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INCOMPRESSIBLE AND IDEAL 2D FLOW AROUND A SMALL OBSTACLE WITH CONSTANT VELOCITY AT INFINITY

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Abstract. This article is concerned with the limiting behavior of incompressible flow past a small obstacle. Previous work on this problem has dealt with flows with vanishing velocity at infinity. We examine this limit for flows that are constant at infinity in the simplest case, that of two-dimensional, ideal flow past an obstacle. This extends the work in Iftimie, Lopes Filho, and Lopes (2003).

1. Introduction. This note is concerned with the limiting behavior of incompressible flows past a small obstacle. This problem has seen much recent activity, beginning with the case of a single obstacle in two-dimensional ideal flow; see [7]. Further work has included the viscous case (see [6, 8]), bounded domains with several holes (see [14]), thin obstacles (see [11, 12]), moving obstacles (see [4]), and the interaction between small obstacles and small viscosity (see [9]). In all the cases where the exterior domain problem was considered, the flow was assumed to vanish at infinity. The purpose of the present note is to consider flows with constant velocity at infinity, in the simplest case of ideal flows past a smooth obstacle in the plane. This is a natural extension of the work in [7].

The main technical issue in this extension is the description of potential flows with constant velocity at infinity in the exterior of a smooth obstacle, a classical problem in fluid mechanics. We will see that, beyond this description, the analysis follows very closely that of [7]. The final conclusion is also the same, namely, that the limiting flow

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satisfies a modified Euler equation in the plane, where the limiting circulation around the obstacle manifests in the limit as background flow generated by a point vortex.

We begin with a precise formulation of our problem. Let Ω be a smooth, bounded, connected and simply connected domain in the plane, with boundary Γ, and consider $\Omega_\epsilon \equiv \epsilon\Omega$ with boundary Γ_ϵ for each $\epsilon > 0$. Denote $\Pi_\epsilon \equiv \mathbb{R}^2 \setminus \overline{\Omega_\epsilon}$.

Fix $u_\infty \in \mathbb{R}^2$, $\gamma \in \mathbb{R}$ and $\omega_0 \in C_c^\infty(\mathbb{R}^2 \setminus \{0\})$. For each $\epsilon > 0$ sufficiently small, we consider an initial divergence-free velocity u_ϵ^0 in Π_ϵ, tangent to Γ_ϵ, with circulation γ around Γ_ϵ, with curl $u_\epsilon^0 = \omega_0$, such that $\lim_{|x|\to\infty} u_\epsilon^0(x) = u_\infty$. We will see later that, given γ, u_∞ and ω_0, there exists a unique such u_ϵ^0. We consider the solution u^ϵ of the initial boundary value problem:

$$\begin{cases} u_t + u\cdot\nabla u = -\nabla p, \text{ in } \Pi_\epsilon, \\ \operatorname{div} u = 0, \\ u(x,0) = u_\epsilon^0, \\ u\cdot\widehat{n} = 0 \text{ on } \Gamma_\epsilon, \\ u(x,t) \to u_\infty \text{ as } |x|\to\infty. \end{cases} \tag{1}$$

Existence of such flows is due to K. Kikuchi; see [5]. Our main result is the existence of a subsequence of u^ϵ that converges to a flow $u = u(x,t)$ such that its vorticity $\omega = \text{curl } u$ satisfies:

$$\begin{cases} \omega_t + u\cdot\nabla\omega = 0, \text{ in } \mathbb{R}^2 \\ u = K_{\mathbb{R}^2}[\omega] + \gamma H + u_\infty \\ \omega(x,0) = \omega_0, \end{cases} \tag{2}$$

where $H = H(x) = x^\perp/2\pi|x|^2$ is the velocity field induced by a unit strength point vortex in the plane and $K_{\mathbb{R}^2}[\omega] = H * \omega$ is the Biot-Savart law in the plane. We denote $(x_1,x_2)^\perp = (-x_2,x_1)$. As a consequence of our main result we obtain the existence of a weak solution for problem (2).

The remainder of this paper is organized as follows. In the next section, we introduce some information on classical potential theory for the Laplacian in an exterior domain. We will use the conformal mappings between Ω_ϵ and the exterior of the unit disk to allow us to write explicit formulas for the harmonic fields and the Biot-Savart law. In Section 3, we present a precise formulation of our problem and we collect *a priori* estimates. In the last section we discuss the passage to the limit, and draw concluding remarks.

2. The Laplacian in an exterior domain.

The purpose of this section is to study harmonic vector fields in an exterior domain which are constant at infinity. Let Ω be a smooth, bounded, connected and simply connected domain in the plane with boundary Γ, and let $\Pi = \mathbb{R}^2 \setminus \overline{\Omega}$. We denote the exterior of the unit closed disk in the plane by $\mathcal{U}$. We begin with a result from [7].

LEMMA 2.1. There exists a smooth biholomorphism $T : \Pi \longrightarrow \mathcal{U}$, extending smoothly up to the boundary, mapping Γ to $S = \{\,|\,z\,| = 1\,\}$. Furthermore, there exists a nonzero real number β and a bounded holomorphic function $h : \Pi \longrightarrow \mathbb{C}$ such that

$$T(z) = \beta z + h(z).$$

Additionally,

$$h'(z) \;=\; \mathcal{O}\left(\frac{1}{z^2}\right) \text{ as } |z| \;\to\; \infty.$$

REMARK. Given any complex number θ with $|\theta| = 1$, we can replace T above by θT and still obtain a biholomorphism between Π and U with the same properties, except that now $T(z) = \beta z + h(z)$, with $\beta = |\beta|\theta$.

We denote the Green's function of the Dirichlet Laplacian in Π by $G_\Pi = G_\Pi(x,y)$. The kernel of the Biot-Savart law on Π is $K_\Pi(x,y) = \nabla_x^\perp G_\Pi(x,y)$. We use the same notation for the associated integral operator $f \mapsto K_\Pi[f] = \int K_\Pi(\cdot,y)f(y)dy$. Also, we denote by H_Π the unique harmonic vector field in Π (i.e. divergence-free and curl-free), tangent to Γ, vanishing at infinity, with unit circulation around Γ. For $y \in \mathbb{R}^2 \setminus \{0\}$ we denote $y^* \equiv y/|y|^2$. We can use the conformal map T above to obtain explicit formulas for K_Π and H_Π, namely:

$$K_\Pi(x,y) = \frac{((T(x)-T(y))DT(x))^\perp}{2\pi|T(x)-T(y)|^2} - \frac{((T(x)-(T(y))^*)DT(x))^\perp}{2\pi|T(x)-(T(y))^*|^2}$$

and

$$H_\Pi(x) = \frac{1}{2\pi}\frac{DT^t(x)(T(x))^\perp}{|T(x)|^2}.$$

We are interested in harmonic vector fields that are tangent to Γ and tend to a constant vector at infinity.

PROPOSITION 2.2. For each $\xi \in \mathbb{R}^2$ there exists a unique harmonic vector field $V_{\Pi,\xi}$ in Π which is tangent to Γ, has circulation zero around Γ and tends to ξ as $|x| \to \infty$.

Proof. The case $\xi = 0$ is Proposition 2.1 in [7]. We assume $\xi \neq 0$.

We begin the proof by explicitly constructing $V_{D,(\lambda,0)}$ where D is the exterior of the unit disk and $\lambda > 0$. If we denote $V_{D,(\lambda,0)} = (V_1,V_2)$, it is well known that

$$V_1 - iV_2 = \lambda\left(1 - \frac{1}{z^2}\right), z \in \mathbb{C}.$$

If $z = x_1 + ix_2$, then:

$$V_1 = \lambda\frac{x_1^4 + 2x_1^2x_2^2 - x_1^2 + x_2^4 + x_2^2}{x_1^4 + 2x_1^2x_2^2 + x_2^4},\; V_2 = \frac{-2\lambda x_1x_2}{x_1^4 + 2x_1^2x_2^2 + x_2^4}.$$

To verify that this flow satisfies all the requirements, first we note that the complex form of (V_1,V_2) easily implies that the limit at infinity is $(\lambda,0)$. The Cauchy-Riemann equations imply that $V_{D,(\lambda,0)}$ is indeed divergence-free and curl-free. Restricting to $|x| = 1$ and taking the scalar product with $x = (x_1,x_2)$, it can be easily verified that $V_{D,(\lambda,0)}$ is tangent to $|x| = 1$. Finally, once again restricting to the unit circle, we find that $(V_1,V_2)\cdot(-x_2,x_1) = -2\lambda x_2$, which integrates to zero on the unit circle.

Next we use the conformal mapping T from Lemma 2.1 to define, for each unimodular complex number $\theta = \theta_1 + i\theta_2$,

$$V_{\Pi,\xi,\theta}(x_1,x_2) = D(\theta T)^t V_{D,(\lambda,0)}(\theta T(x_1,x_2)). \tag{3}$$

The limit of $V_{\Pi,\xi,\theta}$ at infinity is $\lambda\beta(\theta_1,\theta_2)$. We choose $\lambda = |\xi|/\beta$ and $(\theta_1,\theta_2) = \xi/|\xi|$ to guarantee that $\lim_{x\to\infty} V_{\Pi,\xi,\theta}(x) = \xi$. We denote the vector field obtained with these choices of λ and θ by $V_{\Pi,\xi}$.

By a straightforward calculation, using the Cauchy-Riemann equations for T, we have that both the divergence and the curl of $V_{\Pi,\xi}$ vanish. The fact that a biholomorphism is also a diffeomorphism implies that $V_{\Pi,\xi}$ is tangent to Γ, since $V_{D,(\lambda,0)}$ is tangent to the unit circle. Furthermore, the circulation of $V_{\Pi,\xi}$ around Γ is equal to the circulation of $V_{D,(\lambda,0)}$ around the unit circle, which vanishes. This can be seen by expressing the circulation of $V_{\Pi,\xi}$ around Γ as a line integral and performing a change of variables using the restriction of T to Γ, which is a diffeomorphism between Γ and $\{|z| = 1\}$. We have established the existence of $V_{\Pi,\xi}$.

Uniqueness is a consequence of Proposition 2.1 in [7]. $\square$

Remark. Recall the initial velocity field u^0_ϵ introduced in Section 1. Proposition 2.2 implies that, together, the circulation γ, the initial vorticity ω_0 and the velocity at infinity u_∞ determine u^0_ϵ in a unique manner.

3. Flow in an exterior domain and *a priori* estimates. Our objective in this section is both to formulate a precise statement of the small obstacle problem and to obtain the required *a priori* estimates. Let $\omega_0 \in C^\infty_c(\mathbb{R}^2 \setminus \{0\})$, and fix $\gamma \in \mathbb{R}$ and $u_\infty \in \mathbb{R}^2$. We fix Ω to be a smooth, bounded, connected and simply connected domain in the plane with boundary Γ, and we define $\Pi_\epsilon \equiv \mathbb{R}^2 \setminus \epsilon\overline{\Omega}$ and $\Gamma_\epsilon = \epsilon\Gamma$. By Proposition 2.2, for each sufficiently small $\epsilon > 0$, there exists a unique smooth, divergence-free velocity field u^0_ϵ in Π_ϵ with vorticity ω_0, circulation γ around Γ_ϵ, tangent to Γ_ϵ, and such that its limit, as $|x| \to \infty$, is u_∞. By Kikuchi's Theorem, there exists a unique velocity field $u_\epsilon = u_\epsilon(x,t)$ which is a classical solution of (1) with initial velocity u^0_ϵ. Let $\omega_\epsilon = \operatorname{curl} u_\epsilon$ be the vorticity corresponding to u_ϵ. It is easy to prove that the total mass of vorticity is conserved; by Kelvin's Circulations Theorem, it follows that the circulation of $u_\epsilon(\cdot,t)$ around Γ_ϵ is conserved as well, hence equal to γ. Let $U_\epsilon = u_\epsilon - V_{\Pi_\epsilon,u_\infty}$. First observe that, for each fixed time t, $U_\epsilon(\cdot,t)$ is divergence-free, curl $U_\epsilon = \omega_\epsilon$, tangent to Γ_ϵ and vanishes at infinity. In was shown in [7], as a consequence of Lemma 2.2 and Proposition 2.1, that for such a vector field there exists a unique $\alpha \in \mathbb{R}$ such that

$$U_\epsilon = K_{\Pi_\epsilon}[\omega_\epsilon] + \alpha H_{\Pi_\epsilon}.$$

In the proof of Lemma 3.1 in [7] it was shown that

$$\int_{\Gamma_\epsilon} (K_{\Pi_\epsilon}[\omega_\epsilon] + \alpha H_{\Pi_\epsilon}) \cdot ds = \alpha - \int_{\Pi_\epsilon} \omega_\epsilon dx = \alpha - \int_{\mathbb{R}^2} \omega_0 \, dx \equiv \alpha - m,$$

where we used the conservation of mass of vorticity. Hence,

$$\gamma = \int_{\Gamma_\epsilon} u \cdot ds = \int_{\Gamma_\epsilon} U_\epsilon \cdot ds = \alpha - \int_{\Pi_\epsilon} \omega_\epsilon \, dx.$$

We obtain that $\alpha = \gamma + m$ and, therefore, it is independent of ϵ and time. Therefore we have

$$u_\epsilon = K_{\Pi_\epsilon}[\omega_\epsilon] + \alpha H_{\Pi_\epsilon} + V_{\Pi_\epsilon,u_\infty}. \tag{4}$$

The vorticity formulation of the exterior domain problem becomes:

$$\begin{cases} \partial_t \omega_\epsilon + u_\epsilon \cdot \nabla \omega_\epsilon = 0, \\ u_\epsilon = K_{\Pi_\epsilon}[\omega_\epsilon] + \alpha H_{\Pi_\epsilon} + V_{\Pi_\epsilon, u_\infty}, \\ \omega(x,0) = \omega_0. \end{cases} \tag{5}$$

The argument used in [7], which we will follow here, involves passing to the limit $\epsilon \to 0$ in the weak formulation of (5). Since vorticity is transported by a divergence-free vector field, we have the *a priori estimate*: $\|\omega_\epsilon(x,t)\|_{L^p} \leq \|\omega_0\|_{L^p}$, for any $1 \leq p \leq \infty$. By Alaoglu's Theorem, we may consider a subsequence, which we still call $\{\omega_\epsilon\}$, which converges weak-$*$ in $L^\infty([0,T) \times \mathbb{R}^2)$ to a limit ω (we have extended each $\omega_\epsilon(\cdot,t)$ to $\mathbb{R}^2$ by setting it to vanish inside Ω_ϵ).

The key ingredients for the convergence theorem we wish to prove are velocity estimates. Once again following [7], we decompose the velocity as

$$u_\epsilon = v_\epsilon + (\alpha - m) H_{\Pi_\epsilon} + V_{\Pi_\epsilon, u_\infty},$$

where

$$v_\epsilon = K_{\Pi_\epsilon}[\omega_\epsilon] + m H_{\Pi_\epsilon}.$$

Most of our work is already done. By Theorem 4.1 in [7], we have that there exists a constant $C > 0$, independent of ϵ, such that:

$$\|v_\epsilon\|_{L^\infty(\Pi_\epsilon)} \leq C \|\omega_0\|_{L^1}^{1/2} \|\omega_0\|_{L^\infty}^{1/2}.$$

Also, in [7] an adapted cut-off function ϕ_ϵ was introduced, such that $\phi_\epsilon H_{\Pi_\epsilon}$ is divergence-free and converges strongly in $L^1_{loc}(\mathbb{R}^2)$ to $x^\perp/2\pi|x|^2$; see Lemma 4.2.

We are missing estimates on the behavior of $V_{\Pi_\epsilon, u_\infty}$ when $\epsilon \to 0$. This is the subject of the next result.

LEMMA 3.1. We have the following:

(1) There exists a constant $C > 0$ depending only on Ω, such that

$$\|V_{\Pi_\epsilon, u_\infty}\|_{L^\infty(\Pi_\epsilon)} \leq C |u_\infty|.$$

(2) Let ϕ_ϵ be the adapted cut-off introduced in [7]. We have:

$$\phi_\epsilon V_{\Pi_\epsilon, u_\infty} \to u_\infty,$$

strongly in $L^2(\mathbb{R}^2)$.

Proof. Let $T : \Pi_1 \to \mathcal{U}$ be the conformal mapping as obtained in Lemma 2.1. Taking $\theta = (u^1_\infty + i u^2_\infty)/|u_\infty|$, we define a new conformal mapping $S = \theta T$, which still maps Π_1 to $\mathcal{U}$. Using (3), we can write an explicit formula for $V_{\Pi_\epsilon, u_\infty}$ as follows:

$$V_{\Pi_\epsilon, u_\infty}(x) = DS^t\left(\frac{x}{\epsilon}\right) V_{D,(\lambda,0)}\left(S\left(\frac{x}{\epsilon}\right)\right),$$

where, in complex notation, $S(z) = \theta\beta z + h(z)$, with $\beta \in \mathbb{R}$, h bounded and $h'(z) = \mathcal{O}(1/z^2)$ at infinity, $\lambda = |u_\infty|/\beta$, and $V_{D,(\lambda,0)}(z) = \lambda(1 - 1/z^2)$. Clearly, for all $z \in \Pi_\epsilon$, $DS^t(z)$ is bounded by $|\theta\beta| + \sup|h'|$ and $|V_{D,(\lambda,0)}(z)|$ is bounded by $2|u_\infty|/|\beta|$, so that the first statement in the lemma follows.

For the second statement, we split $\mathbb{R}^2$ into three parts, writing:

$$\|\phi_\epsilon V_{\Pi_\epsilon,u_\infty} - u_\infty\|^2_{L^2(\mathbb{R}^2)} = \int_{|x|\le\sqrt{\epsilon}} |\phi_\epsilon V_{\Pi_\epsilon,u_\infty} - u_\infty|^2 dx$$

$$+ \int_{\sqrt{\epsilon}\le|x|\le 1} |\phi_\epsilon V_{\Pi_\epsilon,u_\infty} - u_\infty|^2 dx + \int_{|x|\ge 1} |\phi_\epsilon V_{\Pi_\epsilon,u_\infty} - u_\infty|^2 dx \equiv I_1 + I_2 + I_3.$$

For I_1 we use the uniform boundedness of $\phi_\epsilon V_{\Pi_\epsilon,u_\infty}$ from the first part of the lemma, together with the fact that the measure of the domain of integration goes to zero as $\epsilon \to 0$. To analyze I_2 and I_3, we will use the fact that $\phi_\epsilon \equiv 1$ in the domain of integration. For each $|x| > 0$ we have:

$$DS^t\left(\frac{x}{\epsilon}\right) \to \beta \begin{bmatrix} \theta_1 & -\theta_2 \\ \theta_2 & -\theta_1 \end{bmatrix} \text{ as } \epsilon \to 0,$$

and

$$V_{D,(\lambda,0)}\left(S\left(\frac{x}{\epsilon}\right)\right) \to \left(\frac{|u_\infty|}{\beta}, 0\right).$$

We conclude, in particular, that $V_{\Pi_\epsilon,u_\infty}(x) \to u_\infty$ pointwise in $\{x \in \mathbb{R}^2 \mid \sqrt{\epsilon} < |x| < 1\}$. Once again using the uniform boundedness of $V_{\Pi_\epsilon,u_\infty}(x)$, the fact that the domain has finite measure, and the Dominated Convergence Theorem, it follows that $I_2 \to 0$ and $\epsilon \to 0$.

Finally, we estimate I_3. Note that, in complex notation:

$$|V_{\Pi_\epsilon,u_\infty} - u_\infty| = \left|(\theta\beta + h'(z/\epsilon))\frac{|u_\infty|}{\beta}\left(1 - \frac{1}{S^2(z/\epsilon)}\right) - u_\infty\right|$$

$$\le \left|\frac{u_\infty}{S^2(z/\epsilon)}\right| + \left|h'(z/\epsilon)\frac{|u_\infty|}{\beta}\left(1 - \frac{1}{S^2(z/\epsilon)}\right)\right| \le C\frac{\epsilon^2}{|z|^2},$$

which implies that $I_3 = \mathcal{O}(\epsilon^2)$. This concludes our proof. □

The final ingredients needed to pass to the limit $\epsilon \to 0$ are temporal estimates. The proofs in Section 4.3 of [7] apply without change, as the only difference (in the velocity) is the time-independent potential field $V_{\Pi_\epsilon,u_\infty}$. We conclude that $\partial_t\omega^\epsilon$ is bounded in $L^\infty((0,T); W^{-1,1}_{loc}(\mathbb{R}^2))$ and that $\partial_t(\phi_\epsilon v^\epsilon)$ is bounded in $L^\infty((0,T); H^{-3}_{loc}(\mathbb{R}^2))$.

4. Passing to the limit. This last section is devoted to our main theorem. We return to the argument in [7], which uses a parametrized Div-Curl Lemma and the Aubin-Lions Lemma, to find $v \in L^2_{loc}((0,T)\times\mathbb{R}^2)$ and a subsequence of $\{\phi^\epsilon v^\epsilon\}$ (which we do not relabel) converging strongly in $L^2_{loc}((0,T)\times\mathbb{R}^2)$ to v. The proof presented in [7] works in our case without modification. In addition, passing to a further subsequence if needed, we obtain that there exists $\omega \in L^\infty((0,T)\times\mathbb{R}^2)$ such that $\phi^\epsilon\omega^\epsilon \rightharpoonup \omega$, weak-$*$ in $L^\infty((0,T)\times\mathbb{R}^2)$. The limiting flow velocity will be $u = v + (\alpha - m)H + u_\infty$. The final step in our work is to show that the pair u, ω thus obtained is a weak solution of system (2), in a suitable sense. Let us first define precisely this notion of weak solution.

Definition 4.1. Let $\omega_0 \in L^\infty_c(\mathbb{R}^2)$. Fix $\gamma \in \mathbb{R}$ and $u_\infty \in \mathbb{R}^2$. Let $u \in L^\infty([0,\infty); L^1_{loc}(\mathbb{R}^2))$ and $\omega \in L^\infty([0,\infty); L^\infty(\mathbb{R}^2))$. We say that the pair (u,ω) is a weak solution of system (2), with initial vorticity ω_0 and velocity u_∞ at infinity, if:

(1) for any test function $\varphi \in C_c^\infty([0,\infty)\times\mathbb{R}^2)$ we have
$$\int_0^\infty\int_{\mathbb{R}^2}\varphi_t\,\omega\,dx\,dt+\int_0^\infty\int_{\mathbb{R}^2}\nabla\varphi.\,u\omega\,dx\,dt+\int_{\mathbb{R}^2}\varphi(x,0)\,\omega_0(x)\,dx=0,$$
(2) we have $\operatorname{div} u = 0$ and $\operatorname{curl} u = \omega + \gamma\delta$ in the sense of distributions, where δ denotes the Dirac delta distribution, and

(3) $\lim_{|x|\to\infty} u = u_\infty$.

We are now ready to state and prove the main result of this note.

THEOREM 4.2. Let $\omega_0 \in C_c^\infty(\mathbb{R}^2-\{0\})$. Fix $\gamma\in\mathbb{R}$, $u_\infty\in\mathbb{R}^2$, and $T>0$. Then there exists a velocity field $u\in L^2_{loc}((0,T)\times\mathbb{R}^2)$ with curl $u=\omega\in L^\infty((0,T)\times\mathbb{R}^2)$ such that the pair (u,ω) is a weak solution of system (2), with initial vorticity ω_0 and velocity u_∞ at infinity. Moreover, the velocity u is obtained as a strong limit in $L^2_{loc}((0,T)\times\mathbb{R}^2)$ of a subsequence of solutions of the exterior domain problem (1) having, as initial velocity,
$$u_{0,\epsilon}=K_{\Pi_\epsilon}[\omega_0]+\alpha H_{\Pi_\epsilon}+V_{\Pi_\epsilon,u_\infty}.$$

Proof. Again, the proof is almost the same as Theorem 5.2 in [7]. We outline it briefly in order to highlight the adaptation required. Define the sequence of solutions u^ϵ of system (1), having $u_{0,\epsilon}$ as initial data and with vorticity $\omega^\epsilon=$ curl u^ϵ. We introduce a family of functionals I_ϵ, which, for any fixed test function $\varphi\in C_c^\infty([0,\infty)\times\mathbb{R}^2)$, is given by:
$$I_\epsilon[\varphi]\equiv\int_0^\infty\int_{\mathbb{R}^2}\varphi_t(\phi_\epsilon)^2\omega^\epsilon dxdt+\int_0^\infty\int_{\mathbb{R}^2}\nabla\varphi\cdot(\phi_\epsilon u^\epsilon)(\phi_\epsilon\omega^\epsilon)dxdt,$$
where ϕ_ϵ is the adapted cutoff from [7]. The proof is reduced to two steps: (i) prove that
$$I_\epsilon[\varphi]+\int\varphi(\cdot,0)\omega_0dx\to 0,$$
and (ii) show that
$$I_\epsilon[\varphi]\to\int\varphi_t\omega+\nabla\varphi\cdot u\omega dxdt$$
as $\epsilon\to 0$. To establish the first limit, we use the fact that u^ϵ and ω^ϵ satisfy the vorticity equation in Π^ϵ, so that, after integration by parts, we have:
$$\int_0^\infty\int_{\mathbb{R}^2}\varphi_t(\phi_\epsilon)^2\omega^\epsilon dxdt=-\int_0^\infty\int_{\mathbb{R}^2}\nabla(\varphi(\phi_\epsilon)^2)\cdot u^\epsilon\omega^\epsilon dxdt-\int_{\mathbb{R}^2}\varphi(\cdot,0)(\phi_\epsilon)^2\omega_0dx.$$
Thus,
$$I_\epsilon[\varphi]+\int\varphi(\cdot,0)\omega_0dx=-2\int\varphi\nabla\phi_\epsilon\cdot(\phi_\epsilon u^\epsilon)\omega^\epsilon dxdt$$
$$=-2\int\varphi\nabla\phi_\epsilon\cdot(\phi_\epsilon v^\epsilon)\omega^\epsilon dxdt-2\int\varphi\nabla\phi_\epsilon\cdot(\phi_\epsilon V_{\Pi_\epsilon,u_\infty})\omega^\epsilon dxdt\equiv-2J_\epsilon^1-2J_\epsilon^2.$$
The term J_ϵ^2 is actually the only new feature in this proof, when compared to the proof of Theorem 5.2 in [7]. The term J_ϵ^1 is estimated by $\|\omega^\epsilon\|_{L^\infty}\|\phi_\epsilon v^\epsilon\|_{L^\infty}\|\varphi\phi_\epsilon\|_{L^1}$. The analysis of J_ϵ^1 is concluded by noticing that the first two factors are uniformly bounded, while the last factor tends to zero as $\epsilon\to 0$. We estimate J_ϵ^2 by $\|\omega^\epsilon\|_{L^\infty}\|\phi_\epsilon V_{\Pi_\epsilon,u_\infty}\|_{L^\infty}\|\varphi\phi_\epsilon\|_{L^1}$ and conclude the analysis in a manner similar to J_ϵ^1, given that the middle factor is also bounded; see Lemma 3.1. This concludes the first step. For the second step, we note

that $\nabla\varphi \cdot \phi_\epsilon u^\epsilon$ still converges strongly in $L^1((0,\infty)\times\mathbb{R}^2)$ to $\nabla\varphi\cdot u$, despite the presence of $V_{\Pi_\epsilon,u_\infty}$, whereas everything else in the argument in [7] stays the same. □

A few concluding remarks are in order. First, we note that we assumed $\omega_0 \in C_c^\infty(\mathbb{R}^2 - \{0\})$ for the sake of convenience. It would be easy to adapt Theorem 4.2 for $\omega_0 \in L_c^p(\mathbb{R}^2)$, for any $p > 2$.

Next, as in [7], Theorem 4.2 comprises a new existence result, along with *compactness* of a natural approximating sequence. Convergence of the approximating sequence, which depends on uniqueness of solutions of the limit system, is desirable, as it would open the possibility of describing the error in greater detail.

When [7] was written, and in the case $u_\infty = 0$, uniqueness was known only for $\gamma = 0$, both for smooth initial vorticity (see [2]) and for bounded initial vorticity (see [16]). For $\gamma = 0$, $\omega_0 \in L_c^\infty$ and $u_\infty \neq 0$, uniqueness is a result of P. Serfati in [15]. In the case $\gamma \neq 0$ while $u_\infty = 0$, uniqueness was recently proved by C. Lacave and E. Miot in [13], with the hypothesis that ω_0 vanishes near the origin. This raises several natural problems: (i) to extend Lacave and Miot's result to velocities that do not vanish at infinity, (ii) to relax the hypothesis that ω_0 vanishes near the origin, and (iii) to study asymptotic error terms whenever uniqueness of the limit problem and, therefore, convergence of the approximating sequence, is known.

Finally, another relevant open problem is to consider a viscous version of this work, extending the results in [8] and [6] to the case where velocity is constant at infinity.

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