Incorporating the Pricing Decisions into the Dynamic Fleet Management Problem *

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January 18, 2006

Abstract

This paper presents a model to coordinate the pricing and fleet management decisions of a freight carrier. We consider a setting where the loads faced by the carrier over a certain time horizon are deterministic functions of the prices. We want to find what prices the carrier should charge so that its pricing and fleet management decisions jointly maximize the profits. Our solution approach is an iterative one. At each iteration, we solve the fleet management problem with fixed prices, and then, adjust these prices by using the primal-dual solution to the fleet management problem so as to obtain "better" prices. Computational experiments show that our approach yields high-quality solutions and can efficiently be applied on large problems.
Keywords: Transport, Road Transport, Logistics, Pricing.

There is a rich body of literature on how to manage a fleet of vehicles to serve the loads that occur, possibly randomly, at different locations in a transportation network. However, little attention has been directed to the problem of what prices to charge for serving the loads. Due to the competition in the freight transportation industry, reducing the prices can increase the number of loads, but the pricing and fleet management decisions for different lanes (origin-destination pairs) and different time periods interact with each other, and one should consider the "downstream" effects on the whole transportation network and on the later time periods when making these decisions.

In this paper, we show how to coordinate the pricing and fleet management decisions of a freight carrier. We consider a setting where the loads faced by the carrier over a certain time horizon are deterministic functions of the prices. The objective is to find what prices the carrier should charge so that its pricing and fleet management decisions jointly maximize the profits. Our solution approach is an iterative one and is similar to a subgradient search method. At each iteration, we solve the fleet management problem with fixed prices and obtain a search direction by using the primal-dual solution to the fleet management problem. This search direction indicates how the prices should be adjusted to increase the profits. After adjusting the prices, we resolve the fleet management problem with the adjusted prices at the next iteration.

*Running Title: Pricing in Fleet Management.

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Fleet management models have a long history. Early models formulate the problem over a "state-time network," where the nodes represent the supplies of vehicles at different locations and at different time periods, the arcs represent the vehicle movements and the load availabilities act as upper bounds on the arcs (see Dantzig & Fulkerson (1954), Ferguson & Dantzig (1955), White & Bomberault (1969), White (1972)). These models are referred to as deterministic models because they assume that the future load arrivals are known in advance. The majority of the commercial fleet management models used in practice today are deterministic.

Another class of fleet management models address the randomness in the load arrivals by formulating the problem as a stochastic control problem and using value functions to assess the impact of the current decisions on the future. Due to the large number of decision variables and possible load realizations, it is difficult to compute the value functions exactly, and most of these models seek to approximate the value functions in a tractable manner (see Jordan & Turnquist (1983), Frantzeskakis & Powell (1990), Crainic, Gendreau & Dejax (1993), Carvalho & Powell (2000), Godfrey & Powell (2002), Adelman (2004), Kleywegt, Nori & Savelsbergh (2004), Topaloglu & Powell (2006)).

There has been little work on pricing decisions in the fleet management context. Some of the past work considers the problem of how much the total profit would change when an additional load is introduced into the system. If the underlying fleet management model uses a "state-time network" formulation, then this problem can be solved by using the dual variables associated with the upper bound constraints that represent the load availabilities (see Powell (1985), Powell, Sheffi, Nickerson, Butterbaugh & Atherton (1988), Powell (1989)). However, the same problem becomes much harder when the load arrivals are random, specifically due to the fact that the optimal fleet management policy is not known. Topaloglu & Powell (2004) show how to compute the change in the total expected profit in response to an additional load introduced into the system by assuming that the fleet management decisions are made according to a particular class of suboptimal policies.

Pricing models also appear in the network revenue management literature. Important topics in this literature can be found in Weatherford & Bodily (1992), Talluri & van Ryzin (1998), McGill & van Ryzin (1999), Bitran & Caldentey (2003), Talluri & van Ryzin (2004). The central question in this literature is whether a perishable product should be sold to a customer who is available now but is willing to pay a low price or it should be kept for a customer who may be available in the future but will be willing to pay a higher price. Similar to fleet management systems, revenue management systems are characterized by high level of uncertainty, but one of the most well-known revenue management tools, called bid-price controls, assumes that the future demand is deterministic. This is reminiscent of our pricing approach, which assumes that the loads faced by the carrier are deterministic functions of the prices. (We note that, just like bid-price controls, our pricing approach can be applied on a "rolling-horizon" basis to handle the randomness in the load arrivals.) On the other hand, the pricing problem we consider in this paper is set apart from the ones considered by the revenue management literature by the fact that the revenue management context typically assumes a fixed initial level of inventory that cannot be replenished over the lifetime of the product, whereas the fleet management context provides the ability to reposition the vehicles and to adjust the capacity to cover the loads occurring at different locations.

In this paper, we make the following research contributions. We present a model and an efficient approximate solution method to find what prices a freight carrier should charge so that its pricing and fleet management decisions jointly maximize the profits. This is one of the few attempts in the literature to jointly make the pricing and fleet management decisions. Computational experiments show that our approach yields high-quality solutions when compared with benchmark methods and with upper bounds on the optimal objective values.

The organization of the paper is as follows. Section 1 formulates the problem and Section 2 describes our solution approach. Section 3 builds on Section 2 and develops a pricing algorithm to find what prices the carrier should charge. This section assumes that, for a given lane, the carrier can charge different prices for different time periods. Section 4 modifies the results of Section 3 to address the situation where the carrier has to charge a single price for each lane. Section 5 presents our computational experiments.

1 Problem Formulation

We have a fleet of vehicles to serve the loads occurring at different locations over a finite planning horizon. The number of loads over each lane and at each time period is a deterministic function of the price. We want to find what prices should be charged so that the pricing and fleet management decisions jointly maximize the profits. We define the following.

$\mathcal{T} =$ Set of time periods in the planning horizon. We have $\mathcal{T} = \{1, \dots, T\}$ for finite T.

$\mathcal{I} =$ Set of locations in the transportation network.

$\mathcal{L} =$ Set of lanes that correspond to loaded vehicle movements.

$\mathcal{E} =$ Set of lanes that correspond to empty vehicle movements.

$o_l, d_l =$ Origin and destination locations for lane $l \in \mathcal{L} \bigcup \mathcal{E}$.

$x_{lt} =$ Number of vehicles moving loaded over lane $l \in \mathcal{L}$ at time period $t \in \mathcal{T}$.

$y_{lt} =$ Number of vehicles moving empty over lane $l \in \mathcal{E}$ at time period $t \in \mathcal{T}$.

$p_{lt} =$ Price charged over lane $l \in \mathcal{L}$ at time period $t \in \mathcal{T}$.

Roughly speaking, a location corresponds to a "node" and a lane corresponds to an "arc" in the transportation network. As such, we can have multiple lanes with the same origin and destination locations. For example, if we have $l, l' \in \mathcal{L}$ with $i = o_l = o_{l'}$ and $j = d_l = d_{l'}$, then x_{lt} and $x_{l't}$ may represent the numbers of vehicles serving two different types of loads that need to be carried from location i to j at time period t. For all $i \in \mathcal{I}$, we assume that there exists $l_i \in \mathcal{E}$ such that $i = o_{l_i} = d_{l_i}$. In this case, $y_{l_i t}$ represents the number of vehicles held at location i at time period t. We also define the following.

$D_{lt}(p_{lt}) =$ Given that the price charged is p_{lt}, number of loads over lane $l \in \mathcal{L}$ at time period $t \in \mathcal{T}$. We assume that $D_{lt}(p_{lt})$ is a decreasing function of p_{lt}.

$r_{lt}(p_{lt}) =$ Given that the price charged is p_{lt}, profit from serving a load over lane $l \in \mathcal{L}$ at time period $t \in \mathcal{T}$. We assume that $r_{lt}(p_{lt}) = a_{lt}\, p_{lt} + b_{lt}$, where a_{lt} and b_{lt} are constants and $a_{lt} > 0$.

$c_{lt} =$ Cost of moving a vehicle empty over lane $l \in \mathcal{E}$ at time period t.

$f_{i1} =$ Number of vehicles at location $i \in \mathcal{I}$ at the beginning of the planning horizon.

For notational brevity, we take the travel time between any two locations to be one time period. Without loss of generality, we assume that $D_{lt}(\cdot)$ is obtained by "rounding down" a differentiable "intensity" function $\lambda_{lt}(\cdot)$. That is, we have $D_{lt}(p_{lt}) = \lfloor \lambda_{lt}(p_{lt}) \rfloor$ (see Figure 1). We also assume that the loads that are not covered are lost. The linear form of $r_{lt}(\cdot)$ is not restrictive because the prices are charged on a "per-mile" basis in practice and we have $r_{lt}(p_{lt}) = \Delta(o_l, d_l)\,[p_{lt} - C]$, where $\Delta(o_l, d_l)$ is the distance between locations o_l and d_l, and C is the "per-mile" empty repositioning cost. Finally, by suppressing one or more of the indices in the variables defined above, we denote a vector composed of the components ranging over the suppressed indices. For example, we have $p_t = \{p_{lt} : l \in \mathcal{L}\}$, $p = \{p_{lt} : l \in \mathcal{L},\ t \in \mathcal{T}\}$. The problem can be formulated as

$$\max \quad \sum_{t \in \mathcal{T}} \sum_{l \in \mathcal{L}} r_{lt}(p_{lt})\, x_{lt} - \sum_{t \in \mathcal{T}} \sum_{l \in \mathcal{E}} c_{lt}\, y_{lt} \tag{1}$$

$$\text{subject to} \quad \sum_{l \in \mathcal{L}: o_l = i} x_{l1} + \sum_{l \in \mathcal{E}: o_l = i} y_{l1} = f_{i1} \qquad i \in \mathcal{I} \tag{2}$$

$$\sum_{l \in \mathcal{L}: d_l = i} x_{l,t-1} + \sum_{l \in \mathcal{E}: d_l = i} y_{l,t-1} - \sum_{l \in \mathcal{L}: o_l = i} x_{lt} - \sum_{l \in \mathcal{E}: o_l = i} y_{lt} = 0 \quad i \in \mathcal{I},\ t \in \mathcal{T} \setminus \{1\} \tag{3}$$

$$x_{lt} - D_{lt}(p_{lt}) \le 0 \qquad l \in \mathcal{L},\ t \in \mathcal{T} \tag{4}$$

$$x_{lt}, y_{l't} \in \mathbb{Z}_+,\ p_{lt} \in \mathbb{R} \qquad l \in \mathcal{L},\ l' \in \mathcal{E},\ t \in \mathcal{T}. \tag{5}$$

The objective function above accounts for the profits and costs from the loaded and empty movements. For all $l \in \mathcal{L}$, $t \in \mathcal{T}$, the cost of the movement from location o_l to d_l is implicitly accounted for in $r_{lt}(p_{lt})$. Therefore, we do not need to subtract c_{lt} from $r_{lt}(p_{lt})$ in the objective function coefficient of x_{lt}. Constraints (2)-(3) are the flow balance constraints, whereas constraints (4) are the load availability constraints. Problem (1)-(5) is a difficult problem for several reasons. First, since $D_{lt}(p_{lt}) = \lfloor \lambda_{lt}(p_{lt}) \rfloor$, $D_{lt}(\cdot)$ is not continuous. Furthermore, we have integrality requirements on the decision variables x, y. An immediate approach to find an approximate solution to problem (1)-(5) may be to use a continuous approximation to $D_{lt}(\cdot)$ and relax the integrality requirements. One may argue that the primary purpose of problem (1)-(5) is to make the pricing decisions and the fact that the decision variables x, y take fractional values does not necessarily mean that the prices found in this manner are not "good." We note that since we have $D_{lt}(p_{lt}) = \lfloor \lambda_{lt}(p_{lt}) \rfloor$, $\lambda_{lt}(\cdot)$ is a good candidate for the continuous approximation to $D_{lt}(\cdot)$. Second, even when we approximate $D_{lt}(\cdot)$ by $\lambda_{lt}(\cdot)$ and relax the integrality requirements, the feasible region of problem (1)-(5) is convex if and only if $\lambda_{lt}(\cdot)$ is concave for all $l \in \mathcal{L},\ t \in \mathcal{T}$. Third, due to the nonlinear first term, the objective function is not necessarily concave. Fourth, for realistic applications, problem (1)-(5) is a large nonlinear program with hundreds of locations and tens of time periods.

2 Solution Strategy

For given prices $p = \{p_{lt} : l \in \mathcal{L}\ , t \in \mathcal{T}\}$ and loads $D(p) = \{D_{lt}(p_{lt}) : l \in \mathcal{L},\ t \in \mathcal{T}\}$, we let

$$F(p, D(p)) = \max \quad \sum_{t \in \mathcal{T}} \sum_{l \in \mathcal{L}} r_{lt}(p_{lt})\, x_{lt} - \sum_{t \in \mathcal{T}} \sum_{l \in \mathcal{E}} c_{lt}\, y_{lt} \tag{6}$$

$$\text{subject to} \quad (2), (3), (4) \tag{7}$$

$$x_{lt}, y_{l't} \in \mathbb{Z}_+ \qquad l \in \mathcal{L},\ l' \in \mathcal{E},\ t \in \mathcal{T}. \tag{8}$$

Problem (6)-(8) assumes that the prices are fixed, and hence, it is an integer program. In fact, it is easy to see that this problem is a min-cost network flow problem with integer data, in which case we can relax the integrality requirements and talk about the dual of problem (6)-(8).

Noting the definition of $F(p, D(p))$, problem (1)-(5) is equivalent to

$$\max_p F(p, D(p)). \tag{9}$$

We use a subgradient search-like method to solve problem (9). We emphasize that $F(\cdot, D(\cdot))$ is not continuous and does not possess subgradients, and using a subgradient search-like method to solve problem (9) is simply a heuristic based on the gradient ascent idea. For example, Figure 2 shows typical "cross sections" of $F(\cdot, D(\cdot))$. In this figure, letting e_{lt} be the $|\mathcal{L}||\mathcal{T}|$-dimensional unit vector with a 1 in the element corresponding to $l \in \mathcal{L}$, $t \in \mathcal{T}$ and $\hat{p}$ be given prices, we plot $F(\hat{p} + \varepsilon\, e_{lt}, D(\hat{p} + \varepsilon\, e_{lt}))$ as a function of ε for three different (l,t) pairs. The "jumps" correspond to the points of discontinuity for $D_{lt}(\cdot)$.

Although $F(\cdot, D(\cdot))$ is not continuous and does not possess subgradients, this section shows that there exists a set of prices $\mathcal{P}$ such that $\mathcal{P}$ includes the optimal solution to problem (9), and for all $p \in \mathcal{P}$, there exists $f(p) = \{f_{lt}(p) : l \in \mathcal{L},\ t \in \mathcal{T}\}$ that satisfies

$$F(p + \varepsilon\, e_{lt}, D(p + \varepsilon\, e_{lt})) \leq F(p, D(p)) + \varepsilon\, f_{lt}(p) + \delta_{lt}\, |\varepsilon| \tag{10}$$

for all $\varepsilon \in \mathbb{R}$, where δ_{lt} is a constant. In practice, it turns out that δ_{lt} does not need to be too large and $f(p)$ acts similar to a subgradient of $F(\cdot, D(\cdot))$ at p.

In particular, we define the set $\mathcal{P}$ as

$$\mathcal{P} = \{p \in \mathbb{R}^{|\mathcal{L}||\mathcal{T}|} : \lambda_{lt}(p_{lt}) \in \mathbb{Z}_+ \text{ for all } l \in \mathcal{L},\ t \in \mathcal{T}\}.$$

Therefore, $p \in \mathcal{P}$ if and only if p_{lt} is a point of discontinuity for $D_{lt}(\cdot)$ for all $l \in \mathcal{L},\ t \in \mathcal{T}$. It is easy to show that there exists an optimal solution to problem (9) that is in $\mathcal{P}$. To see this, we assume that $\hat{p}$ is an optimal solution to problem (9) that is not in $\mathcal{P}$. For all $l \in \mathcal{L},\ t \in \mathcal{T}$, we let p^*_{lt} be the smallest price such that $p^*_{lt} \geq \hat{p}_{lt}$ and $\lambda(p^*_{lt}) \in \mathbb{Z}_+$ (see Figure 1). Since we have $D_{lt}(\hat{p}_{lt}) = \lfloor \lambda_{lt}(\hat{p}_{lt}) \rfloor = \lfloor \lambda_{lt}(p^*_{lt}) \rfloor = D_{lt}(p^*_{lt})$ and $r_{lt}(\hat{p}_{lt}) \leq r_{lt}(p^*_{lt})$ for all $l \in \mathcal{L},\ t \in \mathcal{T}$, we obtain $F(\hat{p}, D(\hat{p})) \leq F(p^*, D(p^*))$, and hence, p^* is also an optimal solution to problem (9) and $p^* \in \mathcal{P}$.

To show that (10) holds, we let $x^*(p, D(p)) = \{x^*_{lt}(p, D(p)) : l \in \mathcal{L},\ t \in \mathcal{T}\}$, $y^*(p, D(p)) = \{y^*_{lt}(p, D(p)) : l \in \mathcal{E},\ t \in \mathcal{T}\}$ be an optimal solution to problem (6)-(8) and $u^*(p, D(p)) = \{u^*_{lt}(p, D(p)) : l \in \mathcal{L},\ t \in \mathcal{T}\}$ be the corresponding optimal dual variables associated with constraints (4) in problem (6)-(8). By duality theory, we have

$$F(p, D(p + \varepsilon\, e_{lt})) \leq F(p, D(p)) + u^*_{lt}(p, D(p))\, [D_{lt}(p_{lt} + \varepsilon) - D_{lt}(p_{lt})]$$
$$F(p, D(p + \varepsilon\, e_{lt})) \geq F(p + \varepsilon\, e_{lt}, D(p + \varepsilon\, e_{lt})) + x^*_{lt}(p + \varepsilon\, e_{lt}, D(p + \varepsilon\, e_{lt}))\, [r_{lt}(p_{lt}) - r_{lt}(p_{lt} + \varepsilon)]$$

(see Chapter 10 in Vanderbei (1997)). Letting $\hat{p} = p + \varepsilon\, e_{lt}$ for notational brevity, we obtain

$$\begin{aligned} F(\hat{p}, D(\hat{p})) &\leq F(p, D(p)) + x^*_{lt}(\hat{p}, D(\hat{p}))\, [r_{lt}(\hat{p}_{lt}) - r_{lt}(p_{lt})] + u^*_{lt}(p, D(p))\, [D_{lt}(\hat{p}_{lt}) - D_{lt}(p_{lt})] \\ &= F(p, D(p)) + x^*_{lt}(p, D(p))\, [r_{lt}(\hat{p}_{lt}) - r_{lt}(p_{lt})] + u^*_{lt}(p, D(p))\, [D_{lt}(\hat{p}_{lt}) - D_{lt}(p_{lt})] \\ &\qquad + [x^*_{lt}(\hat{p}, D(\hat{p})) - x^*_{lt}(p, D(p))]\, [r_{lt}(\hat{p}_{lt}) - r_{lt}(p_{lt})]. \end{aligned} \tag{11}$$

Assuming that $p \in \mathcal{P}$ and $\lambda_{lt}(\cdot)$ is concave, we have

$$D_{lt}(\hat{p}_{lt}) = \lfloor \lambda_{lt}(\hat{p}_{lt}) \rfloor \leq \lambda(\hat{p}_{lt}) \leq \lambda_{lt}(p_{lt}) + \dot{\lambda}_{lt}(p_{lt})\, [\hat{p}_{lt} - p_{lt}] = D_{lt}(p_{lt}) + \dot{\lambda}_{lt}(p_{lt})\, [\hat{p}_{lt} - p_{lt}], \tag{12}$$

where we use $\dot{\lambda}_{lt}(\cdot)$ to denote the derivative of $\lambda_{lt}(\cdot)$ and the second equality follows from the fact that $p \in \mathcal{P}$. Furthermore, noting that the total number of available vehicles is $\sum_{i \in \mathcal{I}} f_{i1}$, we have

$$[x^*_{lt}(\hat{p}, D(\hat{p})) - x^*_{lt}(p, D(p))]\,[r_{lt}(\hat{p}_{lt}) - r_{lt}(p_{lt})] \\ = [x^*_{lt}(\hat{p}, D(\hat{p})) - x^*_{lt}(p, D(p))]\,a_{lt}\,[\hat{p}_{lt} - p_{lt}] \leq a_{lt}\,|\hat{p}_{lt} - p_{lt}| \sum_{i \in \mathcal{I}} f_{i1}. \quad (13)$$

Letting $\delta_{lt} = a_{lt} \sum_{i \in \mathcal{I}} f_{i1}$ and noting (12)-(13), (11) implies that

$$F(\hat{p}, D(\hat{p})) \leq F(p, D(p)) + x^*_{lt}(p, D(p))\,a_{lt}\,[\hat{p}_{lt} - p_{lt}] + u^*_{lt}(p, D(p))\,\dot{\lambda}_{lt}(p_{lt})\,[\hat{p}_{lt} - p_{lt}] + \delta_{lt}\,|\hat{p}_{lt} - p_{lt}|,$$

where we use the fact that $u^*_{lt}(p, D(p)) \geq 0$. Therefore, (10) holds when we let

$$f_{lt}(p) = x^*_{lt}(p, D(p))\,a_{lt} + u^*_{lt}(p, D(p))\,\dot{\lambda}_{lt}(p_{lt}). \quad (14)$$

3 Pricing Algorithm

We propose the following algorithm to solve problem (9) approximately. The idea is to solve problem (6)-(8) and adjust the prices iteratively by using the primal-dual solution to this problem.

Step 1. Choose the initial prices $p^1 \in \mathcal{P}$ and let $n = 1$.

Step 2. Solve problem (6)-(8) with the prices p^n and loads $D(p^n)$. Let $x^*(p^n, D(p^n))$, $y^*(p^n, D(p^n))$ be an optimal solution to this problem and $u^*(p^n, D(p^n))$ be the corresponding optimal dual variables associated with constraints (4).

Step 3. For all $l \in \mathcal{L}$, $t \in \mathcal{T}$, let $f_{lt}(p^n) = x^*_{lt}(p^n, D(p^n))\,a_{lt} + u^*_{lt}(p^n, D(p^n))\,\dot{\lambda}_{lt}(p^n_{lt})$.

Step 4. For all $l \in \mathcal{L}$, $t \in \mathcal{T}$, let $q^n_{lt} = p^n_{lt} + \alpha^n_{lt}\,f_{lt}(p^n)$, where α^n_{lt} is a positive step-size parameter.

Step 5. For all $l \in \mathcal{L}$, $t \in \mathcal{T}$, let p^{n+1}_{lt} be the smallest price such that $p^{n+1}_{lt} \geq q^n_{lt}$ and $\lambda_{lt}(p^{n+1}_{lt}) \in \mathbb{Z}_+$.

Step 6. Increase n by 1 and go to Step 2.

Noting (10) and (14), Step 3 finds a search direction and Step 4 adjusts the prices by using this search direction. The prices q^n obtained in Step 4 are not necessarily in $\mathcal{P}$ and Step 5 finds the prices p^{n+1} in $\mathcal{P}$. By the discussion that follows the definition of $\mathcal{P}$ in Section 2, we have $F(q^n, D(q^n)) \leq F(p^{n+1}, D(p^{n+1}))$, and hence, the prices p^{n+1} are trivially "better" than the prices q^n.

4 Charging Uniform Prices over the Planning Horizon

Sections 2 and 3 assume that the prices charged over a particular lane at different time periods can be different. We now consider the situation where the carrier has to charge a single price over each lane. Therefore, we have to have $p_{l1} = p_{l2} = \ldots = p_{lT}$ for all $l \in \mathcal{L}$. Consequently, if the price charged over lane l at time period t is adjusted by ε, then the price charged over this lane for every time period has to be adjusted by ε. In this case, assuming that $\lambda_{l1}(\cdot) = \lambda_{l2}(\cdot) = \ldots = \lambda_{lT}(\cdot)$ for all $l \in \mathcal{L}$, we can use an argument similar to the one in Section 2 to show that if we let

$$g_l(p) = \sum_{t \in \mathcal{T}} x^*_{lt}(p, D(p))\, a_{lt} + \sum_{t \in \mathcal{T}} u^*_{lt}(p, D(p))\, \dot{\lambda}_{lt}(p_{lt}), \tag{15}$$

then we have

$$F(p + \varepsilon \sum_{t \in \mathcal{T}} e_{lt}, D(p + \varepsilon \sum_{t \in \mathcal{T}} e_{lt})) \leq F(p, D(p)) + \varepsilon\, g_l(p) + \delta_l\, |\varepsilon| \tag{16}$$

for all $p \in \mathcal{P}$, $\varepsilon \in \mathbb{R}$, where δ_l is a constant. Using (15)-(16), the algorithm in Section 3 can be modified to find prices that are fixed over the planning horizon. All we need to do is to compute $g_l(p^n)$ in Step 3 and let $q^n_{lt} = p^n_{lt} + \alpha^n_l\, g_l(p^n)$ for all $l \in \mathcal{L}$, $t \in \mathcal{T}$ in Step 4, where α^n_l is a positive step-size parameter. Therefore, if we start the algorithm with the prices satisfying $p^1_{l1} = p^1_{l2} = \ldots = p^1_{lT}$ for all $l \in \mathcal{L}$, then the prices satisfy $p^n_{l1} = p^n_{l2} = \ldots = p^n_{lT}$ for all $l \in \mathcal{L}$ at any iteration n.

5 Computational Experiments

In this section, we show that our pricing approach yields high-quality solutions. We also observe how the prices react to changes in certain problem parameters and make sure that they comply with our expectations.

We assume that the prices are charged on a "per-mile" basis. That is, we have $r_{lt}(p_{lt}) = \Delta(o_l, d_l)\,[p_{lt} - C]$ for all $l \in \mathcal{L}$, $t \in \mathcal{T}$ and $c_{lt} = \Delta(o_l, d_l)\, C$ for all $l \in \mathcal{E}$, $t \in \mathcal{T}$. We assume that $\lambda_{lt}(\cdot)$ has the form

$$\lambda_{lt}(p_{lt}) = \mu_{lt} \left[1 + Q_l - Q_l \left(\frac{p_{lt}}{\rho_l} \right)^{k_l} \right], \tag{17}$$

where $Q_l > 0$, $k_l > 1$, $\mu_{lt} \geq 0$, $\rho_l > 0$. In this expression, ρ_l stands for the prevailing price charged over lane l and μ_{lt} stands for the forecasted number of loads over lane l at time period t (given that we continue charging the prevailing price ρ_l). We have $\lambda_{lt}(\rho_l) = \mu_{lt}$, which means that if we continue charging the prevailing price, then the number of loads is equal to the forecast. For each $l \in \mathcal{L}$, we generate Q_l and k_l respectively from the uniform distributions over $[0.5, 1.5]$ and $[1, 3]$.

We use a multiplicative step-size parameter. In particular, we let h^N_{lt} be the number of sign changes of $f_{lt}(p^n)$ in the first N iterations. That is, we have $h^N_{lt} = \sum_{n=2}^{N} \mathbf{1}(f_{lt}(p^{n-1})\, f_{lt}(p^n) < 0)$, where $\mathbf{1}(\cdot)$ is the indicator function. We adjust the prices in Step 4 of the algorithm in Section 3 by

$$q^n_{lt} = \begin{cases} \left(1 + \frac{1}{h^n_{lt}+2}\right) p^n_{lt} & \text{if } f_{lt}(p^n) \geq 0 \\ \left(1 - \frac{1}{h^n_{lt}+2}\right) p^n_{lt} & \text{if } f_{lt}(p^n) < 0. \end{cases}$$

Although quite robust, a disadvantage of this approach is that if the initial prices are too low, then it may take a large number of iterations to recover. For this reason, we do not start with prices less than \$0.10/mile on any lane. Setup runs showed that the prices found by our pricing approach stabilizes after about 40-50 iterations. To be safe, we carry out the price adjustments for 100 iterations.

5.1 Behavior analysis

In this section, we observe our pricing approach from a qualitative viewpoint and make sure that its behavior complies with our expectations.

Price reactions to increasing fleet size. One would expect the optimal prices to have a tendency to decline as the vehicles become more abundant. To test whether the prices found by our pricing approach meet this expectation, we apply the algorithm in Section 3 to problems with different fleet sizes. Table 1 shows the results. In all tables in this section, we describe the characteristics of the test problems by the triplets $(|\mathcal{I}|, T, f)$, where $|\mathcal{I}|$ is the number of locations, T is the length of the planning horizon and f is the fleet size. Letting p^* be the prices found by our pricing approach, Table 1 gives the average, and 20-th and 80-th percentiles of $\{p^*_{lt} : l \in \mathcal{L},\ t \in \mathcal{T}\}$. The results confirm our expectation that the optimal prices should be lower when we utilize higher number of vehicles.

Checking for an optimality condition. We let p^* be an optimal solution to problem (9) and $x^*(p^*, D(p^*))$, $y^*(p^*, D(p^*))$ be an optimal solution to problem (6)-(8) when it is solved with prices p^* and loads $D(p^*)$. In this case, if $x^*_{lt}(p^*, D(p^*)) > 0$ for some $l \in \mathcal{L}$, $t \in \mathcal{T}$, then we must have $x^*_{lt}(p^*, D(p^*)) = D_{lt}(p^*_{lt})$. To see this, if, on the contrary, we have $0 < x^*_{lt}(p^*, D(p^*)) < D_{lt}(p^*_{lt})$, then we can choose $\varepsilon > 0$ such that $D_{lt}(p^*_{lt} + \varepsilon) = D_{lt}(p^*_{lt}) - 1$, in which case the solution $x^*(p^*, D(p^*))$, $y^*(p^*, D(p^*))$ is feasible to problem (6)-(8) when it is solved with prices $p^* + \varepsilon\, e_{lt}$ and loads $D(p^* + \varepsilon\, e_{lt})$. Therefore, since $r_{lt}(p^*_{lt} + \varepsilon) > r_{lt}(p^*_{lt})$, we obtain $F(p^* + \varepsilon\, e_{lt}, D(p^* + \varepsilon\, e_{lt})) > F(p^*, D(p^*))$ and this contradicts the fact that p^* is an optimal solution to problem (9).

To check whether the prices found by our pricing approach satisfy the property above, we let $\hat{p}^*$ be the prices found by our pricing approach and $\mathcal{Q} = \{(l,t) : x^*_{lt}(\hat{p}^*, D(\hat{p}^*)) > 0\}$, and compute $x^*_{lt}(\hat{p}^*, D(\hat{p}^*))/D_{lt}(\hat{p}^*_{lt})$ for all $(l,t) \in \mathcal{Q}$. By the discussion above, if $\hat{p}^*$ is indeed an optimal solution to problem (9), then we must have $x^*_{lt}(\hat{p}^*, D(\hat{p}^*))/D_{lt}(\hat{p}^*_{lt}) = 1$ for all $(l,t) \in \mathcal{Q}$. Table 2 shows the average and standard deviation of $\{x^*_{lt}(\hat{p}^*, D(\hat{p}^*))/D_{lt}(\hat{p}^*_{lt}) : (l,t) \in \mathcal{Q}\}$ and indicates that $x^*_{lt}(\hat{p}^*, D(\hat{p}^*))/D_{lt}(\hat{p}^*_{lt})$ is almost always equal to 1 whenever we have $(l,t) \in \mathcal{Q}$.

We note that the converse of the statement "if $x^*_{lt}(p^*, D(p^*)) > 0$ for some $l \in \mathcal{L}$, $t \in \mathcal{T}$, then we must have $x^*_{lt}(p^*, D(p^*)) = D_{lt}(p^*_{lt})$" is the statement "if $x^*_{lt}(p^*, D(p^*)) < D_{lt}(p^*_{lt})$ for some $l \in \mathcal{L}$, $t \in \mathcal{T}$, then we must have $x^*_{lt}(p^*, D(p^*)) = 0$." Therefore, letting $\hat{p}^*$ be the prices found by our pricing approach and $\mathcal{R} = \{(l,t) : x^*_{lt}(\hat{p}^*, D(\hat{p}^*)) < D_{lt}(\hat{p}^*_{lt})\}$, we can check whether $x^*_{lt}(\hat{p}^*, D(\hat{p}^*)) = 0$ for all $(l,t) \in \mathcal{R}$ instead of checking whether $x^*_{lt}(\hat{p}^*, D(\hat{p}^*)) = D_{lt}(\hat{p}^*_{lt})$ for all $(l,t) \in \mathcal{Q}$.

Price reactions to diminishing differences in regional market conditions. We consider problems where $\lambda_{lt}(\cdot) = \lambda_{l't'}(\cdot)$ and $r_{lt}(\cdot) = r_{l't'}(\cdot)$ for all $l, l' \in \mathcal{L}$, $t, t' \in \mathcal{T}$. In this case, since the loads and profits over different lanes and at different time periods react to price changes in the same manner, we expect the optimal prices over different lanes and at different time periods to be similar. Letting p^* be the prices found by our pricing approach, Table 3 shows the 10-th and 90-th percentiles of $\{p^*_{lt} : l \in \mathcal{L},\ t \in \mathcal{T}\}$. For every test problem, the initial prices over different lanes and at different time periods range between 0.10 and 3.00 with a standard deviation of 0.87, but Table 3 indicates that the final prices over different lanes and at different time periods are almost identical.

Price reactions to different initial prices. The objective function of problem (9) is not concave (it is not even continuous) and our pricing approach does not have a convergence guarantee. Potentially, if we start with different initial prices, then our pricing approach may find different prices. To make sure that this is not a major issue, we start our pricing approach with three different initial prices, say $\hat{p}^1$, $\tilde{p}^1$ and $\bar{p}^1$. We let $\hat{p}^*$, $\tilde{p}^*$ and $\bar{p}^*$ be the prices that our pricing approach finds by starting respectively from the initial prices $\hat{p}^1$, $\tilde{p}^1$ and $\bar{p}^1$. The first three rows of Table 4 show the coefficients of correlation among $\{\hat{p}^1_{lt} : l \in \mathcal{L},\ t \in \mathcal{T}\}$, $\{\tilde{p}^1_{lt} : l \in \mathcal{L},\ t \in \mathcal{T}\}$ and $\{\bar{p}^1_{lt} : l \in \mathcal{L},\ t \in \mathcal{T}\}$, and indicate that the prices $\hat{p}^1$, $\tilde{p}^1$ and $\bar{p}^1$ are either uncorrelated or strongly negatively correlated. The next three rows of Table 4 show $F(\hat{p}^*, D(\hat{p}^*))$, $F(\tilde{p}^*, D(\tilde{p}^*))$ and $F(\bar{p}^*, D(\bar{p}^*))$, and indicate that the performances of the prices found by starting from the initial prices $\hat{p}^1$, $\tilde{p}^1$ and $\bar{p}^1$ are very similar. Finally, the last three rows show the coefficients of correlation among $\{\hat{p}^*_{lt} : l \in \mathcal{L},\ t \in \mathcal{T}\}$, $\{\tilde{p}^*_{lt} : l \in \mathcal{L},\ t \in \mathcal{T}\}$ and $\{\bar{p}^*_{lt} : l \in \mathcal{L},\ t \in \mathcal{T}\}$, and indicate that the prices $\hat{p}^*$, $\tilde{p}^*$ and $\bar{p}^*$ are strongly positively correlated. Therefore, the prices found by starting from different initial prices are in close agreement. Also, Figure 3 shows $\{(\hat{p}^1_{lt}, \tilde{p}^1_{lt}) : l \in \mathcal{L},\ t \in \mathcal{L}\}$ and $\{(\hat{p}^*_{lt}, \tilde{p}^*_{lt}) : l \in \mathcal{L},\ t \in \mathcal{L}\}$ for problem $(20, 28, 100)$, and indicates that although the initial prices $\hat{p}^1$ and $\tilde{p}^1$ are substantially different, the final prices $\hat{p}^*$ and $\tilde{p}^*$ are almost the same.

5.2 Solution quality

In this section, we compare the prices found by our pricing approach with those found by a benchmark method and with upper bounds on the optimal objective value of problem (9).

Comparisons with the "relaxed" problem. Since we have $D_{lt}(p_{lt}) = \lfloor \lambda_{lt}(p_{lt}) \rfloor$, replacing $D_{lt}(\cdot)$ by $\lambda_{lt}(\cdot)$ and relaxing the integrality requirements on the decision variables x, y in problem (1)-(5), we can obtain an upper bound on the optimal objective value of problem (1)-(5) by solving

$$\max \quad \sum_{t \in \mathcal{T}} \sum_{l \in \mathcal{L}} r_{lt}(p_{lt})\, x_{lt} - \sum_{t \in \mathcal{T}} \sum_{l \in \mathcal{E}} c_{lt}\, y_{lt} \tag{18}$$

$$\text{subject to} \quad (2), (3) \tag{19}$$

$$x_{lt} - \lambda_{lt}(p_{lt}) \leq 0 \qquad l \in \mathcal{L},\ t \in \mathcal{T} \tag{20}$$

$$x_{lt}, y_{l't} \in \mathbb{R}_+,\ p_{lt} \in \mathbb{R} \qquad l \in \mathcal{L},\ l' \in \mathcal{E},\ t \in \mathcal{T}. \tag{21}$$

Due to our choice of the parameters in (17), the feasible region of problem (18)-(21) is convex. However, since the objective function is not necessarily concave, we solve problem (18)-(21) by using 30 different initial solutions in a nonlinear programming package. We let p^r be the prices found by solving problem (18)-(21).

Letting p^* be the prices found by our pricing approach, Table 5 shows $F(p^*, D(p^*))$, $F(p^r, D(p^r))$ and the optimal objective value of problem (18)-(21). The last three problems involve 3 time periods because our nonlinear programming package did not find solutions in reasonable time to problems with 28 time periods. In Table 5, comparing $F(p^*, D(p^*))$ with the optimal objective value of problem (18)-(21) indicates that our pricing approach finds prices whose performances are close to the upper bounds on the optimal objective value. Also, comparing $F(p^*, D(p^*))$ with $F(p^r, D(p^r))$ indicates that the performances of the prices p^r may be up to 40-50% worse than those of the prices p^*. Therefore, solving a "relaxed" version of problem (1)-(5) may not provide "good" prices. Finally, letting p^n be the prices found by our pricing approach at iteration n, Figure 4 shows $F(p^n, D(p^n))$ as a function of the iteration number n for problem $(3, 28, 100)$. After 40-50 iterations, the performances of the prices stabilize. This has been the case for all of our test problems.

Comparisons with exhaustive numerical search. We choose $l_1, l_2 \in \mathcal{L}$ and $t_1, t_2 \in \mathcal{T}$, and assume that only the prices over lane l_1 at time period t_1 and over lane l_2 at time period t_2 are decision variables. The other prices are fixed at predetermined levels, say $\hat{p} = \{\hat{p}_{lt} : l \in \mathcal{L},\ t \in \mathcal{T}\}$. In this case, since the number of decision variables is only two, we can solve problem (9) through exhaustive numerical search. In particular, for a small mesh size $\xi > 0$, our exhaustive numerical search tests the performances of the prices $\hat{p} + k_1\, \xi\, e_{l_1 t_1} + k_2\, \xi\, e_{l_2 t_2}$ for different values of $(k_1, k_2) \in \mathbb{Z}^2$. We test every value of (k_1, k_2) in the set

$$\begin{aligned} \mathcal{N} = \{(n_1, n_2) : \hat{p}_{l_1 t_1} + n_1\, \xi \geq 0,\ & D_{l_1 t_1}(\hat{p}_{l_1 t_1} + n_1\, \xi) \geq 0 \\ & \hat{p}_{l_2 t_2} + n_2\, \xi \geq 0,\ D_{l_2 t_2}(\hat{p}_{l_2 t_2} + n_2\, \xi) \geq 0\}. \end{aligned}$$

If we let

$$(k_1^*, k_2^*) = \operatorname{argmax}_{(k_1, k_2) \in \mathcal{N}} F(\hat{p} + k_1\, \xi\, e_{l_1 t_1} + k_2\, \xi\, e_{l_2 t_2}, D(\hat{p} + k_1\, \xi\, e_{l_1 t_1} + k_2\, \xi\, e_{l_2 t_2})),$$

then $\hat{p} + k_1^* \xi e_{l_1 t_1} + k_2^* \xi e_{l_2 t_2}$ corresponds to the best prices found by exhaustive numerical search.

We let $\hat{p}^* = \hat{p} + k_1^* \xi e_{l_1 t_1} + k_2^* \xi e_{l_2 t_2}$ and p^* be the prices found by our pricing approach by starting from the initial prices p^1. Table 6 shows $p^1_{l_1 t_1}$, $p^1_{l_2 t_2}$, $F(p^1, D(p^1))$, $p^*_{l_1 t_1}$, $p^*_{l_2 t_2}$, $F(p^*, D(p^*))$, $\hat{p}^*_{l_1 t_1}$, $\hat{p}^*_{l_2 t_2}$ and $F(\hat{p}^*, D(\hat{p}^*))$, and indicates that the prices found by our pricing approach perform almost as well as the ones found by exhaustive numerical search.

6 Conclusion

We developed a model and a solution approach to find what prices a freight carrier should charge so that its pricing and fleet management decisions jointly maximize the profits. Computational experiments showed that our approach yields high-quality solutions and the manner in which the prices change in response to certain problem parameters complies with our expectations.

An attractive feature of our pricing approach is that the mechanism used to adjust the prices does not interfere with the mechanism used to solve the fleet management problem. The former simply takes the primal-dual solution from the latter and suggests improved prices. This relative independence enables the carriers that are already solving the fleet management problem to incorporate our pricing approach with small overhead.

Clearly, assuming that the load arrivals are deterministic functions of the prices is not realistic. However, this criticism applies to all deterministic fleet management models and our pricing approach can be applied on a "rolling-horizon" basis to handle the randomness in the load arrivals. One path of research we are investigating is to assume that the loads arrivals are random and the rates of the load arrivals depend on the prices. Further research should also incorporate multiple vehicle types, load time windows and advance load bookings into the underlying fleet management model.

7 Acknowledgements

This work was supported in part by National Science Foundation grant DMI-0422133.

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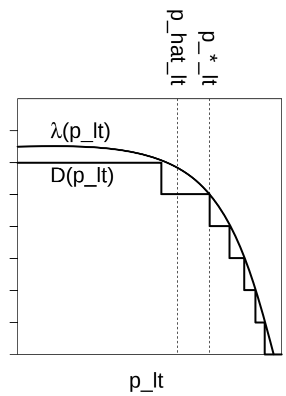

Figure 1: Relationship between $D_{lt}(\cdot)$ and $\lambda_{lt}(\cdot)$.

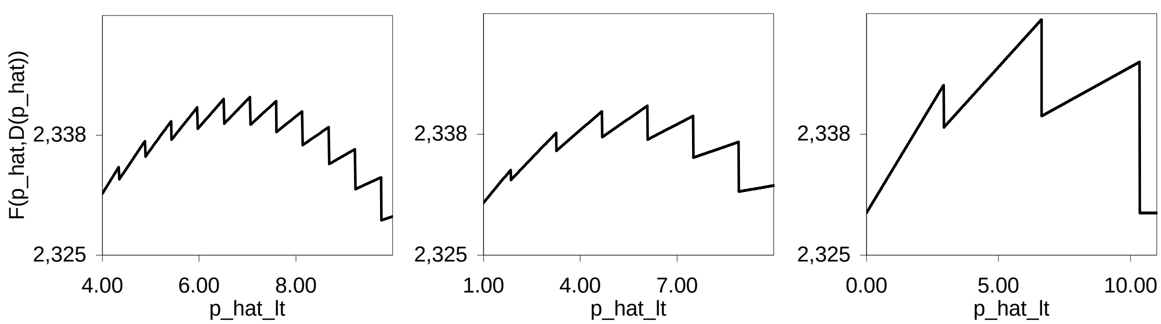

Figure 2: Three "cross sections" of $F(\cdot, D(\cdot))$.

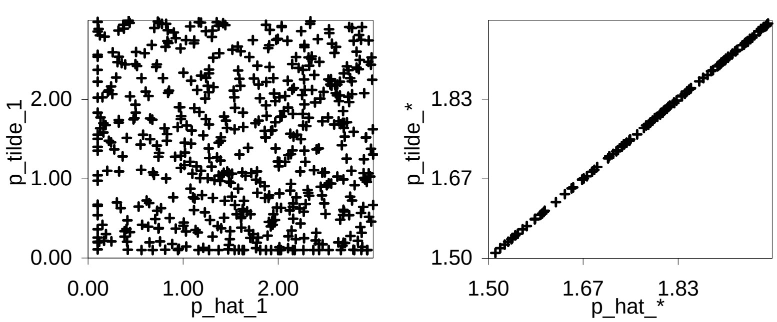

Figure 3: Plots of $(\hat{p}^1, \tilde{p}^1)$ and $(\hat{p}^*, \tilde{p}^*)$.

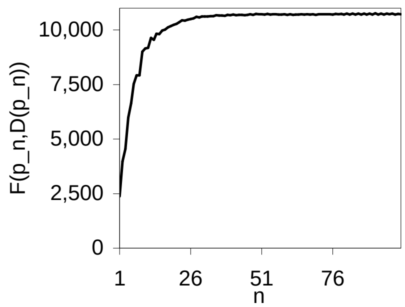

Figure 4: Plot of $F(p^n, D(p^n))$ as a function of the iteration number n.

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Problem	(20,28,10)	(20,28,25)	(20,28,50)	(20,28,100)	(20,28,250)	(20,28,500)	(20,28,1000)
Avg.	6.44	6.20	5.97	5.69	5.19	4.63	4.10
20-th perc.	5.44	5.17	4.76	4.56	4.00	3.54	3.03
80-th perc.	7.61	7.30	7.06	6.70	6.09	5.73	5.02

Table 1: Comparison of the prices for different fleet sizes.

Problem	(3,28,10)	(3,28,100)	(3,28,1000)	(5,28,10)	(5,28,100)	(5,28,1000)	(10,28,10)	(10,28,100)	(10,28,1000)	(20,28,10)	(20,28,100)	(20,28,1000)
Avg.	0.99	1.00	1.00	0.98	1.00	1.00	0.97	0.99	1.00	1.00	1.00	1.00
Std. dev.	0.19	0.01	0.00	0.16	0.03	0.00	0.13	0.06	0.00	0.00	0.00	0.00

Table 2: Comparison of $x^*_{lt}(\hat{p}^*, D(\hat{p}^*))$ with $D_{lt}(\hat{p}^*_{lt})$.

Problem	(3,28,100)	(5,28,100)	(10,28,100)	(20,28,100)
10-th perc.	3.41	3.31	2.97	2.83
90-th perc.	3.42	3.31	2.98	2.84

Table 3: Comparison of the 10-th percentile with the 90-th percentile of the prices.

Problem	(3,28,100)	(5,28,100)	(10,28,100)	(20,28,100)
Corr. coeff. btw. $(\hat{p}^1, \tilde{p}^1)$	-0.03	-0.01	-0.02	-0.04
Corr. coeff. btw. $(\hat{p}^1, \bar{p}^1)$	-1.00	-1.00	-1.00	-1.00
Corr. coeff. btw. $(\tilde{p}^1, \bar{p}^1)$	0.03	0.01	0.05	0.04
$F(\hat{p}^*, D(\hat{p}^*))$	10,787	8,223	5,609	538
$F(\tilde{p}^*, D(\tilde{p}^*))$	10,788	8,227	5,630	538
$F(\bar{p}^*, D(\bar{p}^*))$	10,793	8,234	5,601	538
Corr. coeff. btw. $(\hat{p}^*, \tilde{p}^*)$	1.00	1.00	0.98	1.00
Corr. coeff. btw. $(\hat{p}^*, \bar{p}^*)$	1.00	1.00	0.99	1.00
Corr. coeff. btw. $(\tilde{p}^*, \bar{p}^*)$	1.00	0.99	0.98	1.00

Table 4: Comparison of the prices found by starting from different initial prices.

Problem	(3,28,10)	(3,28,100)	(3,28,1000)	(5,28,10)	(5,28,100)	(5,28,1000)	(10,28,10)	(10,28,100)	(10,28,1000)	(20,3,10)	(20,3,100)	(20,3,1000)
$F(p^*, D(p^*))$	3,305	10,787	12,030	2,532	8,223	9,375	1,471	5,609	6,115	431	3,343	15,290
$F(p^r, D(p^r))$	2,981	10,739	11,966	1,500	7,733	8,770	949	3,729	4,518	219	2,807	13,459
Opt. obj. of pr. (18)-(21)	3,550	11,073	12,295	2,866	9,016	10,161	1,899	6,153	7,092	563	3,600	15,856

Table 5: Comparison of the prices found by our pricing approach with those found by solving problem (18)-(21).

Problem	(3,28,100)	(3,28,100)	(3,28,100)	(5,28,100)	(5,28,100)	(5,28,100)	(10,28,100)	(10,28,100)	(10,28,100)	(20,3,100)	(20,3,100)	(20,3,100)
$p^1_{l_1 t_1}$	7.81	7.81	2.27	2.27	2.27	2.11	2.27	2.27	5.67	1.32	1.32	1.90
$p^1_{l_2 t_2}$	2.27	5.88	5.88	2.11	2.62	2.62	5.67	3.94	3.94	1.90	3.25	3.25
$F(p^1, D(p^1))$	4,990	4,990	4,990	8,539	8,539	8,539	11,790	11,790	11,790	15,459	15,459	15,459
$p^*_{l_1 t_1}$	4.87	3.91	5.26	5.02	5.12	5.00	4.83	4.83	3.30	5.25	5.25	5.58
$p^*_{l_2 t_2}$	5.90	3.58	3.82	4.82	3.61	3.65	3.87	5.69	5.69	5.57	3.86	3.86
$F(p^*, D(p^*))$	6,273	6,146	6,280	8,699	8,655	8,676	12,049	12,055	12,028	16,401	16,394	16,400
$\hat{p}^*_{l_1 t_1}$	4.90	4.03	5.30	5.03	5.13	4.98	4.83	4.83	3.53	5.23	5.23	5.75
$\hat{p}^*_{l_2 t_2}$	6.03	3.58	3.88	4.95	3.60	3.63	4.10	5.68	5.68	5.78	3.83	3.90
$F(\hat{p}^*, D(\hat{p}^*))$	6,276	6,146	6,283	8,701	8,653	8,685	12,051	12,055	12,028	16,402	16,394	16,401

Table 6: Comparison of the prices found by our pricing approach with those found by exhaustive numerical search.

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