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INCOMPRESSIBLE FLOWS OF AN IDEAL FLUID WITH VORTICITY IN BORDERLINE SPACES OF BESOV TYPE

BY MISHA VISHIK^(*)

ABSTRACT. – We prove a uniqueness theorem for the Euler equations for an ideal incompressible fluid under the condition that vorticity belongs to a space of Besov type. We also prove an existence theorem in dimension two. © Elsevier, Paris

RÉSUMÉ. – Nous démontrons un théorème d'unicité pour le système d'Euler pour un fluide incompressible idéal sous l'hypothèse que le tourbillon appartient à un espace de type Besov. Nous démontrons également un théorème d'existence en dimension deux. © Elsevier, Paris

0. Introduction

In this paper we study the nonstationary Euler equations of an ideal incompressible fluid

$$(0.1) \quad \begin{cases} \dot{v}_j(x, t) = - \sum_{i=1}^n v_i \partial_i v_j - \partial_j p, & 1 \leq j \leq n, \quad x \in \mathbb{R}^n, \\ \operatorname{div} v = \sum_{j=1}^n \partial_j v_j = 0, \\ v(x, 0) = v_0(x). \end{cases}$$

Here $v(x, t) = (v_1, \dots, v_n)$ is the Eulerian velocity of a fluid flow. For incompressible fluids the key characteristic of the flow is vorticity

$$\omega_{ij}(x, t) = \partial_j v_i - \partial_i v_j, \quad 1 \leq i, j \leq n.$$

The mathematical theory of Euler equations (0.1) is an old subject. In [C1] J.-Y. Chemin develops the theory in detail and gives an account of more recent results including his work [C2] on the regularity of vortex patches for the two-dimensional Euler equations.

Existence and uniqueness theorems are obtained for the problem (0.1) (locally in time for $n \geq 3$) for vorticity in various function spaces with *supercritical* smoothness. Here

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we refer to the condition $s > n/p$ where s is smoothness of vorticity in the sense of a particular scale of function spaces based on L^p .

The papers of L. Lichtenstein [L], N. Gunther [G], Wolibner [W] that started the subject deal with Hölder classes. Some of the more recent papers include those of D. Ebin and J. Marsden [EM], J. Bourguignon and H. Brezis [BB], R. Temam [T], T. Kato and G. Ponce [KP] (Sobolev spaces).

In [V] we proved the global existence for $n = 2$ where vorticity belongs to the Besov space $B_{2/s,1}^s$.

In the present paper we continue to investigate the *borderline* case $s = n/p$ which corresponds to the critical case of Sobolev (-Besov-Lizorkin-Triebel) embedding.

V. Yudovich [Y1] proved the basic uniqueness theorem for the weak solutions of (0.1) under the condition (bounded domain in \mathbb{R}^n) $\omega \in L^\infty$. For $n = 2$ he was able to construct a weak solution in this class that exists globally in time. The existence of a weak solution in two dimensions (possibly without uniqueness) was proved for a wider class L^p , $1 < p < \infty$, by V. Yudovich [Y1] and by R. DiPerna and A. Majda [DM]. D. Chae [C] proved the existence theorem for vorticity in $L \log L(\mathbb{R}^2)$. J.-M. Delort [De] constructed a weak solution with initial vorticity arbitrary sign definite measure. A Bourbaki talk by P. Gérard [Ge] gives an account of this result as well as the results of [C2]. A different proof of J.-M. Delort's result was given by L.C. Evans and S. Müller [EvM] and by A. Majda [M].

V. Scheffer [S] and A. Shnirelman [Sh] constructed the first examples of *nonuniqueness* for weak solutions with $v(\cdot, \cdot) \in L_{loc}^2(\mathbb{R}^2 \times \mathbb{R})$.

Recently V. Yudovich [Y2] further improved his uniqueness theorem [Y1] allowing (case of a bounded domain) $\omega \in \bigcap_{p_0 \leq p < \infty} L^p$ so that $\|\omega\|_{L^p} \leq C\theta(p)$ and $\theta(p)$ grows *moderately* in p . For example $\theta(p) = \log p$ guarantees uniqueness.¹ More precisely, V. Yudovich proved that for $Z(\alpha) = \inf_{\varepsilon \in (0, p_0^{-1})} \{\varepsilon^{-1} \alpha^\varepsilon \theta(\frac{1}{\varepsilon})\}$ the condition $\int_1^\infty \alpha^{-1} Z(\alpha) d\alpha = \infty$ implies uniqueness. This result holds for arbitrary n while the existence theorem was proved in [Y2] (see also [Y3]) only for $n = 2$. The proof of uniqueness in [Y2] is based on the energy method. In case the growth condition for $\theta(p)$ fails V. Yudovich constructed *counterexamples to this method* of proving uniqueness (if not to uniqueness per se).

We introduce here a different uniqueness class which is a variant of a *borderline* Besov space. To describe a particular case of the results of the paper we define

$$B_\Pi = \left\{ f \in S'(\mathbb{R}^n) \mid \sum_{j=-1}^N \|\Delta_j f\|_{L^\infty} = O(\Pi(N)) \right\}.$$

Here the increasing function Π satisfies certain conditions (see (i)-(iii) below), $\Delta_j f$ denotes the terms in Littlewood-Paley decomposition of f . One possible choice is $\Pi(N) = (N + 2) \log_2^\kappa(N + 3)$, $0 \leq \kappa \leq 1$. Notice $f \in B_\Pi$ means the norm $\|f\|_{B_{\infty,1}^0}$ is generally divergent but in a controlled way.

We prove uniqueness for the problem (0.1), n arbitrary, under the condition

$$(0.2) \quad \int_1^\infty \Pi(\alpha)^{-1} d\alpha = \infty,$$

¹ "Not much stronger than linear" on p. 28 of [Y2] is a misprint. Logarithmic singularities that lead to linear in p growth of $\|\omega\|_{L^p}$ are in fact not allowed but singularities of the kind $\log \log |x|$ that produce $\log p$ growth of $\|\omega\|_{L^p}$ as $p \rightarrow \infty$ are covered by the results of [Y2].

$\omega \in L^{p_0}$, $p_0 \in (1, n)$ and the norm $\|\omega\|_{B_\Pi}$ is bounded. By the choice $\Pi(N) = N + 2$ this implies uniqueness for flows with vorticity in Hölder space $\Lambda^0 = B_{\infty, \infty}^0$. As a corollary, a solution with vorticity bounded in bmo is unique. Indeed, $\text{bmo} \hookrightarrow \Lambda^0$.

For Π as above the (unique) solution exists globally in time ($n = 2$) for $0 < \kappa \leq \frac{1}{2}$ and (possibly only) locally in time for $\frac{1}{2} < \kappa \leq 1$. The initial vorticity belongs to the space B_Γ with $\Gamma(N) = \log_2^\kappa(N + 3)$.

The existence theorem is proved in §9.

The proof of the uniqueness theorem given in §8 uses a “paradifferential analogue” of the Osgood uniqueness argument and seems *unrelated to the energy method*. The condition (0.2) in fact guarantees that the Lagrangian flow satisfies conditions of the Osgood uniqueness theorem. It also looks similar to the growth condition in [Y2]. On the other hand there are functions in B_Γ with a *compact support* that fail to belong to L^p for *all* sufficiently large p (see §2). Therefore the growth condition for $\theta(p)$ is violated.

The main tool we use to get an a priori estimate in B_Π are *wavelet decompositions* of vorticity. In fact we obtain a complete description of all spaces of this type in terms of wavelet coefficients. This allows to get the information about the action of a volume preserving homeomorphism (given by the fluid flow) on B_Γ .

We use the method of H. Bahouri and J.-Y. Chemin [BC] to obtain the necessary “paradifferential estimates” of the terms in Littlewood-Paley decomposition of the vorticity ω .

Some of the immediate problems left open include whether the time of existence is indeed finite for e.g., $\kappa = 1$ or this restriction is due to the weakness of the method. Also the relation with the transport theory developed by R. DiPerna and P.-L. Lions [DL], B. Desjardins [D1], H. Bahouri and J.-Y. Chemin [BC], J.-Y. Chemin and N. Lerner [CL], B. Desjardins [D2], especially for the flows that are not quasi Lipschitz, seems of interest.

1. Function spaces

Let $\Gamma : \mathbb{R} \rightarrow [1, \infty)$ be a locally Lipschitz continuous monotonically nondecreasing function satisfying the following conditions:

- (i) $\Gamma(\alpha) = 1$ for $\alpha \in (-\infty, -1]$, $\lim_{\alpha \rightarrow \infty} \Gamma(\alpha) = \infty$.
- (ii) There is a constant $C > 0$ such that $C^{-1}\Gamma(\beta) \leq \Gamma(\alpha) \leq C\Gamma(\beta)$ for $\alpha, \beta \in [-1, \infty)$, $|\alpha - \beta| \leq 1$.
- (iii) There is a constant $C > 0$ such that

$$C\Gamma(\alpha) \geq \int_\alpha^\infty 2^{-(\xi-\alpha)}\Gamma(\xi) \, d\xi, \alpha \in [-1, \infty).$$

EXAMPLE 1.1. $\Gamma(\alpha) = (N + 2)^\varepsilon \log_2^\kappa(\alpha + 3)$ for $\alpha \geq -1$, where $\varepsilon, \kappa \in [0, \infty)$, except $\varepsilon = \kappa = 0$.

We choose radial $\Phi \in S(\mathbb{R}^n)$ such that $\text{supp}\hat{\Phi} \subset \{\xi \in \mathbb{R}^n \mid |\xi| \leq 1\}$ and $|\hat{\Phi}(\xi)| \geq C > 0$ on $\{\xi \in \mathbb{R}^n \mid |\xi| \leq \frac{5}{6}\}$. We also choose radial $\varphi \in S(\mathbb{R}^n)$ with $\text{supp}\hat{\varphi} \subset \{\xi \in \mathbb{R}^n \mid \frac{1}{2} \leq |\xi| \leq 2\}$, $|\hat{\varphi}(\xi)| \geq C > 0$ on $\{\xi \in \mathbb{R}^n \mid \frac{3}{5} \leq |\xi| \leq \frac{5}{3}\}$. Let $\varphi_j(x) = 2^{jn}\varphi(2^jx)$ [i.e., $\hat{\varphi}_j(\xi) = \hat{\varphi}(2^{-j}\xi)$], $j \in \mathbb{Z}$.

DEFINITION 1.2. – Let for $f \in S'(\mathbb{R}^n)$

$$\begin{aligned}\Delta_{-1}f &= \hat{\Phi} \left(\frac{1}{i} \frac{\partial}{\partial x} \right) f = (\hat{\Phi} \cdot \hat{f})^\vee = \Phi * f ; \\ \Delta_j f &= \hat{\varphi}_j \left(\frac{1}{i} \frac{\partial}{\partial x} \right) f = (\hat{\varphi}_j \cdot \hat{f})^\vee = \varphi_j * f , \quad j \geq 0 ; \\ \Delta_j f &= 0 , \quad j \leq -2 ; \\ S_k f &= \sum_{j \leq k} \Delta_j f .\end{aligned}$$

DEFINITION 1.3. – Let Γ be the function satisfying (i)–(iii) above. Let

$$(1.1) \quad B_\Gamma = \left\{ f \in S'(\mathbb{R}^n) \mid \sum_{j=-1}^N \|\Delta_j f\|_{L^\infty} \leq C\Gamma(N) , N \geq -1 \right\} .$$

The norm on B_Γ is defined as the best constant in (1.1):

$$\|f\|_\Gamma = \sup_{N \geq -1} \left(\sum_{j=-1}^N \|\Delta_j f\|_{L^\infty} \right) \Gamma(N)^{-1} .$$

We give the characterization of B_Γ via the wavelet expansions. For standard Besov spaces see [FJ], [LM], [M].

Let radial $\psi \in S(\mathbb{R}^n)$ satisfy

$$\text{supp } \hat{\psi} \subset \left\{ \xi \in \mathbb{R}^n \mid \frac{1}{2} \leq |\xi| \leq 2 \right\}, \quad |\hat{\psi}(\xi)| \geq C > 0 \text{ on } \left\{ \xi \in \mathbb{R}^n \mid \frac{3}{5} \leq |\xi| \leq \frac{5}{3} \right\} .$$

Choose radial $\Psi \in S(\mathbb{R}^n)$ so that $\text{supp } \hat{\Psi} \subset \{ \xi \in \mathbb{R}^n \mid |\xi| \leq \pi \}$, $|\hat{\Psi}(\xi)| \geq C > 0$ on $\{ \xi \in \mathbb{R}^n \mid |\xi| \leq 1 \}$. We define the set of dyadic cubes as follows: for $m \in \mathbb{Z}$, $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$

$$Q_{m,k} = \{ x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid k_i 2^{-m} \leq x_i < (k_i + 1) 2^{-m} ; i = 1, \dots, n \} .$$

We set for any dyadic cube $Q = Q_{m,k}$

$$\psi_{m,k}(x) = \psi(2^m x - k) .$$

PROPOSITION 1.4. – Let $f \in B_\Gamma$. There is the following representation

$$(1.2) \quad f(x) = \sum_{k \in \mathbb{Z}^n} a_k \Psi(x - k) + \sum_{m=0}^{\infty} \sum_{k \in \mathbb{Z}^n} a_{m,k} \psi_{m,k}(x) .$$

The series in (1.2) is convergent in $S'(\mathbb{R}^n)$. The coefficients a_k , $a_{m,k}$ satisfy the following estimate

$$(1.3) \quad C^{-1} \|f\|_\Gamma \leq \sup_{N \geq -1} \left(\sum_{m=-1}^N \sup_{k \in \mathbb{Z}^n} |a_{m,k}| \right) \Gamma(N)^{-1} \leq C \|f\|_\Gamma ,$$

where the constant $C > 0$ does not depend on $f \in B_\Gamma$. For $m = -1$ we set $a_{-1,k} = a_k$, $k \in \mathbb{Z}^n$.

Proof. – It is possible to make the above choices of $\varphi, \Phi, \psi, \Psi$ so that

$$\hat{\Phi}(\xi)\hat{\Psi}(\xi) + \sum_{m=0}^{\infty} \hat{\varphi}(2^{-m}\xi)\hat{\psi}(2^{-m}\xi) = 1, \quad \xi \in \mathbb{R}^n .$$

Then, following [FJ]

$$(1.4) \quad f(\cdot) = \sum_{k \in \mathbb{Z}^n} (\Phi * f)(k)\Psi(\cdot - k) + \sum_{m=0}^{\infty} 2^{-mn}(\varphi_m * f)(2^{-m}k)\psi_m(\cdot - 2^{-m}k)$$

where $\psi_m(x) = 2^{mn}\psi(2^m x)$, $m \geq 0$. The series in the right side of (1.4) is convergent in $S'(\mathbb{R}^n)$. Therefore, we define $a_k = \Phi * f(k)$; $a_{m,k} = \varphi_m * f(2^{-m}k)$. By construction

$$\begin{aligned} |a_k| &\leq \|\Phi * f\|_{L^\infty} = \|\Delta_{-1}f\|_{L^\infty} , \\ |a_{m,k}| &\leq \|\varphi_m * f\|_{L^\infty} = \|\Delta_m f\|_{L^\infty} \end{aligned}$$

for $m \geq 0, k \in \mathbb{Z}^n$. This proves the second inequality in 1.3. It remains to prove the first inequality. We need the following estimate from [FJ].

LEMMA 1.5 ([FJ]). – *The following two inequalities hold true for any $M \in \mathbb{Z}_+, j, m \geq 0$*

$$(1.5) \quad |(\Delta_j \psi_{m,k})(x)| \leq C 2^{-(m-j)(n+1)}(1 + 2^j|x - k2^{-m}|)^{-M} \text{ for } j \leq m, \text{ and}$$

$$(1.6) \quad |(\Delta_j \psi_{m,k})(x)| \leq C 2^{-(j-m)}(1 + 2^m|x - k2^{-m}|)^{-M} \text{ for } m \leq j .$$

The same inequalities are valid for $j = -1, m \geq 0$ in case (1.5) and for $m = -1, j \geq 0$ in case (1.6) where $\psi_{-1,k}(\cdot) = \Psi(\cdot - k)$.

To prove Proposition 1.4 we have to estimate $\|\Delta_j f\|_{L^\infty}$ where f is given by (1.2). We have from Lemma 1.5

$$\begin{aligned} \|\Delta_{-1}f\|_{L^\infty} &\leq \left\| \Delta_{-1} \sum_{k \in \mathbb{Z}^n} a_k \Psi(\cdot - k) \right\|_{L^\infty} + \sum_{m=0}^{\infty} \left\| \Delta_{-1} \sum_{k \in \mathbb{Z}^n} a_{m,k} \psi_{m,k} \right\|_{L^\infty} \\ &\leq C \left\| \sum_{k \in \mathbb{Z}^n} a_k \Psi(\cdot - k) \right\|_{L^\infty} + \sum_{m=0}^{\infty} \left\| \sum_{k \in \mathbb{Z}^n} a_{m,k} \Delta_{-1} \psi_{m,k} \right\|_{L^\infty} \\ &\leq C \sup_{k \in \mathbb{Z}^n} |a_k| \cdot \sup_x \sum_{k \in \mathbb{Z}^n} (1 + |x - k|)^{-M} \\ &\quad + C \sum_{m=0}^{\infty} \sup_{k \in \mathbb{Z}^n} |a_{m,k}| \cdot 2^{-m(n+1)} \sup_x \sum_k (1 + |x - k2^{-m}|)^{-M} \\ &\leq C \sup_{k \in \mathbb{Z}^n} |a_k| + C \sum_{m=0}^{\infty} 2^{-m-1} \sup_{k \in \mathbb{Z}^n} |a_{m,k}| . \end{aligned}$$

Likewise, for $j \geq 0$, applying Lemma 1.5

$$\begin{aligned}
\|\Delta_j f\|_{L^\infty} &\leq \left\| \sum_{k \in \mathbb{Z}^n} a_k \Delta_j \Psi(\cdot - k) \right\|_{L^\infty} \\
&\quad + \sum_{m=0}^j \left\| \sum_{k \in \mathbb{Z}^n} a_{m,k} \Delta_j \psi_{m,k} \right\|_{L^\infty} \\
&\quad + \sum_{m=j+1}^{\infty} \left\| \sum_{k \in \mathbb{Z}^n} a_{m,k} \Delta_j \psi_{m,k} \right\|_{L^\infty} \\
&\leq C 2^{-j} \sup_{k \in \mathbb{Z}^n} |a_k| \left\| \sum_{k \in \mathbb{Z}^n} (1 + |\cdot - k|)^{-M} \right\|_{L^\infty} \\
&\quad + C \sum_{m=0}^j 2^{-(j-m)} \sup_{k \in \mathbb{Z}^n} |a_{m,k}| \left\| \sum_{k \in \mathbb{Z}^n} (1 + 2^m |\cdot - k 2^{-m}|)^{-M} \right\|_{L^\infty} \\
&\quad + C \sum_{m=j+1}^{\infty} 2^{-(m-j)(n+1)} \sup_{k \in \mathbb{Z}^n} |a_{m,k}| \left\| \sum_{k \in \mathbb{Z}^n} (1 + 2^j |\cdot - k 2^{-m}|)^{-M} \right\|_{L^\infty} \\
&\leq C 2^{-j-1} \sup_{k \in \mathbb{Z}^n} |a_k| + C \sum_{m=0}^j 2^{-(j-m)} \sup_{k \in \mathbb{Z}^n} |a_{m,k}| \\
&\quad + C \sum_{m=j+1}^{\infty} 2^{-(m-j)} \sup_{k \in \mathbb{Z}^n} |a_{m,k}|.
\end{aligned}$$

Let

$$\begin{aligned}
\sup_{k \in \mathbb{Z}^n} |a_{m,k}| &= b_m, \quad m \geq -1; \\
\sum_{j=-1}^m b_j &= d_m, \quad m \geq -1; \quad d_{-2} = 0.
\end{aligned}$$

Then, for every $j \geq -1$, $N \geq -1$

$$\begin{aligned}
\|\Delta_j f\|_{L^\infty} &\leq C \sum_{m=-1}^j 2^{-(j-m)} b_m + C \sum_{m=j+1}^{\infty} 2^{-(m-j)} b_m; \\
\sum_{j=-1}^N \|\Delta_j f\|_{L^\infty} &\leq C \sum_{m=-1}^N \left(\sum_{j=-1}^{m-1} 2^{-(m-j)} + \sum_{j=m}^N 2^{-(j-m)} \right) b_m \\
&\quad + C \sum_{m=N+1}^{\infty} \left(\sum_{j=-1}^N 2^{-(m-j)} \right) b_m \\
&\leq C \sum_{m=-1}^N b_m + C \sum_{m=N+1}^{\infty} 2^{-(m-N)} (d_m - d_{m-1}) \\
&\leq C d_N + C 2^N \sum_{m=N+1}^{\infty} (2^{-m} - 2^{-(m+1)}) d_m \\
&\leq C d_N + C \sum_{m=N+1}^{\infty} 2^{-(m-N)} d_m.
\end{aligned}$$

Suppose for any $m \geq -1$, $d_m \leq C\Gamma(m)$. Then the right side of the last inequality is

$$\leq C\Gamma(N) + C \int_N^\infty 2^{-(\xi-N)}\Gamma(\xi) d\xi \leq C\Gamma(N)$$

because of the conditions (ii) and (iii) describing Γ . This concludes the proof of the proposition.

We now construct the predual space to B_Γ . Let H_Γ be defined as follows:

$$H_\Gamma = \left\{ f \in S'(\mathbb{R}^n) \mid \exists \{d_j\}_{j=-1}^\infty, d_j \geq 0, \sum_{j=-1}^\infty d_j < \infty \right. \\ \left. \text{such that } \|\Delta_m f\|_{L^1} \leq \sum_{j \geq m} d_j \Gamma(j)^{-1}, m \geq -1 \right\}.$$

We set

$$\|f\|_{\mathbb{T}} = \inf_{\{d_j\}} \sum_{j=-1}^\infty d_j.$$

PROPOSITION 1.6. – *The dual space to H_Γ is isomorphic to B_Γ :*

$$H'_\Gamma \approx B_\Gamma.$$

Proof. – We omit the proof which is fairly standard (cf. [P]).

PROPOSITION 1.7. – *Let $f \in H_\Gamma$. There is the following representation:*

$$(1.7) \quad f(x) = \sum_{k \in \mathbb{Z}^n} a_k \Psi(x - k) + \sum_{m=0}^\infty \sum_{k \in \mathbb{Z}^n} a_{m,k} 2^{mn} \psi_{m,k}(x).$$

The series in (1.7) is convergent in $S'(\mathbb{R}^n)$. Moreover there is a constant $C > 0$ which is independent of f such that

$$(1.8) \quad C^{-1} \|f\|_{\mathbb{T}} \leq \inf_{\{d_j\}_{j=-1}^\infty, d_j \geq 0, \sum_{j=-1}^\infty d_j < \infty} \sum_{j=-1}^\infty d_j \leq C \|f\|_{\mathbb{T}}. \\ \sum_{k \in \mathbb{Z}^n} |a_{m,k}| \leq \sum_{j=m}^\infty d_j \Gamma(j)^{-1}, m \geq -1$$

We omit the proof which is similar to the proof of Proposition 1.4.

REMARK 1.8. – In particular predual H_Γ of the Banach space B_Γ is separable, and $S \hookrightarrow H_\Gamma$ is a dense embedding.

REMARK 1.9. – Since the function spaces B_Γ and H_Γ do not depend upon the choice of Φ, φ satisfying standard conditions (see e.g., [P], [FJ]) Propositions 1.4, 1.7 provide a complete description of these spaces in terms of wavelet decompositions with respect to band-limited wavelets constructed by P. Lemarié and Y. Meyer [LM], [M2].

REMARK 1.10. – If $\Gamma(N) = N + 2$, $N \geq -1$ then obviously $L^\infty \hookrightarrow B_\Gamma$. In case $\Gamma(N)$ grows slower than $(N + 2)$, that is $\Gamma(N) = o(N + 2)$ as $N \rightarrow \infty$, the above embedding is no longer valid. It means for such a $\Gamma(N)$ the space B_Γ is not comparable with the space L^∞ . As we mentioned in the introduction V. Yudovich proved the global existence and uniqueness theorem [Y1] in dimension 2 for the Euler equations with vorticity in L^∞ (uniqueness for arbitrary $n \geq 2$).

2. A counterexample

PROPOSITION 2.1. – For any $\rho \in (1, \infty)$ there exists a function f such that $f \in B_\Gamma$, $f \in \bigcap_{p \in (1, \rho)} L^p(\mathbb{R}^n)$ and $f \notin L^p(\mathbb{R}^n)$ for any $p \geq \rho$.

Proof. – We use the following fundamental characterization of L^p -spaces [LM], [M2]. To simplify the notation we give the construction in dimension $n = 1$. Let ψ, Ψ be obtained from the MRA procedure [M2]. Let

$$f(x) = \sum_{k \in \mathbb{Z}} a_k \Psi(x - k) + \sum_{m=0}^{\infty} \sum_{k \in \mathbb{Z}} a_{m,k} \psi(2^m x - k).$$

THEOREM 2.2 ([LM], [M2]). – Let $p \in (1, \infty)$. Then $f \in L^p$ if and only if the following two conditions are satisfied:

$$(2.1) \quad \{a_k\}_{k \in \mathbb{Z}} \in \ell^p;$$

$$(2.2) \quad g = \left(\sum_{m=0}^{\infty} \sum_{k \in \mathbb{Z}} |a_{m,k}|^2 \chi_{Q_{m,k}} \right)^{1/2} \in L^p.$$

We note that the function g in (2.2) is a discrete analogue of the Littlewood-Paley quadratic function; $\chi_{Q_{m,k}}$ denotes the characteristic function of a dyadic cube $Q_{m,k}$.

To construct the example in Proposition 2.1 let $a_k = 0$ for $k \in \mathbb{Z}$ so that (2.1) is satisfied.

Since $\Gamma(\alpha) \rightarrow \infty$ as $\alpha \rightarrow \infty$ there is a sequence $\{c_m\}_{m=0}^{\infty}$ satisfying the following 3 conditions:

$$(2.3) \quad c_m \rightarrow 0 \text{ as } m \rightarrow \infty;$$

$$(2.4) \quad \sum_{m=0}^N |c_m| \leq C\Gamma(N), \quad N \geq 0;$$

$$(2.5) \quad \sum_{m=0}^{\infty} |c_m|^2 = \infty.$$

Indeed we proceed by induction.

We construct a strictly monotonic sequence of nonnegative integers $\{n_i\}$, $n_1 = 0$ and set

$$c_m = \begin{cases} i^{-1/2}, & m = n_i \\ 0, & m \neq n_i \text{ for all } i \geq 1. \end{cases}$$

Thus (2.3) and (2.5) are satisfied. We set $C = \Gamma(0)^{-1}$ and assume n_1, \dots, n_q are already constructed so that (2.4) is satisfied for all $N \leq n_q$. Choose $n_{q+1} > n_q$ so that

$$\sum_{i=1}^{q+1} i^{-1/2} \leq C\Gamma(n_{q+1}).$$

Such a choice could be made since $\Gamma(\alpha) \rightarrow \infty$. This takes care of (2.4). We fix a sequence $\{c_m\}_{m=0}^\infty$ satisfying (2.3)–(2.5) and let

$$f = \sum_{m=0}^\infty c_m \sum_{k=0}^{2^m-1} \varepsilon_{m,k} \psi(2^m x - k) ,$$

where $\varepsilon_{m,k}$ are taking on values 0 and 1.

We now assert that we can choose $\varepsilon_{m,k}$ in such a way that

$$(2.6) \quad g(x) = \begin{cases} 0 , & x \notin Q_{0,0} = [0, 1) \\ x^{-\rho^{-1}} , & x \in Q_{0,0} . \end{cases}$$

Indeed we start with

$$\varepsilon_{0,0} = \begin{cases} 1 , & \text{if } |c_0|^2 \leq 1 \\ 0 , & \text{otherwise.} \end{cases}$$

Assume we constructed $\varepsilon_{m,k}$, $m \leq \ell$ so that

$$(2.7) \quad \sum_{m=0}^\ell |c_m|^2 \sum_{k=1}^{2^m-1} \varepsilon_{m,k} \chi_{Q_{m,k}}(x) \leq x^{-2\rho^{-1}} , \quad x \in Q_{0,0} .$$

To do the induction step we simply set $\varepsilon_{\ell+1,\kappa} = 1$ in case

$$(2.8) \quad \sum_{m=0}^\ell |c_m|^2 \sum_{k=1}^{2^m-1} \varepsilon_{m,k} \chi_{Q_{m,k}}(x) + |c_{\ell+1,\kappa}|^2 \chi_{Q_{\ell+1,\kappa}} \leq x^{-2\rho^{-1}}$$

for $x \in Q_{\ell+1,\kappa}$, $0 \leq \kappa \leq 2^{\ell+1} - 1$ and $\varepsilon_{\ell+1,\kappa} = 0$ otherwise.

Then (2.7) is satisfied on the $(\ell + 1)$ -st step because of (2.8) and (2.6) follows from (2.3) and (2.5). Obviously $g(x)$ in (2.6) satisfies the conditions $g \in \bigcap_{p \in (1, \rho)} L^p$, $g \notin L^p$ for any $p \geq \rho$.

We conclude from the Theorem 2.2 that f satisfies the same properties. Furthermore, from Proposition 1.4 and (2.4) $f \in B_\Gamma$. This concludes the proof.

REMARK 2.3. – The function f in Proposition 2.1 could be chosen to have *compact support*. Indeed, using sufficiently regular wavelets with compact support [D], [M2] and repeating the same construction word for word produces such a function.

3. Estimates for the Biot-Savart law

Let the vorticity $\omega \in L^{p_0} \cap B_\Gamma$, $1 < p_0 < n$. We define the velocity as

$$u = \mathcal{K} * \omega ,$$

\mathcal{K} being the kernel in the Biot-Savart law.

THEOREM 3.1. – *The function u is continuous and bounded on \mathbb{R}^n with the following modulus of continuity:*

$$(3.1) \quad |u(x) - u(y)| \leq C(\|\omega\|_{L^{p_0}} + \|\omega\|_{\Gamma} \Gamma(-\log_2 |x - y|)) |x - y|, \quad x \neq y;$$

$$(3.2) \quad \|u\|_{L^\infty} \leq C(\|\omega\|_{L^{p_0}} + \|\omega\|_{\Gamma}), \quad 1 < p_0 < n.$$

Proof. – For any $N \geq -1$, $x \neq y$

$$(3.3) \quad \begin{aligned} |u(x) - u(y)| &\leq \sum_{j=-1}^N \|\Delta_j \nabla u\|_{L^\infty} |x - y| + 2 \sum_{j=N+1}^{\infty} \|\Delta_j u\|_{L^\infty} \\ &\leq C \sum_{j=-1}^N \|\Delta_j \nabla u\|_{L^\infty} |x - y| + C \sum_{j=N+1}^{\infty} 2^{-j} \|\Delta_j \omega\|_{L^\infty}. \end{aligned}$$

We estimate the second term in the right side of (3.3) as follows. Let

$$d_m = \sum_{j=-1}^m \|\Delta_j \omega\|_{L^\infty}.$$

Then,

$$(3.4) \quad \begin{aligned} \sum_{j=N+1}^{\infty} 2^{-j} \|\Delta_j \omega\|_{L^\infty} &\leq \sum_{j=N+1}^{\infty} 2^{-j} (d_j - d_{j-1}) \\ &= -2^{-(N+1)} d_N + \sum_{j=N+1}^{\infty} d_j (2^{-j} - 2^{-(j+1)}) \\ &\leq -2^{-(N+1)} d_N + \|\omega\|_{\Gamma} \sum_{j=N+1}^{\infty} \Gamma(j) 2^{-j} \\ &\leq C 2^{-N} \Gamma(N) \|\omega\|_{\Gamma} \end{aligned}$$

because of the conditions (ii) and (iii). We estimate now the first term in the right side of (3.3)

$$(3.5) \quad \begin{aligned} \sum_{j=-1}^N \|\Delta_j \nabla u\|_{L^\infty} &\leq \|\Delta_{-1} \nabla u\|_{L^\infty} + C \sum_{j=0}^N \|\Delta_j \omega\|_{L^\infty} \\ &\leq C \|\Delta_{-1} \nabla u\|_{L^{p_0}} + C \Gamma(N) \|\omega\|_{\Gamma} \\ &\leq C \|\Delta_{-1} \omega\|_{L^{p_0}} + C \Gamma(N) \|\omega\|_{\Gamma} \\ &\leq C \|\omega\|_{L^{p_0}} + C \Gamma(N) \|\omega\|_{\Gamma}. \end{aligned}$$

Here we used Bernstein's inequality and boundedness of the Calderón-Zygmund operator $\omega \mapsto \nabla u$ in L^{p_0} . The inequalities (3.3)–(3.5) yield

$$|u(x) - u(y)| \leq C(\|\omega\|_{L^{p_0}} + \|\omega\|_{\Gamma} \Gamma(N)) |x - y| + C 2^{-N} \Gamma(N) \|\omega\|_{\Gamma}.$$

We choose $N = \lceil -\log_2 |x - y| \rceil$ (in case $|x - y| \leq 2$, otherwise choose $N = -1$). This proves (3.1). It remains to prove (3.2). We have

$$\begin{aligned} \|u\|_{L^\infty} &\leq \|\Delta_{-1}u\|_{L^\infty} + \sum_{j=0}^{\infty} \|\Delta_j u\|_{L^\infty} \\ &\leq \|\Delta_{-1}u\|_{L^\infty} + C \sum_{j=0}^{\infty} 2^{-j} \|\Delta_j \omega\|_{L^\infty} . \end{aligned}$$

The second term is $\leq C\|\omega\|_\Gamma$ as follows from (3.4) for $N = -1$. To estimate the first term we write for $\chi \in \mathcal{D}(\mathbb{R}^n)$, $\chi \equiv 1$ near the origin,

$$\begin{aligned} (3.6) \quad \|\Delta_{-1}u\|_{L^\infty} &\leq \|(\chi\mathcal{K}) * \Delta_{-1}\omega\|_{L^\infty} + \|((1 - \chi)\mathcal{K}) * \Delta_{-1}\omega\|_{L^\infty} \\ &\leq \|\chi\mathcal{K}\|_{L^1} \|\Delta_{-1}\omega\|_{L^\infty} + \|(1 - \chi)\mathcal{K}\|_{L^{q_0}} \|\Delta_{-1}\omega\|_{L^{p_0}} \\ &\leq C\|\Delta_{-1}\omega\|_{L^{p_0}} \end{aligned}$$

since $\chi\mathcal{K} \in L^1$ and $(1 - \chi)\mathcal{K} \in L^{q_0}$ ($\frac{1}{q_0} + \frac{1}{p_0} = 1$). This concludes the proof.

PROPOSITION 3.2. – *Let $\omega \in L^{p_0} \cap L^{p_1}$ where $1 < p_0 < n$, $n < p_1 < \infty$. Then for all $x, y \in \mathbb{R}^n$, $|x - y| \leq 1$ we have*

$$(3.7) \quad |u(x) - u(y)| \leq C|x - y|^{1 - \frac{n}{p_1}} \|\omega\|_{L^{p_1}} ,$$

where $C > 0$ depends on n and p_1 ,

$$(3.8) \quad \|u\|_{L^\infty} \leq C\|\omega\|_{L^{p_0}}^{\frac{p_0(p_1 - n)}{n(p_1 - p_0)}} \|\omega\|_{L^{p_1}}^{\frac{p_1(n - p_0)}{n(p_1 - p_0)}} .$$

Proof. – For any $N \geq -1$

$$\begin{aligned} |u(x) - u(y)| &\leq \sum_{j=-1}^N |\Delta_j u(x) - \Delta_j u(y)| + 2 \sum_{j=N+1}^{\infty} \|\Delta_j u\|_{L^\infty} \\ &\leq \sum_{j=-1}^N \|\Delta_j \nabla u\|_{L^\infty} |x - y| + C \sum_{j=N+1}^{\infty} 2^{-j} \|\Delta_j \omega\|_{L^\infty} \\ &\leq \|\Delta_{-1} \nabla u\|_{L^\infty} |x - y| + C \sum_{j=0}^N \|\Delta_j \omega\|_{L^\infty} |x - y| \\ &\quad + C \sum_{j=N+1}^{\infty} 2^{-j + \frac{nj}{p_1}} \|\Delta_j \omega\|_{L^{p_1}} \\ &\leq C\|\Delta_{-1}\omega\|_{L^{p_1}} |x - y| + \sum_{j=0}^N 2^{nj/p_1} \|\omega\|_{L^{p_1}} |x - y| \\ &\quad + C2^{-N(1 - \frac{n}{p_1})} \|\omega\|_{L^{p_1}} \\ &\leq C\|\omega\|_{L^{p_1}} \left(|x - y| + 2^{Nn/p_1} |x - y| + 2^{-N(1 - \frac{n}{p_1})} \right) . \end{aligned}$$

We choose $N = \lceil -\log_2 |x - y| \rceil$. This proves (3.7).

To estimate $\|u\|_{L^\infty}$ we proceed as in (3.6)

$$\|u\|_{L^\infty} \leq \|(1 - \chi)\mathcal{K}\|_{L^{q_0}} \|\omega\|_{L^{p_0}} + \|\chi\mathcal{K}\|_{L^{q_1}} \|\omega\|_{L^{p_1}} .$$

Replacing $\chi(\cdot)$ by $\chi(\cdot/R)$ and choosing the appropriate R yields (3.8).

PROPOSITION 3.3. – Let $1 < p_0 < n$, $\omega \in L^{p_0}$. Then $u = \mathcal{K} * \omega \in L^{np_0/(n-p_0)}$ and

$$(3.9) \quad \|u\|_{L^{np_0/(n-p_0)}} \leq C \|\omega\|_{L^{p_0}} .$$

Proof. – Follows immediately from the Hardy-Littlewood-Sobolev inequality.

4. Action of volume preserving homeomorphisms on B_Γ

Let $g : \mathbb{R}^n \xrightarrow{\text{onto}} \mathbb{R}^n$ be a volume preserving homeomorphism. Assume there are decreasing functions $\sigma, \rho : \mathbb{R} \rightarrow (0, \infty)$, $\lim_{\xi \rightarrow \infty} \sigma(\xi) = 0$, $\lim_{\xi \rightarrow \infty} \rho(\xi) = 0$, satisfying the following

$$(4.1) \quad |g^{-1}(x) - g^{-1}(y)| \leq c_{g^{-1}} \sigma(-\log_2 |x - y|) ;$$

$$(4.2) \quad |g(x) - g(y)| \leq c_g \rho(-\log_2 |x - y|) ;$$

$$(4.3) \quad \sigma(\xi) = 2^{-\xi} \quad , \quad \rho(\xi) = 2^{-\xi} \quad \text{for } \xi \leq 0 ;$$

$$(4.4) \quad \log_2 \sigma(\xi) \quad , \quad \log_2 \rho(\xi) \quad \text{are convex.}$$

It follows from (4.3), (4.4) that

$$\sigma(\xi) \geq 2^{-\xi} \quad , \quad \rho(\xi) \geq 2^{-\xi} \quad , \quad \xi \in \mathbb{R} .$$

We need the following technical statement which is the main ingredient of the proof of the a priori estimate.

PROPOSITION 4.1. – Assume g, ρ, σ satisfy the above properties and for $m \geq -1$

$$(4.5) \quad f_m(x) = \sum_{k \in \mathbb{Z}^n} a_{m,k} \psi(2^m x - k) ,$$

where

$$(4.6) \quad \sup_{k \in \mathbb{Z}^n} |a_{m,k}| \leq 1 ,$$

$$\left[f_{-1}(x) = \sum_{k \in \mathbb{Z}^n} a_k \Psi(x - k) \right] .$$

Then there is a constant $\gamma = \gamma(n)$ such that for $j \leq m$

$$(4.7) \quad \|\Delta_j(f_m \circ g^{-1})\|_{L^\infty} \leq C c_{g^{-1}}^n c_g^{n+1} 2^{\gamma j} \rho(m) .$$

Proof. – We have

$$(4.8) \quad \Delta_j(f_m \circ g^{-1})(x) = \sum_k a_{m,k} 2^{nj} \int \varphi(2^j y) \psi(2^m g^{-1}(x - y) - k) dy .$$

Since the statement is evident for $j = m = -1$,

$$\left[\left\| \sum_k a_k \Psi(x - k) \right\|_{L^\infty} \leq C \sup_k |a_k| \right]$$

we will assume that $m \geq 0$.

Let

$$(4.9) \quad \begin{aligned} h_{j,m,k}(x) &= 2^{nj} \int \varphi(2^j y) \psi(2^m g^{-1}(x - y) - k) dy \\ &= \int \psi(2^m g^{-1}(x - 2^{-j} y) - k) \varphi(y) dy , \end{aligned}$$

where we made a change of variables $y \rightarrow 2^{-j} y$. Then since $\int \psi(y) dy = 0$,

$$(4.10) \quad \begin{aligned} h_{j,m,k}(2^{-j} x + g(2^{-m} k)) &= \int \psi(2^m g^{-1}(2^{-j}(x - y) + g(2^{-m} k)) - k) \varphi(y) dy \\ &= \left(\int_{|x-y| < \frac{|x|}{2}} + \int_{|x-y| \geq \frac{|x|}{2}} \right) q(x - y) \int_0^1 \nabla \varphi(x + \tau(y - x)) \cdot (y - x) d\tau dy \\ &= I + J , \end{aligned}$$

where

$$q(z) = \psi(2^m g^{-1}(2^{-j} z + g(2^{-m} k)) - k) .$$

We estimate I first.

Since $\varphi \in S(\mathbb{R}^n)$ we have for any real M

$$(4.11) \quad \begin{aligned} |I| &\leq \int_{|x-y| < \frac{|x|}{2}} |q(x - y)| |x - y| \int_0^1 |\nabla \varphi(x + \tau(y - x))| d\tau dy \\ &\leq C \int_{|z| < \frac{|x|}{2}} |q(z)| |z| dz (1 + |x|)^{-M} . \end{aligned}$$

We substitute

$$(4.12) \quad z = 2^j (g(2^{-m}(w + k)) - g(2^{-m} k)) ;$$

$$g(2^{-m}(w + k)) = 2^{-j} z + g(2^{-m} k) ;$$

$$(4.13) \quad w = 2^m g^{-1}(2^{-j} z + g(2^{-m} k)) - k .$$

Since g is a volume preserving homeomorphism, for any real M_1, M_2 and since ρ is a decreasing function,

$$\begin{aligned}
 (4.14) \quad \int_{|z| < \frac{|x|}{2}} |q(z)| |z| dz &= 2^{(j-m)n+j} \int_{|z| < \frac{|x|}{2}} |\psi(w)| |g(2^{-m}(w+k)) - g(2^{-m}k)| dw \\
 &\leq 2^{(j-m)n+j} c_g \left(\int_{|w| < 2^m} + \int_{|w| \geq 2^m} \right) |\psi(w)| \rho(m - \log_2 |w|) dw \\
 &\leq 2^{(j-m)n+j} c_g \int_{|w| \geq 2^m} |\psi(w)| 2^{-m} |w| dw \\
 &\quad + 2^{(j-m)n+j} C c_g \sum_{\ell=-m}^{\infty} \rho(m+\ell) 2^{-n\ell} (1+2^{-\ell})^{-M_1} \\
 &\leq C c_g 2^{(j-m)(n+1)-mM_2} + C c_g 2^{(j-m)n+j} \left(\rho(0) 2^{-(M_1-n)m} \right. \\
 &\quad \left. + \rho(1) 2^{-(M_1-n)(m-1)} + \dots + \rho(m) 2^0 \right. \\
 &\quad \left. + \rho(m+1) 2^{-n} + \rho(m+2) 2^{-2n} \right. \\
 &\quad \left. + \dots + \rho(m+\ell) 2^{-\ell n} + \dots \right) \\
 &\leq C c_g 2^{(j-m)(n+1)-mM_2} + C c_g 2^{(j-m)n+j} \left(\rho(0) 2^{-(M_1-n)m} \right. \\
 &\quad \left. + \rho(1) 2^{-(M_1-n)(m-1)} + \dots + \rho(m) 2^0 \right).
 \end{aligned}$$

LEMMA 4.2. – Let ρ be any function that satisfies (4, 3), (4.4). Let $M_3 > M_4 > 0$. Then

$$\rho(0)^{M_4} 2^{-M_3 m} + \rho(1)^{M_4} 2^{-M_3(m-1)} + \dots + \rho(m)^{M_4} 2^0 \leq C \rho(m)^{M_4}.$$

Proof. – Since $\log_2 \rho$ is convex,

$$\begin{aligned}
 &\rho(0)^{M_4} 2^{-mM_3} + \rho(1)^{M_4} 2^{-(m-1)M_3} + \dots + \rho(m)^{M_4} 2^0 \\
 &\leq \rho(0)^{M_4} 2^{-mM_3} + \rho(0)^{\frac{m-1}{m}M_4} \rho(m)^{\frac{1}{m}M_4} 2^{-(m-1)M_3} + \dots + \rho(m)^{M_4} 2^0 \\
 &\leq \rho(m)^{M_4} \left\{ \left(\frac{\rho(0)}{\rho(m)} \right)^{\frac{m}{m}M_4} 2^{-M_3 m} + \left(\frac{\rho(0)}{\rho(m)} \right)^{\frac{m-1}{m}M_4} 2^{-M_3(m-1)} + \dots + \left(\frac{\rho(0)}{\rho(m)} \right)^0 2^0 \right\} \\
 &\leq \rho(m)^{M_4} \left(1 - 2^{-M_3} \rho(m)^{-M_4/m} \right)^{-1}.
 \end{aligned}$$

But $\rho(m) \geq 2^{-m}$; therefore, $\rho(m)^{1/m} \geq \frac{1}{2}$. This implies the statement.

REMARK 4.3. – In particular,

$$\sigma(0)^{M_4} 2^{-M_3 m} + \sigma(1)^{M_4} 2^{-M_3(m-1)} + \dots + \sigma(m)^{M_4} 2^0 \leq C \sigma(m)^{M_4}.$$

To continue the proof of the proposition we note that the right side of (4.14) is

$$\leq C c_g 2^{(j-m)(n+1)-mM_2} + C c_g 2^{(j-m)n+j} \rho(m) \leq C c_g 2^{(j-m)n+j} \rho(m)$$

in case $M_2 \geq 0$.

Choosing $M_2 = 0$, $M_1 = n + 2$ we obtain from (4.11), (4.14)

$$|I| \leq Cc_g 2^{(j-m)n+j} \rho(m)(1 + |x|)^{-M} .$$

We now turn to J in the right side of (4.10)

$$(4.15) \quad |J| \leq \int_{|x-y| \geq \frac{|x|}{2}} |q(x-y)| \int_0^1 d\tau |\nabla\varphi(x + \tau(y-x))| |y-x| dy$$

$$\leq C \int_{|x-y| \geq \frac{|x|}{2}} |q(x-y)| |x-y|(1 + |x-y|)^{-1} dy$$

since

$$\int_0^1 |\nabla\varphi(x + \tau(y-x))| d\tau \leq C(1 + |x-y|)^{-1} .$$

The right side of (4.15) after the substitution (4.12), (4.13) becomes

$$(4.16) \quad \leq C \int_{|z| \geq \frac{|x|}{2}} |q(z)| |z|(1 + |z|)^{M-1} dz (1 + |x|)^{-M}$$

$$\leq C \int_{|z| \geq \frac{|x|}{2}} |q(z)| (|z| + |z|^M) dz (1 + |x|)^{-M}$$

$$\leq C 2^{(j-m)n} \int |\psi(w)| \{ 2^j |g(2^{-m}(w+k)) - g(2^{-m}k)|$$

$$+ 2^{Mj} |g(2^{-m}(w+k)) - g(2^{-m}k)|^M \} dw (1 + |x|)^{-M}$$

$$\leq C 2^{(j-m)n} \left(\int_{|w| \geq 2^m} + \int_{|w| < 2^m} \right) |\psi(w)| \left((c_g 2^j \rho(m - \log_2 |w|) \right.$$

$$+ c_g^M 2^{Mj} \rho(m - \log_2 |w|)^M \Big) dw (1 + |x|)^{-M}$$

$$\leq C 2^{(j-m)n} \int_{|w| \geq 2^m} |\psi(w)| \left(2^{j-m} c_g |w| + 2^{M(j-m)} c_g^M |w|^M \right) dw (1 + |x|)^{-M}$$

$$+ C 2^{(j-m)n} \left(\sum_{\ell=-m}^{\infty} \left\{ 2^{-n\ell} (1 + 2^{-\ell})^{-M_5} 2^j c_g \rho(m + \ell) \right. \right.$$

$$\left. \left. + 2^{-n\ell} (1 + 2^{-\ell})^{-M_5} 2^{Mj} c_g^M \rho(m + \ell)^M \right\} \right) (1 + |x|)^{-M}$$

$$\leq C c_g^M 2^{(j-m)(n+1) - M_6 m} (1 + |x|)^{-M}$$

$$+ \left(C c_g 2^{(j-m)n+j} \rho(m) + C c_g^M 2^{(j-m)n+Mj} \rho(m)^M \right) (1 + |x|)^{-M}$$

$$\leq C 2^{(j-m)n} \left(c_g 2^j \rho(m) + c_g^M 2^{Mj} \rho(m)^M \right) (1 + |x|)^{-M}$$

for appropriate choice of M_5, M_6 . Combining the estimates for I and J and using (4.10) we obtain ($m \geq j$)

$$(4.17) \quad \|\Delta_j(f_m \circ g^{-1})\|_{L^\infty} \leq \sup_x \sum_k |a_{m,k}| |h_{j,m,k}(x)| \\ \leq \kappa_{j,m} \sup_k |a_{m,k}| \sup_x \sum_k (1 + 2^j |x - g(2^{-m}k)|)^{-M} \\ \leq \kappa_{j,m} \sup_x \sum_k (1 + 2^j |x - g(2^{-m}k)|)^{-M},$$

where

$$(4.18) \quad \kappa_{j,m} = C 2^{(j-m)n} \left(c_g 2^j \rho(m) + c_g^M 2^{Mj} \rho(m)^M \right).$$

Next we have to take care of the sum in the right side of (4.17). Let $x \in \mathbb{R}^n$ be fixed. Let

$$(4.19) \quad N(x, \lambda) = \# \{k \in \mathbb{Z}^n \mid |x - g(2^{-m}k)| \leq \lambda\}.$$

Then,

$$(4.20) \quad \sum_k \left(1 + 2^j |x - g(2^{-m}k)|\right)^{-M} = \int_{-\infty}^{\infty} (1 + 2^j \lambda)^{-M} dN(x, \lambda) \\ = M 2^j \int_0^{\infty} (1 + 2^j \lambda)^{-M-1} N(x, \lambda) d\lambda.$$

We need the estimate for $N(x, \lambda)$.

$$(4.21) \quad N(x, \lambda) \leq \# \{k \in \mathbb{Z}^n \mid |g^{-1}(x) - 2^{-m}k| \leq c_{g^{-1}} \sigma(-\log_2 \lambda)\} \\ = \# \{k \in \mathbb{Z}^n \mid |2^m g^{-1}(x) - k| \leq 2^m c_{g^{-1}} \sigma(-\log_2 \lambda)\} \\ \leq C(1 + 2^{mn} c_{g^{-1}}^n \sigma(-\log_2 \lambda)^n).$$

Combining (4.19)–(4.21) we obtain

$$(4.22) \quad \sum_k (1 + 2^j |x - g(2^{-m}k)|)^{-M} \\ \leq C 2^j \int_0^{\infty} (1 + 2^j \lambda)^{-M-1} (1 + 2^{mn} c_{g^{-1}}^n \sigma(-\log_2 \lambda)^n) d\lambda \\ \leq C + C c_{g^{-1}}^n 2^{mn} \int_0^{\infty} (1 + \lambda)^{-M-1} \sigma(j - \log_2 \lambda)^n d\lambda.$$

To estimate the integral we split

$$(4.23) \quad \left(\int_0^{2^j} + \int_{2^j}^{\infty} \right) (1 + \lambda)^{-M-1} \sigma(j - \log_2 \lambda)^n d\lambda \\ \leq 2^{-jn} \int_{2^j}^{\infty} (1 + \lambda)^{-M-1} \lambda^n d\lambda + C \sum_{\ell=-j}^{\infty} \sigma(j + \ell)^n 2^{-\ell} (1 + 2^{-\ell})^{-M-1} \\ \leq C 2^{-jM} + C \sigma(j)^n \leq C \sigma(j)^n,$$

provided M is sufficiently large ($M > n$). From (4.17)–(4.23) and since $\sigma(j) \geq 2^{-j}$

$$(4.24) \quad \|\Delta_j(f_m \circ g^{-1})\|_{L^\infty} \leq C 2^{(j-m)n} (c_g 2^j \rho(m) + c_g^M 2^{Mj} \rho(m)^M) (1 + 2^{mn} c_{g^{-1}}^n \sigma(j)^n) \\ \leq C c_{g^{-1}}^n 2^{jn} \sigma(j)^n (c_g 2^j \rho(m) + c_g^M 2^{Mj} \rho(m)^M) .$$

We now choose $M = (n + 1)$. The right side of (4.24) is

$$\leq C c_{g^{-1}}^n c_g^{n+1} 2^{(2n+1)j} \rho(m) .$$

We set $\gamma = 2n + 1$. This concludes the proof.

5. A priori estimate

Let Γ be a function $\mathbb{R} \rightarrow [1, \infty)$ satisfying the conditions (i)–(iii) above. Let $\Gamma_1(\alpha) = (\alpha + 2)\Gamma(\alpha)$ for $\alpha \in [-1, \infty)$, $\Gamma_1(\alpha) = 1$ for $\alpha \in (-\infty, -1)$. We assume

(iv) Γ_1 satisfies the condition (iii);

(v) Γ_1 is convex;

(vi) $\int_1^\infty \Gamma_1(\alpha)^{-1} d\alpha = \infty$.

The conditions (i), (ii) are automatically satisfied for Γ_1 . It is clear also that (ii)–(iv) follow from one condition

$$\lim \operatorname{ess}_{\alpha \rightarrow \infty} \Gamma'(\alpha) \Gamma(\alpha)^{-1} = 0 .$$

Indeed, assuming this last condition, (ii) follows immediately while (iii) and (iv) are verified using integration by parts in the right side of (iii) (also Γ being replaced by Γ_1). For example, $\Gamma(\alpha) = \log_2^\kappa(\alpha + 3)$, $\alpha \geq -1$, where $\kappa \in (0, 1]$ satisfies all the conditions (i)–(vi). So does $\Gamma(\alpha) = \log_2(\alpha + 3) \log_2 \log_2(\alpha + 5)$, etc.

We fix $f \in B_\Gamma \cap L^{p_0} \cap L^{p_1}$ where $1 < p_0 < 2 < p_1 < \infty$ and consider the Euler equation for the vorticity of an incompressible fluid:

$$(5.1) \quad \dot{\omega}(x, t) = -(v, \nabla)\omega \equiv - \sum_{i=1}^2 v_i \partial_i \omega ; (x, t) \in \mathbb{R}^2 \times [0, T_1] ,$$

$$(5.2) \quad v = \mathcal{K} * \omega ,$$

$$(5.3) \quad \omega(\cdot, 0) = f(\cdot) \in B_\Gamma \cap L^{p_0} \cap L^{p_1} ,$$

where \mathcal{K} is the kernel in Biot-Savart law. We fix $T_1 > 0$ and assume we are given a flow

$$(5.4) \quad v(\cdot) \in \mathcal{K} * C([0, T_1]; B_{\Gamma_1} \cap L^{p_0} \cap L^{p_1}) .$$

According to Theorem 3.1 the classical trajectories are well defined. Indeed, the conditions of the Osgood uniqueness theorem are satisfied as follows from (vi). We refer to [CL] and

[C1] for an appropriate version of the Osgood uniqueness theorem. Since $\operatorname{div} v = 0$, the corresponding flow is volume preserving.

We denote the flow by $g_v(t) = g(t)$, i.e.

$$(5.5) \quad \dot{g}_v(x, t) = v(g_v(x, t), t); \quad g_v(x, 0) = x, \quad x \in \mathbb{R}^2.$$

For any $t \in [0, T_1]$, $g_v(\cdot, t)$ is a volume preserving homeomorphism $\mathbb{R}^2 \xrightarrow{\text{onto}} \mathbb{R}^2$.

Let $\lambda(t)$, $t \in [0, T]$, $0 < T \leq T_1$ be a continuous positive nondecreasing function defined as follows:

$$(5.6) \quad \dot{\lambda} = C\lambda^2,$$

$$(5.7) \quad \lambda(0) = \max(\|\omega(0)\|_{B_{\Gamma_1}}, 1)$$

The constant $C > 0$ will be chosen later.

THEOREM 5.1. – *Let $f \in B_{\Gamma} \cap L^{p_0} \cap L^{p_1}$. Let $v(\cdot)$ be a regular solution to (5.1)–(5.3) satisfying (5.4). Let*

$$(5.8) \quad (\alpha + 2)\Gamma'(\alpha) \leq C \text{ for a.e. } \alpha \in [-1, \infty);$$

$$(5.9) \quad \|f\|_{B_{\Gamma} \cap L^{p_0} \cap L^{p_1}} \leq C.$$

Then there is a $T > 0$, $T \leq T_1$ such that

$$(5.9') \quad \|\omega(t)\|_{B_{\Gamma_1}} \leq \lambda(t), \quad t \in [0, T].$$

Both $T > 0$ and the constant C in (5.6) that defines λ depend only on the constant in (5.9) and on Γ . In addition, $\omega(t)$ is equimeasurable with f .

REMARK 5.2. – Obviously (5.8) implies (vi).

Proof. – We define first $\lambda(\cdot)$ on the whole interval $[0, T_1]$ as follows:

$$\lambda(t) = \max\left(\sup_{0 \leq \tau \leq t} \|\omega(\tau)\|_{B_{\Gamma_1}}, 1\right), \quad t \in [0, T_1].$$

Let C_0 be a sufficiently large constant such that (see (3.7))

$$|w(x) - w(y)| \leq C_0|x - y|, \quad |x - y| \geq \frac{1}{2}$$

for any $w = \mathcal{K} * f_1$ where $f_1 \in L^{p_0} \cap L^{p_1}$ is equimeasurable with f . In particular

$$(5.10) \quad |v(x, t) - v(y, t)| \leq C_0|x - y|, \quad |x - y| \geq \frac{1}{2}, \quad t \in [0, T_1].$$

Also from (3.1) for C_0 sufficiently large

$$(5.11) \quad |v(x, t) - v(y, t)| \leq C_0\lambda(t)\Gamma_1(-\log_2|x - y|)|x - y|, \quad |x - y| \leq \frac{1}{2}.$$

We define

$$\tilde{\Gamma}_1(m, t) = \begin{cases} \lambda(t)^{-1}, & -\infty < m \leq m_1 \\ 1 + (m + 1)\Gamma'_1(-1+), & m_1 \leq m \leq -1 \\ \Gamma_1(m), & -1 < m < \infty \end{cases}$$

with $m_1 = -1 - (1 - \lambda(t)^{-1})\Gamma'_1(-1+)^{-1}$.

Since $\Gamma'_1(-1+) \geq 1$, we have $-2 \leq m_1 \leq 1$. It is clear that $\tilde{\Gamma}_1$ is *convex* on \mathbb{R} . It follows from (5.10), (5.11) that

$$(5.12) \quad |v(x, t) - v(y, t)| \leq C_0 \lambda(t) \tilde{\Gamma}_1(-\log_2 |x - y|, t) |x - y|, \quad x \neq y.$$

We estimate stretching in the flow given by the vector-field $v(x, t)$, $t \in [0, T_1]$.

We denote by $\mu(t) = \mu(m, t)$ solution to the Cauchy problem

$$(5.13) \quad \dot{\mu}(m, t) = -C_0 (\log_2 e) \lambda(t) \tilde{\Gamma}_1(\mu(m, t), t), \quad \mu(m, 0) = m \in \mathbb{R}.$$

The solution exists and is unique for all $t \in [0, T_1]$ as follows easily from our construction of $\tilde{\Gamma}_1$.

PROPOSITION 5.2. – *Let $x, y \in \mathbb{R}^n$, $x \neq y$. Then*

$$|g_v(t)x - g_v(t)y| \leq 2^{-\mu(m, t)}, \quad t \in [0, T_1]$$

where $m = -\log_2 |x - y|$.

Proof. – Let $\xi(t) = |g_v(t)x - g_v(t)y|$. Then (5.12) yields

$$\begin{aligned} \frac{d}{dt} \xi(t) &\leq |v(g_v(t)x, t) - v(g_v(t)y, t)| \\ &\leq C_0 \lambda(t) \tilde{\Gamma}_1(-\log_2 \xi(t), t) \xi(t). \end{aligned}$$

Therefore,

$$\frac{d}{dt} (-\log_2 \xi(t)) \geq -C_0 \lambda(t) (\log_2 e) \tilde{\Gamma}_1(-\log_2 \xi(t), t).$$

Since the right side is *nonincreasing* on \mathbb{R} as a function of $(-\log_2 \xi(t))$ it follows from the Gronwall type of argument that

$$-\log_2 \xi(t) \geq \mu(m, t), \quad t \in [0, T_1].$$

This proves the proposition.

PROPOSITION 5.3. – *For any fixed $t \in [0, T_1]$ the function*

$$m \longmapsto \mu(m, t)$$

is concave.

Proof. – To simplify the notation we change time and reduce the equation (5.13) to

$$\dot{\mu}(m, t) = -\tilde{\Gamma}_1(\mu(m, t), t), \quad \mu(m, 0) \equiv m.$$

Let $m_1, m_2 \in \mathbb{R}$, $m_1 \neq m_2$. For any $\theta \in (0, 1)$

$$\begin{aligned} (\theta\mu(m_1, t) + (1 - \theta)\mu(m_2, t))' &= -\theta\tilde{\Gamma}_1(\mu(m_1, t), t) \\ &- (1 - \theta)\tilde{\Gamma}_1(\mu(m_2, t), t) \leq -\tilde{\Gamma}_1(\theta\mu(m_1, t) + (1 - \theta)\mu(m_2, t), t) . \end{aligned}$$

Hence,

$$-(\theta\mu(m_1, t) + (1 - \theta)\mu(m_2, t))' \geq \tilde{\Gamma}_1(\theta\mu(m_1, t) + (1 - \theta)\mu(m_2, t), t) .$$

Since the right side is a *nonincreasing* function of $-(\theta\mu(m_1, t) + (1 - \theta)\mu(m_2, t))$ employing the same Gronwall's argument as before yields

$$-(\theta\mu(m_1, t) + (1 - \theta)\mu(m_2, t)) \geq -\mu(\theta m_1 + (1 - \theta)m_2, t) .$$

This completes the proof.

We now set

$$\tilde{\rho}(m, t) = 2^{-\mu(m, t)} .$$

It follows from Proposition 5.3 that $\log_2 \tilde{\rho}(m, t)$ is convex. For $m \leq -2$ $\mu(m, t) = m - C_0(\log_2 e)t$ thus $\tilde{\rho}(m, t) = e^{C_0 t} 2^{-m}$.

We now set

$$\begin{aligned} \rho(m, t) &= \frac{1}{4} e^{-C_0 t} \tilde{\rho}(m - 2, t) ; \\ c_{g_v(t)} &= 4e^{C_0 t} . \end{aligned}$$

Then all the conditions for ρ at the beginning of §4 are satisfied for $g = g_v(t)$, $\rho(m) = \rho(m, t)$. In particular, (4.2) follows from Proposition 5.2.

There are similar statements concerning σ . We just replace $v(t)$ by $-v(T_1 - t)$, $\tilde{\Gamma}_1(m, t)$ by $\tilde{\Gamma}_1(m, T_1 - t)$, etc. Let $\eta(m, t)$ solve the Cauchy problem

$$\dot{\eta}(m, t) = -C_0(\log_2 e)\lambda(T_1 - t)\tilde{\Gamma}_1(\eta(m, t), T_1 - t) , \quad \eta(m, 0) = m .$$

PROPOSITION 5.2'. - Let $x, y \in \mathbb{R}^n$, $x \neq y$. Then

$$|g_{-v(T_1 - \cdot)}(t)x - g_{-v(T_1 - \cdot)}(t)y| \leq 2^{-\eta(m, t)} , \quad t \in [0, T_1]$$

where $m = -\log_2 |x - y|$.

PROPOSITION 5.3'. - For any fixed $t \in [0, T_1]$ the function

$$m \mapsto \eta(m, t)$$

is concave.

We set

$$\sigma(m, T_1) = \frac{1}{4} e^{-C_0 T_1} 2^{-\eta(m - 2, T_1)} .$$

Then (4.1), (4.3) and (4.4) are satisfied for $g = g_v(T_1)$, $\sigma(m) = \sigma(m, T_1)$, $C_{g^{-1}} = 4e^{C_0 T_1}$. Replacing T_1 by arbitrary $t \in [0, T_1]$ we recover the desirable estimates for the homeomorphism $(g_v(t))^{-1}$.

To continue the proof of Theorem 5.1 we have to estimate $\|f \circ g_v(t)^{-1}\|_{B_{\Gamma_1}}$ for $t \in [0, T_1]$.

PROPOSITION 5.4. – *Let $f, g_v(t)$ $t \in [0, T_1]$ be as above. Then*

$$\|f \circ g_v(t)^{-1}\|_{\Gamma_1} \leq C \|f\|_{\Gamma} 2^C \int_0^t \lambda(\tau) d\tau, \quad t \in [0, T_1].$$

Proof. – Let $N \geq -1$. We use the “wavelet” decomposition (1.2) and Proposition 4.1. We have for arbitrary $m \geq N$ (see (4.5)):

$$\begin{aligned} (5.14) \quad & \sum_{j=-1}^N \|\Delta_j(f \circ g_v(t)^{-1})\|_{L^\infty} \\ & \leq \sum_{j=-1}^N \sum_{\ell=-1}^\infty \|\Delta_j(f_\ell \circ g_v(t)^{-1})\|_{L^\infty} \\ & = \sum_{j=-1}^N \left(\sum_{\ell=-1}^m + \sum_{\ell=m+1}^\infty \right) \|\Delta_j(f_\ell \circ g_v(t)^{-1})\|_{L^\infty} \\ & \leq C(N+2) \sum_{\ell=-1}^m \|f_\ell\|_{L^\infty} + \sum_{j=-1}^N C 2^{\gamma j} \sum_{\ell=m+1}^\infty \rho(\ell, t) \sup_{k \in \mathbb{Z}^n} |a_{\ell, k}| \\ & \leq C(N+2)\Gamma(m) \|f\|_{\Gamma} + C 2^{\gamma N} \sum_{\ell=m+1}^\infty \rho(\ell, t) \sup_{k \in \mathbb{Z}^n} |a_{\ell, k}|. \end{aligned}$$

We used in (5.14) that the constants $c_{g_v(t)^{\pm 1}}$ are uniformly bounded with respect to $t \in [0, T_1]$.

The last sum in (5.14) requires some attention. We proceed as follows. Let

$$d_\ell = \sum_{j=-1}^\ell \sup_{k \in \mathbb{Z}^n} |a_{j, k}|.$$

Then,

$$\begin{aligned} (5.15) \quad & \sum_{\ell=m+1}^\infty \rho(\ell, t) \sup_{k \in \mathbb{Z}^n} |a_{\ell, k}| \\ & = \sum_{\ell=m+1}^\infty \rho(\ell, t) (d_\ell - d_{\ell-1}) \\ & = -d_m \rho(m+1, t) + \sum_{\ell=m+1}^\infty (-\rho(\ell+1, t) + \rho(\ell, t)) d_\ell \\ & \leq -d_m \rho(m+1, t) + \|f\|_{\Gamma} \sum_{\ell=m+1}^\infty (-\rho(\ell+1, t) + \rho(\ell, t)) \Gamma(\ell) \\ & \leq -d_m \rho(m+1, t) + \|f\|_{\Gamma} \sum_{\ell=m+1}^\infty \int_\ell^{\ell+1} -\partial_\xi \rho(\xi, t) d\xi \Gamma(\ell) \\ & \leq -d_m \rho(m+1, t) + C \|f\|_{\Gamma} \int_{m+1}^\infty -\partial_\xi \rho(\xi, t) \Gamma(\xi) d\xi \end{aligned}$$

since Γ satisfies (ii). Since (5.8) is satisfied we can estimate the integral in the right side of (5.15) as follows:

$$\begin{aligned} & \int_{m+1}^{\infty} -\partial_{\xi} \rho(\xi, t) \Gamma(\xi) d\xi \\ & \leq C \int_{m+1}^{\infty} \partial_{\xi} \mu(\xi - 2, t) 2^{-\mu(\xi-2, t)} \Gamma(\xi) d\xi \\ & = C \int_{m-1}^{\infty} \partial_{\xi} \mu(\xi, t) 2^{-\mu(\xi, t)} \Gamma(\xi + 2) d\xi \\ & \leq C \int_{m-1}^{\infty} \partial_{\xi} \mu(\xi, t) 2^{-\mu(\xi, t)} \Gamma(\xi) d\xi \\ & = C \int_{\mu(m-1, t)}^{\infty} 2^{-\mu} \Gamma(\xi(\mu, t)) d\mu . \end{aligned}$$

We have from (5.13)

$$(5.16) \quad \Gamma(\xi) - \Gamma(\mu) = C_0(\log_2 e) \int_0^t \lambda(\tau) \Gamma'(\mu(\tau)) \tilde{\Gamma}_1(\mu(\tau), \tau) d\tau .$$

This combined with (5.8) yields

$$\Gamma(\xi) \leq \Gamma(\mu) 2^C \int_0^t \lambda(\tau) d\tau .$$

Hence,

$$(5.17) \quad \begin{aligned} & \sum_{\ell=m+1}^{\infty} \rho(\ell, t) \sup_{k \in \mathbb{Z}^n} |a_{\ell, k}| \\ & \leq C \|f\|_{\Gamma} 2^C \int_0^t \lambda(\tau) d\tau \Gamma(m) 2^{-\mu(m-1, t)} \\ & \leq C \|f\|_{\Gamma} 2^C \int_0^t \lambda(\tau) d\tau \Gamma(m) \rho(m+1, t) . \end{aligned}$$

The choice of m in (5.14) so far was arbitrary. We now choose m so that (assuming $N \geq 1$)

$$(5.18) \quad \mu(m-1, t) \geq \gamma N > \mu(m-2, t) .$$

As follows from (5.17) the second term in the right side of (5.14) is

$$(5.19) \quad \leq C \|f\|_{\Gamma} 2^C \int_0^t \lambda(\tau) d\tau \Gamma(m) .$$

We have to estimate $\Gamma(m)$ as above

$$\Gamma(m) \leq \Gamma(\mu(m, t)) 2^C \int_0^t \lambda(\tau) d\tau .$$

Furthermore, in general, for all m

$$\partial_m \mu(m, t) \leq 1 .$$

Therefore, using (5.8)

$$(5.20) \quad \begin{aligned} \Gamma(m) &\leq C\Gamma(\gamma N)2^C \int_0^t \lambda(\tau) \, d\tau \\ &\leq C\Gamma(N)2^C \int_0^t \lambda(\tau) \, d\tau . \end{aligned}$$

It follows from (5.14), (5.19), (5.20) that for $N \geq 1$

$$(5.21) \quad \sum_{j=-1}^N \|\Delta_j(f \circ g_v^{-1}(t))\|_{L^\infty} \leq C\Gamma_1(N)\|f\|_{\Gamma} 2^C \int_0^t \lambda(\tau) \, d\tau .$$

This concludes the proof of the Proposition.

We can now finish the proof of the Theorem. Let $\tilde{\lambda}(\cdot)$ satisfy the following ordinary differential equation:

$$(5.22) \quad \dot{\tilde{\lambda}} = C\tilde{\lambda}^2, \quad \tilde{\lambda}(0) = \lambda(0), \quad t \in [0, T],$$

where the constant C is sufficiently large. The time $T > 0$ is chosen to be less than the blow-up time for (5.22). Then (5.9') is satisfied because (5.21) implies $\lambda(t) \leq \tilde{\lambda}(t)$, $t \in [0, T]$. In addition to this $f \circ g_v(t)^{-1}$ is equimeasurable with f .

This concludes the proof.

We now give an a priori estimate on an infinite interval.

THEOREM 5.5. — *Let $f \in B_\Gamma \cap L^{p_0} \cap L^{p_1}$ where Γ satisfies (i)–(v). Let $v(t)$ be a regular solution to (5.1)–(5.3) such that*

$$v(\cdot) \in \mathcal{K} * C_{\text{loc}}([0, \infty); B_{\Gamma_1} \cap L^{p_0} \cap L^{p_1}) .$$

Assume

$$(5.23) \quad \Gamma'(\alpha)\Gamma_1(\alpha) \leq C \text{ a.e. } \alpha \geq -1 ;$$

$$(5.24) \quad \|f\|_{B_\Gamma \cap L^{p_0} \cap L^{p_1}} \leq C .$$

Then there exists a positive continuous nondecreasing function $\lambda(\cdot)$ on $[0, \infty)$ such that

$$(5.25) \quad \|\omega(t)\|_{B_{\Gamma_1}} \leq \lambda(t), \quad t \in [0, \infty)$$

This function depends only on the constant in (5.24) and on Γ .

Proof. — We have from (5.16), (5.23)

$$(5.26) \quad \Gamma(\xi) \leq \Gamma(\mu) + C \int_0^t \lambda(\tau) \, d\tau .$$

Therefore,

$$(5.27) \quad \int_{m+1}^\infty -\partial_\xi \rho(\xi, t) \Gamma(\xi) \, d\xi \leq C\rho(m+1, t) \left\{ \Gamma(\mu(m, t)) + \int_0^t \lambda(\tau) \, d\tau \right\} .$$

This yields

$$\sum_{\ell=m+1}^{\infty} \rho(\ell, t) \sup_{k \in \mathbb{Z}^n} |a_{\ell, k}| \leq C \|f\|_{\Gamma} \rho(m+1, t) \left\{ \Gamma(\mu(m, t)) + \int_0^t \lambda(\tau) d\tau \right\}.$$

We choose m so that (5.18) is satisfied. Then the second term in the right side of (5.14) is

$$\begin{aligned} (5.28) \quad & \leq C \|f\|_{\Gamma} \left\{ \Gamma(\mu(m, t)) + \int_0^t \lambda(\tau) d\tau \right\} \\ & \leq C \|f\|_{\Gamma} \left\{ \Gamma(N) + \int_0^t \lambda(\tau) d\tau \right\}. \end{aligned}$$

Using (5.26), (5.28), (5.14) we get for $N \geq 1$

$$\begin{aligned} \sum_{j=-1}^N \|\Delta_j(f \circ g_v(t)^{-1})\|_{L^\infty} & \leq C \|f\|_{\Gamma} N \left(\Gamma(N) + \int_0^t \lambda(\tau) d\tau \right) \\ & \leq C \left(1 + \int_0^t \lambda(\tau) d\tau \right) \|f\|_{\Gamma} \Gamma_1(N). \end{aligned}$$

We fix an arbitrary $T > 0$ and choose $\tilde{\lambda}$ to satisfy the initial value problem

$$\dot{\tilde{\lambda}} = C \tilde{\lambda}, \quad \tilde{\lambda}(0) = C, \quad t \in [0, T],$$

where C is a sufficiently large constant. Then

$$\|f \circ g_v(t)^{-1}\|_{B_{r_1}} \leq \lambda(t) \leq \tilde{\lambda}(t), \quad t \in [0, T].$$

This concludes the proof.

EXAMPLE 5.6. – The function

$$\Gamma(\alpha) = \log_2^\kappa(\alpha + 3), \quad \alpha \geq -1$$

satisfies (5.23) for $0 < \kappa \leq \frac{1}{2}$.

6. Commutator estimates

Let u be a vector field, $\operatorname{div} u = 0$, $u \in L^\infty$, $\nabla u \in B_\Pi$. Let the function $\Pi : \mathbb{R} \rightarrow [1, \infty)$ satisfies the conditions (i)–(iii) above. Let (e.g.) $w \in L^\infty$. We define as in [BC] for $j \geq 1$

$$R_j(u, w) = \Delta_j(u, \nabla)w - (S_{j-2}u, \nabla)\Delta_j w.$$

We use the normalization $\hat{\Phi}(\xi) + \sum_{j=0}^{\infty} \hat{\varphi}(2^{-j}\xi) = 1$ with the same conditions on the support as in §1. Then $-\hat{\Phi}(\xi/2) + \hat{\Phi}(\xi) = \hat{\varphi}(\xi)$.

For $|j - j'| \geq 2$ we have $\Delta_j \Delta_{j'} = 0$. We also use the standard notation (when a and b are functions or distributions)

$$T_a b = \sum_{j=-1}^{\infty} S_{j-2} a \Delta_j b ;$$

$$R(a, b) = \sum_{|j-j'| \leq 1} \Delta_j a \Delta_{j'} b .$$

Then Bony's [B] formula (see also [M1]) reads

$$a \cdot b = T_a b + T_b a + R(a, b) .$$

THEOREM 6.1. – *There is an absolute constant M_0 so that the following inequality holds true:*

$$(6.1) \quad \|R_j(u, w)\|_{L^\infty} \leq C \sum_{|j'-j| \leq M_0} \left\{ \|S_{j'-2} \nabla w\|_{L^\infty} \|\Delta_{j'} u\|_{L^\infty} + \|S_{j'-2} \nabla u\|_{L^\infty} \|\Delta_{j'} w\|_{L^\infty} \right\} \\ + C 2^j \sum'_{\substack{j' \geq j - M_0 \\ |j' - j''| \leq 1}} 2^{-j'} \|\Delta_{j'} \nabla u\|_{L^\infty} \|\Delta_{j''} w\|_{L^\infty} .$$

When $j' = -1$ the factor $\|\Delta_{j'} \nabla u\|_{L^\infty}$ in the last sum ought to be replaced by $\|\Delta_{-1} u\|_{L^\infty}$.

Here and below the notation \sum' is to indicate this convention: $\|\Delta_{-1} u\|_{L^\infty}$ instead of $\|\Delta_{-1} \nabla u\|$ (also for u_1, w in the next section).

REMARK 6.2. – The first term in the first sum could be estimated as follows:

$$\|S_{j'-2} \nabla w\|_{L^\infty} \|\Delta_{j'} u\|_{L^\infty} \leq C \|S_{j'-2} w\|_{L^\infty} \|\Delta_{j'} \nabla u\|_{L^\infty} , \quad j' \geq -1 .$$

Notice that both the left side and the right side vanish for $j' \leq 0$.

Proof. – The proof will follow very closely the arguments of H. Bahouri and J.-Y. Chemin [BC]. We have

$$(6.2) \quad R_j(u, w) = \sum_{\ell=1}^4 R_j^\ell(u, w) ,$$

where

$$(6.3) \quad \left\{ \begin{array}{l} R_j^1(u, w) = \sum_{k=1}^n \Delta_j T_{\partial_k w} u_k \\ R_j^2(u, w) = - \sum_{k=1}^n [T_{u_k} \partial_k, \Delta_j] w \\ R_j^3(u, w) = \sum_{k=1}^n T_{u_k - S_{j-2} u_k} \partial_k \Delta_j w \\ R_j^4(u, w) = \sum_{k=1}^n \{ \Delta_j R(u_k, \partial_k w) - R(S_{j-2} u_k, \Delta_j \partial_k w) \} . \end{array} \right.$$

Indeed,

$$\begin{aligned}
 \Delta_j(u, \nabla)w &= \Delta_j \sum_{k=1}^n u_k \partial_k w \\
 &= \Delta_j \sum_{k=1}^n \{T_{\partial_k w} u_k + T_{u_k} \partial_k w + R(u_k, \partial_k w)\} \\
 &= \sum_{\ell=1}^2 R_j^\ell(u, w) + \sum_{k=1}^n \{T_{u_k} \partial_k \Delta_j w + \Delta_j R(u_k, \partial_k w)\} \\
 &= \sum_{\ell=1}^3 R_j^\ell(u, w) + \sum_{k=1}^n \{T_{S_{j-2}u_k} \partial_k \Delta_j w + \Delta_j R(u_k, \partial_k w)\} \\
 &= \sum_{\ell=1}^4 R_j^\ell(u, w) + \sum_{k=1}^n \left\{ T_{S_{j-2}u_k} \partial_k \Delta_j w + T_{\partial_k \Delta_j w} S_{j-2}u_k \right. \\
 &\quad \left. + R(S_{j-2}u_k, \Delta_j \partial_k w) - T_{\partial_k \Delta_j w} S_{j-2}u_k \right\} \\
 &= \sum_{\ell=1}^4 R_j^\ell + (S_{j-2}u, \nabla) \Delta_j w - \sum_{k=1}^n T_{\partial_k \Delta_j w} S_{j-2}u_k .
 \end{aligned}$$

We claim the last term is vanishing.

Indeed,

$$\sum_{k=1}^n T_{\partial_k \Delta_j w} S_{j-2}u_k = \sum_{k=1}^n \sum_{j'} S_{j'-2} \partial_k \Delta_j w \cdot \Delta_{j'} S_{j-2}u_k .$$

The second factor vanishes for $j' \geq j$. The first factor vanishes for $j' \leq j$. This proves (6.2). We estimate the terms in the right side of (6.2) in L^∞ . We have

$$R_j^1(u, w) = \sum_{k=1}^n \Delta_j \sum_{j'} S_{j'-2} \partial_k w \Delta_{j'} u_k .$$

All the terms in this sum with $|j - j'| > M_0$ vanish. Hence,

$$(6.4) \quad \|R_j^1(u, w)\|_{L^\infty} \leq \sum_{k=1}^n \sum_{|j'-j| \leq M_0} \|S_{j'-2} \partial_k w\|_{L^\infty} \|\Delta_{j'} u_k\|_{L^\infty} .$$

We may assume $j' \geq 0$ in the right side of (6.4) since otherwise $S_{j'-2} \partial_k w = 0$. Hence,

$$(6.5) \quad \|\Delta_{j'} u_k\|_{L^\infty} \leq C 2^{-j} \|\Delta_{j'} \nabla u\|_{L^\infty} .$$

Also

$$(6.6) \quad \|S_{j'-2} \partial_k w\|_{L^\infty} \leq C 2^j \|S_{j-2} w\|_{L^\infty} .$$

The combination of (6.4)–(6.6) yields

$$\begin{aligned}
 (6.7) \quad \|R_j^1(u, w)\|_{L^\infty} &\leq C \sum_{|j'-j| \leq M_0} \|S_{j'-2} \nabla w\|_{L^\infty} \|\Delta_{j'} u\|_{L^\infty} \\
 &\leq C \sum_{|j'-j| \leq M_0} \|S_{j'-2} w\|_{L^\infty} \|\Delta_{j'} \nabla u\|_{L^\infty} , \quad j \geq -1 .
 \end{aligned}$$

We now study $R_j^2(u, w)$

$$\begin{aligned} R_j^2(u, w) &= - \sum_{j'} \sum_{k=1}^n [S_{j'-2} u_k \partial_k \Delta_{j'}, \Delta_j] w \\ &= - \sum_{k=1}^n \sum_{j'} [S_{j'-2} u_k, \Delta_j] \partial_k \Delta_{j'} w \\ &= - \sum_{k=1}^n \sum_{|j'-j| \leq M_0} [S_{j'-2} u_k, \Delta_j] \partial_k \Delta_{j'} w . \end{aligned}$$

We use as in [BC] the following explicit representation of the operator in the right side

$$\begin{aligned} R_j^2(u, w)(x) &= - \sum_{k=1}^n \sum_{|j'-j| \leq M_0} 2^{jn} \int \varphi(2^j(x-y)) (S_{j'-2} u_k(x) - S_{j'-2} u_k(y)) \partial_k \Delta_{j'} w(y) dy \\ &= - \sum_{|j'-j| \leq M_0} \sum_{k=1}^n 2^{j(n+1)} \int \partial_k \varphi(2^j(x-y)) (S_{j'-2} u_k(x) - S_{j'-2} u_k(y)) \Delta_{j'} w(y) dy \end{aligned}$$

since $\operatorname{div} S_{j'-2} u(y) = S_{j'-2} \operatorname{div} u(y) = 0$.

Applying the Taylor's formula we obtain

$$\begin{aligned} R_j^2(u, w)(x) &= - \sum_{|j'-j| \leq M_0} \sum_{k=1}^n 2^{j(n+1)} \int \partial_k \varphi(2^j(x-y)) \\ &\quad \sum_{m=1}^n \int_0^1 S_{j'-2} \partial_m u_k(x + \tau(y-x)) \cdot (x_m - y_m) d\tau \Delta_{j'} w(y) dy \\ &= - \sum_{|j'-j| \leq M_0} \sum_{k=1}^n \sum_{m=1}^n \int \partial_k \varphi(z) z_m \int_0^1 S_{j'-2} \partial_m u_k(x - \tau 2^{-j} z) \Delta_{j'} w(x - 2^{-j} z) d\tau dz . \end{aligned}$$

Minkowski integral inequality yields:

$$\begin{aligned} (6.8) \quad \|R_j^2(u, w)\|_{L^\infty} &\leq C \sum_{m=1}^n \sum_{k=1}^n \sum_{|j'-j| \leq M_0} \|S_{j'-2} \partial_m u_k\|_{L^\infty} \|\Delta_{j'} w\|_{L^\infty} \\ &\leq C \sum_{|j'-j| \leq M_0} \|S_{j'-2} \nabla u\|_{L^\infty} \|\Delta_{j'} w\|_{L^\infty} . \end{aligned}$$

Next we estimate the term $R_j^3(u, w)$. The argument similar to the one used for R_j^1 leads to the estimate

$$(6.9) \quad \|R_j^3(u, w)\| \leq C \sum_{m=j-1}^j \|\Delta_m \nabla u\|_{L^\infty} \|\Delta_j w\|_{L^\infty} , \quad j \geq -1 ,$$

except when $m = -1$ the factor $\|\Delta_{-1}\nabla u\|_{L^\infty}$ ought to be replaced by $\|\Delta_{-1}u\|_{L^\infty}$. We now turn to $R_j^4(u, w)$:

$$(6.10) \quad R_j^4(u, w) = R_j^{4,1}(u, w) + R_j^{4,2}(u, w) ,$$

$$(6.11) \quad R_j^{4,1}(u, w) = \sum_{k=1}^n \Delta_j \partial_k R(u_k - S_{j-2}u_k, w) ,$$

$$(6.12) \quad R_j^{4,2}(u, w) = \sum_{k=1}^n \Delta_j R(S_{j-2}u_k, \partial_k w) - R(S_{j-2}u_k, \Delta_j \partial_k w) .$$

We have:

$$R_j^{4,1}(u, w) = \sum_{k=1}^n \partial_k \Delta_j \sum_{|j'-j''|\leq 1} \Delta_{j'}(u_k - S_{j-2}u_k) \Delta_{j''} w .$$

Therefore,

$$(6.13) \quad \begin{aligned} \|R_j^{4,1}(u, w)\|_{L^\infty} &\leq C 2^j \sum_{j' \geq j-M_0} \sum_{|j'-j''|\leq 1} \|\Delta_{j'} u\|_{L^\infty} \|\Delta_{j''} w\| \\ &\leq C 2^j \sum_{j' \geq j-M_0} \sum_{|j'-j''|\leq 1} 2^{-j'} \|\Delta_{j'} \nabla u\|_{L^\infty} \|\Delta_{j''} w\|_{L^\infty} , \end{aligned}$$

except when $j' = -1$ the factor $\|\Delta_{j'} \nabla u\|_{L^\infty}$ ought to be replaced by $\|\Delta_{-1}u\|_{L^\infty}$. We now estimate the term $R_j^{4,2}(u, w)$:

$$(6.14) \quad \begin{aligned} R_j^{4,2}(u, w) &= \sum_{k=1}^n \sum_{|j'-j''|\leq 1} \left\{ \Delta_j ((\Delta_{j'} S_{j-2}u_k) \cdot \Delta_{j''} \partial_k w) \right. \\ &\quad \left. - (\Delta_{j'} S_{j-2}u_k) (\Delta_{j''} \Delta_j \partial_k w) \right\} \\ &= \sum_{k=1}^n \sum_{j-1 \geq j' \geq j-M_0} \sum_{|j'-j''|\leq 1} [\Delta_j, \Delta_{j'} S_{j-2}u_k] \Delta_{j''} \partial_k w . \end{aligned}$$

Using the explicit integral representation as above and proceeding as with the term R_j^2 we arrive at

$$(6.15) \quad \begin{aligned} \|R_j^{4,2}(u, w)\|_{L^\infty} &\leq C \sum_{\substack{|j'-j|\leq M_0 \\ |j'-j''|\leq 1}} \|\Delta_{j'} S_{j-2} \nabla u\|_{L^\infty} \|\Delta_{j''} w\|_{L^\infty} \\ &\leq C \sum'_{\substack{|j'-j|\leq M_0 \\ |j'-j''|\leq 1}} \|\Delta_{j'} \nabla u\|_{L^\infty} \|\Delta_{j''} w\|_{L^\infty} \end{aligned}$$

Adding up the estimates (6.7), (6.8), (6.9), (6.13) and (6.15) and using (6.2), (6.3), (6.10)–(6.12) yield the statement of the Theorem.

7. Uniqueness theorem

Let Π be a function $\mathbb{R} \rightarrow [1, \infty)$ that satisfies the conditions (i)–(iii) above. In this section the dimension $n \geq 2$ is arbitrary.

THEOREM 7.1 (Uniqueness theorem). – *Let the following conditions be satisfied:*

$$(7.1) \quad \int_1^\infty \Pi(\alpha)^{-1} d\alpha = \infty ;$$

$$(7.1') \quad \Pi(\alpha)2^{-\alpha} \text{ is nonincreasing for } \alpha \geq C, \quad \lim_{\alpha \rightarrow \infty} \Pi(\alpha)2^{-\alpha} = 0 .$$

Let for $t \in [0, T]$, $u_1(x, t)$, $u_2(x, t)$ be two vector fields in \mathbb{R}^n and $\omega_{1,2} = \text{curl } u_{1,2}$. Assume the following:

$$(7.2) \quad \omega_{1,2} \in L^\infty([0, T]; L^{p_0}), \quad \|\omega_{1,2}(\cdot)\|_\Pi \in L^\infty([0, T]) ;$$

$$(7.3) \quad u_{1,2} = \mathcal{K} * \omega_{1,2} ;$$

$$(7.4) \quad \text{div} u_{1,2} = 0 ;$$

$$(7.5) \quad \dot{u}_{1,2} = -(u_{1,2}, \nabla)u_{1,2} - \nabla p_{1,2} ;$$

$$(7.6) \quad \omega_{1,2}(\cdot, 0) = f(\cdot) \in B_\Pi \cap L^{p_0} ,$$

where $p_0 \in (1, n)$.

Then, $u_1 = u_2$ on $[0, T]$.

Proof. – Let $w = u_1 - u_2$, $\omega = \text{curl } w$, $p = p_1 - p_2$. Then,

$$(7.7) \quad \dot{w} = -(u_1, \nabla)w - (w, \nabla)u_2 - \nabla p, \quad t \in [0, T] ;$$

$$(7.8) \quad \text{div} w = 0 ;$$

$$(7.9) \quad w|_{t=0} = 0 .$$

Note that $w \in L^\infty(\mathbb{R}^n \times [0, T])$ as follows from (7.2), (7.3) and Theorem 3.1.

It follows from (7.7) that for any $j \geq -1$

$$(7.10) \quad \begin{aligned} \Delta_j \dot{w} = & -(S_{j-2}u_1, \nabla)\Delta_j w - R_j(u_1, w) \\ & - (S_{j-2}w, \nabla)\Delta_j u_2 - R_j(w, u_2) \\ & - \Delta_j \nabla p . \end{aligned}$$

We have according to Theorem 6.1

$$(7.11) \quad \|R_j(u_1, w)\|_{L^\infty} \leq C \sum_{|j'-j| \leq M_0} \left\{ \|S_{j'-2} \nabla u_1\|_{L^\infty} \|\Delta_{j'} w\|_{L^\infty} \right. \\ \left. + \|S_{j'-2} \nabla w\|_{L^\infty} \|\Delta_{j'} u_1\|_{L^\infty} \right\} \\ + C 2^j \sum'_{\substack{j' \geq j - M_0 \\ |j' - j''| \leq 1}} 2^{-j'} \|\Delta_{j'} \nabla u_1\|_{L^\infty} \|\Delta_{j''} w\|_{L^\infty} ,$$

where \sum' indicates the same agreement as in Theorem 6.1.

Likewise,

$$(7.12) \quad \|R_j(w, u_2)\|_{L^\infty} \leq C \sum_{|j'-j| \leq M_0} \left\{ \|S_{j'-2} \nabla w\|_{L^\infty} \|\Delta_{j'} u_2\|_{L^\infty} \right. \\ \left. + \|S_{j'-2} \nabla u_2\|_{L^\infty} \|\Delta_{j'} w\|_{L^\infty} \right\} \\ + C 2^j \sum'_{\substack{j' \geq j - M_0 \\ |j' - j''| \leq 1}} 2^{-j'} \|\Delta_{j'} \nabla w\|_{L^\infty} \|\Delta_{j''} u_2\|_{L^\infty} .$$

We have to estimate $\Delta_j \nabla p$.

Taking div of both sides of (7.10) we get

$$(7.12') \quad \Delta_j \Delta p = -\operatorname{div} R_j(u_1, w) - \operatorname{div} R_j(w, u_2) \\ - \operatorname{tr}(\nabla \Delta_j w \cdot \nabla S_{j-2} u_1) - \operatorname{tr}(\nabla \Delta_j u_2 \cdot \nabla S_{j-2} w) .$$

Here Δ in the left side stands for Laplacian and we have used the solenoidality condition for w and u_2 .

We consider two cases.

Case I. $j \geq 0$. Then from (7.10), (7.8), (7.12') since

$$-\nabla \Delta_j p(x) = \mathcal{F}_{\xi \rightarrow x}(i\xi|\xi|^{-2}(\Delta_j \Delta p)^\wedge(\xi)) ,$$

we can use a standard Littlewood-Paley argument in conjunction with the Bernstein's inequality and arrive at

$$(7.13) \quad \|\Delta_j \nabla p\|_{L^\infty} \leq C \|R_j(u_1, w)\|_{L^\infty} + C \|R_j(w, u_2)\|_{L^\infty} \\ + C 2^{-j} \|S_{j-2} \nabla u_1\|_{L^\infty} \|\Delta_j \nabla w\|_{L^\infty} \\ + C 2^{-j} \|S_{j-2} \nabla w\|_{L^\infty} \|\Delta_j \nabla u_2\|_{L^\infty} \\ \leq C \|R_j(u_1, w)\|_{L^\infty} + C \|R_j(w, u_2)\|_{L^\infty} \\ + C \|S_{j-2} \nabla u_1\|_{L^\infty} \|\Delta_j w\|_{L^\infty} \\ + C \|S_{j-2} \nabla w\|_{L^\infty} \|\Delta_j u_2\|_{L^\infty} .$$

Case II. $j = -1$. Then from (7.7) applying the Bernstein's inequality, choosing arbitrary $p_2 \in [\frac{np_0}{n-p_0}, \infty)$ and using boundedness of H. Weyl projection onto gradients we get

$$\begin{aligned}
 (7.14) \quad \|\Delta_{-1}\nabla p\|_{L^\infty} &\leq C\|\Delta_{-1}\nabla p\|_{L^{p_2}} \\
 &\leq C\left\|-\sum_{k=1}^n\partial_k\Delta_{-1}\{u_{1k}w+w_ku_2\}\right\|_{L^{p_2}} \\
 &\leq C\|\Delta_{-1}(u_1\otimes w)\|_{L^{p_2}}+C\|\Delta_{-1}(w\otimes u_2)\|_{L^{p_2}} \\
 &\leq C\sum_{|j'-j''|\leq M_0}(\|\Delta_{j'}u_1\|_{L^{p_2}}+\|\Delta_{j'}u_2\|_{L^{p_2}})\|\Delta_{j''}w\|_{L^\infty}.
 \end{aligned}$$

We now fix an $N \geq -1$ and estimate the sum

$$\begin{aligned}
 (7.15) \quad \sum_{j=-1}^N\|R_j(u_1,w)\|_{L^\infty}+\sum_{j=-1}^N\|R_j(w,u_2)\|_{L^\infty} \\
 +\sum_{j=-1}^N\|\Delta_j\nabla p\|_{L^\infty}+\sum_{j=-1}^N\|(S_{j-2}w,\nabla)\Delta_ju_2\|_{L^\infty}\equiv Q_1+Q_2+Q_3+Q_4.
 \end{aligned}$$

We will prove below that

$$Q_j \leq C\Pi(N)\|w\|_{B_{\infty,1}^0}, \quad j = 1, \dots, 4.$$

Here the norm in Besov space $B_{\infty,1}^0$ is defined as follows:

$$B_{\infty,1}^0 = \left\{ w \in S' \mid \|w\|_{B_{\infty,1}^0} = \sum_{j=-1}^\infty \|\Delta_j w\|_{L^\infty} < \infty \right\}.$$

The inequality (7.11) yields (see Remark 6.2)

$$\begin{aligned}
 (7.16) \quad Q_1 \leq C \sum_{j=-1}^N \sum_{|j'-j|\leq M_0} \left\{ \|S_{j'-2}\nabla u_1\|_{L^\infty} \|\Delta_{j'}w\|_{L^\infty} + \|S_{j'-2}w\|_{L^\infty} \|\Delta_{j'}\nabla u_1\|_{L^\infty} \right\} \\
 + C \sum_{j=-1}^N 2^j \sum'_{\substack{j' \geq j-M_0 \\ |j'-j''|\leq 1}} 2^{-j'} \|\Delta_{j'}\nabla u_1\|_{L^\infty} \|\Delta_{j''}w\|_{L^\infty}.
 \end{aligned}$$

The first sum in (7.16) can be taken care of as follows. Using (7.2), (7.3) and Proposition 3.3 we obtain:

$$\begin{aligned}
 (7.17) \quad \sum_{j=-1}^N \sum_{|j'-j|\leq M_0} \left\{ \|S_{j'-2}\nabla u_1\|_{L^\infty} \|\Delta_{j'}w\|_{L^\infty} + \|S_{j'-2}w\|_{L^\infty} \|\Delta_{j'}\nabla u_1\|_{L^\infty} \right\} \\
 \leq C \left(\sup_{-1 \leq j' \leq N+M_0} \|S_{j'-2}\nabla u_1\|_{L^\infty} \right) \sum_{j'=-1}^{N+M_0} \|\Delta_{j'}w\|_{L^\infty}
 \end{aligned}$$

$$\begin{aligned}
& + C \left(\sum_{j'=-1}^{N+M_0} \|\Delta_{j'} \nabla u_1\|_{L^\infty} \right) \sup_{-1 \leq j' \leq N+M_0} \|S_{j'-2} w\|_{L^\infty} \\
& \leq C \|\omega_1\|_{B_{\Pi} \cap L^{p_0}} \Pi(N+M_0) \sum_{j'=-1}^{N+M_0} \|\Delta_{j'} w\|_{L^\infty} \\
& \leq C \Pi(N) \sum_{j'=-1}^{N+M_0} \|\Delta_{j'} w\|_{L^\infty} .
\end{aligned}$$

We now turn to the second sum in the right side of (7.16)

$$\begin{aligned}
(7.18) \quad & \sum_{j=-1}^N 2^j \sum'_{\substack{j' \geq j-M_0 \\ |j'-j''| \leq 1}} 2^{-j'} \|\Delta_{j'} \nabla u_1\|_{L^\infty} \|\Delta_{j''} w\|_{L^\infty} \\
& \leq C \sum_{j''=-1}^{\infty} \|\Delta_{j''} w\|_{L^\infty} \left(\sum_{j=-1}^{\min(N, j''+M_0+1)} 2^{j-j''} \right) \sum'_{|j'-j''| \leq 1} \|\Delta_{j'} \nabla u_1\|_{L^\infty} \\
& \leq C 2^{\min(N, j'')-j''} \sum'_{|j'-j''| \leq 1} \|\Delta_{j'} \nabla u_1\|_{L^\infty} \left(\sum_{j''=-1}^{\infty} \|\Delta_{j''} w\|_{L^\infty} \right) .
\end{aligned}$$

We estimate the coefficient in front of $\sum_{j''=-1}^{\infty} \|\Delta_{j''} w\|_{L^\infty}$. We consider two cases. In case $j'' \leq N$

$$(7.19) \quad 2^{\min(N, j'')-j''} \sum'_{|j'-j''| \leq 1} \|\Delta_{j'} \nabla u_1\|_{L^\infty} \leq C \Pi(N) \|\omega_1\|_{B_{\Pi} \cap L^{p_0}} \leq C \Pi(N) .$$

In case $j'' > N$

$$\begin{aligned}
(7.20) \quad & 2^{N-j''} \sum'_{|j'-j''| \leq 1} \|\Delta_{j'} \nabla u_1\|_{L^\infty} \\
& \leq C 2^{N-j''} \Pi(j'') \|\omega_1\|_{B_{\Pi} \cap L^{p_0}} \\
& \leq C 2^{N-j''} \Pi(j'') \leq C \int_N^{\infty} 2^{N-\xi} \Pi(\xi) d\xi \\
& \leq C \Pi(N) .
\end{aligned}$$

The combination of (7.18)–(7.20) yields

$$(7.21) \quad \sum_{j=-1}^N 2^j \sum'_{\substack{j' \geq j-M_0 \\ |j'-j''| \leq 1}} 2^{-j'} \|\Delta_{j'} \nabla u_1\|_{L^\infty} \|\Delta_{j''} w\|_{L^\infty} \leq C \Pi(N) \sum_{j=-1}^{\infty} \|\Delta_j w\|_{L^\infty} .$$

It follows from (7.16), (7.17), (7.21) that

$$(7.22) \quad Q_1 \leq C \Pi(N) \sum_{j=-1}^{\infty} \|\Delta_j w\|_{L^\infty} .$$

We now estimate Q_2 . From (7.12)

$$(7.23) \quad Q_2 \leq C \sum_{j=-1}^N \sum_{|j'-j| \leq M_0} \left\{ \|S_{j'-2} w\|_{L^\infty} \|\Delta_{j'} \nabla u_2\|_{L^\infty} + \|S_{j'-2} \nabla u_2\|_{L^\infty} \|\Delta_{j'} w\|_{L^\infty} \right\} + C \sum_{j=-1}^N 2^j \sum'_{\substack{j' \geq j - M_0 \\ |j' - j''| \leq 1}} 2^{-j'} \|\Delta_{j'} \nabla w\|_{L^\infty} \|\Delta_{j''} u_2\|_{L^\infty} .$$

We take care of the first sum in the right side of (7.23) as follows:

$$(7.24) \quad \sum_{j=-1}^N \sum_{|j'-j| \leq M_0} \left\{ \|S_{j'-2} w\|_{L^\infty} \|\Delta_{j'} \nabla u_2\|_{L^\infty} + \|S_{j'-2} \nabla u_2\|_{L^\infty} \|\Delta_{j'} w\|_{L^\infty} \right\} \leq C \left(\sup_{-1 \leq j' \leq N+M_0} \|S_{j'-2} w\|_{L^\infty} \right) \sum_{j'=-1}^{N+M_0} \|\Delta_{j'} \nabla u_2\|_{L^\infty} + C \left(\sup_{-1 \leq j' \leq N+M_0} \|S_{j'-2} \nabla u_2\|_{L^\infty} \right) \sum_{j'=-1}^{N+M_0} \|\Delta_{j'} w\|_{L^\infty} \leq C \Pi(N) \sum_{j'=-1}^{N+M_0} \|\Delta_{j'} w\|_{L^\infty}$$

using the same argument as in (7.17). To estimate the second sum in (7.23) we notice that

$$(7.25) \quad \sum_{j=-1}^N 2^j \sum'_{\substack{j' \geq j - M_0 \\ |j' - j''| \leq 1}} 2^{-j'} \|\Delta_{j'} \nabla w\|_{L^\infty} \|\Delta_{j''} u_2\|_{L^\infty} \leq C \sum_{j=-1}^N 2^j \sum_{\substack{j' \geq j - M_0 \\ |j' - j''| \leq 1}} \|\Delta_{j'} w\|_{L^\infty} \|\Delta_{j''} u_2\|_{L^\infty} \leq C \sum_{j'=-1}^{\infty} \|\Delta_{j'} w\|_{L^\infty} \left(\sum_{j=-1}^{\min(N, j' + M_0)} 2^j \right) \sum_{|j'' - j'| \leq 1} \|\Delta_{j''} u_2\|_{L^\infty} \leq C \sum_{j=-1}^{\infty} \|\Delta_{j'} w\|_{L^\infty} 2^{\min(N, j')} \sum_{|j'' - j'| \leq 1} \|\Delta_{j''} u_2\|_{L^\infty} .$$

In case $j' \leq N$,

$$(7.26) \quad 2^{\min(N, j')} \sum_{|j'' - j'| \leq 1} \|\Delta_{j''} u_2\|_{L^\infty} \leq C 2^{j'} 2^{-j'} \Pi(j') \|\omega_2\|_{B_{\Pi} \cap L^{p_0}} \leq C \Pi(N) .$$

In the opposite case $j' > N$,

$$\begin{aligned}
 2^{\min(N, j')} \sum_{|j'' - j'| \leq 1} \|\Delta_{j''} u_2\|_{L^\infty} &\leq C 2^{N - j'} \Pi(j') \|\omega_2\|_{B_{\Pi} \cap L^{p_0}} \\
 (7.27) \qquad \qquad \qquad &\leq C 2^{N - j'} \Pi(j') \leq C \int_N^\infty 2^{N - \xi} \Pi(\xi) \, d\xi \\
 &\leq C \Pi(N) .
 \end{aligned}$$

We conclude from (7.23)–(7.27) that

$$(7.28) \qquad \qquad \qquad Q_2 \leq C \Pi(N) \sum_{j=-1}^\infty \|\Delta_j w\|_{L^\infty} .$$

We turn now to Q_3 in the right side of (7.15). From (7.13), (7.14), (7.15), (7.22), (7.28) [notice that the last two terms in the right side of (7.13) are already present in the estimates (7.11), (7.12) for $\|R_j(u_1, w)\|_{L^\infty}$, $\|R_j(w, u_2)\|_{L^\infty}$] we obtain:

$$\begin{aligned}
 (7.29) \qquad \qquad \qquad Q_3 &\leq C \Pi(N) \sum_{j=-1}^\infty \|\Delta_j w\|_{L^\infty} + C \|\Delta_{-1} \nabla p\|_{L^\infty} \\
 &\leq C \Pi(N) \sum_{j=-1}^\infty \|\Delta_j w\|_{L^\infty} .
 \end{aligned}$$

Indeed, the factor in the right side on (7.14) is taken care of as follows:

$$\|\Delta_{j'} u_1\|_{L^{p_2}} + \|\Delta_{j'} u_2\|_{L^{p_2}} \leq C \|u_1\|_{L^{p_2}} + C \|u_2\|_{L^{p_2}} .$$

We choose $p_2 = \frac{np_0}{n-p_0}$. Then from (3.9)

$$\begin{aligned}
 (7.29') \qquad \qquad \qquad &\|u_{1,2}\|_{L^{p_2}} \leq C \|\omega_{1,2}\|_{L^{p_0}} \leq C . \\
 Q_4 &\leq \sum_{j=-1}^N \|S_{j-2} w\|_{L^\infty} \|\Delta_j \nabla u_2\|_{L^\infty} \\
 &\leq \sup_{-1 \leq j \leq N} \|S_{j-2} w\|_{L^\infty} \sum_{j=-1}^N \|\Delta_j \nabla u_2\|_{L^\infty} \\
 &\leq C \Pi(N) \sum_{j=-1}^{N-2} \|\Delta_j w\|_{L^\infty} \\
 &\leq C \Pi(N) \sum_{j=-1}^\infty \|\Delta_j w\|_{L^\infty} .
 \end{aligned}$$

It follows from (7.15), (7.22), (7.28), (7.29), (7.29') that

$$\begin{aligned}
 (7.30) \qquad \qquad \qquad &\sum_{j=-1}^N \|R_j(u_1, w)\|_{L^\infty} + \sum_{j=-1}^N \|R_j(w, u_2)\|_{L^\infty} \\
 &+ \sum_{j=-1}^N \|\Delta_j \nabla p\|_{L^\infty} + \sum_{j=-1}^N \|(S_{j-2} w, \nabla) \Delta_j u_2\|_{L^\infty} \\
 &\leq C \Pi(N) \sum_{j=-1}^\infty \|\Delta_j w\|_{L^\infty} .
 \end{aligned}$$

Solving the equation (7.10) along characteristics and using (7.9) $[\Delta_j w]_{t=0} = 0$ we obtain, for any $t \in [0, T]$ and any $j \geq -1$,

$$(7.31) \quad \|\Delta_j w(t)\|_{L^\infty} \leq C \int_0^t \left\{ \|R_j(u_1(\tau), w(\tau))\|_{L^\infty} + \|R_j(w(\tau), u_2(\tau))\|_{L^\infty} \right. \\ \left. + \|\Delta_j \nabla p(\tau)\|_{L^\infty} + \|(S_{j-2} w(\tau), \nabla) \Delta_j u_2(\tau)\|_{L^\infty} \right\} d\tau .$$

We have for arbitrary $t \in [0, T]$, $N \geq -1$

$$(7.32) \quad \sum_{j=-1}^\infty \|\Delta_j w(t)\|_{L^\infty} = \sum_{j=-1}^N \|\Delta_j w(t)\|_{L^\infty} + \sum_{j=N+1}^\infty \|\Delta_j w(t)\|_{L^\infty} \\ \leq \sum_{j=-1}^N \|\Delta_j w(t)\|_{L^\infty} + C \sum_{j=N+1}^\infty 2^{-j} \|\Delta_j \nabla w(t)\|_{L^\infty} .$$

We estimate the second term in the right side of (7.32). Let

$$(7.33) \quad d_m = \sum_{k=-1}^m \|\Delta_k \nabla w(t)\|_{L^\infty} .$$

Then, from (7.2)

$$(7.34) \quad \sum_{j=N+1}^\infty 2^{-j} \|\Delta_j \nabla w(t)\|_{L^\infty} = \sum_{j=N+1}^\infty 2^{-j} (d_j - d_{j-1}) \\ = -2^{-(N+1)} d_N + \sum_{j=N+1}^\infty (2^{-j} - 2^{-(j+1)}) d_j \\ \leq C \sum_{j=N+1}^\infty 2^{-j} \Pi(j) \|\omega(t)\|_{B_\Pi \cap L^{p_0}} \\ \leq C 2^{-N} \Pi(N) .$$

Using (7.30)–(7.34) we get:

$$(7.35) \quad \sum_{j=-1}^\infty \|\Delta_j w(t)\|_{L^\infty} \leq C \Pi(N) \int_0^t \sum_{j=-1}^\infty \|\Delta_j w(\tau)\|_{L^\infty} d\tau + C 2^{-N} \Pi(N) .$$

We denote by $\zeta(t)$ the following function on $[0, T]$

$$(7.36) \quad \zeta(t) = \int_0^t \sum_{j=-1}^\infty \|\Delta_j w(\tau)\|_{L^\infty} d\tau .$$

Since

$$\left\| \sum_{j=-1}^\infty \|\Delta_j w(\cdot)\|_{L^\infty} \right\|_{L^\infty([0, T])} \leq C \left\| \|\omega(\cdot)\|_\Pi \right\|_{L^\infty([0, T])} + C \|\omega\|_{L^\infty([0, T]; L^{p_0})} \leq C ,$$

$\zeta(t)$ is monotonically nondecreasing absolutely continuous function. Since $\zeta(0) = 0$ there exists a $t_0 \in [0, T]$ such that

$$\zeta(t) \equiv 0 \text{ on } [0, t_0] \quad ; \quad \zeta(t) > 0 \text{ on } (t_0, T] .$$

If $t_0 = T$ then the uniqueness follows. Therefore, we assume $t_0 < T$. Let ε be small enough so that $t_0 + \varepsilon < T$ and $\zeta(t) < 2^{-M_1-1}$ on $(t_0, t_0 + \varepsilon)$. We will choose M_1 later. For any $t \in (t_0, t_0 + \varepsilon)$ we choose N in (7.35) as follows:

$$N = [-\log_2 \zeta(t)],$$

where the square bracket means the entire part of a real number. Then (7.35) implies:

$$(7.37) \quad \dot{\zeta}(t) \leq C\Pi(-\log_2 \zeta(t)) \cdot \zeta(t) , \quad \zeta(0) = 0 .$$

Notice that from (7.1)

$$\int_0^{1/2} \zeta^{-1} \Pi(-\log_2 \zeta)^{-1} d\zeta = (\log_2 e)^{-1} \int_1^\infty \Pi(\alpha)^{-1} d\alpha = \infty .$$

Therefore, the Osgood uniqueness theorem applies to the initial value problem

$$(7.38) \quad \dot{\eta}(t, \delta) = C\Pi(-\log_2 \eta)\eta , \quad \eta(t_0, \delta) = \delta ,$$

where $\delta > 0$ is small. The solution η exists on $(t_0, t_0 + \varepsilon)$ and depends continuously on the parameter δ . The simple Gronwall's argument now works for comparison of ζ and η . Then $\zeta(t) < \eta(t, \delta)$ for sufficiently small δ and for all $t \in [t_0, t_0 + \varepsilon)$. For otherwise let $t_1 = \min_{t \in [t_0, t_0 + \varepsilon)} \{t \mid \zeta(t) = \eta(t, \delta)\}$. From (7.37), (7.38)

$$(7.39) \quad \begin{aligned} \zeta(t_1) &\leq C \int_{t_0}^{t_1} \Pi(-\log_2 \zeta(\tau)) \zeta(\tau) d\tau \\ &< \delta + C \int_{t_0}^{t_1} \Pi(-\log_2 \eta(\tau, \delta)) \eta(\tau, \delta) d\tau = \eta(t_1, \delta) . \end{aligned}$$

We have to use in (7.39) that the function $\zeta \mapsto \Pi(-\log_2 \zeta)\zeta$ is *monotonically* decreasing to 0 as $\zeta \rightarrow 0+$. This is just a reformulation of (7.1').

We choose M_1 in such a way that $\Pi(\alpha)2^{-\alpha}$ is nonincreasing for $\alpha \geq M_1$. Now (7.39) contradicts the definition of t_1 . Passing to the limit as $\delta \rightarrow 0+$ we have to conclude that $\zeta \equiv 0$ on $[t_0, t_0 + \varepsilon)$. But this contradicts the definition of t_0 . The proof of uniqueness is now completed.

COROLLARY 7.2. – Assume (7.3), (7.4), (7.5). Let, in addition to this,

$$\begin{aligned} \omega_{1,2} &\in L^\infty([0, T]; L^{p_0}) ; \\ \|\omega_{1,2}(\cdot)\|_{\text{bmo}} &\in L^\infty([0, T]) ; \\ \omega_{1,2}(\cdot, 0) &= f(\cdot) \in \text{bmo} \cap L^{p_0} , \end{aligned}$$

where $p_0 \in (1, n)$. Then $u_1 = u_2$ on $[0, T]$.

Proof. – Let $\Pi(N) = (N + 2)$, $N \geq -1$. Then $\text{bmo} \hookrightarrow B_\Pi$. But

$$\int_1^\infty (N + 2)^{-1} dN = \infty$$

so (7.1) is satisfied. Condition (7.1') is satisfied as well. Therefore uniqueness follows from the Theorem 7.1.

REMARK 7.3. – The same proof works for the space $\Lambda^0 = B_{\infty, \infty}^0$ replacing the space bmo .

8. Construction of the flow

In this section $n = 2$. Let Γ satisfy the conditions (i)–(v).

THEOREM 8.1. – Assume (5.8). Let

$$(8.1) \quad f \in B_\Gamma \cap L^{p_0} \cap L^{p_1}, \quad 1 < p_0 < 2 < p_1 < \infty.$$

Then there exists a $T > 0$ and a solution $\omega(\cdot)$ to the Euler equations

$$(8.2) \quad \dot{u} = -(u, \nabla)u - \nabla p, \quad \omega(0) = f,$$

$$(8.3) \quad u = \mathcal{K} * \omega, \quad \operatorname{div} u = 0,$$

satisfying the following condition

$$(8.4) \quad \omega(\cdot) \in L^\infty([0, T]; L^{p_0} \cap L^{p_1}) \cap C_{w^*}([0, T]; B_{\Gamma_1}),$$

where $C_{w^*}([0, T]; B_{\Gamma_1})$ stands for the space of weak* continuous functions with values in B_{Γ_1} in the sense of duality $H_{\Gamma_1}^t = B_{\Gamma_1}$.

THEOREM 8.2. – Assume (5.23), (8.1). Then there exists a solution

$$\omega(\cdot) \in L_{\text{loc}}^\infty([0, \infty); L^{p_0} \cap L^{p_1}) \cap C_{w^*}([0, \infty); B_{\Gamma_1})$$

to the problem (8.2), (8.3).

REMARK 8.3. – The solution $\omega(\cdot)$ in both theorems is unique provided (7.1') is satisfied. Indeed, (5.8) implies (7.1) and uniqueness follows from the Theorem 7.1 with $\Pi = \Gamma_1$. Likewise (5.23) implies (7.1) with the same Π .

REMARK 8.4. – See the definition of H_{Γ_1} in Section 1 (also the Proposition 1.6).

To construct the solution in Theorems 8.1, 8.2 we use approximation by regular solutions. We need therefore the following result.

THEOREM 8.5 ([C1]). – Let $n = 2$, $p \in (1, \infty)$, $f \in \cap_{r>0} \Lambda^r \cap L^p$. Then there exists a (unique) solution

$$\omega(\cdot) \in \cap_{r>0} L_{\text{loc}}^\infty([0, \infty); \Lambda^r \cap L^p)$$

to the problem (8.2), (8.3).

Here Λ^r , $r \in \mathbb{R}$ denotes the Hölder space. For convenience we recall the definition.

DEFINITION 8.6. – Let $r \in \mathbb{R}$.

$$\Lambda^r = B_{\infty, \infty}^r = \left\{ f \in S' \mid \|f\|_{\Lambda^r} = \sup_{j \geq -1} 2^{rj} \|\Delta_j f\|_{L^\infty} < \infty \right\}.$$

In particular for $r > 0$, $\Lambda^r \subset L^\infty$.

PROPOSITION 8.7. – For $r > 0$, $\Lambda^r \subset B_\Gamma$.

Proof.

$$\sum_{j=-1}^N \|\Delta_j f\|_{L^\infty} \leq \sum_{j=-1}^N 2^{-rj} \|f\|_{\Lambda^r} \leq (1 - 2^{-r})^{-1} \|f\|_{\Lambda^r} \leq CT(N), \quad N \geq -1.$$

Proof of Theorem 8.1. – We construct for any $m \geq 1$ the solution

$$(8.5) \quad \omega_m(\cdot) \in \cap_{r>0} L_{\text{loc}}^\infty([0, \infty); \Lambda^r \cap L^{p_0})$$

provided by Theorem 8.5 such that

$$(8.6) \quad \omega_m(0) = S_m f \in \cap_{r>0} \Lambda^r.$$

It follows from Littlewood-Paley theory and (8.1) that

$$(8.7) \quad \|\omega_m(0)\|_{L^{p_0}} \leq C, \quad \|\omega_m(0)\|_{L^{p_1}} \leq C.$$

We also have

$$(8.8) \quad \|\omega_m(0)\|_{\Gamma} = \sup_{N \geq -1} \sum_{j=-1}^N \|\Delta_j \omega_m(0)\|_{L^\infty} \Gamma(N)^{-1}.$$

But $\Delta_j S_m = 0$ for $j \geq m + 2$; $\Delta_j S_m = \Delta_j$ for $j \leq m - 2$ and in any case

$$(8.9) \quad \|\Delta_j S_m f\|_{L^\infty} = \|S_m \Delta_j f\|_{L^\infty} \leq C \|\Delta_j f\|_{L^\infty}.$$

Using properties of Γ we conclude from (8.8) that

$$(8.10) \quad \|\omega_m(0)\|_{\Gamma} \leq C \|f\|_{\Gamma}.$$

In addition (8.5) together with Proposition 8.7 imply

$$(8.11) \quad \omega_m(\cdot) \in L_{\text{loc}}^\infty([0, \infty); B_{\Gamma_1}).$$

From Theorem 5.1 using (8.7), (8.10), (8.11) we find a $T > 0$ and a constant $C > 0$ independent of m such that

$$(8.12) \quad \|\omega_m\|_{L^\infty([0, T]; L^{p_0} \cap L^{p_1})} \leq C, \quad \|\omega_m\|_{L^\infty([0, T]; B_{\Gamma_1})} \leq C.$$

Let $m > \ell \geq 1$ are two indices. We use the estimates of §7 where by abusing notation we set $\Pi = \Gamma_1$,

$$(8.13) \quad u_m = \mathcal{K} * \omega_m, \quad u_\ell = \mathcal{K} * \omega_\ell,$$

$$(8.14) \quad w = u_m - u_\ell = \mathcal{K} * (\omega_m - \omega_\ell).$$

The estimate (7.31) ought to be replaced by

$$(8.15) \quad \begin{aligned} \|\Delta_j w(t)\|_{L^\infty} \leq & \|\Delta_j w(0)\|_{L^\infty} + C \int_0^t \left\{ \|R_j(u_m(\tau), w(\tau))\|_{L^\infty} \right. \\ & + \|R_j(w(\tau), u_\ell(\tau))\|_{L^\infty} + \|\Delta_j \nabla p(\tau)\|_{L^\infty} \\ & \left. + \|(S_{j-2} w(\tau), \nabla) \Delta_j u_\ell(\tau)\|_{L^\infty} \right\} d\tau \end{aligned}$$

since $\Delta_j w(0)$ does not have to vanish. Combining (7.30), (8.15), (7.32)–(7.34) yields:

$$(8.16) \quad \sum_{j=-1}^{\infty} \|\Delta_j w(t)\|_{L^\infty} \leq \sum_{j=-1}^N \|\Delta_j w(0)\|_{L^\infty} + C\Pi(N) \int_0^t \sum_{j=-1}^{\infty} \|\Delta_j w(\tau)\|_{L^\infty} d\tau + C2^{-N}\Pi(N)$$

for any $N \geq -1$.

We use the same function $\zeta(t)$ as in (7.36). The estimate (8.16) implies

$$(8.17) \quad \dot{\zeta}(t) \leq \sum_{j=-1}^{\infty} \|\Delta_j w(0)\|_{L^\infty} + C\Gamma_1(-\log_2 \zeta(t))\zeta(t)$$

for $t \in [0, T]$ by making the same choice of N as in §7.

We now observe that

$$(8.18) \quad \begin{aligned} \kappa_{m,\ell} &= \sum_{j=-1}^{\infty} \|\Delta_j w(0)\|_{L^\infty} = \sum_{j=-1}^{\infty} \|\Delta_j \mathcal{K} * (S_m - S_\ell) f\|_{L^\infty} \\ &= \sum_{j=-1}^{\infty} \|\Delta_j \mathcal{K} * \sum_{q=\ell+1}^m \Delta_q f\|_{L^\infty} \\ &\leq \sum_{q=\ell+1}^m \sum_{|j-q|\leq 1} \|\mathcal{K} * \Delta_j \Delta_q f\|_{L^\infty} \\ &\leq C \sum_{q=\ell+1}^{\infty} \|\mathcal{K} \Delta_q f\|_{L^\infty} \\ &\leq C \sum_{q=\ell+1}^{\infty} 2^{-q} \|\Delta_q f\|_{L^\infty} . \end{aligned}$$

Let $d_m = \sum_{j=-1}^m \|\Delta_j f\|_{L^\infty}$. Then (8.18) implies

$$(8.19) \quad \begin{aligned} \kappa_{m,\ell} &\leq C \sum_{q=\ell+1}^{\infty} 2^{-q} (d_q - d_{q-1}) \\ &= -C2^{-(\ell+1)} d_\ell + C \sum_{q=\ell+1}^{\infty} 2^{-q-1} d_q \\ &\leq C \sum_{q=\ell+1}^{\infty} 2^{-q} \Gamma(q) \leq C \int_{\ell+1}^{\infty} 2^{-\xi} \Gamma(\xi) d\xi \\ &\leq C2^{-\ell} \Gamma(\ell) . \end{aligned}$$

In (8.19) we used properties of Γ . Integrating both sides of (8.17) and using (8.18), (8.19) we obtain

$$\zeta(t) \leq C2^{-\ell} \Gamma(\ell) + C \int_0^t \Gamma_1(-\log_2 \zeta(\tau)) \zeta(\tau) d\tau .$$

Using monotonicity of $\Gamma_1(-\log \zeta) \cdot \zeta$ for small $\zeta \geq 0$ as in §8 we apply a similar Gronwall's argument to conclude

$$(8.20) \quad \zeta(t) \leq \eta(t, C2^{-\ell}\Gamma(\ell)) , \quad t \in [0, T] .$$

Going back to (8.17) and using again (8.18)–(8.19) and monotonicity of $\Gamma_1(-\log \zeta) \cdot \zeta$ for small $\zeta \geq 0$ we obtain

$$\begin{aligned} \dot{\zeta}(t) &\leq C2^{-\ell}\Gamma(\ell) + C \int_0^t \Gamma_1(-\log_2 \eta(\tau, C2^{-\ell}\Gamma(\ell))) \\ &\quad \cdot \eta(\tau, C2^{-\ell}\Gamma(\ell)) d\tau , \quad t \in [0, T] . \end{aligned}$$

Therefore, $\{u_m(\cdot)\}_{m=1}^\infty$ is a Cauchy sequence in the Banach space $L^\infty([0, T]; B_{\infty,1}^0)$. We remind that the norm in the Besov space $B_{\infty,1}^0$ is defined as follows

$$B_{\infty,1}^0 = \left\{ w \in S' \mid \|w\|_{B_{\infty,1}^0} = \sum_{j=-1}^\infty \|\Delta_j w\|_{L^\infty} < \infty \right\} .$$

Therefore, there exists a strong limit in $L^\infty([0, T]; B_{\infty,1}^0)$

$$(8.21) \quad u_m \rightarrow u \in L^\infty([0, T]; B_{\infty,1}^0) \text{ as } m \rightarrow \infty .$$

We claim now that in fact $\omega = \text{curl}u$ satisfies

$$(8.22) \quad \|\omega(\cdot)\|_{\Gamma_1} \in L^\infty([0, T]) .$$

Let $N \geq -1$ be an arbitrary integer.

Consider the seminorm on $L^\infty([0, T]; B_{\infty,1}^0)$

$$\rho_N(v) = \left\| \sum_{j=-1}^N \|\Delta_j v(\cdot)\|_{L^\infty} \right\|_{L^\infty([0, T])} .$$

Obviously $\rho_N(u_m - u) \rightarrow 0$ as $m \rightarrow \infty$. On the other hand, from Bernstein's inequality

$$(8.23) \quad \left\| \sum_{j=-1}^N \|\Delta_j(\omega_m - \omega)(\cdot)\|_{L^\infty} \right\|_{L^\infty([0, T])} \leq C2^N \rho_N(u_m - u) .$$

Hence,

$$\begin{aligned} (8.24) \quad &\left\| \sum_{j=-1}^N \|\Delta_j \omega_m(\cdot)\|_{L^\infty} - \sum_{j=-1}^N \|\Delta_j \omega(\cdot)\|_{L^\infty} \right\|_{L^\infty([0, T])} \\ &\leq \left\| \sum_{j=-1}^N \left| \|\Delta_j \omega_m(\cdot)\|_{L^\infty} - \|\Delta_j \omega(\cdot)\|_{L^\infty} \right| \right\|_{L^\infty([0, T])} \\ &\leq \left\| \sum_{j=-1}^N \|\Delta_j(\omega_m - \omega)(\cdot)\|_{L^\infty} \right\|_{L^\infty([0, T])} \\ &\leq C2^N \rho_N(u_m - u) \rightarrow 0 \text{ as } m \rightarrow \infty , \end{aligned}$$

where on the last step we used (8.23). From (8.12) we have

$$(8.25) \quad \sum_{j=-1}^N \|\Delta_j \omega_m(t)\|_{L^\infty} \leq CT_1(N) ,$$

where the constant C does not depend on m . Passing to the limit as $m \rightarrow \infty$ in (8.24) and using (8.25) yields

$$\left\| \sum_{j=-1}^N \|\Delta_j \omega(\cdot)\|_{L^\infty} \right\|_{L^\infty([0,T])} \leq CT_1(N) .$$

This implies (8.22).

In fact since $\{\omega_m\}$ is a Cauchy sequence in $C([0, T]; B_{\infty,1}^0)$ this argument shows that $\omega(t) \in B_{\Gamma_1}$ for all $t \in [0, T]$. We note that (8.21) implies

$$(8.26) \quad u_m \rightarrow u \text{ strongly in } L^\infty(\mathbb{R}^2 \times [0, T]) \cap C(\mathbb{R}^2 \times [0, T]) .$$

We have in addition the following facts after possibly choosing a subsequence

$$(8.27) \quad \omega_m \rightarrow \omega \text{ weak* in } L^\infty([0, T]; L^{p_0}) \text{ and in } L^\infty([0, T]; L^{p_1}) ;$$

$$(8.28) \quad \dot{u}_m \rightarrow \dot{u} \text{ weak* in } L^\infty([0, T]; L^{p_0}) \text{ and in } L^\infty([0, T]; L^{p_1}) .$$

Let $\rho \in S$, $\operatorname{div} \rho = 0$ is a test function. Also let $\theta \in \mathcal{D}([0, T])$. Then, by construction

$$\langle u_m(0), \rho \rangle \theta(0) + \int_0^T \langle u_m(\tau), \rho \rangle \dot{\theta}(\tau) \, d\tau + \int_0^T \langle u_m(\tau), (u_m(\tau), \nabla) \rho \rangle \theta(\tau) \, d\tau = 0 .$$

We have $\langle u_m(0), \rho \rangle \rightarrow \langle \mathcal{K} * f, \rho \rangle$ as $m \rightarrow \infty$ because of (8.1), (8.6). Also from (8.26)

$$\begin{aligned} \int_0^T \langle u_m(\tau), \rho \rangle \dot{\theta}(\tau) \, d\tau &\rightarrow \int_0^T \langle u(\tau), \rho \rangle \dot{\theta}(\tau) \, d\tau ; \\ \int_0^T \langle u_m(\tau), (u_m(\tau), \nabla) \rho \rangle \theta(\tau) \, d\tau &\rightarrow \int_0^T \langle u(\tau), (u(\tau), \nabla) \rho \rangle \theta(\tau) \, d\tau \text{ as } m \rightarrow \infty . \end{aligned}$$

Therefore, the limit u satisfies the Euler equation and the initial condition

$$\begin{cases} \dot{u} = -(u, \nabla)u - \nabla p , \\ \operatorname{div} u = 0 , \quad u(0) = \mathcal{K} * f . \end{cases}$$

The remaining part of the proof is contained in the following Lemma.

LEMMA 8.4. – *The vorticity constructed above $\omega(\cdot)$ is weak* continuous with values in $B_{\Gamma_1} = H_{\Gamma_1}^1$.*

Proof. – Since $\|u - u_m\|_{C([0,T]; B_{\infty,1}^0)} \rightarrow 0$ as $m \rightarrow \infty$ we have

$$(8.29) \quad \|\omega - \omega_m\|_{C([0,T]; B_{\infty,1}^{-1})} \rightarrow 0 \text{ as } m \rightarrow \infty .$$

Let $h \in H_{\Gamma_1}$. Consider the function $t \mapsto \langle \omega(t), h \rangle = \varphi(t)$ on $[0, T]$. Let $\varphi_m(t) = \langle \omega_m(t), h \rangle$. Then, for any $t_0 \in [0, T]$

$$(8.30) \quad \varphi(t) - \varphi(t_0) = (\varphi - \varphi_m)(t) - (\varphi - \varphi_m)(t_0) + (\varphi_m(t) - \varphi_m(t_0)) .$$

For a fixed m $\varphi_m(t) - \varphi_m(t_0) \rightarrow 0$ as $t \rightarrow t_0$ since $t \mapsto \omega_m(t)$ is (strongly) continuous with values in B_{Γ_1} . To obtain the statement of the lemma we estimate the first and the second term in (8.30). We have

$$|(\varphi - \varphi_m)(t)| \leq | \langle (\omega - \omega_m)(t), h - \tilde{h} \rangle | + | \langle (\omega - \omega_m)(t), \tilde{h} \rangle | ,$$

where $\tilde{h} \in H_{\Gamma_1}$.

We choose $\tilde{h} \in B_{1,1+\delta}^1$ so that

$$(8.31) \quad \|\tilde{h} - h\|_{\Gamma_1} < \varepsilon .$$

Obviously $B_{1,1+\delta}^1 \subset B_{1,\infty}^1 \subset H_{\Gamma_1}$. Indeed, if

$$\|\Delta_j f\|_{L^1} \leq C 2^{-j} = C 2^{-j} \Gamma_1(j) \Gamma_1(j)^{-1} ,$$

this implies

$$\|f\|_{\Gamma_1} \leq C \sum_{j=-1}^{\infty} 2^{-j} \Gamma_1(j) \leq C$$

according to (ii)-(iv). It is clear that $B_{1,1+\delta}^1$ is dense in H_{Γ_1} since it contains all functions from L^1 with a bounded Fourier spectrum.

Since $\|\omega(t)\|_{\Gamma_1}, \|\omega_m(t)\|_{\Gamma_1}$ are uniformly bounded on $[0, T]$,

$$(8.32) \quad | \langle (\omega - \omega_m)(t), h - \tilde{h} \rangle | < C\varepsilon .$$

Also, $B_{\infty,1}^{-1} \hookrightarrow B_{\infty,1+\delta}^{-1}$ and thus

$$(8.33) \quad | \langle (\omega - \omega_m)(t), \tilde{h} \rangle | \leq C \|\omega - \omega_m\|_{C([0,T]; B_{\infty,1}^{-1})} \|\tilde{h}\|_{B_{1,1+\delta}^1} .$$

We need here the duality $(B_{1,1+\delta}^1)' = B_{\infty,1+\delta}^{-1}$ (see e.g., [P], [Tr]). We now fix any $\delta > 0$ (say $\delta = 1$). For any $\varepsilon > 0$ we choose \tilde{h} to satisfy (8.31) and using (8.29) choose m so that the right side of (8.33) becomes $< \varepsilon$.

Then from (8.31)–(8.33) $\overline{\lim}_{t \rightarrow t_0} |\varphi(t) - \varphi(t_0)| \leq C\varepsilon$. This concludes the proof of the lemma.

Theorem 8.1 is proved.

Proof of Theorem 8.2. – We repeat the proof of Theorem 8.1 word for word except now the choice of $T > 0$ is arbitrary. Here we have to refer to Theorem 5.5 instead of Theorem 5.1. Thus condition (5.23) has to be satisfied for this argument to work. This completes the proof.

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Last minute addition: A result very similar to [Y2] was obtained independently in Philippe Serfati, Pertes de régularité pour le laplacien et l'équation d'Euler sur \mathbb{R}^n , preprint, 15 pp., 1994.