8, $\mathrm{Spec}(B\otimes_A \kappa(M))$ is a finite set for each $M\in\mathrm{Max}\,A$. However, the converse of each statement is false as is seen in the following example.

EXAMPLE 3.5. Let (A, M) be a local domain which is not a field. Let a be a non-zero element of M, and put $B=A[X, Y]/(aY-1)$. We have obviously $c(aY-1)=A$. Since $MB=B$, $\mathrm{Spec}(B\otimes_A \kappa(M))$ is an empty set. On the other hand, $A\to B$ is not an incomparable morphism obviously.

COROLLARY 3.6. *Let* $I_1, I_2,\dots, I_n$ *be ideals of* $A[X]$ *and put* $B=\bigotimes_{i=1}^{n} A[X]/I_i$. *Then the following statements are equivalent.*

(1) $A\to B$ *is an incomparable morphism.*

(2) *Let* $Q\in\mathrm{Spec}\,B$. *If* $Q\cap A[X]$ *contains all* I_i, *then* $c(I_i)\not\subset Q\cap A$ *for each* i.

(3) *Let* $P\in\mathrm{Spec}\,A$. *If there exists a prime ideal of* B *lying over* P, *then* $\mathrm{Spec}(A[X]/I_i\otimes_A\kappa(P))$ *is a finite set for each* i.

PROOF. The equivalence between (2) and (3) follows immediately from Theorem 3.1.

Let $P\in\mathrm{Spec}\,A$. We put $B_i=A[X]/I_i\otimes_A \kappa(P)$.

(1)$\Rightarrow$(3). Let $P\in\mathrm{Spec}\,A$ and assume that there exists a prime ideal of B lying over P. Since $B\otimes_A \kappa(P)\neq 0$, $\bigotimes_{j\neq i}{}_{\kappa(P)} B_j\neq 0$ for each i. That is, $\kappa(P)\to \bigotimes_{j\neq i}{}_{\kappa(P)} B_j$ satisfies LO. On the other hand, since $\kappa(P)\to \bigotimes_{j=1}^{n}{}_{\kappa(P)} B_j$ is an incomparable morphism by Proposition 1.6, $\bigotimes_{j\neq i}{}_{\kappa(P)} B_j\to \bigotimes_{j=1}^{n}{}_{\kappa(P)} B_j$ is an incomparable morphism by (2) of Proposition 1.2. Therefore, $\kappa(P)\to B_i$ is an incomparable morphism by (3) of Proposition 1.5. Thus, $\mathrm{Spec}\,B_i$ is a finite set by Proposition 1.7.

(3)$\Rightarrow$(1). Let $P\in\mathrm{Spec}\,A$ and assume that $\mathrm{Spec}(B\otimes_A \kappa(P))\neq\phi$. By the assumption and Proposition 1.7, B_i is integral over $\kappa(P)$, and hence $B\otimes_A \kappa(P)$ is integral over $\kappa(P)$. Therefore, $\kappa(P)\to B\otimes_A \kappa(P)$ is a universally incomparable morphism by Theorem 2.9. Thus, $A\to B$ is a universally incomparable morphism by Theorem 2.11.

REMARK 3.7 (cf. (6.11.5) in [2]). Let $A\to B_i$ be a morphism of finite type for $i=1, 2,\dots, n$. If every $A\to B_i$ is an incomparable morphism, then $A\to \bigotimes_{i=1}^{n} B_i$ is an incomparable morphism by Corollary 2.18. In particular, if $f_1(X), f_2(X),\dots, f_n(X)$ are polynomials of $A[X]$ with $c(f_i)=A$ for all i, then $A\to \bigotimes_{i=1}^{n} A[X]/(f_i(X))$ is an incomparable morphism. However, the converse is not true as is seen in the following example.

EXAMPLE 3.8. Let k be an algebraically closed field and let X, Y, Z be three indeterminates. Let $A=k[X]$ and $B=A[Y]/(XY-1)\otimes_A A[Z]/(XZ)$. By Corollary 3.2, $A\to A[Y]/(XY-1)$ is an incomparable morphism, but $A\to A[Z]/(XZ)$ is not an incomparable morphism. On the other hand, $B=k[X, Y, Z]/(XY-1, XZ)$. Since k is algebraically closed, we can readily see that $A\to B$ is an incomparable morphism.

Let $A\to B$ be a morphism. We consider a condition (*) that $\mathrm{Spec}(B\otimes_A \kappa(M))$ is a finite set for each $M\in \mathrm{Max}\,A$. In Remark 3.4, we pointed out the following fact: (*) does not necessarily imply that $A\to B$ is an incomparable morphism. In the following proposition, we give a condition for (*) to imply that $A\to B$ is an incomparable morphism.

PROPOSITION 3.9. *With the notation of Corollary* 3.6, *assume that for each* $P\in \mathrm{Spec}\,A$ *which is the contraction of a prime ideal of* B *into* A, *there exists a maximal ideal* $M\in \mathrm{Max}\,A$ *containing* P *such that* M *is the contraction of a prime ideal of* B *into* A. *Then the following statements are equivalent.*

(1) $A\to B$ *is an incomparable morphism.*

(2) *Let* $Q\in \mathrm{Spec}\,B$. *If* $Q\cap A[X]$ *contains all* I_i, *then* $\mathrm{c}(I_i)\not\subset Q\cap A$ *for each* i.

(3) *For each* $M\in \mathrm{Max}\,A$, $\mathrm{Spec}(B\otimes_A \kappa(M))$ *is a finite set.*

PROOF. (1)⇔(2) and (1)⇒(3). These implications follow from Corollary 3.6.
(3)⇒(2). Let $P\in \mathrm{Spec}\,A$ and assume that $\mathrm{Spec}(B\otimes_A \kappa(P))\neq\phi$. Then there exists a maximal ideal $M\in \mathrm{Max}\,A$ such that $P\subset M$ and $\mathrm{Spec}(B\otimes_A \kappa(M))\neq\phi$. By the assumption (3), $\mathrm{Spec}(B\otimes_A \kappa(M))$ is a finite set. In the same manner as (1)⇒(3) in Corollary 3.6, $\mathrm{Spec}(A[X]/I_i\otimes_A \kappa(M))$ is a finite set for each i. By Theorem 3.1, $\mathrm{c}(I_i)\not\subset M$, hence $\mathrm{c}(I_i)\not\subset P$. This completes the proof.

REMARK 3.10. If $A\to B$ satisfies LO, then the assumption of Proposition 3.9 is satisfied.

PROPOSITION 3.11. *Let* A *be a Prüfer domain and let* Ω *be the algebraic closure of the quotient field* F *of* A. *Then for each* $\alpha\in\Omega$, $A\to A[\alpha]$ *is an incomparable morphism.*

PROOF. Let $P\in \mathrm{Spec}\,A$. Since A_P is a valuation ring, there is a polynomial $f(X)$ in $A_P[X]$ such that $f(\alpha)=0$, $\mathrm{c}(f)=A_P$ and $f(X)$ is irreducible over F. By Theorem A in [5], $f(X)A_P[X]$ is a prime ideal, hence $A_P[\alpha]=A_P[X]/(f(X))$. Therefore, $A_P\to A_P[\alpha]$ is an incomparable morphism by Corollary 3.3. Thus, $A\to A[\alpha]$ is an incomparable morphism by Proposition 1.4.

COROLLARY 3.12. *With the notation of Proposition* 3.11, *let* $\alpha_1, \alpha_2,\dots, \alpha_n$

$\in \Omega$. *Then* $A \to A[\alpha_1, \alpha_2, \dots, \alpha_n]$ *is an incomparable morphism.*

PROOF. By Proposition 3.11, $A \to A[\alpha_i]$ is an incomparable morphism for each i, and hence $A \to \bigotimes_{i=1}^{n} {}_A A[\alpha_i]$ is an incomparable morphism by Remark 3.7. Thus, $A \to A[\alpha_1, \alpha_2, \dots, \alpha_n]$ is an incomparable morphism by (2) of Proposition 1.1 and (1) of Proposition 1.2.

References

[1] N. Bourbaki, *Algèbre Commutative, Chapitres* 1 *et* 2, Hermann, Paris, 1961.
[2] A. Grothendieck et J. Dieudonné, *Éléments de Géométrie Algébrique* I, Springer-Verlag, Berlin-Heidelberg-New York, 1971.
[3] I. Kaplansky, *Commutative rings*, University of Chicago Press, 1974.
[4] S. McAdam, *Going down in polynomial rings*, Can. J. Math., **23** (1971), 704–711.
[5] H. T. Tang, *Gauss' lemma*, Proc. Amer. Math. Soc., **35** (1972), 372–376.

Faculty of Education,
Miyazaki University