Incomplete Multigranulation Rough Set

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***Abstract*—The original rough-set model is primarily concerned with the approximations of sets described by a single equivalence relation on a given universe. With granular computing point of view, the classical rough-set theory is based on a single granulation. This correspondence paper first extends the rough-set model based on a tolerance relation to an incomplete rough-set model based on multigranulations, where set approximations are defined through using multiple tolerance relations on the universe. Then, several elementary measures are proposed for this rough-set framework, and a concept of approximation reduct is introduced to characterize the smallest attribute subset that preserves the lower approximation and upper approximation of all decision classes in this rough-set model. Finally, several key algorithms are designed for finding an approximation reduct.**

***Index Terms*—Attribute reduction, granular computing, information systems (ISs), rough set.**

I. Introduction

Rough-set theory, proposed by Pawlak and Skowron [24], [26], has become a well-established mechanism for uncertainty management in a wide variety of applications related to artificial intelligence [4], [11], [12], [17], [43], [44], [58]. In this framework, an attribute set is viewed as a granular space, which partitions the universe into some knowledge granules or elemental concepts. Partition, granulation, and approximation are the methods widely used in human's reasoning [55], [56]. Rough-set methodology presents a novel paradigm to deal with uncertainty and has been applied to feature selection [18], [48], [49], knowledge reduction [9], [16], [51], rule extraction [1], [8], [35], [50], [59], uncertainty reasoning [20], [25], [31], decision evaluation [36]–[38], and granular computing [2], [3], [19], [21], [29], [30], [52], [57].

Knowledge representation in the rough-set model is done via *information systems* (ISs) which are a tabular form of an OBJECT → ATTRIBUTE VALUE relationship, similar to relational databases. An IS is an ordered triplet $S = (U, AT, f)$, where U is a finite nonempty set of objects, AT is a finite nonempty set of attributes (predictor features), and $f_a : U \to V_a$ for any $a \in AT$, where V_a is the domain of an attribute a. For any $x \in U$, an information vector of x is given by $Inf(x) = (a, f_a(x))|a \in AT$. In particular, a target IS is given by $S = (U, AT, f, D, g)$, where D is a finite nonempty set of decision attributes and $g_d : U \to V_d$ for any $d \in D$, where V_d is the domain of a decision attribute d.

In the past ten years, under different conditions, several extensions of the rough-set model have been accomplished [26]–[28], which include variable precision rough-set model [60], rough-set model based on tolerance relation [13], [14], [32], [45], [47], Bayesian rough-set model [46], fuzzy rough-set model [5], rough fuzzy set model [5], and fuzzy probabilistic rough-set model [10]. In the view of granular computing proposed by Zadeh [56], a target concept in these models is always characterized via the so-called upper and lower approximations under a single granulation, i.e., the concept is depicted by available knowledge induced from a single relation (such as equivalence relation, tolerance relation, and reflexive relation) on the universe. This approach in describing a target concept relies mainly on the following assumption.

If P and Q are two sets of predictor features and $X \subseteq U$ is a target concept, then the rough set of X is derived from the quotient set $U/(P \cup Q)$. In fact, the quotient set is equivalent to the formula

$$\widehat{P \cup Q} = \{P_i \cap Q_j : P_i \in U/P, Q_j \in U/Q\}.$$

This assumption implies the following conditions.

1) Any two attributes must be independent in ISs.
2) An intersection operation between any P_i and Q_j can be performed.
3) The target concept is approximately described by using the quotient set $U/(P \cup Q)$.

In fact, the target concept is described by using a finer granulation (partitions) formed through combining two known granulations (partitions) induced by two-attribute subsets. Although it generates a much finer granulation and more knowledge granules, the combination/fining destroys the original granulation structure/partitions [34].

However, this assumption, in general, cannot always be satisfied in many practical issues. For example, in some decision-making processes, for the same object (or sample, project, and element), there is a contradiction/inconsistent relationship between its values under one-attribute set P and those under another attribute set Q. In other words, we cannot perform intersection operations between their quotient sets, and the target concept cannot be approximated by the quotient set $U/(P \cup Q)$. In this case, we often need to describe concurrently the target concept through multiple binary relations (e.g., equivalence relation, tolerance relation, reflexive relation, and neighborhood relation) on the universe according to user requirements or targets of problem solving [34].

In view of granular computing, an equivalence relation (or a tolerance relation) on the universe can be regarded as a granulation, and a partition (or a cover) on the universe can be regarded as a granulation space [19], [53], [54]. Hence, the classical rough-set theory is based on a single granulation (only one equivalence relation). Note that any attribute set can induce a certain equivalence relation in a complete IS. In the literature, to more widely apply the rough-set theory in practical applications, Qian and Liang [34] extended Pawlak's single-granulation rough-set model to a *multigranulation rough-set model* (MGRS), where the set approximations are defined by multiple equivalence relations on the universe. In the literature [39]–[42], Rasiowa *et al.* also investigated the approaches to approximation based on many indiscernibility relations for rough approximations. However, in essence, the approximations in these approaches are still based on a singleton granulation induced from an indiscernibility relation, which can be applied to knowledge representation in distributive systems

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and groups of intelligent agents. Furthermore, in the literature [33], Qian *et al.* gave several basic views for establishing an MGRS model in incomplete ISs.

The main objective of this correspondence paper is to fully establish a rough-set model based on multiple tolerance relations in incomplete ISs. The rest of this paper is organized as follows. Some basic concepts in complete MGRS are briefly reviewed in Section II. In Section III, a rough-set model based on multiple tolerance relations, called incomplete MGRS, is proposed in incomplete ISs, some of its important properties are investigated, and several elementary measures for this rough-set model are presented, which are accuracy measure, quality of approximation, and precision of approximation. In Section IV, we first introduce a concept of approximation reduct to the incomplete MGRS, which is based on the so-called upper approximation reduct and lower approximation reduct, and then design two algorithms to compute the upper/underapproximation reducts for applications of this theory in practical issues. In Section V, an illustrative example shows the actual applicability of the proposed approach. Finally, Section VI concludes with some remarks.

II. Preliminaries

Throughout this correspondence paper, we assume that the universe U is a finite nonempty set. Suppose that $U/IND(P)$ is a partition of U induced by the attribute set P in an IS. For $x \in U$, let $[x]_P$ be the class containing x in $U/IND(P)$ and θ_P the equivalence relation associated with $U/IND(P)$, i.e., $x\theta_P y \Leftrightarrow [x]_P = [y]_P$.

In the rough-set model MGRS, unlike Pawlak's rough-set theory, a target concept is approximated through multiple partitions induced by multiple equivalence relations [34]. Suppose that $S = (U, AT, f)$ is a complete IS, then $X \subseteq U$, and $P_1, P_2, \ldots, P_m$ are m-attribute subsets. We define a lower approximation and an upper approximation of X related to $P_1, P_2, \ldots, P_m$ by the following:

$$\underline{\sum_{i=1}^{m} P_i} X = \bigcup \{x | [x]_{P_i} \subseteq X, \qquad \text{for some } i \leq m\} \tag{1}$$

$$\overline{\sum_{i=1}^{m} P_i} X = \sim \underline{\sum_{i=1}^{m}} P_i(\sim X). \tag{2}$$

Similarly, the *boundary region* in MGRS can be extended as $Bn_{\sum_{i=1}^{m} P_i}(X) = \overline{\sum_{i=1}^{m} P_i} X \setminus \underline{\sum_{i=1}^{m} P_i} X$.

Fig. 1 shows the difference between Pawlak's rough-set model and the MGRS model.

In the figure, the bias region is the lower approximation of a set X obtained by a single granulation $P \cup Q$, which are expressed by the equivalence classes in the quotient set $U/(P \cup Q)$, and the shadow region is the lower approximation of X induced by two granulations $P + Q$, which are characterized by the equivalence classes in the quotient set U/P and the quotient set U/Q together.

III. MGRS in Incomplete ISs

In this section, we extend MGRS in complete ISs to MGRS in incomplete ISs, which is just called incomplete MGRS.

A. Incomplete ISs

For an IS, any attribute domain V_a may contain special symbol "$*$" to indicate that the value of an attribute is unknown. Here, we assume that an object $x \in U$ possesses only one value for an attribute a, $a \in AT$. Thus, if the value of an attribute a is missing, then the real value must be from the set $V_a \setminus \{*\}$. Any domain value different from "$*$" will be called regular. A system in which values of all attributes for all objects from U are regular (known) is called complete and is called incomplete, otherwise [13]–[15], [18]. In particular, $S = (U, AT, f, D, g)$ is called an incomplete target IS if values of some attributes in AT are missing and those of all attributes in D are regular (known), where AT is called the conditional attributes and D is the decision attributes.

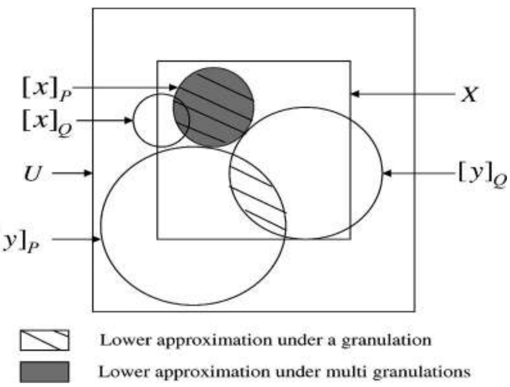

Fig. 1. Difference between Pawlak's rough-set model and MGRS.

TABLE I
Incomplete Target Is About an Emporium Investment Project

Project	Locus	Investment	Population density	Decision
e_1	common	high	0.88	Yes
e_2	bad	high	*	Yes
e_3	bad	*	0.33	No
e_4	bad	low	0.40	No
e_5	bad	low	0.37	No
e_6	bad	*	0.60	Yes
e_7	common	high	0.65	No
e_8	good	*	0.62	Yes

Example 1: Here, we employ an example to illustrate some concepts of an incomplete target IS and computations involved in our proposed incomplete MGRS. Table I depicts an incomplete target IS containing some information about an emporium investment project. Locus, Investment, and Population density are the conditional attributes of the system, and Decision is the decision attribute (in the sequel, L, I, P, and D will stand for Locus, Investment, Population density, and Decision, respectively). The attribute domains are as follows: $V_L = \{good, common, bad\}$, $V_I = \{high, low\}$, $V_P = \{0.88, 0.33, 0.40, 0.37, 0.60, 0.65, 0.62\}$, and $V_D = \{Yes, No\}$.

Let $S = (U, AT, f)$ be an incomplete IS. Each subset of attributes $P \subseteq AT$ determines a binary relation $SIM(P)$ on the universe U [13]

$$SIM(P) = \{(u, v) \in U \times U \,|\, \forall\, a \in P, a(u) = a(v) \text{ or } a(u) = * \text{ or } a(v) = *\}.$$

The relation $SIM(P)$, $P \subseteq AT$, is a tolerance relation. If the attributes $P \subseteq AT$ are numerical attributes, we define another tolerance relation as follows:

$$SIM(P) = \{(u, v) \in U \times U \,|\, \forall\, a \in P, |a(u) - a(v)| \leq \delta_a \text{ or } a(u) = * \text{ or } a(v) = *, \delta_a \geq 0\}.$$

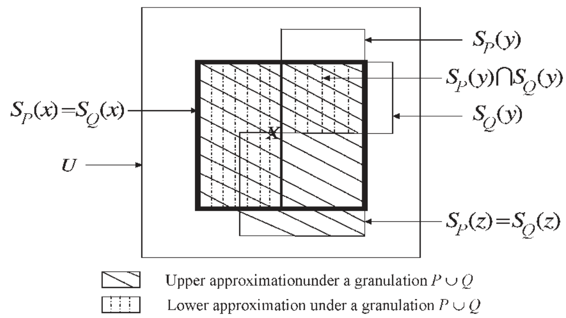

Fig. 2. Set approximation in Kryszkiewicz's rough-set model.

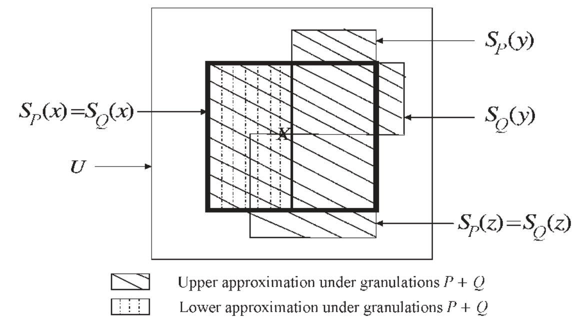

Fig. 3. Set approximation in incomplete MGRS.

It can be shown that $SIM(P) = \bigcup_{a \in P} SIM(\{a\})$. Let $S_P(u)$ denote the set $\{v \in U | (u, v) \in SIM(P)\}$. Clearly, $S_P(u)$ is the maximal set of objects which are possibly indistinguishable by P with u. Let $U/SIM(P)$ denote the family of sets $\{S_P(u) | u \in U\}$ called the classification or knowledge induced by the attributes P. A member $S_P(u)$ from $U/SIM(P)$ is called a tolerance class or an information granule. Note that the tolerance classes in $U/SIM(P)$ cannot constitute a partition of the universe U in general. They constitute a cover of U, i.e., $S_P(u) \neq \varnothing$ for every object $u \in U$ and $\bigcup_{u \in U} S_P(u) = U$. In particular, the *identity partition* is the cover that each of the tolerance classes contains only a singleton set, and the *universal partition* is the cover that each of tolerance classes is equal to the universe set. The former is the finest cover on any nonempty set, and the latter is the roughest cover on the universe U. For an incomplete target IS $S = (U, AT, f, D, g)$, if $SIM(AT) \subseteq \theta_D$, we say S is consistent, and otherwise, S is inconsistent [18].

Given the earlier discussion, we can then define a partial order on the set of all classifications of U. Let $S = (U, AT, f)$ be an incomplete IS and $P, Q \in AT$. One says that P is finer than an attribute set Q (or Q is coarser than P) if and only if $S_P(u_i) \subseteq S_Q(u_i)$ for any $i \in \{1, 2, \ldots, |U|\}$, denoted by $P \preceq Q$. If $P \preceq Q$ and $U/SIM(P) \neq U/SIM(Q)$, one says that P is strictly finer than an attribute set Q (or Q is strictly coarser than P), denoted by $P \prec Q$[33]. In fact, $P \prec Q \Leftrightarrow S_P(u_i) \subseteq S_Q(u_i) \forall i \in \{1, 2, \ldots, |U|\}$, and there exists at least one $j \in \{1, 2, \ldots, |U|\}$ such that $S_P(u_i) \subset S_Q(u_i)$.

B. Incomplete MGRS on Two Granulation Spaces

Simply, we first investigate the approximation of a target set under two tolerance relations on the universe, i.e., how to approximate a target concept through using two granulation spaces.

Definition 1: Let $S = (U, AT, f)$ be an incomplete IS, $P, Q \subseteq AT$ two-attribute subsets, and $X \subseteq U$. A lower approximation and upper approximation of X in U are defined by the following:

$$\underline{P+Q}X = \bigcup \{x \,|\, S_P(x) \subseteq X \text{ or } S_Q(x) \subseteq X\} \tag{3}$$

$$\overline{P+Q}X = \sim \underline{P+Q}(\sim X). \tag{4}$$

The order pair $\langle \underline{P+Q}X, \overline{P+Q}X \rangle$ is called a rough set of X with respect to $P + Q$. The area of uncertainty or boundary region of this rough set is defined as

$$Bn_{P+Q}(X) = \overline{P+Q}X \setminus \underline{P+Q}X.$$

One can understand the rough-set approximation based on multiple tolerance relations and show the difference between the incomplete MGRS and the classical rough-set framework based on a tolerance relation proposed by Kryszkiewicz [13] through Figs. 2 and 3 and Example 2.

In Fig. 2, the dashed region is the lower approximation of a set X obtained by a single granulation $P \cup Q$, and the bias region is the upper approximation of X induced by the granulation $P \cup Q$ in Kryszkiewicz's incomplete rough-set model. However, in Fig. 3, the dashed region is the lower approximation of a set X obtained by two granulations $P + Q$, and the bias region is the upper approximation of X induced by the granulations $P + Q$ in incomplete MGRS.

Example 2 (Continued From Example 1): Let $X = \{e_1, e_2, e_6, e_8\}$ and $\delta_P = 0.1$. Three covers can be induced from Table I as follows:

$$\begin{aligned}
U/SIM(L) = \{&\{e_1, e_7\}, \{e_2, e_3, e_4, e_5, e_6\},\\
&\{e_2, e_3, e_4, e_5, e_6\}, \{e_2, e_3, e_4, e_5, e_6\},\\
&\{e_2, e_3, e_4, e_5, e_6\}, \{e_2, e_3, e_4, e_5, e_6\},\\
&\{e_1, e_7\}, \{e_8\}\}\\
U/SIM(P) = \{&\{e_1, e_2\}, \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\},\\
&\{e_2, e_3, e_4, e_5\}, \{e_2, e_3, e_4, e_5\},\\
&\{e_2, e_3, e_4, e_5\}, \{e_2, e_6, e_7, e_8\},\\
&\{e_2, e_6, e_7, e_8\}, \{e_2, e_6, e_7, e_8\}\}\\
U/SIM(L \cup P) = \{&\{e_1\}, \{e_2, e_3, e_4, e_5, e_6\}, \{e_2, e_3, e_4, e_5\},\\
&\{e_2, e_3, e_4, e_5\}, \{e_2, e_3, e_4, e_5\},\\
&\{e_2, e_6\}, \{e_7\}, \{e_8\}\}.
\end{aligned}$$

From Definition 1, one can obtain that

$$\begin{aligned}
\underline{L+P}X &= \bigcup \{x \,|\, S_L(x) \subseteq X \text{ or } S_P(x) \subseteq X\}\\
&= \{e_8\} \cup \{e_1\} = \{e_1, e_8\}\\
\overline{L+P}X &= \sim \underline{L+P}(\sim X)\\
&= \sim \{\varnothing \cup \varnothing\} = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}.
\end{aligned}$$

However, the lower approximation and the upper approximation of X in the classical rough-set model based on a single tolerance relation are calculated as follows:

$$\begin{aligned}
\underline{L \cup P}X &= \bigcup \{x \,|\, S_{L \cup P}(x) \subseteq X\} = \{e_1, e_6, e_8\}\\
\overline{L \cup P}X &= \bigcup \{x \,|\, S_{L \cup P}(x) \cap X \neq \varnothing\}\\
&= \{e_1, e_2, e_3, e_4, e_5, e_6, e_8\}.
\end{aligned}$$

Clearly, it follows from the earlier computations that

$$\begin{aligned}
\underline{L+P}X &= \{e_1, e_8\} \subseteq \{e_1, e_6, e_8\} = \underline{L \cup P}X\\
\overline{L+P}X &= \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}\\
&\supseteq \{e_1, e_2, e_3, e_4, e_5, e_6, e_8\} = \overline{L \cup P}X.
\end{aligned}$$

The difference between the two kinds of set approximations can be easily understood by the following theorem.

Theorem 1: Let $S = (U, AT, f)$ be an incomplete IS and $X \subseteq U$ and $P, Q \subseteq AT$ be two-attribute subsets. Then, the following properties hold.

1) $\underline{P + Q}X \subseteq \underline{P \cup Q}X$.
2) $\overline{P + Q}X \supseteq \overline{P \cup Q}X$.

Proof:

1) For any $x \in \underline{P + Q}X$, from Definition 1, it follows that $S_P(x) \subseteq X$ or $S_Q(x) \subseteq X$. Hence, $x \in S_P(x) \cap S_Q(x)$. But $S_P(x) \cap S_Q(x) \in S_{P \cup Q}(x) \forall x \in U$. Therefore, $x \in S_{P \cup Q}(x)$, i.e., $\underline{P + Q}X \subseteq \underline{P \cup Q}X$.
2) From the classical rough-set model based on a tolerance relation, we know that $\overline{P \cup Q}X = \sim \underline{P \cup Q}(\sim X)$. According to the result of 1), one can obtain that $\overline{P \cup Q}X = \sim \underline{P \cup Q}(\sim X) \subseteq \sim \underline{P + Q}(\sim X) = \overline{P + Q}X$.

Corollary 1: $Bn_P(X) \subseteq Bn_{P+Q}(X)$ and $Bn_Q(X) \subseteq Bn_{P+Q}(X)$.

From the definition of set approximations under two granulation spaces, one can get the following properties of the lower approximation and the upper approximation.

Theorem 2: Let $S = (U, AT, f)$ be an incomplete IS and $X \subseteq U$ and $P, Q \subseteq AT$ be two-attribute subsets. Then, the following properties hold.

1) $\underline{P + Q}X \subseteq X \subseteq \overline{P + Q}X$.
2) $\underline{P + Q}\varnothing = \overline{P + Q}\varnothing = \varnothing$, $\underline{P + Q}U = \overline{P + Q}U = U$.
3) $\underline{P + Q}(\sim X) = \sim \overline{P + Q}X$, $\underline{P + Q}(\sim X) = \sim \overline{P + Q}X$.
4) $\underline{P + Q}X = \underline{P}X \cup \underline{Q}X$.
5) $\overline{P + Q}X = \overline{P}X \cap \overline{Q}X$.
6) $\underline{P + Q}X = \underline{Q + P}X$, $\overline{P + Q}X = \overline{Q + P}X$.

To establish the relationship between the approximation of a single set and that of two sets approximated through using two granulations, the following properties are given.

Theorem 3: Let $S = (U, AT, f)$ be an incomplete IS and $X, Y \subseteq U$ and $P, Q \subseteq AT$ be two-attribute subsets. Then, the following properties hold.

1) $\underline{P + Q}(X \cap Y) = (\underline{P}X \cap \underline{P}Y) \cup (\underline{Q}X \cap \underline{Q}Y)$.
2) $\overline{P + Q}(X \cup Y) = (\overline{P}X \cup \overline{P}Y) \cap (\overline{Q}X \cup \overline{Q}Y)$.
3) $\underline{P + Q}(X \cap Y) \subseteq \underline{P + Q}X \cap \underline{P + Q}Y$.
4) $\overline{P + Q}(X \cup Y) \supseteq \overline{P + Q}X \cup \overline{P + Q}Y$.
5) $X \subseteq Y \Rightarrow \underline{P + Q}X \subseteq \underline{P + Q}Y$.
6) $X \subseteq Y \Rightarrow \overline{P + Q}X \subseteq \overline{P + Q}Y$.
7) $\underline{P + Q}(X \cup Y) \supseteq \underline{P + Q}X \cup \underline{P + Q}Y$.
8) $\overline{P + Q}(X \cap Y) \subseteq \overline{P + Q}X \cap \overline{P + Q}Y$.

C. Incomplete MGRS on Multiple Granulation Spaces

Based on the earlier conclusions, we can then extend the rough-set model based on a single tolerance relation to a rough-set model based on multigranulations in the context of incomplete ISs.

Definition 2: Let $S = (U, AT, f)$ be an incomplete IS and $X \subseteq U$ and $P_1, P_2, \ldots, P_m \subseteq AT$ be m- attribute subsets. We define a lower approximation of X and an upper approximation of X with respect to $P_1, P_2, \ldots, P_m$ by the following:

$$\underline{\sum_{i=1}^{m}} P_i X = \bigcup \{x \mid S_{P_i}(x) \subseteq X, \qquad \text{for some } i \leq m\} \tag{5}$$

$$\overline{\sum_{i=1}^{m}} P_i X = \sim \underline{\sum_{i=1}^{m}} P_i(\sim X). \tag{6}$$

Similarly, the area of uncertainty or boundary region in incomplete MGRS can be represented as

$$Bn_{\underline{\sum_{i=1}^{m} P_i}}(X) = \overline{\sum_{i=1}^{m}} P_i X \Big\backslash \underline{\sum_{i=1}^{m}} P_i X.$$

From this definition, we obtain the following interpretations.

1) The lower approximation of a set X with respect to $\sum_{i=1}^{m} P_i$ is the set of all objects, which can be certainly classified as X using $\sum_{i=1}^{m} P_i$ (are certainly X in view of $\sum_{i=1}^{m} P_i$).
2) The upper approximation of a set X with respect to $\sum_{i=1}^{m} P_i$ is the set of all objects, which can be possibly classified as X using $\sum_{i=1}^{m} P_i$ (are possibly X in view of $\sum_{i=1}^{m} P_i$).
3) The boundary region of a set X with respect to $\sum_{i=1}^{m} P_i$ is the set of all objects, which can be classified neither as X nor as $\sim X$ using $\sum_{i=1}^{m} P_i$.

To apply this approach in practical issues, we present here an algorithm for computing a lower approximation of a set X in this rough-set model based on multiple tolerance relations.

Algorithm I: Let $S = (U, AT, f)$ be an incomplete IS and $X \subseteq U$ and $P \subseteq AT$, where $P = \{P_1, P_2, \ldots, P_m\}$.

This algorithm gives the lower approximation of X by P: $\underline{\sum_{i=1}^{m}} P_i X = \bigcup\{x | S_{P_i}(x) \subseteq X, i \leq m\}$.

We use the following pointers.

1) $i = 1, 2, \ldots, m$ points to P_i.
2) $j = 1, 2, \ldots, |U|$ points to $S_{P_i}(u_j) \in U/SIM(P_i)$.
3) L records the computation of the lower approximation.

For every i and j, we check whether $S_{P_i}(u_j) \cap X = S_{P_i}(u_j)$. If $S_{P_i}(u_j) \cap X = S_{P_i}(u_j)$, then we put u_j into the lower approximation of X: $L \leftarrow L \cup \{u_j\}$.

(I1) Compute m covers: $U/SIM(P_1)$, $U/SIM(P_2), \ldots, U/SIM(P_m)$;
(I2) Set $i \leftarrow 1$, $j \leftarrow 1$, $L = \varnothing$;
(I3) For $i = 1$ to m Do
  For $j = 1$ to $|U|$ Do
    If $S_{P_i}(u_j) \cap X = S_{P_i}(u_j)$, then
      let $L \leftarrow L \cup \{u_j\}$,
    Endif
  Endfor
  Set $j \leftarrow 1$,
Endfor
(I4) The computation of the lower approximation X by P is completed. Output the set L.

We know that the time complexity of computing m covers is $O(m|U|^2)$. The time complexity of (I3) is also $O(m|U|^2)$ as there are $\sum_{i=1}^{m} |P_i|$ intersections $Y_i^j \cap X$ $(\leq |U| \times |U|)$ to be calculated. Hence, the time complexity of Algorithm I is $\mathrm{O}(m|U|^2)$.

This algorithm can be run in parallel mode to compute concurrently all corresponding covers and intersections from many attributes. Its time complexity will be $\mathrm{O}(|U|^2)$. Like this idea, the algorithm for computing the upper approximation of a set can also be designed correspondingly.

Directly from Definition 2, one can obtain the following properties of the lower approximation and the upper approximation in incomplete MGRS.

Theorem 4: Let $S = (U, AT, f)$ be an incomplete IS and $X \subseteq U$ and $P_1, P_2, \ldots, P_m \subseteq AT$ be m-attribute subsets. Then, the following properties hold.

1) $\underline{\sum_{i=1}^m P_i}X = \bigcup_{i=1}^m \underline{P_i}X$.
2) $\overline{\sum_{i=1}^m P_i}X = \bigcap_{i=1}^m \overline{P_i}X$.
3) $\overline{\sum_{i=1}^m P_i}(\sim X) =\sim \underline{\sum_{i=1}^m P_i}X$.
4) $\underline{\sum_{i=1}^m P_i}(\sim X) =\sim \overline{\sum_{i=1}^m P_i}X$.

Proof:

1) From (4) in Theorem 2, it can be easily proved.
2) From (3) and 1) in this theorem, one can have

$$\overline{\sum_{i=1}^m P_i}X = \sim \underline{\sum_{i=1}^m P_i}(\sim X) =\sim \bigcup_{i=1}^m \underline{P_i}(\sim X)$$
$$= \sim \bigcup_{i=1}^m (\sim \overline{P_i}X) = \bigcap_{i=1}^m \overline{P_i}X.$$

3) From (3), it is straightforward.
4) Let $X =\sim X$ in (3). Then, we have $\underline{\sum_{i=1}^m P_i}(\sim X) =\sim \overline{\sum_{i=1}^m P_i}X$.

Theorem 5: Let $S = (U, AT, f)$ be an incomplete IS and $X_1, X_2, \ldots, X_n \subseteq U$ be n subsets on U and $P_1, P_2, \ldots, P_m \subseteq AT$ be m-attribute subsets. Then, the following properties hold.

1) $\underline{\sum_{i=1}^m P_i}(\bigcap_{j=1}^n X_j) = \bigcup_{i=1}^m (\bigcap_{j=1}^n \underline{P_i}X_j)$.
2) $\overline{\sum_{i=1}^m P_i}(\bigcup_{j=1}^n X_j) = \bigcap_{i=1}^m (\bigcup_{j=1}^n \overline{P_i}X_j)$.
3) $\underline{\sum_{i=1}^m P_i}(\bigcap_{j=1}^n X_j) \subseteq \bigcap_{j=1}^n (\underline{\sum_{i=1}^m P_i}X_j)$.
4) $\overline{\sum_{i=1}^m P_i}(\bigcup_{j=1}^n X_j) \supseteq \bigcup_{j=1}^n (\overline{\sum_{i=1}^m P_i}X_j)$.
5) $\underline{\sum_{i=1}^m P_i}(\bigcup_{j=1}^n X_j) \supseteq \bigcup_{j=1}^n (\underline{\sum_{i=1}^m P_i}X_j)$.
6) $\overline{\sum_{i=1}^m P_i}(\bigcap_{j=1}^n X_j) \subseteq \bigcap_{j=1}^n (\overline{\sum_{i=1}^m P_i}X_j)$.

Theorem 6: Let $S = (U, AT, f)$ be an incomplete IS and $X_1, X_2, \ldots, X_n \subseteq U$ with $X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n$ be n subsets on U and $P_1, P_2, \ldots, P_m \subseteq AT$ be m-attribute subsets. Then, the following properties hold.

1) $\underline{\sum_{i=1}^m P_i}X_1 \subseteq \underline{\sum_{i=1}^m P_i}X_2 \subseteq \cdots \subseteq \underline{\sum_{i=1}^m P_i}X_n$.
2) $\overline{\sum_{i=1}^m P_i}X_1 \subseteq \overline{\sum_{i=1}^m P_i}X_2 \subseteq \cdots \subseteq \overline{\sum_{i=1}^m P_i}X_n$.

Proof: Suppose $1 \le i \le j \le n$, then $X_i \subseteq X_j$.

1) Clearly, $X_i \cap X_j = X_i$. Hence, it follows from 3) in Theorem 5 that

$$\underline{\sum_{i=1}^m P_i}X_i = \underline{\sum_{i=1}^m P_i}(X_i \cap X_j) \subseteq \underline{\sum_{i=1}^m P_i}X_i \cap \underline{\sum_{i=1}^m P_i}X_j.$$

Thus, $\underline{\sum_{i=1}^m P_i}X_i = \underline{\sum_{i=1}^m P_i}X_i \cap \underline{\sum_{i=1}^m P_i}X_j$. Therefore, we have that $\underline{\sum_{i=1}^m P_i}X_i \subseteq \underline{\sum_{i=1}^m P_i}X_j$. Therefore, it follows that $\underline{\sum_{i=1}^m P_i}X_1 \subseteq \underline{\sum_{i=1}^m P_i}X_2 \subseteq \cdots \subseteq \underline{\sum_{i=1}^m P_i}X_n$.

2) Clearly, $X_i \cup X_j = X_j$. Hence, it follows from 4) in Theorem 5 that

$$\overline{\sum_{i=1}^m P_i}X_j = \overline{\sum_{i=1}^m P_i}(X_i \cup X_j) \supseteq \overline{\sum_{i=1}^m P_i}X_i \cup \overline{\sum_{i=1}^m P_i}X_j.$$

Thus, $\overline{\sum_{i=1}^m P_i}X_j = \overline{\sum_{i=1}^m P_i}X_i \cup \overline{\sum_{i=1}^m P_i}X_j$. Therefore, we have that $\overline{\sum_{i=1}^m P_i}X_i \subseteq \overline{\sum_{i=1}^m P_i}X_j$. Therefore, it follows that $\overline{\sum_{i=1}^m P_i}X_1 \subseteq \overline{\sum_{i=1}^m P_i}X_2 \subseteq \cdots \subseteq \overline{\sum_{i=1}^m P_i}X_n$.

Theorem 7: Let $S = (U, AT, f)$ be an incomplete IS and $X \subseteq U$ and $P = \{P_1, P_2, \ldots, P_m\}$ with $P_1 \preceq P_2 \preceq \cdots \preceq P_m \forall P_i \subseteq A$ $(i \le m)$. Then, the following properties hold.

1) $\underline{\sum_{i=1}^m P_i}X = \underline{P_1}X$.
2) $\overline{\sum_{i=1}^m P_i}X = \overline{P_1}X$.

Proof: Suppose $1 \le j \le k \le m$ with $P_j \preceq P_k$. From the definition of $\preceq$, we know that for any $S_{P_j}(x) \in U/SIM(P_j)$, there exits $S_{P_k}(x) \in U/SIM(P_k)$ such that $S_{P_j}(x) \subseteq S_{P_k}(x)$. Therefore, one can obtain that

$$\underline{P_j}X = \left\{x \,|\, S_{P_j}(x) \subseteq X\right\} \supseteq \underline{P_k}X = \left\{x \,|\, S_{P_k}(x) \subseteq X\right\}$$

i.e., $\underline{P_j + P_k}X = \underline{P_j}X \cup \underline{P_k}X = \underline{P_j}X$. Since $P_1 \preceq P_2 \preceq \cdots \preceq P_m$, one can get that $\underline{\sum_{i=1}^m P_i}X = \bigcup_{i=1}^m \underline{P_i}X = \underline{P_1}X$.

Similarly, we also have that

$$\overline{P_j}X = \left\{x \,|\, S_{P_j}(x) \cap X \neq \varnothing\right\}$$
$$\supseteq \overline{P_k}X = \left\{x \,|\, S_{P_k}(x) \cap X \neq \varnothing\right\}$$

i.e., $\overline{P_j + P_k}X = \overline{P_j}X \cap \overline{P_k}X = \overline{P_j}X$. Hence, one can obtain that $\overline{\sum_{i=1}^m P_i}X = \bigcap_{i=1}^m \overline{P_i}X = \overline{P_1}X$.

D. Several Elementary Measures in Incomplete MGRS

In this section, we investigate several elementary measures in incomplete MGRS and their properties.

Uncertainty of a set (category) is due to the existence of a borderline region. The bigger the borderline region of a set is, the lower the accuracy of the set is (and vice versa). To more precisely express this idea, we introduce the accuracy measure to incomplete MGRS as follows.

Definition 3: Let $S = (U, AT, f)$ be an incomplete IS and $X \subseteq U$ and $P = \{P_1, P_2, \ldots, P_m\} \forall P_i \subseteq AT$. Approximation measure of X by P is defined as

$$\alpha_P(X) = \frac{\left|\underline{\sum_{i=1}^m P_i}X\right|}{\left|\overline{\sum_{i=1}^m P_i}X\right|} \tag{7}$$

where $X \neq \varnothing$, $|X|$ denotes the cardinality of set X.

From this definition, one can derive the following theorem.

Theorem 8: Let $S = (U, AT, f)$ be an incomplete IS and $X \subseteq U$, $P = \{P_1, P_2, \ldots, P_m\} \forall P_i \subseteq AT$, and $P' \subseteq P$ be a subset of P. Then

$$\alpha_P(X) \ge \alpha_{P'}(X) \ge \alpha_{P_i}(X), \qquad (i \le m).$$

TABLE II
DATA SETS DESCRIPTION

Data sets	Samples	Numerical features	Symbolic features	Decision classes
breast-cancer-wisconsin	699	0	9	2
audiology	200	0	69	24
hepatitis	155	6	13	2
crx	690	6	9	2

Proof: Since $P' \subseteq P$, it follows from Definition 2 that

$$\bigcup_{i=1}^{m} \underline{P_i}X \supseteq \bigcup_{P_i \in P, P_i \notin P'} \underline{P_i}X$$

$$\bigcap_{i=1}^{m} \overline{P_i}X \subseteq \bigcap_{P_i \in P, P_i \notin P'} \overline{P_i}X.$$

Then, it is clear that $|\bigcup_{i=1}^{m} \underline{P_i}X| \geq |\bigcup_{P_i \in P, P_i \notin P'} \underline{P_i}X|$ and $|\bigcap_{i=1}^{m} \overline{P_i}X| \leq |\bigcap_{P_i \in P, P_i \notin P'} \overline{P_i}X|$. Hence

$$\begin{aligned}\alpha_P(X) &= \frac{\left|\underline{\sum_{i=1}^{m} P_i}X\right|}{\left|\overline{\sum_{i=1}^{m} P_i}X\right|} = \frac{\left|\bigcup_{i=1}^{m} \underline{P_i}X\right||}{\left|\bigcap_{i=1}^{m} \overline{P_i}X\right|} \\ &\geq \frac{\left|\bigcup_{P_i \in P, P_i \notin P'} \underline{P_i}X\right|}{\left|\bigcap_{P_i \in P, P_i \notin P'} \overline{P_i}X\right|} = \frac{\left|\underline{\sum_{P_i \in P, P_i \notin P'} P_i}X\right|}{\left|\overline{\sum_{P_i \in P, P_i \notin P'} P_i}X\right|} \\ &= \alpha_{P'}(X).\end{aligned}$$

Similarly, we have $\alpha_{P'}(X) \geq \alpha_{P_i}(X)$ $(i \leq m)$. Thus, $\alpha_P(X) \geq \alpha_{P'}(X) \geq \alpha_{P_i}(X)$ $(i \leq m)$.

Theorem 8 shows that the approximation measure of a target concept enlarges as the number of granulations for describing the concept increases.

Example 3 (Continued From Example 2): Suppose $A = \{L, P\}$. Computing the approximation measures, it follows that

$$\begin{aligned}\alpha_A(X) &= \frac{|\underline{L+P}X|}{|\overline{L+P}X|} = \frac{1}{4} \\ \alpha_L(X) &= \frac{|\underline{L}X|}{|\overline{L}X|} = \frac{1}{8} \\ \alpha_P(X) &= \frac{|\underline{P}X|}{|\overline{P}X|} = \frac{1}{8}.\end{aligned}$$

Clearly, it follows from the earlier computations that $\alpha_A(X) > \alpha_L(X)$ and $\alpha_A(X) > \alpha_P(X)$.

In particular, $\alpha_A(X) > \alpha_L(X)$ if $L \preceq P$, which can be easily derived from Theorem 7.

Note that the approximation measure of a target concept approximated by using multiple granulations is always much better than that approximated by using a single granulation, which is suitable for more precisely characterizing a target concept and problem solving according to user requirements.

Definition 4: Let $S = (U, AT, f, D, g)$ be an incomplete target IS, $U/IND(D)$ be a decision induced by the decision attributes D, and $P = \{P_1, P_2, \ldots, P_m\}$ be m-attribute sets. Approximation quality of D by P, also called a degree of dependence, is defined as

$$\gamma(P, D) = \frac{\sum \left\{ \left|\underline{\sum_{i=1}^{m} P_i}Y\right| : Y \in U/IND(D) \right\}}{|U|}. \quad (8)$$

This measure can be used to evaluate the deterministic part of the rough-set description of $U/IND(D)$ by counting those objects which can be reclassified as blocks of $U/IND(D)$ with the knowledge given by $\sum_{i=1}^{m} P_i$. As a result of the earlier definition, we come to the following two theorems.

Theorem 9: Let $P = \{P_1, P_2, \ldots, P_m\}$ be m-attribute sets and D_1, D_2 with $D_1 \preceq D_2$ be two decisions, then $\gamma(\sum_{i=1}^{m} P_i, D_1) \leq \gamma(\sum_{i=1}^{m} P_i, D_2)$.

Theorem 10: Let $P = \{P_1, P_2, \ldots, P_m\}$ be m-attribute sets and D be a decision. If $P' \subseteq P$, then $\gamma(\sum_{i=1}^{m} P_i, D) \geq \gamma(\sum_{P_i \in P'} P_i, D) \geq \gamma(P_i, D)$.

Gediga and Düntsch [6] introduced a simple statistic $\pi(R, X) = |\underline{R}X|/|X|$ for the precision of (deterministic) approximation of $X \subseteq U$ given $U/IND(R)$, which is not affected by the approximation of $\sim X$. This is just the relative number of elements in X which can be approximated by R. Clearly, $\pi(R, X) \geq \alpha(R, X)$. It is important to point out that $\pi(R, X)$ requires complete knowledge of X, whereas α does not, since the latter uses only the rough set $(\underline{R}X, \overline{R}X)$. In incomplete MGRS, it can be extended to be the formula

$$\pi\left(\sum_{i=1}^{m} P_i, X\right) = \frac{\left|\underline{\sum_{i=1}^{m} P_i}X\right|}{|X|}. \quad (9)$$

It is clear that $\pi(\sum_{i=1}^{m} P_i, X) \geq \alpha(\sum_{i=1}^{m} P_i, X)$. In fact, this measure denotes the relative number of objects in X which can be approximated by $\sum_{i=1}^{m} P_i$. Then, the following theorem can be easily proved.

Theorem 11: Let $P = \{P_1, P_2, \ldots, P_m\}$ be m-attribute sets and X be a target concept. If $P' \subseteq P$, then $\pi(\sum_{i=1}^{m} P_i, X) \geq \pi(\sum_{P_i \in P'} P_i, X) \geq \pi(P_i, X)$.

E. Experimental Analysis

In the following, through experimental analyses, we illustrate the difference between the incomplete MGRS and Kryszkiewicz's rough-set model. We have downloaded four public data sets (incomplete target ISs) from UCI Repository of machine learning databases, which are described in Table II.

Here, we compare the degree of dependence in incomplete MGRS with that in Kryszkiewicz's rough-set model on these two practical data sets. The comparisons of values of two measures with the numbers of features in these two data sets are shown in Figs. 4–7.

In Figs. 4–7, the term MGRS is the incomplete MGRS framework proposed in this correspondence paper, and the term SGRS is Kryszkiewicz's rough-set model. It is shown in Figs. 4–7 that the value of the degree of dependence in incomplete MGRS is not bigger than that in Kryszkiewicz's rough-set model for the same number of selected features, and this value increases as the number of selected features does in the same data set. In particular, from Fig. 6, it is easy to see that the values of the degree of dependence in incomplete MGRS are equal to zero when the number of features falls in between one and five. In this situation, the lower approximation of the target decision equals an empty set in the incomplete decision table. In essence, it is because that the tolerance classes induced by a singleton attribute are all coarser than those induced by all attributes. One can draw the same conclusion from the other figures. Further illustrations and applications are shown in Section V.

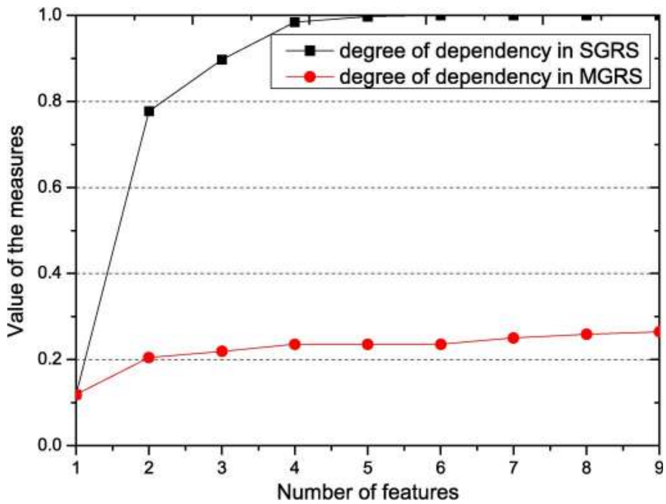

Fig. 4. Variation of the 2° of dependence with the numbers of features (data set breast-cancer-wisconsin).

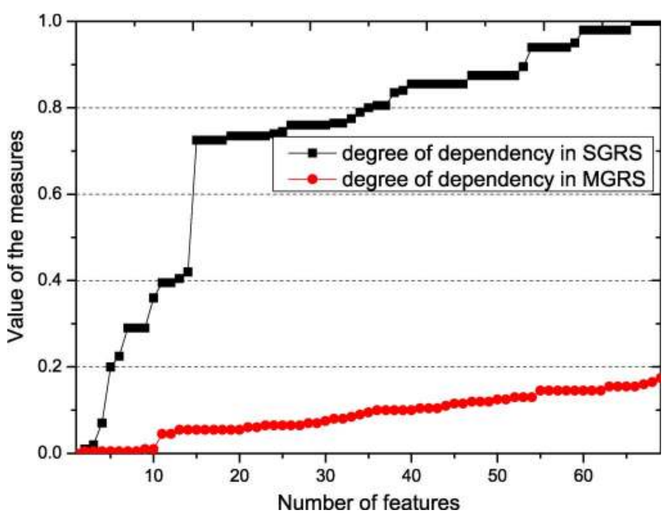

Fig. 5. Variation of the 2° of dependence with the numbers of features (data set audiology).

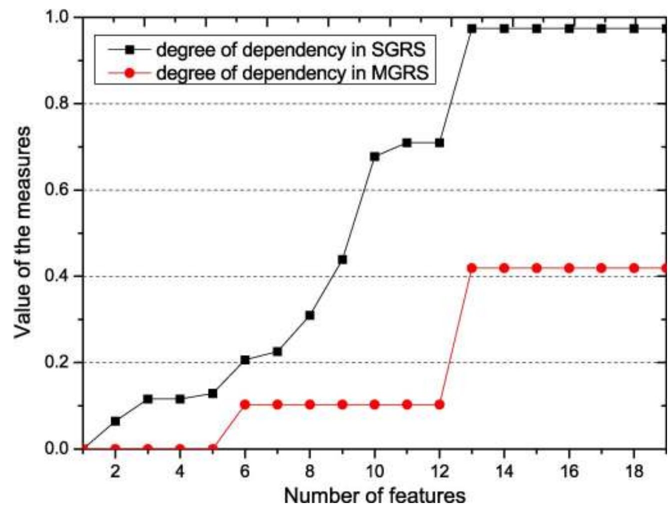

Fig. 6. Variation of the 2° of dependence with the numbers of features (data set hepatitis).

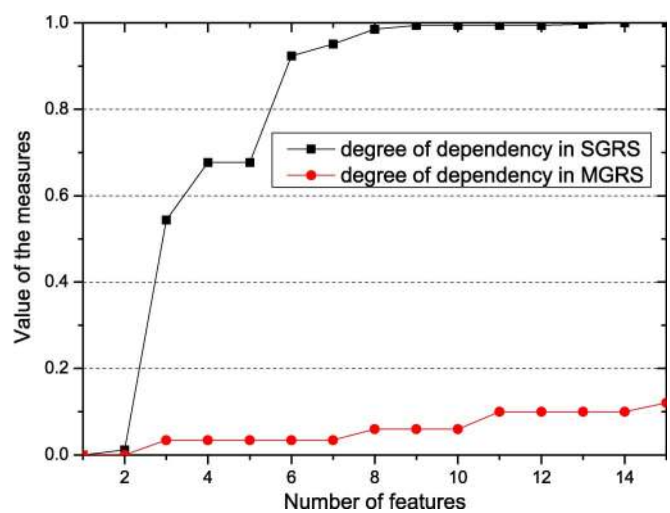

Fig. 7. Variation of the 2° of dependence with the numbers of features (data set crx).

IV. Attribute Reduction in Incomplete MGRS

Reduct is a minimal attribute subset of the original data which is independent and has the same discernibility power as all of the attributes in the classical rough-set framework. Obviously, reduction is a feature-subset selection process, where the selected feature subset not only retains the representation power but also has the minimal redundancy [22], [23]. In this section, we deal with attribute reduction in incomplete MGRS.

We first introduce the notions of two approximation distribution functions. Let $S = (U, AT, f, D, g)$ be an incomplete target IS, $P \subseteq AT$, and the decision $U/IND(D) = \{Y_1, Y_2, \cdots, Y_r\}$. Lower approximation distribution function and upper approximation function are defined as

$$\underline{D}_P = \left(\underline{\sum_{P_i \in P} P_i Y_1}, \underline{\sum_{P_i \in P} P_i Y_2}, \ldots, \underline{\sum_{P_i \in P} P_i Y_r}\right)$$

$$\overline{D}_P = \left(\overline{\sum_{P_i \in P} P_i Y_1}, \overline{\sum_{P_i \in P} P_i Y_2}, \ldots, \overline{\sum_{P_i \in P} P_i Y_r}\right).$$

Through using these two approximation distribution functions, three new reducts can be defined in the following, which are lower approximation reduct, upper approximation reduct, and approximation reduct.

Definition 5: Let $S = (U, AT, f, D, g)$ be an incomplete target IS and P be a nonempty subset of AT.

1) If $\underline{D}_P = \underline{D}_{AT}$, we say that P is a *lower approximation consistent set* of S. If P is a lower approximation consistent set and no proper subset of P is lower approximation consistent, then P is called a *lower approximation reduct* of S.
2) If $\overline{D}_P = \overline{D}_{AT}$, we say that P is an *upper approximation consistent set* of S. If P is an upper approximation consistent set and no proper subset of P is upper approximation consistent, then P is called an *upper approximation reduct* of S.
3) If P is not only a lower approximation reduct but also an upper approximation reduct, then P is called an *approximation reduct* of S.

It is easy to prove that an upper approximation consistent set must be a lower approximation consistent set. However, the converse relationship cannot be satisfied in an inconsistent incomplete target IS. From the earlier definition, it is clear that P is a lower approximation consistent set if and only if P is an upper approximation consistent set

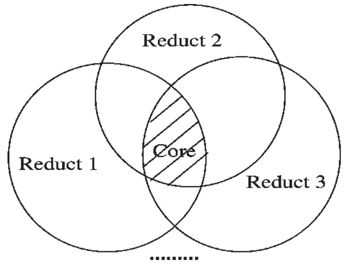

Fig. 8. Relationship between approximation reducts and approximation core.

in a consistent incomplete target IS. In particular, if $U/IND(D) = X$, we can regard P as a lower approximation reduct, an upper approximation reduct, and an approximation reduct of a target concept X, respectively.

Let $\mathbf{A}$ be the set of all lower approximation reducts and $\mathbf{B}$ be the set of all upper approximation reducts. Then, it is obvious that the set of all approximation reducts $\mathbf{C} = \mathbf{A} \cap \mathbf{B}$.

Suppose that $S = (U, AT, f, D, g)$ be an incomplete target IS, where $U = \{e_1, e_2, \ldots, e_{|U|}\}$, $AT = \{P_1, P_2, \ldots, P_{|AT|}\}$, and $U/IND(D) = \{Y_1, Y_2, \ldots, Y_r\}$. We denote all lower approximation reducts of $Y \in U/IND(D)$ by $\mathbf{A}(Y)$, all upper approximation reducts of $Y \in U/IND(D)$ by $\mathbf{B}(Y)$, and all approximation reducts of $Y \in U/IND(D)$ by $\mathbf{C}(Y)$, respectively. Moreover, we call $Core(\mathbf{A}(Y))$ the lower approximation core of Y, $Core(\mathbf{B}(Y))$ the upper approximation core of Y, and $Core(\mathbf{C}(Y))$ the approximation core of Y, respectively.

Theorem 12: Let $S = (U, AT, f, D, g)$ be an incomplete target IS and $U/IND(D) = \{Y_1, Y_2, \ldots, Y_r\}$. Then

$$\mathbf{A} = \bigcap_{k=1}^{r} \mathbf{A}(Y_k) \quad \mathbf{B} = \bigcap_{k=1}^{r} \mathbf{B}(Y_k).$$

We call $Core(\mathbf{A}) = \bigcap \mathbf{A}_i(\mathbf{A}_i \in \mathbf{A})$, $Core(\mathbf{B}) = \bigcap \mathbf{B}_i(\mathbf{B}_i \in \mathbf{B})$, and $Core(S) = \bigcap \mathbf{C}_i(\mathbf{C}_i \in \mathbf{C})$ as the lower approximation core, the upper approximation core, and the core of an incomplete target IS S, respectively.

Theorem 13: Let $S = (U, AT, f, D, g)$ be an incomplete target IS and $U/IND(D) = \{Y_1, Y_2, \ldots, Y_r\}$. Then

$$Core(\mathbf{A}) = \bigcap_{k=1}^{r} Core\left(\mathbf{A}(Y_k)\right)$$

$$Core(\mathbf{B}) = \bigcap_{k=1}^{r} Core\left(\mathbf{B}(Y_k)\right).$$

Clearly, $Core(S) = Core(\mathbf{A}) \cap Core(\mathbf{B})$. In fact, the core is indispensable to construct an approximation reduct. Fig. 8 shows the relationship between the approximation reducts and the approximation core of a target IS.

Now, we consider how to find attribute reducts from an incomplete target IS in the framework of incomplete MGRS. Let $S = (U, AT, f, d, g)$ be an incomplete target IS, where $U = \{e_1, e_2, \ldots, e_U\}$, $AT = \{P_1, P_2, \ldots, P_{AT}\}$, and $U/IND(\{d\}) = \{Y_1, Y_2, \ldots, Y_r\}$. As follows, in the framework of incomplete MGRS, we design an algorithm for computing all lower approximation reducts, i.e., all subsets $AT_0 : AT_{01}, AT_{02}, \ldots, AT_{0s}$ of AT such that we have as follows: 1) $\underline{d}_{AT_0} = \underline{d}_{AT}$ and 2) if $AT' \subset AT_0$, then $\underline{d}_{AT_0} \neq \underline{d}_{AT}$.

Algorithm II: This algorithm gives all lower approximation reducts of the incomplete target IS S (similar to the idea in [7, Algorithm K]).

Let us denote the binomial coefficients by $C^k_{|AT|} = |AT|!/k!(|AT| - k)!$.

1) Let us denote $C^1_{|AT|} = |AT|$ one-attribute subsets by

$$AT_{11} = \{P_1\}$$
$$AT_{12} = \{P_2\}, \ldots, AT_{1j} = \{P_j\}, \ldots, AT_{1C^1_{|AT|}} = \left\{P_{|AT|}\right\}.$$

2) Let us denote $C^2_{|AT|} = |AT|(|AT| - 1)/2!$ two-attribute subsets by

$$AT_{21} = \{P_1, P_2\}, \ldots, AT_{2j} = \{P_1, P_j\}, \ldots, AT_{2C^2_{|AT|}} = \left\{P_{|AT|-1}, P_{|AT|}\right\}.$$

3) Generally, let us denote $C^k_{|AT|} = |AT|!/k!(|AT| - k)!$ k-attribute subsets by

$$AT_{k1} = \{P_1, P_2, \ldots, P_k\}, \ldots, AT_{kj}, \ldots, AT_{kC^k_{|AT|}} = \left\{P_{|AT|-k+1}, \ldots, P_{|AT|-1}, P_{|AT|}\right\}.$$

4) Notice that $C^{|AT|}_{|AT|} = 1$ and the $|AT|$-attribute subset is $AT_{|AT|1} = AT$.

The algorithm is to search subsets of AT as follows: singletons, two-attribute subsets, …, t-attribute subsets, and so on. Continue up to the unique $|AT|$-attribute subset AT itself.

We use the following variables.

1) s—The number of the lower approximation reducts we have already found.
2) t—Counting from 1 to s.
3) k—We are currently searching k-attribute subset AT_{kj}.
4) j—We are currently searching the jth subset AT_{kj} in all k-attribute subsets $AT_{k1}, \ldots, AT_{kj}, \ldots, AT_{kC^k_{|AT|}}$.

(II1) Set $j \leftarrow 1$, $s \leftarrow 0$, $k \leftarrow 1$;
(II2) While $k \leq |AT|$ Do
  $j \leftarrow 1$;
  While $j \leq C^k_{|AT|}$ Do
    for $t = 1$ to s Do
      If $AT_{0t} \subset AT_{kj}$, then break;
      Endif
    Endfor
    if $\underline{d}_{AT_{kj}} = \underline{d}_{AT}$, then
      $s \leftarrow s + 1$, $AT_{0s} \leftarrow AT_{kj}$;
    Endif
    $j \leftarrow j + 1$;
  Endwhile
  $k \leftarrow k + 1$;
Endwhile
(II3) Output $AT_{01}, AT_{02}, \ldots, AT_{0s}$ (s lower approximation reducts).

The time complexity of this algorithm for all lower approximation reducts is exponential since it checks all subsets in 2^{AT}, and $|2^{AT}| = 2^{|AT|}$. We know that the time complexity of computing $|AT|$ covers is $O(|AT||U|^2)$, and the time complexity of computing a lower approximation of every $Y \in U/IND(\{d\})$ by AT_{kj} ($k \leq |AT|$) is $O(|AT||U|^3)$. Thus, the time complexity of Algorithm II is

$$2^{|AT|} \times O\left(|AT||U|^2 + |AT||U|^3\right) = O\left(2^{|AT|}|AT||U|^3\right).$$

Through Algorithm II, one can obtain that the attribute set $\{L, P\}$ is only one lower approximation reduct of Table I. However, the time complexity of Algorithm II is exponential so that it cannot be applied efficiently in practical applications. To reduce the time complexity of computing an approximation reduct, we introduce, in the following, two heuristic functions, which are an importance measure of lower approximation and an importance measure of upper approximation.

Let $S = (U, AT, f, D, g)$ be an incomplete target IS and P be a nonempty subset of AT. Given a condition attribute $a \in P$ and $Y \in U/IND(D)$, we first give two preliminary definitions in the following, which will be helpful for constructing heuristic functions.

Definition 6: We say that a is lower approximation significant in P with respect to X if

$$\underline{\sum_{i=1}^{|P|} P_i X} \supset \underline{\sum_{i=1, P_i \neq a}^{|P|} P_i X}, \qquad (P_i \in P)$$

and a is not lower approximation significant in P with respect to X if

$$\underline{\sum_{i=1}^{|P|} P_i X} = \underline{\sum_{i=1, P_i \neq a}^{|P|} P_i X}, \qquad (P_i \in P)$$

where $|P|$ is the cardinality of attribute set P.

Definition 6 shows that if a is lower approximation significant with respect to X, then the lower approximation of X will become smaller; if a is not lower approximation significant with respect to X, then the lower approximation of X will be keep unchanged.

Definition 7: We say that a is upper approximation significant in P with respect to X if

$$\overline{\sum_{i=1}^{|P|} P_i X} \subset \overline{\sum_{i=1, P_i \neq a}^{|P|} P_i X}, \qquad (P_i \in P)$$

and a is not lower approximation significant in P with respect to X if

$$\overline{\sum_{i=1}^{|P|} P_i X} = \overline{\sum_{i=1, P_i \neq a}^{|P|} P_i X}, \qquad (P_i \in P)$$

where $|P|$ is the cardinality of attribute set P.

Analogously to Definition 6, Definition 7 states that if a is upper approximation significant with respect to X, then the upper approximation of X will become bigger; if a is not upper approximation significant with respect to X, then the upper approximation of X will be keep unchanged.

Through these two definitions, one can easily construct two heuristic functions. An important measure of lower approximation of $P \subseteq AT$ with respect to D is defined as

$$S_P(D) = \frac{\sum_{Y \in U/D} \left| \underline{\sum_{i=1}^{m} P_i Y} \setminus \underline{\sum_{i=1, P_i \notin P}^{m} P_i Y} \right|}{|U|} \tag{10}$$

and an important measure of upper approximation of $P \subseteq AT$ with respect to D is defined as

$$S^P(D) = \frac{\sum_{Y \in U/D} \left| \overline{\sum_{i=1, P_i \notin P}^{m} P_i Y} \setminus \overline{\sum_{i=1}^{m} P_i Y} \right|}{|U|}. \tag{11}$$

In particular, when $P = \{a\}$, $S_a(D)$ and $S^a(D)$ can be regard as the importance measure of lower approximation and the importance measure of upper approximation of the attribute $a \in AT$ with respect to D, respectively.

From Algorithm I, we know that the time complexity of computing the lower approximation of Y by $P = \{P_1, P_2, \ldots, P_m\}$ is $O(m|U|^2)$. For computing the measure of importance of a, we need to calculate the lower approximations for at most $|U|$ times. Therefore, the time complexity of computing an importance measure of lower/upper approximation of an attribute with respect to D is $O(m|U|^3)$.

From the earlier denotations, we come to the following conclusions.

1) $S_P(D) \geq 0$ and $S^P(D) \geq 0$.
2) P is not lower approximation significant with respect to D if and only if $S_P(D) = 0$.
3) P is not upper approximation significant with respect to D if and only if $S^P(D) = 0$.

As follows, we provide a heuristic algorithm based on the importance measure of lower approximation of a condition attribute with respect to the decision attribute d to find a lower approximation reduct in an incomplete target IS.

Algorithm III: Let $S = (U, AT, f, d, g)$ be a complete target IS, where $U = \{e_1, e_2, \ldots, e_{|U|}\}$, $AT = \{P_1, P_2, \ldots, P_{|AT|}\}$, and $U/IND(\{d\}) = \{Y_1, Y_2, \ldots, Y_r\}$. This algorithm finds a lower approximation reduct through using a heuristic.

The following variables will be used in the algorithm.

1) AT_0—To record a lower approximation reduct.
2) i—We are currently searching the *i*th condition attribute AT_i' in the sequence given.

(III1) Compute $|AT|$ covers and a decision partition $U/IND(\{d\})$;
(III2) Sort $AT = \{P_1', P_2', \ldots, P_{|AT|}'\}$, where $S_{P_i'}(d) \geq S_{P_{i+1}'}(d)$;
(III3) Set $i \leftarrow 1$, $AT_0 = \varnothing$.
(III4) If $\underline{d}_{AT_0} \neq \underline{d}_{AT}$, then
$AT_0 \leftarrow AT_0 \cup P_i'$,
$i \leftarrow i + 1$;
(III5) Found a lower approximation reduct: AT_0. Output the set AT_0.

The time complexity of this algorithm for computing $|AT|$ covers, and a decision partition $U/IND(\{d\})$ is $O(|AT||U|^2)$. The time complexity of computing $|AT|$ importance measures is $O(|AT||U|^3)$, and the time complexity of sorting is $O(|AT| \log_2 |AT|)$. Moreover, the time complexity for running $|AT|$ comparisons $\underline{d}_{AT_0} = \underline{d}_{AT}$ is $O(|AT||U|^3)$. Thus, the time complexity of Algorithm III is

$$O\left(|AT||U|^2 + |AT||U|^3 + |AT| \log_2 |AT| + |AT||U|^3\right) = O\left(|AT||U|^3\right).$$

Through Algorithm III, a lower approximation reduct can be found, which keeps the lower approximation distribution function of this incomplete target IS. Analogously, we can design a heuristic algorithm to find an upper approximation reduct of an incomplete target IS through using a heuristic function $S^P(D)$.

V. Application to Venture Investment

Venture capital has become an increasingly important source of financing for new companies, particularly when such companies are operating on the frontier of emerging technologies and markets. It plays an essential role in the entrepreneurial process. For an investor or decision maker, he may need to adopt a better one from some possible investment projects or find some directions from existing successful investment projects before investing. The purpose of this section is, through a venture-investment issue, to illustrate the mechanism of incomplete MGRS and its applications.

TABLE III
INCOMPLETE EVALUATION TABLE ABOUT VENTURE INVESTMENT

U	E_1	E_2	E_3	E_4	E_5	D
x_1	2	3	3	2	3	High
x_2	1	3	3	2	2	High
x_3	1	1	1	1	1	Low
x_4	1	1	1	1	1	Low
x_5	1	1	1	1	1	Low
x_6	$*$	2	1	2	2	High
x_7	2	2	2	2	2	Low
x_8	3	2	2	3	3	High
x_9	2	3	2	3	1	High
x_{10}	1	1	$*$	1	2	Low
x_{11}	3	2	3	3	3	High
x_{12}	2	3	3	2	$*$	High
x_{13}	1	2	3	1	2	Low
x_{14}	$*$	3	1	3	3	High
x_{15}	2	1	$*$	1	1	Low
x_{16}	2	2	2	2	1	Low
x_{17}	2	1	2	2	2	Low
x_{18}	1	1	3	1	2	Low
x_{19}	3	3	3	3	3	High
x_{20}	2	2	1	1	2	High
x_{21}	2	2	$*$	1	1	Low
x_{22}	1	3	1	1	2	High
x_{23}	$*$	2	2	2	$*$	High
x_{24}	1	1	2	2	1	Low
x_{25}	1	1	2	2	2	Low
x_{26}	1	3	1	2	2	High
x_{27}	1	2	3	1	2	Low
x_{28}	2	3	3	1	1	High
x_{29}	2	3	3	2	1	High
x_{30}	2	2	1	1	1	High
x_{31}	2	3	1	1	1	High
x_{32}	2	1	3	2	2	Low
x_{33}	3	2	3	3	2	High
x_{34}	1	1	3	1	1	Low
x_{35}	2	2	2	1	1	Low
x_{36}	2	2	2	1	1	Low
x_{37}	2	3	$*$	2	2	High
x_{38}	3	3	1	3	3	High
x_{39}	1	1	1	1	2	Low
x_{40}	3	3	1	3	3	High
x_{41}	1	2	2	1	1	Low
x_{42}	1	1	2	2	2	Low
x_{43}	1	2	3	2	2	Low
x_{44}	2	3	3	2	$*$	High
x_{45}	3	3	2	3	3	High
x_{46}	3	2	2	3	3	High
x_{47}	2	1	1	1	1	Low
x_{48}	2	1	1	1	2	Low
x_{49}	3	3	2	3	3	High
x_{50}	1	1	2	1	2	Low

Let us consider a real investment issue of a venture-investment company (here, we conceal the company's name and the details of investment projects). There are 50 investment projects x_i $(i = 1, 2, \ldots, 50)$ to be considered, which are evaluated by five evaluation experts. Venture level is classified to three: classes 1, 2, and 3. The bigger the value of venture level is, the higher the venture of investment project is. Table III is an incomplete evaluation table about venture investment given by these five experts, in which the symbol "$*$" means that an expert cannot decide the venture level of a project. In the evaluation process, each of the evaluation experts makes a decision independently, i.e., one does not perform intersection operations between any two evaluation results. For this situation, the classical Kryszkiewicz's rough-set model will be helpless. In the following, we apply incomplete MGRS proposed in this correspondence paper for decision-making.

From Table III, it is easy to see that $U/IND(D) = \{\{x_1, x_2, x_6, x_8, x_9\}, \{x_3, x_4, x_5, x_7, x_{10}\}\}$. Suppose that $Y_1 = \{x_1, x_2, x_6, x_8, x_9\}$ and $Y_2 = \{x_3, x_4, x_5, x_7, x_{10}\}$.

To acquire certain decision rules, we only calculate the lower approximation of the decision D with respect to the five granulation spaces determined by the five experts. It easily follows from Definition 2 that

$$\underline{D}_{AT} = \{\{x_1, x_2, x_8, x_9\}, \{x_3, x_4, x_5, x_{10}\}\}.$$

From Definition 5, one can obtain the following lower approximation reducts of Table III, which are as follows:

$$\mathbf{A}(D) = \{\{E_1, E_2\}, \{E_2, E_4\}, \{E_2, E_5\}\}.$$

Thus, there is only a core E_2. That is to say, the decision given by the second expert is indispensable.

From the earlier three reducts, one can extract three groups of certain decision rules as follows:

$$\begin{aligned}(E_1 = 3) \vee (E_2 = 3) &\Rightarrow (D = High),\\ (E_2 = 1) &\Rightarrow (D = Low);\\ (E_2 = 3) \vee (E_4 = 3) &\Rightarrow (D = High),\\ (E_2 = 1) \vee (E_4 = 1) &\Rightarrow (D = Low);\\ (E_2 = 3) \vee (E_5 = 3) &\Rightarrow (D = High),\\ (E_2 = 1) \vee &\Rightarrow (D = Low).\end{aligned}$$

In addition, from (10), one can calculate the importance measure of lower approximation of each expert's decision. Through using the sequence and Algorithm III, we obtain one of the reducts from Table III, which is $\{E_2, E_1\}$.

Remark: The incomplete MGRS model does not attempt to keep or reduce the uncertainty induced by the classical Kryszkiewicz's model but aims at concept representation and rule extraction on the basis of keeping the original granulation structures. The incomplete MGRS has several useful applications.

1) It can deal with intelligent decision-making under multiple granulations. For example, the earlier evaluation issue demands that each of the evaluation experts makes a decision independently, i.e., one does not perform intersection operations between any two evaluation results.
2) To extract decision rules from distributive decision systems using rough-set approaches, the incomplete MGRS can largely reduce the time complexity of rule extraction when the increase of uncertainty is tolerable, in which there is no need to perform the intersection operations in between all the sites.

VI. CONCLUSION

The contribution of this correspondence paper is twofold. On one side, the incomplete single-granulation rough-set theory has been extended, and an incomplete MGRS model has been obtained. In this extension, the approximations of sets are defined by using multiple tolerance relations on the universe. These tolerance relations can be chosen according to user requirements or targets of problem solving. The theoretical analyses show that some properties of the original incomplete rough-set model become special instances of incomplete MGRS. Under the incomplete MGRS, we also have developed several important measures, such as the accuracy measure, the quality of approximation, and the precision of approximation. On the other side, to acquire a brief representation for the approximation of a target decision, the attribute reduction has been discussed in incomplete ISs. A concept of approximation reduct has been used to characterize the smallest attribute subset that preserves the lower approximation and upper approximation of all decision classes in incomplete MGRS. Two key attribute reduction algorithms have been designed, which will be helpful for applying this theory in practical issues. The MGRS framework proposed in this correspondence paper maybe lead to a mechanism for more widely applying the incomplete rough-set theory in real-world applications. Above all, the incomplete MGRS can be applied to concept representation, rule extraction, and data analysis from incomplete data set under multigranulation spaces and has much wider applicability ranges than Kryszkiewicz's rough-set model.

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