Incompressibility through Colors and IDs

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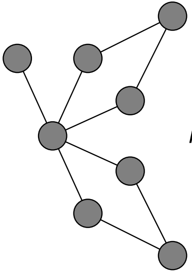
ICALP 2009

Problem Kernels

Polynomial-time preprocessing for NP-hard problems.
Idea: Use data reduction rules to decrease the instance size.

Example:

Vertex Cover
Input: *A graph $G = (V, E)$ and a positive integer k.*
Question: *Is there a vertex set $V' \subseteq V$ with $|V'| \leq k$ that covers all edges?*



$k = 3$

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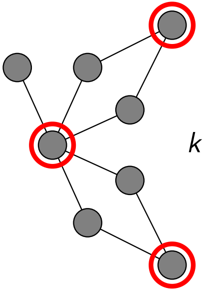

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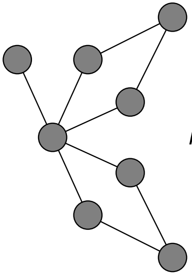
- If there is a vertex with degree $> k$, delete it and decrease k.



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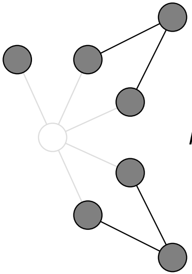

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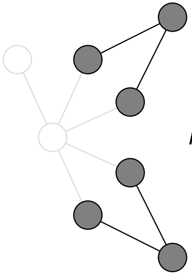


$k = 2$

Finally: $O(k^2)$ edges and vertices.

Problem Kernels

We deal with parameterized problems: Instances of the form (x, k).

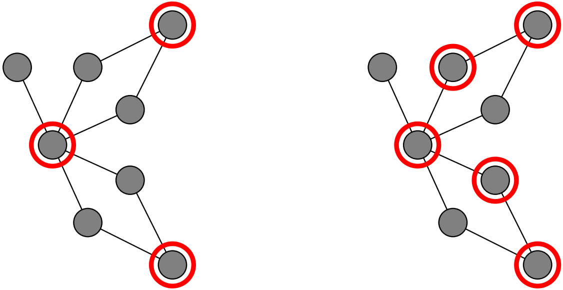
- Polynomial-time data reduction leads to a small problem instance (the *kernel*):

$$\boxed{(x, k) \rightsquigarrow (x', k')}$$

- (x, k) is *yes*-instance $\Leftrightarrow$ (x', k') is *yes*-instance
- $|x'| \leq f_1(k)$
- $k' \leq f_2(k)$

A problem is in FPT $\Leftrightarrow$ it has a problem kernel.
But: Is the kernel of *polynomial size*?

Problem Kernels



- Vertex Cover has a polynomial kernel (kernels with $2k$ vertices are known).
- Open problem [e. g. Guo et al., *Theory Comput. Syst.*, 2007]: Does Connected Vertex Cover has a polynomial kernel?

Non-Existence of Poly. Kernels

Theorem ([Bodlaender et al., *ICALP '08*; Fortnow, Santhanam, *STOC '08*])

Let L be a parameterized problem whose unparameterized version is NP*-complete.*
If L has a composition algorithm, then there is no polynomial kernel for L unless $\mathrm{PH} = \Sigma_p^3$.

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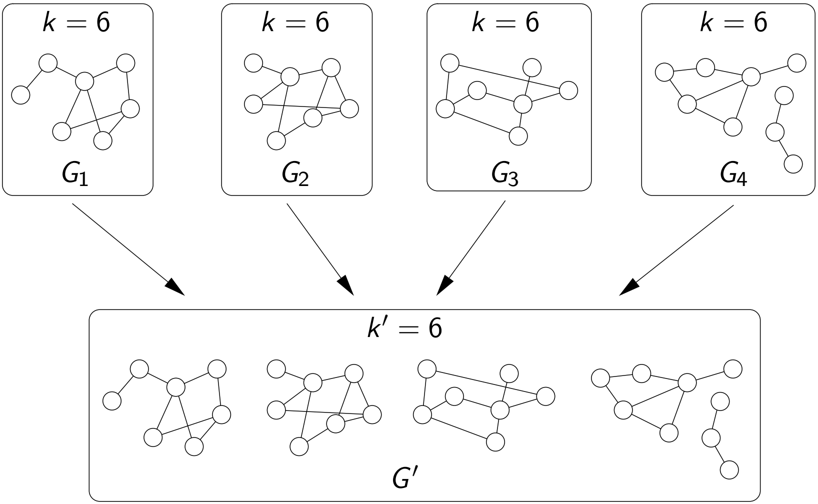
Definition ([Bodlaender et al., *ICALP '08*])

A *composition algorithm* combines problem instances:

- $(x_1, k), (x_2, k), \ldots, (x_t, k) \rightsquigarrow (x', k')$
- (x', k') is *yes*-instance $\Leftrightarrow$ at least one (x_i, k) is *yes*-instance
- $k' \leq \text{poly}(k)$
- (x', k') can be computed in $\text{poly}(\sum_{i=1}^{t} |x_i| + k)$ time

Non-Existence of Poly. Kernels

Example: Composition algorithm for Longest Path



Non-Existence of Poly. Kernels

Theorem ([Bodlaender et al., technical report, Utrecht University, 2008])

Let P and Q be parameterized problems such that Q's unparameterized version is in NP *and P's unparameterized version is* NP*-hard.*

If there is a polynomial parameter transformation from P to Q and if P has no polynomial kernel, then Q also has no polynomial kernel.

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Definition ([Bodlaender et al., technical report, Utrecht University, 2008])

A *polynomial parameter transformation* is a special kind of polynomial-time many-one reduction:

- instance (x, k) of $P \rightsquigarrow$ instance (y, k') of Q
- (x, k) is *yes*-instance $\Leftrightarrow$ (y, k') is *yes*-instance
- $k' \leq \text{poly}(k)$

Our Results

- A general framework for showing "No Polynomial Kernel"
- Non-existence of polynomial kernels for natural parameterizations of
  - Connected Vertex Cover
  - Capacitated Vertex Cover
  - Steiner Tree
  - Red-Blue Dominating Set (=Set Cover/Hitting Set)
  - Dominating Set
  - Unique Coverage
  - Small Subset Sum

Structure of the Talk

- Introduction
- **No Polynomial Kernel for Red-Blue Dominating Set, Parameterization I**
- A General Framework for Showing "No Polynomial Kernel"
- Consequences for Some Other Problems
- No Polynomial Kernel for Red-Blue Dominating Set, Parameterization II

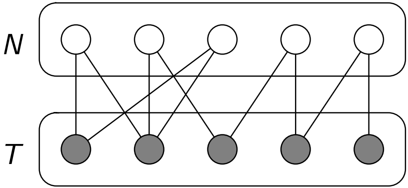
No Poly. Kernel for RBDS with Parameter $(|T|, k)$

Red-Blue Dominating Set (RBDS)

Input: *A bipartite graph $G = (T \cup N, E)$ and a positive integer k.*

Question: *Is there a set $N' \subseteq N$ with $|N'| \leq k$ such that every vertex from T has at least one neighbor in N'?*

$k = 3$



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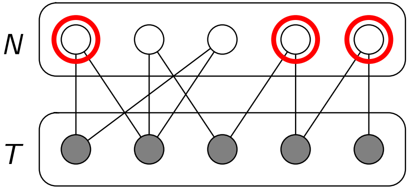

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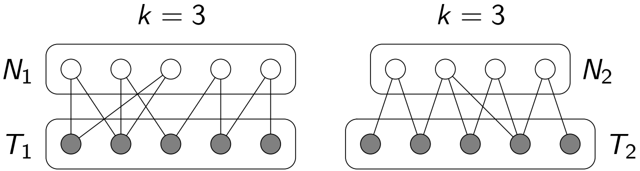
Red-Blue Dominating Set is

- W[2]-complete for the parameter k,
- in FPT for the parameter $|N|$,
- in FPT for the parameter $|T|$.

We show:
No polynomial kernel for the parameters $(|T|, k)$ and $(|N|, k)$.

No Poly. Kernel for RBDS with Parameter $(|T|, k)$



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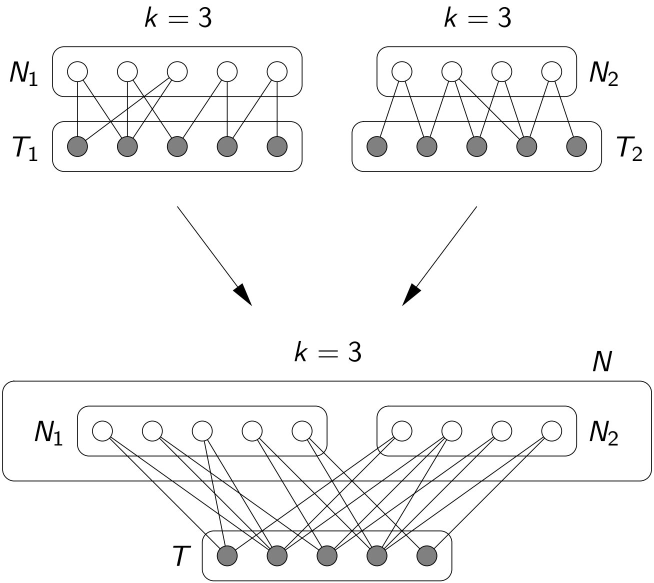

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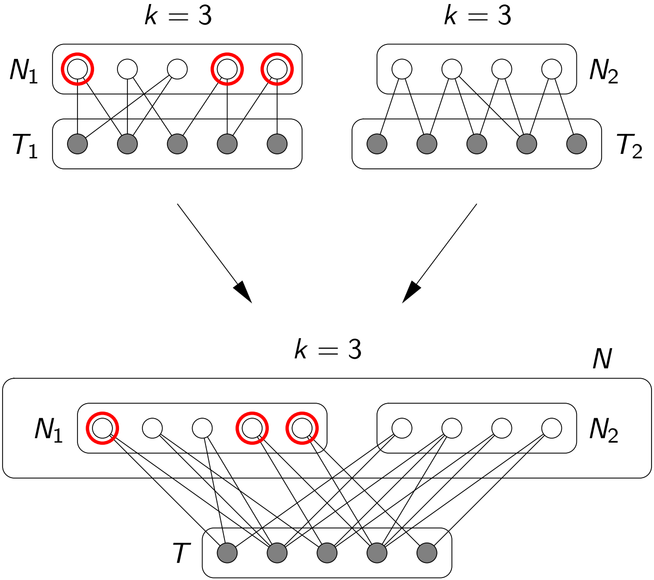

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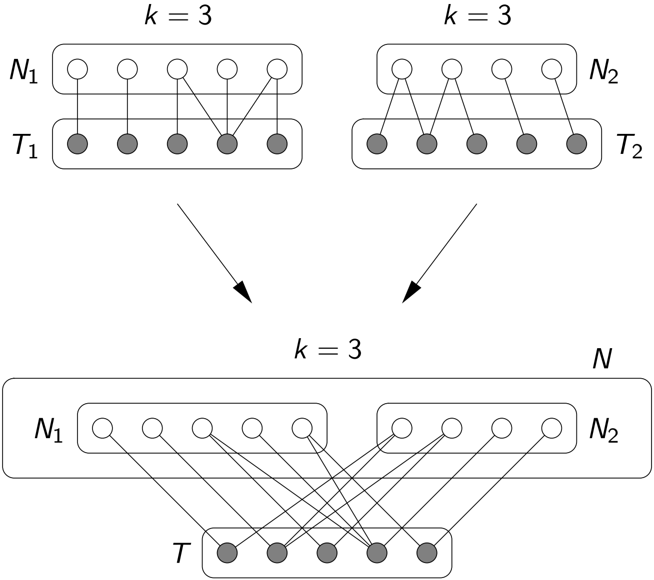

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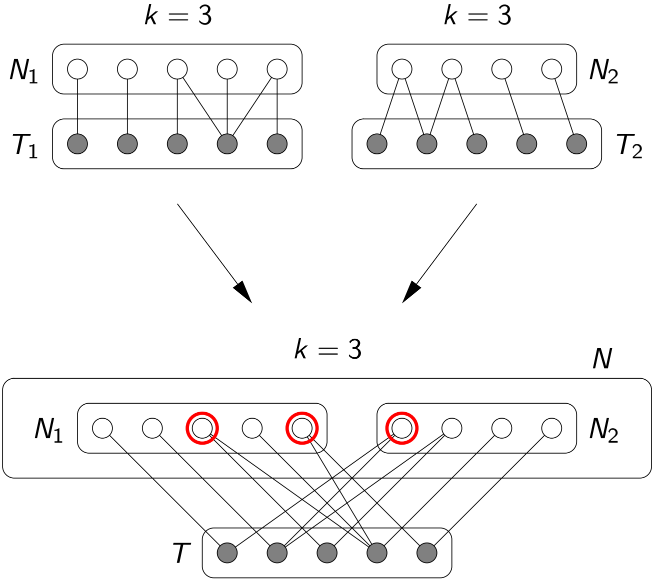


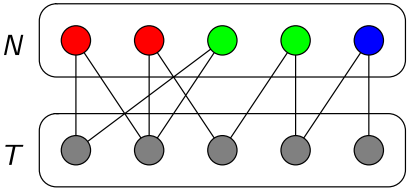
No Poly. Kernel for RBDS with Parameter $(|T|, k)$

Colored Version of RBDS

Input: *A bipartite graph* $G = (T \cup N, E)$ *and a* k*-coloring for* N.

Question: *Is there a set* $N' \subseteq N$ *containing exactly one vertex of each color such that every vertex from* T *has at least one neighbor in* N'?

$k = 3$



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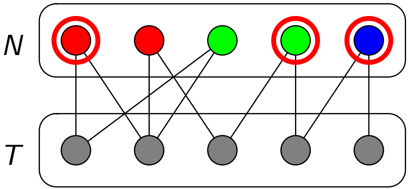

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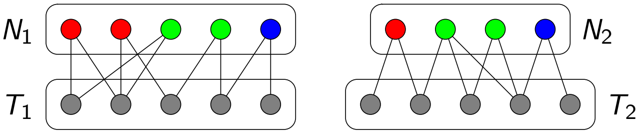

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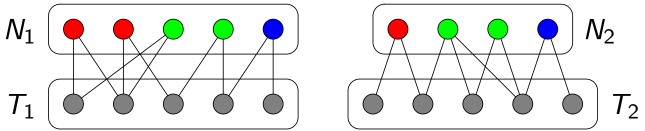

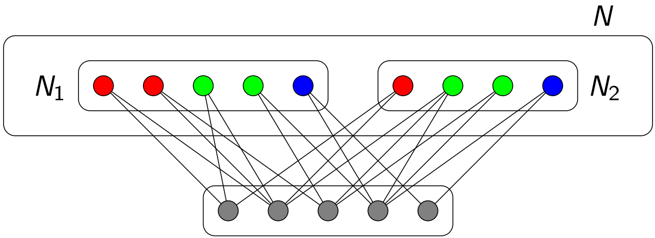

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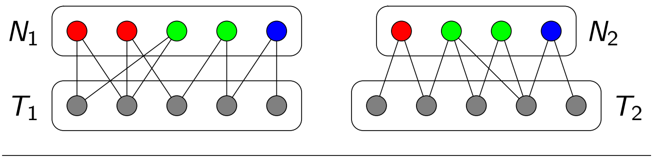
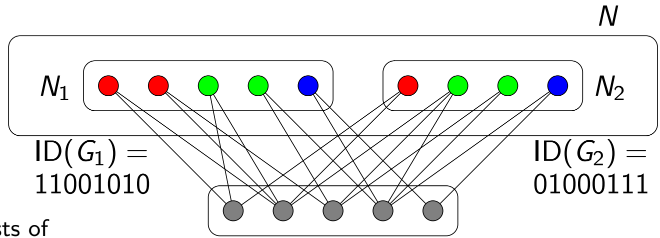
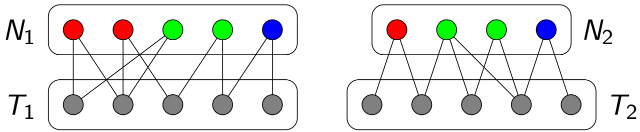


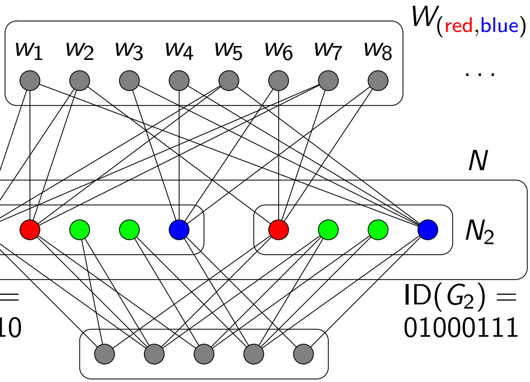


ID consists of $|T| + k$ bits.

No Poly. Kernel for RBDS with Parameter $(|T|, k)$



correspondg. bit $= 1$
$\Rightarrow$ conn. to red vertices.
correspondg. bit $= 0$
$\Rightarrow$ conn. to blue vertices.



$\text{ID}(G_1) =$
11001010

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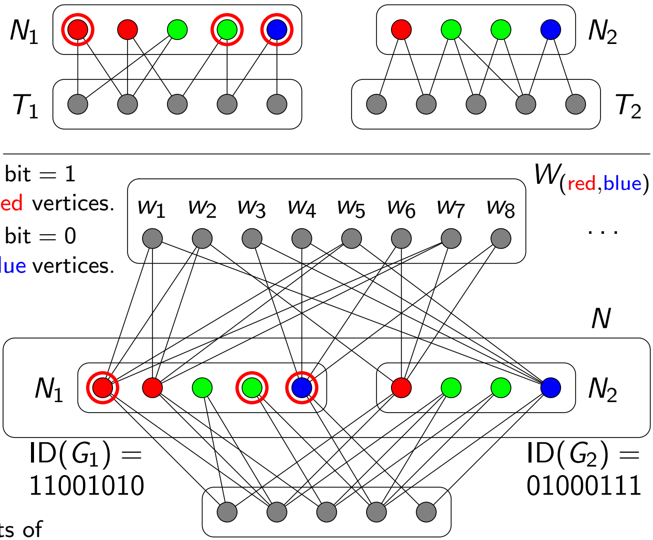

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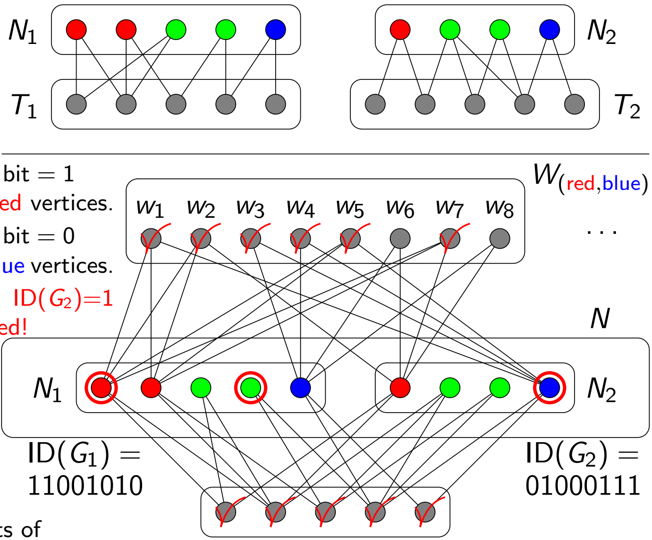

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ID(G_1)=0 $\wedge$ ID(G_2)=1
is undominated!

ID(G_1) $=$
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ID consists of
$|T| + k$ bits.

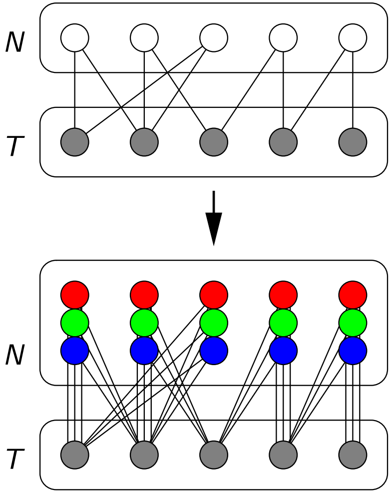
General Framework for Showing "No Polynomial Kernel"

0. Find a suitable parameterization.
1. Define a colored version of the problem.
2. Use IDs to show that the colored version has a composition algorithm.
3. Show that the colored version is solvable in time $2^{k^c} \cdot n^{O(1)}$.
4. Show that the unparameterized colored version is NP-hard (and that the unparameterized uncolored version is in NP).
5. Give a polynomial parameter transformation from the colored to the uncolored version.

No Poly. Kernel for RBDS with Parameter $(|T|, k)$

The unparameterized colored version of RBDS is NP-complete: Reduction from RBDS.

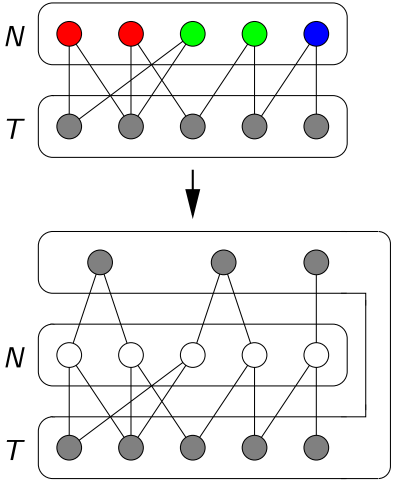
$k = 3$



No Poly. Kernel for RBDS with Parameter $(|T|, k)$

Polynomial parameter transformation from the colored to the uncolored version of RBDS:

$k = 3$



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- No Polynomial Kernel for Red-Blue Dominating Set, Parameterization I
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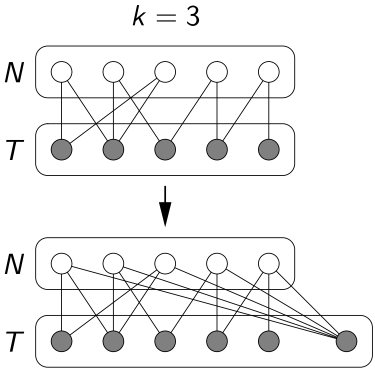
No Poly. Kernel for Steiner Tree with Parameter $(|T|, k)$

Steiner Tree

Input: *A bipartite graph $G = (T \cup N, E)$ and a positive integer k.*

Question: *Is there a set $N' \subseteq N$ with $|N'| \leq k$ such that $G[T \cup N']$ is connected?*

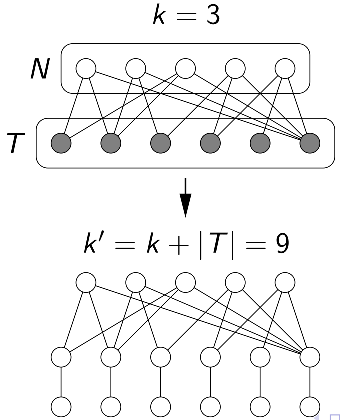
Polynomial parameter transformation from RBDS:



No Poly. Kernel for Connected Vertex Cover w. Param. k

Connected Vertex Cover (ConVC)
Input: *A graph $G = (V, E)$ and a positive integer k.*
Question: *Is there a connected vertex cover $V' \subseteq V$ with $|V'| \leq k$?*

Polynomial parameter transformation from Steiner Tree:



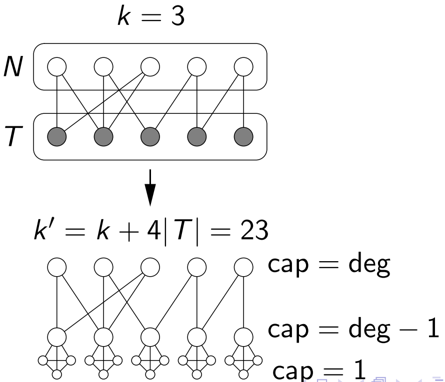
No Poly. Kernel for Capacitated Vertex Cover w. Param. k

Capacitated Vertex Cover (CapVC)

Input: *A graph $G = (V, E)$ with vertex capacities, and a positive integer k.*

Question: *Is there a capacitated vertex cover $V' \subseteq V$ with $|V'| \leq k$?*

Polynomial parameter transformation from RBDS:



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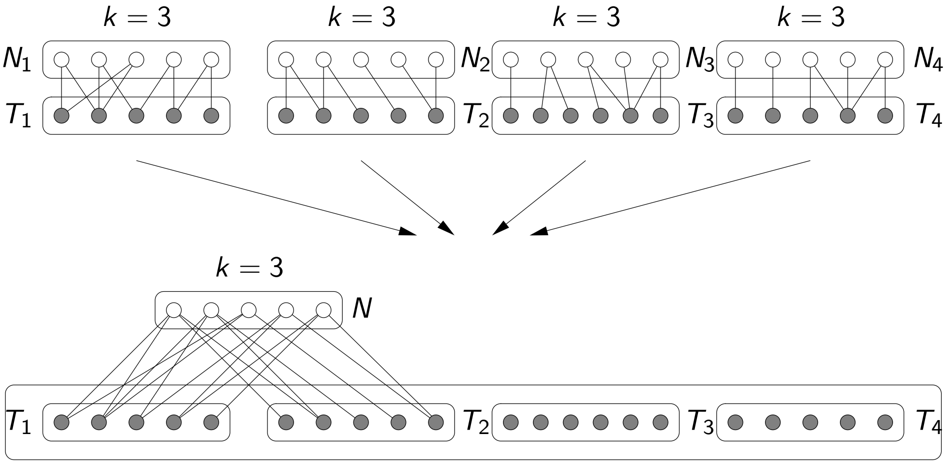

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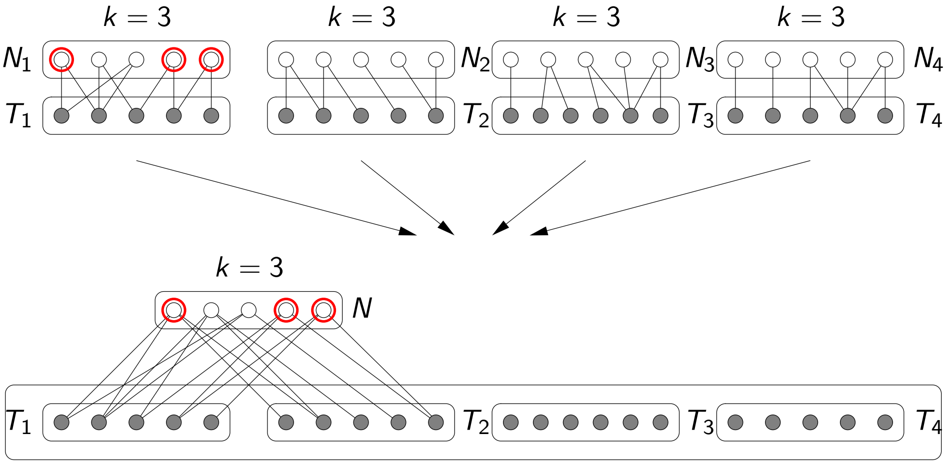

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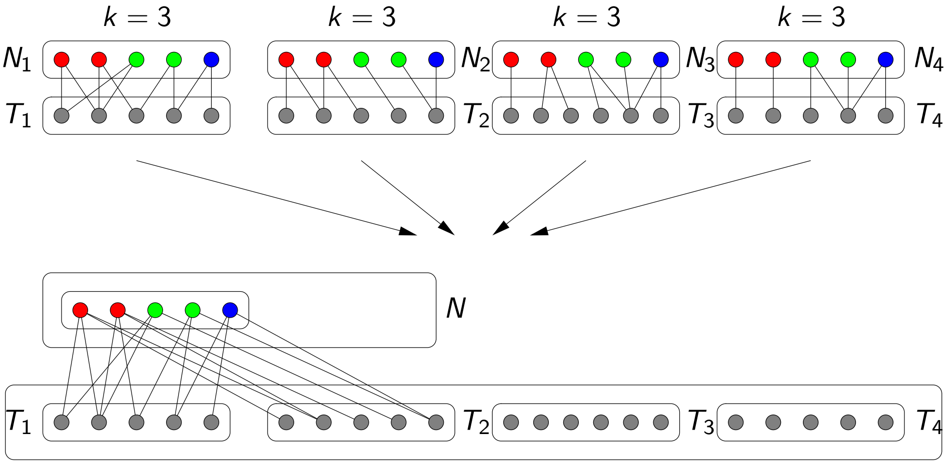

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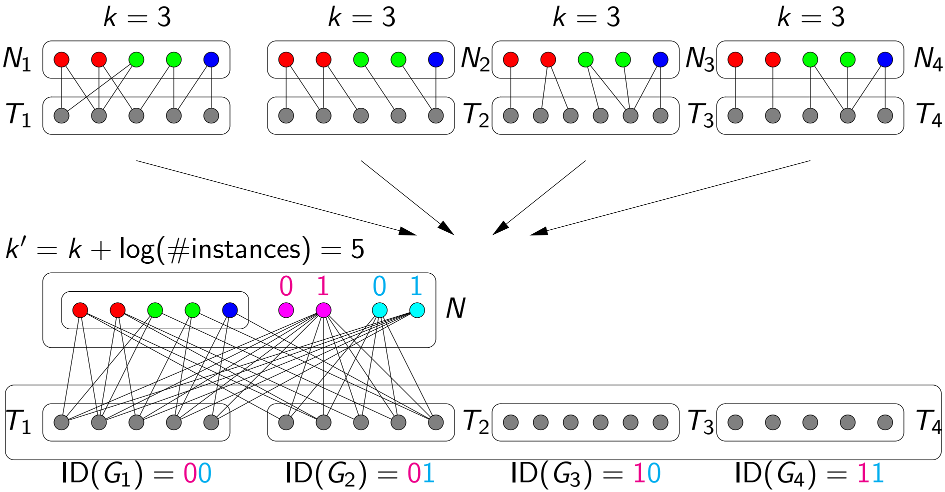

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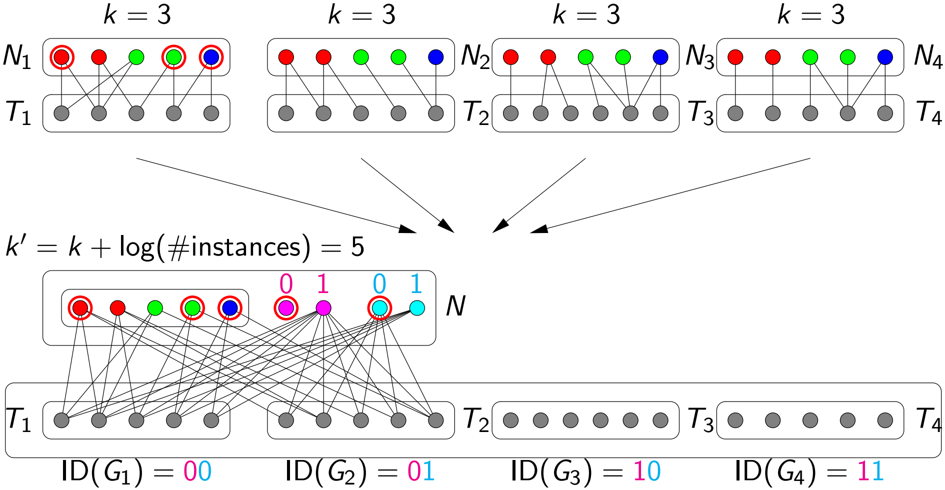

Further Results / Open Questions

There is no polynomial kernel for...

- Dominating Set with parameter $(k, \mathrm{VC}(G))$,
- Dominating Set in H-Minor Free Graphs with parameter $(k, |H|)$,
- Unique Coverage (task: cover $\geq k$ elements from a universe U uniquely) with parameter k,
- Small Subset Sum (task: select $\leq k$ d-bit numbers whose sum is t) with parameter (k, d).

Open:

- RBDS has kernels of size $k^{\deg(G[T])}$ and $k^{\deg(G[N])}$.
  What about kernels of size $f(\deg(\ldots)) \cdot k^{O(1)}$?
- Dominating Set in H-Minor Free Graphs has a kernel of size $k^{f(|H|)}$.
  What about a kernel of size $f(|H|) \cdot k^{O(1)}$?
- Which problems without a polynomial kernel admit a kernelization to *more than one* polynomial kernel ("Turing Kernelization")?

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Thank you.