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INCREASING SEQUENCES OF INDEPENDENT POINTS ON THE PLANAR LATTICE

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In 1977 Vershik and Kerov deduced the asymptotic normalized length of the longest increasing sequence among independent points uniformly distributed on the unit square. We solve the analogous problem for points on the planar square lattice that are present independently of each other.

1. The result. The following question is known as Ulam's problem: Consider a rate 1 Poisson point process on the plane. Let L_n be the maximal number of points on an increasing path of these points in the square $(0, n]^2$. Superadditivity and simple moment bounds imply that for some constant c,

$$\lim_{n\to\infty} \frac{1}{n} L_n = c \quad \text{a.s.}$$

What is the exact value of c? The original problem concerned the longest increasing subsequence of a random permutation. This reformulation in terms of a planar Poisson point process is due to Hammersley [6].

In 1977 Vershik and Kerov derived the answer $c = 2$ [13]. In the same year, Logan and Shepp independently showed that $c \geq 2$ [8]. Both proofs are combinatorial and make use of Young diagrams. Recently two proofs have appeared (Aldous and Diaconis [1] and Seppäläinen [12]) that proceed by embedding the increasing sequences of points in an interacting particle system.

In this paper we use the approach of [12] to solve the analogous problem on the planar square lattice. Fix a parameter $p \in (0, 1)$. For each site of the lattice $\mathbf{Z}^2$ let a point be present (the site is occupied) with probability p and absent (the site is vacant) with probability $q = 1 - p$, independently of all the other sites. For $-\infty < a < b < \infty$ and $0 \leq s < t$, let $L((a, s), (b, t))$ equal the number of points on a longest strictly increasing path of points in the rectangle $(a, b] \times (s, t]$. We emphasize that strictly increasing means the following: an admissible path $(x_1, y_1), (x_2, y_2), \dots, (x_m, y_m)$ of points on $\mathbf{Z}^2$ satisfies $x_1 < x_2 < \cdots < x_m$ and $y_1 < y_2 < \cdots < y_m$.

We have the obvious deterministic bounds,

$$0 \leq L((a, s), (b, t)) \leq \min\{[b] - [a], [t] - [s]\}, \tag{1.1}$$

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superadditivity,

$$L((a,r),(b,s)) + L((b,s),(c,t)) \le L((a,r),(c,t))$$

for $a < b < c$ and $r < s < t$, and independence of points in disjoint sets. Hence by Kingman's theorem [7] the constant limit

$$\Psi(x,y) = \lim_{n\to\infty} \frac{1}{n} L((0,0),(nx,ny)) \quad \text{a.s.} \tag{1.2}$$

exists for $x, y \ge 0$. Our result is the following exact formula for the limit.

THEOREM 1. *The limiting function is given by*

$$\Psi(x,y) = \begin{cases} x, & \text{if } x \le py, \\ y, & \text{if } y \le px, \\ q^{-1}\left[2\sqrt{pxy} - p(x+y)\right], & \text{if } py < x < p^{-1}y. \end{cases} \tag{1.3}$$

Once the exact limit is known, it is not hard to locate the longest paths asymptotically. The choice of optimizing paths can be cast in the form of an extremal problem. Fix a rectangle $A = [0,a] \times [0,b]$. A nondecreasing curve in A is a function $\gamma(s) = (\gamma_1(s), \gamma_2(s))$ from $s \in [0,1]$ into A, where both γ_1 and γ_2 are nondecreasing. Discontinuities are allowed. The derivatives $\gamma_1'(s)$ and $\gamma_2'(s)$ exist a.e. as nonnegative, measurable functions, and it makes sense to define

$$J(\gamma) = \int_0^1 \Psi(\gamma_1'(s), \gamma_2'(s))\, ds. \tag{1.4}$$

For C^1-parametrizations, the quantity $J(\gamma)$ is independent of choice of parametrization. This follows from the homogeneity of Ψ. Let $U_\delta(\gamma)$ be a δ-neighborhood of γ in A, in the topology of the plane.

THEOREM 2. *For any nondecreasing curve γ in A and $\delta > 0$,*

$$\lim_{n\to\infty} P\{nU_\delta(\gamma) \text{ contains a strictly increasing path of at least } n(J(\gamma) - \delta) \text{ points}\} = 1. \tag{1.5}$$

If additionally $\gamma(s)$ is Lipschitz continuous, then for any $\delta > 0$ there exists a $\delta_1 > 0$ such that

$$\lim_{n\to\infty} P\{nU_{\delta_1}(\gamma) \text{ contains a strictly increasing path of at least } n(J(\gamma) + \delta) \text{ points}\} = 0. \tag{1.6}$$

If $pb \le a \le p^{-1}b$, the diagonal $\phi(s) = (as, bs)$, $0 \le s \le 1$, uniquely maximizes $J(\gamma)$ over nondecreasing curves in A. If $a < pb$, $J(\gamma)$ is maximized by the set of nondecreasing curves $\gamma = (\gamma_1, \gamma_2)$ for which γ_1 is absolutely continuous, $\gamma_1(0) = 0$, $\gamma_1(1) = a$ and $\gamma_1'(s) \le p\gamma_2'(s)$ a.e. A symmetric statement holds for $a > p^{-1}b$. In all cases, longest paths of points concentrate around the $J(\gamma)$-maximizing curves: if U_δ, $\delta > 0$, is the δ-neighborhood of the set of

$J(\gamma)$-maximizers, then

$$\lim_{n\to\infty} P\{\textit{there is a strictly increasing path of } L((0,0),(na,nb)) \text{ } \textit{points in } nA \textit{ not wholly contained in } nU_\delta\} = 0. \tag{1.7}$$

Of course, unique maximization in the theorem means uniqueness up to changes in parametrization. Some technical restrictions are necessary for statement (1.6), for otherwise one could parametrize the diagonal as $\gamma(s) = (f(s), f(s))$ where f is the Cantor–Lebesgue function so that $f' = 0$ a.e. and $J(\gamma) = 0$.

Write $\Psi_p(x, y)$ for the function in (1.3) when dependence on p is relevant. One can recover the rate 1 planar Poisson point process by putting the points on the lattice $p^{1/2}\mathbf{Z}^2$, independently with probability p, and letting $p \searrow 0$. As expected,

$$\lim_{p\to 0} \Psi_p\big(p^{-1/2}x, p^{-1/2}y\big) = 2\sqrt{xy},$$

which is the limiting function for the increasing paths problem for Poisson points. The functional $J(\gamma)$ also converges to the corresponding functional of Ulam's problem (see [5]).

Longest common subsequences. We do not assert a rigorous connection between the LCS (longest common subsequence) model and our model of independent points, but the numerical evidence is intriguing enough to merit attention here. First, here is a particular version of the LCS model: Fix an integer $k \ge 1$, and let $(a_i, b_j\colon i, j \ge 1)$ be i.i.d. random variables that take each of the values $1, \dots, k$ with equal probability $1/k$. Put a point at site $(i, j) \in \mathbf{N}^2$ if $a_i = b_j$; otherwise leave site (i, j) vacant. The maximal length $\tilde{L}_n$ of a strictly increasing path of points in $\{1, \dots, n\} \times \{1, \dots, n\}$ equals the length of the longest common subsequence of the strings $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$. Superadditivity gives again the limit

$$c_k = \lim_{n\to\infty} \frac{1}{n} \tilde{L}_n.$$

The constants c_k, sometimes called the Chvátal–Sankoff constants after [4], are not known, but rigorous upper and lower bounds exist. A recent review of topics related to the LCS model and numerous references appear in [9].

In the LCS model, each site is occupied with probability $p = 1/k$, but the sites are not independent of each other. (However, along any path that moves only up and right the sites are independently occupied.) Let us compare the LCS model with our Bernoulli model. The quantity to compare with c_k is $\Psi_{1/k}(1,1)$, the asymptotic length with occupation probability $p = 1/k$, and from (1.3) this is

$$\Psi_{1/k}(1,1) = \frac{2}{k^{1/2} + 1}.$$

TABLE 1
Rigorous upper and lower bounds[1] *and Monte Carlo estimates*[2] *for* c_k, *and the values of* $\Psi_{1/k}(1,1)$, *for* $2 \le k \le 15$

k	Rigorous lower bound for c_k	$\Psi_{1/k}(1,1)$	Rigorous upper bound for c_k	Monte Carlo estimate for c_k
2	0.77391	0.82843	0.83763	0.8082
3	0.61538	0.73205	0.76581	0.6855
4	0.54545	0.66667	0.70824	0.6242
5	0.50615	0.61803	0.66443	0.5778
6	0.47169	0.57980	0.62932	0.5332
7	0.44502	0.54858	0.60019	0.5065
8	0.42237	0.52241	0.57541	0.4812
9	0.40321	0.50000	0.55394	0.4593
10	0.38656	0.48051	0.53486	0.4423
11	0.37196	0.46332	0.51785	0.4268
12	0.35899	0.44802	0.50260	0.4126
13	0.34737	0.43426	0.48880	0.4003
14	0.33687	0.42179	0.47620	0.3827
15	0.32732	0.41043	0.46462	0.3712

[1] From page 134 in [9].
[2] From page 314 in [4].

The connection we wish to highlight is that for each of the values $2 \le k \le 15$, $\Psi_{1/k}(1,1)$ lies between the best known upper and lower bounds for c_k. See Table 1. Furthermore, Sankoff and Mainville [11] have conjectured that $k^{1/2}c_k \to 2$ as $k \to \infty$, a property that $\Psi_{1/k}(1,1)$ satisfies.

A percolation model. Theorem 1 can be reformulated to describe the asymptotic shape of the following directed bond percolation model. Liquid percolates from the origin into the positive quadrant of the plane along three types of bonds: from (i,j) to $(i+1,j)$ (horizontal), from (i,j) to $(i,j+1)$ (vertical), and from (i,j) to $(i+1,j+1)$ (diagonal). The horizontal and vertical bonds have a deterministic cost a_0, while the diagonal bonds have independent random costs with two possible values $a_1 < a_2$, the smaller value a_1 taken with probability p. Declare that site $(i+1, j+1)$ contains a point if the bond $(i,j) \to (i+1,j+1)$ costs a_1. Then, if a_0, a_1, a_2 are appropriately chosen, finding a cost-minimizing bond path from $(0,0)$ to (m,n) is equivalent to finding a longest strictly increasing paths of points in $(0,m] \times (0,n]$. We leave the details of this conversion to the reader.

There is of course a dependent version of this percolation model similarly related to the LCS model. For more on this connection, see [2] and [3].

Nondecreasing instead of strictly increasing paths. The closest variant of the present model is the one where, instead of requiring that successive points (x_i, y_i) and (x_{i+1}, y_{i+1}) satisfy $x_i < x_{i+1}$ and $y_i < y_{i+1}$, we allow $x_i \le x_{i+1}$ and $y_i \le y_{i+1}$. The first part of our proof works again: these nondecreasing paths can be embedded in a particle system that resembles the one we use in the present proof. However, we have not been able to directly apply the second part of the method of this paper because the steady state behavior of the new process appears harder to identify.

2. Proof of Theorem 1. As already indicated, we follow the strategy of [12] in embedding the points in an interacting particle system. For $a \in \mathbf{R}$, $0 \le s \in \mathbf{Z}$, and $k, t \ge 0$, define an inverse of L by

$$\Gamma((a, s), t, k) = \inf\{h \ge 0\colon L((a, s), (a + h, s + t)) \ge k\}. \tag{2.1}$$

Usually $s = 0$, and a, t and k are integers. Then $\Gamma((a, 0), t, k) = l$ means that there is a strictly increasing path of k points in the sites of the set

$$\{a + 1, a + 2, \dots, a + l\} \times \{1, 2, \dots, t\}$$

and no such path in the set

$$\{a + 1, a + 2, \dots, a + l - 1\} \times \{1, 2, \dots, t\}.$$

In particular, the path realizing $\Gamma((a, 0), t, k)$ ends in a point on the vertical line $\{a + l\} \times \{1, 2, \dots, t\}$, but may or may not use a point on any particular horizontal line. In picturing the situation, it is helpful to remember that an admissible, strictly increasing path never has two points on the same horizontal or vertical line.

Next we construct a totally asymmetric exclusion process on the sites of $\mathbf{Z}$. The state of the process is a configuration $z = (z_i)_{i \in \mathbf{Z}}$ of labeled particles, satisfying $z \in \mathbf{Z}^{\mathbf{Z}}$ and

$$z_i < z_{i+1} \quad \text{for all } i. \tag{2.2}$$

The particles move only to the left. Given an initial configuration (z_i), the state at time $t = 1, 2, 3, \dots$ is defined by

$$z_k(t) = \inf_{i \le k}\{z_i + \Gamma((z_i, 0), t, k - i)\}, \qquad k \in \mathbf{Z}. \tag{2.3}$$

In words: The potential locations of z_k at time t are computed by following an increasing path of $k - i$ points from $(z_i, 0)$, for all $i \le k$. Of these potential locations, z_k chooses the leftmost.

Some observations follow.

1. Obviously $z_k(t) \le z_k$, so particles move to the left. Also $z_k(t) < z_{k+1}(t)$, so both the exclusion rule and the labeling convention (2.2) are preserved. Let

   $$b_k = \inf\{j \ge z_k + 1\colon (j, 1) \text{ is occupied by a point}\}.$$

   Then an easy argument (or picture) shows that

   $$z_k(1) = \min\{z_k, b_{k-1}\} \quad \text{for all } k, \tag{2.4}$$

   and thus the dynamics is well defined from any initial $z \in \mathbf{Z}^{\mathbf{Z}}$.
2. A semigroup rule holds: if $z(s)$ and $z(t)$, $0 < s < t$, are computed by rule (2.3), then it is also true that

   $$z_k(t) = \inf_{i \le k}\{z_i(s) + \Gamma((z_i(s), s), t - s, k - i)\}.$$

From this it follows that $z(\cdot)$ is a time-homogeneous Markov chain in the space $\mathbf{Z}^{\mathbf{Z}}$.

We need to understand the steady state of the dynamics $z(\cdot)$. Let us focus on a single time step, from an initial configuration z to a new configuration $z' = z(1)$ as defined by (2.3) or, equivalently, by (2.4). Let $\eta_i = z_i - z_{i-1}$ and $\eta'_i = z'_i - z'_{i-1}$ be the interparticle distances and $x_i = z_i - z'_i$ the amount z_i jumps. These quantities are connected by

$$\eta'_i = \eta_i - x_i + x_{i-1}. \tag{2.5}$$

LEMMA 2.1. *Suppose the $(\eta_i)_{i\in\mathbf{Z}}$ are i.i.d. with common distribution*

$$P[\eta_i = n] = (1-r)r^{n-1}, \qquad n \in \{1, 2, 3, \dots\}, \tag{2.6}$$

for a parameter $r \in [0, 1)$. Then the $(\eta'_i)_{i\in\mathbf{Z}}$ also have the same distribution.

PROOF. Rule (2.4) implies

$$P[x_i = k | \eta_i = n] = \begin{cases} q^{n-1}, & k = 0, \\ pq^{n-k-1}, & 1 \le k \le n-1 \end{cases}$$

for $n \ge 1$. Thus if η_i is distributed as in (2.6), we get

$$\begin{aligned} P[x_i = k] &= \sum_{n=1}^{\infty} P[x_i = k | \eta_i = n] P[\eta_i = n] \\ &= \begin{cases} (1-rq)^{-1}(1-r), & k = 0, \\ (1-rq)^{-1}(1-r)pr^k, & k \ge 1. \end{cases} \end{aligned} \tag{2.7}$$

Now assume (η_i) are i.i.d. Note that, given η_i, x_i is independent of $(\eta_j, x_j)_{j\ne i}$ because distinct x_j are determined by the occupancies of disjoint sets of sites:

$$\begin{aligned} P[\eta'_i = n, x_i = k] &= P[\eta_i - x_i + x_{i-1} = n, x_i = k] \\ &= P[\eta_i + x_{i-1} = n + k, x_i = k] \\ &= \sum_{j=1}^{n+k} P[x_{i-1} = n + k - j] P[\eta_i = j] P[x_i = k | \eta_i = j] \\ &= P[\eta_i = n] P[x_i = k]. \end{aligned}$$

The last step above involves algebra and simplification. For example, in the case $k \ge 1$, the second last line becomes

$$\begin{aligned} &\sum_{j=k+1}^{n+k-1} \frac{(1-r)pr^{n+k-j}}{1-rq}(1-r)r^{j-1}pq^{j-k-1} + \frac{1-r}{1-rq}(1-r)r^{n+k-1}pq^{n-1} \\ &\quad = (1-r)r^{n-1}\frac{(1-r)pr^k}{1-rq}. \end{aligned}$$

This tells us that η_i' is distributed like η_i and is independent of x_i. By (2.5) the quantities $(\eta_{i+1}', \eta_{i+2}', \dots)$ are computable from $(x_i, (\eta_{i+1}, x_{i+1}), (\eta_{i+2}, x_{i+2}), \dots)$ while η_i' is a function of (x_{i-1}, η_i, x_i). Thus, given x_i, $(\eta_{i+1}', \eta_{i+2}', \dots)$ and η_i' are independent. We now proceed by induction:

$$
\begin{aligned}
&P[\eta_i' = n_0, \eta_{i+1}' = n_1, \dots, \eta_{i+l}' = n_l] \\
&\quad = \sum_{k=0}^{\infty} P[\eta_i' = n_0, x_i = k] P[\eta_{i+1}' = n_1, \dots, \eta_{i+l}' = n_l | x_i = k] \\
&\quad = P[\eta_i = n_0] P[\eta_{i+1}' = n_1, \dots, \eta_{i+l}' = n_l] \\
&\quad = \cdots \\
&\quad = P[\eta_i = n_0] P[\eta_{i+1} = n_1] \cdots P[\eta_{i+l} = n_l].
\end{aligned}
$$

The proof is complete. $\square$

It will be convenient to parametrize the equilibrium by $u = E[\eta_i] = (1-r)^{-1}$. In u-equilibrium, the amount particle z_i jumps to the left at each time step has expectation

$$
f(u) \equiv E[x_i] = (pu + q)^{-1} pu(u-1). \tag{2.8}
$$

The function $f(u)$ is defined for $u \in [1, \infty)$, and strictly convex because $p \in (0, 1)$.

This is all we need about the process $z(\cdot)$, so let us return to the increasing paths. Given $x, y > 0$, we can always construct an increasing path in $(0, nx] \times (0, ny]$ with the following trivial strategy: Start at $(1, 1)$. When you are at (i, j), move to $(i+1, j+1)$ if (i, j) is occupied, and move to $(i, j+1)$ if (i, j) is vacant. In other words, collect along each diagonal as many points as are available until you meet a vacant site, then take a step up and repeat. Each diagonal gives an independent Geom(q)-distributed number of points, and forces you to take $1 + \text{Geom}(q)$ many steps up. Thus with expected q^{-1} steps up, pq^{-1} many points are added to the increasing path, on average. If $y \ge xp^{-1}$ and n is large so that a law of large numbers takes over, this trivial strategy gives an increasing path of length roughly nx, which is an upper bound for $L((0,0), (nx, ny))$ [recall (1.1)]. This argument, which the reader can easily make rigorous, shows that $\Psi(x, y) = x$ if $y \ge xp^{-1}$. A symmetric situation holds with x and y interchanged, and we have derived the first two lines of (1.3).

It is easy to see that $\Psi(x, y)$ is superadditive and homogeneous. From this it follows that $\Psi(x, y)$ is concave and consequently continuous on $(0, \infty)^2$. Equation (1.1) implies $0 \le \Psi(x, y) \le \min\{x, y\}$ and $\Psi(x, 0) = \Psi(0, y) = 0$, so in fact Ψ is continuous on $[0, \infty)^2$. Let

$$
h(x) = \Psi(x, 1), \qquad x \ge 0.
$$

Then $h(x)$ is concave, nondecreasing, $h(0) = 0$, $h(x) = x$ for $x \le p$, and $h(x) = 1$ for $x \ge p^{-1}$. Concavity implies that h is strictly increasing for $0 \le x \le a_0$, with $a_0 = \inf\{x\colon h(x) = 1\} \le p^{-1}$. Thus there is a well-defined inverse $g(x) = h^{-1}(x)$ defined for $0 \le x \le 1$. From the properties of h it follows that g is strictly increasing, convex, continuous, $g(x) = x$ for $x \le p$, and $\lim_{x \nearrow 1} g(x) = g(1) = a_0$. Extend g to a lower semicontinuous convex function on $[0, \infty)$ by setting $g(x) = \infty$ for $x > 1$. Let

$$g^+(u) = \sup_{x \ge 0} \{ux - g(x)\}, \qquad u \ge 0, \tag{2.9}$$

be the monotone conjugate of g (see page 111 in [10]).

From the convergence (1.2) and the continuity of g follows that

$$\lim_{n \to \infty} \frac{1}{n} \Gamma((r_n, 0), nt, nx) = tg\left(\frac{x}{t}\right) \tag{2.10}$$

holds in probability for any $x, t \ge 0$ and any sequence $r_n \in \mathbf{R}$. [Notice that if $t < x$, $\Gamma((r_n, 0), nt, nx) = \infty$ for large enough n by definition (2.1), so the convention $g \equiv \infty$ on $(1, \infty)$ is appropriate.]

It remains to calculate $g(x)$ for $p < x \le 1$. Fix $u \ge 1$ for a moment. Define an initial configuration (z_i) for the process as follows: $z_0 = 0$, and $(\eta_i = z_i - z_{i-1}$: $i \in \mathbf{Z})$ are i.i.d. with distribution (2.6) and expectation $E[\eta_i] = u$. Then by the earlier calculation,

$$E\left[n^{-1} z_0(nt)\right] = -tf(u) \tag{2.11}$$

for positive integers n and t. On the other hand, by (2.3),

$$\frac{1}{n} z_0(nt) = \inf_{i \le 0} \left\{ \frac{1}{n} z_i + \frac{1}{n} \Gamma((z_i, 0), nt, -i) \right\}. \tag{2.12}$$

LEMMA 2.2. *We have*

$$\lim_{n \to \infty} \frac{1}{n} z_0(nt) = -tg^+(u) \tag{2.13}$$

in probability.

PROOF. By g's lower semicontinuity, the supremum in (2.9) is attained at some $x = -r/t$. With $i = [nr]$ in (2.12),

$$\frac{1}{n} z_0(nt) \le \frac{1}{n} z_{[nr]} + \frac{1}{n} \Gamma\big((z_{[nr]}, 0), nt, -[nr]\big),$$

and the right-hand side converges to $ur + tg(-r/t) = -tg^+(u)$, by (2.10) and the choice of (z_i).

For the converse inequality, note first that we only need to consider $[-nt] \le i \le 0$ in (2.12). Now pick a fine enough partition of $(-t, 0)$,

$$-t = r_0 < r_1 < \cdots < r_s = 0,$$

and approximate the right-hand side of (2.12) from below while simultaneously restricting i to the finitely many points $[nr_l]$, $l = 0, \ldots, s$. The details are exactly the same as those in the proof of Lemma 8.13 in [12]. $\square$

Since $z_0(nt)$ is a sum of nt steps, each distributed as $-x_0$ in (2.7), a uniform bound

$$\sup_n E\Big[\big(n^{-1}z_0(nt)\big)^2\Big] < \infty$$

is immediate, and then comparison of (2.11) and (2.13) yields

$$f(u) = g^+(u) \quad \text{for } u \ge 1.$$

By Theorem 12.4 of [10],

$$g(x) = g^{++}(x) \equiv \sup_{u \ge 0}\{xu - g^+(u)\}, \qquad x \ge 0.$$

However, $g(x) \ge x$ for all x implies $g^+(u) = 0$ for $0 \le u \le 1$, and consequently the supremum above need only be taken over $u \ge 1$. We conclude that

$$g(x) = \sup_{u \ge 1}\{xu - f(u)\}. \tag{2.14}$$

Now Theorem 1 is proved: use (2.8) to find $g(x)$ from (2.14), then invert $g(x)$ to get $h(x)$, and finally by homogeneity $\Psi(x, y) = yh(x/y)$.

3. Proof of Theorem 2. First we prove (1.5) assuming $\gamma(s)$ is continuous. Let $\varepsilon > 0$ be arbitrary. Given the neighborhood $U_\delta(\gamma)$, use the uniform continuity of $\gamma(s)$ to pick m large enough so that if $s_i = i/m$, each rectangle

$$R_i = [\gamma_1(s_i), \gamma_1(s_{i+1})] \times [\gamma_2(s_i), \gamma_2(s_{i+1})], \qquad i = 0, \dots, m-1,$$

lies inside $U_\delta(\gamma)$. Write $\Delta s = s_{i+1} - s_i = 1/m$. Set

$$f_m(s) = \sum_{i=0}^{m-1} I_{[s_i, s_{i+1})}(s)\Psi\left(\frac{\gamma_1(s_{i+1}) - \gamma_1(s_i)}{\Delta s}, \frac{\gamma_2(s_{i+1}) - \gamma_2(s_i)}{\Delta s}\right).$$

By the a.e. differentiability of γ_1 and γ_2, $f_m(s) \to \Psi(\gamma_1'(s), \gamma_2'(s))$ as $m \to \infty$ for a.e. s. By Fatou's lemma, we may increase m further so that

$$\int_0^1 f_m(s)\, ds \ge J(\gamma) - \delta/3.$$

Having fixed m, pick n large enough so that each rectangle nR_i has a strictly increasing path of at least

$$n\Psi(\gamma_1(s_{i+1}) - \gamma_1(s_i), \gamma_2(s_{i+1}) - \gamma_2(s_i)) - n\delta/(3m)$$

points, with probability at least $1 - \varepsilon/m$. Join together the paths in the rectangles, and then $nU_\delta(\gamma)$ has a path of at least

$$\begin{aligned} n\sum_{i=0}^{m-1} &\Psi(\gamma_1(s_{i+1}) - \gamma_1(s_i), \gamma_2(s_{i+1}) - \gamma_2(s_i)) - n\delta/3 \\ &= n\int_0^1 f_m(s)\, ds - n\delta/3 \\ &\ge n(J(\gamma) - 2\delta/3) \end{aligned}$$

points, and this happens with probability at least $1 - \varepsilon$. This proves statement (1.5) for continuous $\gamma(s)$.

Before completing the proof of (1.5), we argue (1.6). The additional assumption of Lipschitz continuity implies that we may use dominated convergence instead of Fatou's lemma to get

$$\lim_{m \to \infty} \int_0^1 f_m(s)\, ds = J(\gamma).$$

Thus m can be fixed so that the quantity

$$\int_0^1 f_m(s)\, ds = \sum_{i=0}^{m-1} \Psi\big(\gamma_1(s_{i+1}) - \gamma_1(s_i), \gamma_2(s_{i+1}) - \gamma_2(s_i)\big)$$

is within $\delta/4$ of $J(\gamma)$. Next, pick a slightly larger rectangle $S_i = [a_i, a'_i] \times [b_i, b'_i]$ that contains a neighborhood of R_i, $i = 0, \dots, m-1$, but so that

$$\sum_{i=0}^{m-1} \Psi(a'_i - a_i, b'_i - b_i)$$

is within $\delta/2$ of $J(\gamma)$. Pick $\delta_1 > 0$ small enough to have

$$U_{\delta_1}(\gamma) \subset \bigcup_{i=0}^{m-1} S_i.$$

Given $\varepsilon > 0$, the limit (1.2) guarantees that for large enough n, any nS_i has a strictly increasing path exceeding $n\Psi(a'_i - a_i, b'_i - b_i) + n\delta/(4m)$ points with probability at most ε/m. Add these numbers up to see that $nU_{\delta_1}(\gamma)$ has a path exceeding $n(J(\gamma) + 3\delta/4)$ with probability at most ε. We have proved (1.6).

Now return to (1.5), this time for a discontinuous $\gamma(s)$. Monotonicity implies that $\gamma(s)$ has at most countably many discontinuities $\{s_i\}$. Define a new curve λ by connecting the components of the image of γ with horizontal and vertical line segments. Specifically, for each discontinuity s_i, let ρ_i be the curve that first travels horizontally from $(\gamma_1(s_i-), \gamma_2(s_i-))$ to $(\gamma_1(s_i+), \gamma_2(s_i-))$, then vertically from $(\gamma_1(s_i+), \gamma_2(s_i-))$ to $(\gamma_1(s_i+), \gamma_2(s_i+))$. Patch the segments together in the nondecreasing order, with continuous pieces of γ alternating with successive ρ_i's, and the parametrization switching between the parametrization of γ and a parametrization of the ρ_i's by arc length. This gives a continuous curve parametrized by $[0, 1+S]$ where $S \in [0, a+b]$ is the sum of the lengths of the ρ_i's. Normalize the parametrization by dividing by $1 + S$, and let this curve be $\lambda(s)$. Note that $\Psi(\rho'_i(s)) = 0$ so the new pieces do not contribute to $J(\lambda)$, and consequently $J(\lambda) = J(\gamma)$.

Given $\varepsilon > 0$ and $\delta > 0$, we need to show that for large enough n, $nU_\delta(\gamma)$ has a path of at least $n(J(\gamma) - \delta)$ points with probability at least $1 - \varepsilon$. Let $0 < \delta_1 < \delta/3$. There are only finitely many ρ_i's that do not satisfy $U_{\delta_1}(\rho_i) \subseteq U_\delta(\gamma)$. Let these be $\rho_1, \dots, \rho_m$, after a suitable relabeling. Each ρ_i is Lipschitz, so by (1.6) we can shrink δ_1 further and pick n large enough so that longest increasing paths in $nU_{\delta_1}(\rho_1), \dots, nU_{\delta_1}(\rho_m)$ together have at most

$n\delta/2$ points with probability at least $1-\varepsilon/2$. Increase n further so that, by the continuous case of (1.5) proved above, $nU_{\delta_1}(\lambda)$ contains a path of at least $n(J(\gamma)-\delta_1)$ points with probability at least $1-\varepsilon/2$. Now put everything together. To get a path in $nU_\delta(\gamma)$, we need remove from a path in $nU_{\delta_1}(\lambda)$ no more than the subpaths in $nU_{\delta_1}(\rho_1),\dots,nU_{\delta_1}(\rho_m)$ because for $i\notin\{1,\dots,m\}$ $nU_{\delta_1}(\rho_i)$ lies inside $nU_\delta(\gamma)$. Thus we get a path of at least $n(J(\gamma)-\delta_1)-n\delta/2$ points in $nU_\delta(\gamma)$ with probability at least $1-\varepsilon/2-\varepsilon/2$. Statement (1.5) is now proved for all nondecreasing curves γ.

Let $\phi(s)=(as,bs)$, $s\in[0,1]$, be the diagonal in $A=[0,a]\times[0,b]$ as in the statement of the theorem, and let γ be an arbitrary nondecreasing curve in A.

$$\begin{aligned} J(\gamma) &= \int_0^1 \Psi(\gamma'(s))\,ds \le \Psi\left(\int_0^1 \gamma'(s)\,ds\right)\\ &\le \Psi(\gamma(1)-\gamma(0)) \le \Psi(a,b)\\ &= J(\phi). \end{aligned} \tag{3.1}$$

The first inequality above follows from Jensen's inequality and the concavity of Ψ. The second comes from the fact that for a general nondecreasing function $f(s)$,

$$\int_0^1 f'(s)\,ds \le f(1)-f(0) \tag{3.2}$$

with equality if and only if f is absolutely continuous. Equation (3.1) tells us that the supremum of $J(\gamma)$ over nondecreasing curves is $\Psi(a,b)$ and that this value is achieved at $\gamma=\phi$.

Now assume $pb\le a\le p^{-1}b$ and that $J(\gamma)=J(\phi)$. By (1.3), $\Psi(x,y)<\Psi(a,b)$ if $(a,b)\ne(x,y)\in A$, so equality in the last two inequalities in (3.1) forces

$$\int_0^1 \gamma'(s)\,ds = \gamma(1)-\gamma(0) = (a,b). \tag{3.3}$$

Concavity and homogeneity of Ψ give

$$\begin{aligned} \Psi(\gamma'(s)) &\le \Psi(a,b) + \nabla\Psi(a,b)[\gamma'(s)-(a,b)]\\ &= \nabla\Psi(a,b)\gamma'(s). \end{aligned}$$

Equality in (3.1) forces equality above for a.e. s, so that

$$\gamma'(s) \in W \equiv \{(x,y): \Psi(x,y) = \nabla\Psi(a,b)(x,y)\} \quad \text{for a.e. } s. \tag{3.4}$$

If $pb<a<p^{-1}b$, explicit manipulations with Ψ show that W equals the ray through the origin spanned by (a,b), so γ must be some parametrization of the diagonal. If $a=pb$, then $W=\{(x,y)\colon 0\le x\le py\}$, and (3.3) and (3.4) imply

$$a = \int_0^1 \gamma_1'(s)\,ds \le p\int_0^1 \gamma_2'(s)\,ds = pb.$$

Then $a = pb$ forces $\gamma_1'(s) = p\gamma_2'(s)$ for a.e. s, and again γ is the diagonal. A similar argument works for $a = p^{-1}b$.

In the case $a < pb$, $\Psi(x, b)$ is strictly increasing in $x \in [0, a]$. By the second and third inequalities in (3.1), a maximizing γ must have

$$\int_0^1 \gamma_1'(s)\, ds = \gamma_1(1) - \gamma_1(0) = a = \Psi(a, b).$$

To have equality in (3.2) for $f = \gamma_1$, the γ_1-coordinate must be absolutely continuous. Note that the last line of (1.3) can be written $\Psi(x, y) = x - q^{-1}(\sqrt{x} - \sqrt{py})^2$. Then

$$\begin{aligned} J(\gamma) = &\int_{\gamma_1' \le p\gamma_2'} \gamma_1'(s)\, ds \\ &+ \int_{p\gamma_2' < \gamma_1' < \gamma_2'/p} \left[\gamma_1'(s) - q^{-1}\left(\sqrt{\gamma_1'(s)} - \sqrt{p\gamma_2'(s)}\right)^2\right] ds \\ &+ \int_{\gamma_1' \ge \gamma_2'/p} \gamma_2'(s)\, ds \\ = \Psi(a, b) &- \int_{p\gamma_2' < \gamma_1' < \gamma_2'/p} q^{-1}\left(\sqrt{\gamma_1'(s)} - \sqrt{p\gamma_2'(s)}\right)^2 ds \\ &- \int_{\gamma_1' \ge \gamma_2'/p} (\gamma_1'(s) - \gamma_2'(s))\, ds, \end{aligned}$$

which shows that a maximizing γ must also have $\gamma_1' \le p\gamma_2'$ a.e.

Finally, to prove (1.7), first observe that, given $\delta_1 > 0$, there are $\varepsilon > 0$ and $\delta_2 > 0$ such that

$$\begin{aligned} \Psi(a, b) - \Psi(x + \varepsilon/2, y + \varepsilon/2) \\ - \Psi(a - x + \varepsilon/2, b - y + \varepsilon/2) \ge \delta_2 \end{aligned} \tag{3.5}$$

for all $(x, y) \in A \setminus U_{\delta_1}$. Next, note that if there is a longest path of points that does not lie in nU_δ, then for some smaller $\delta_1 > 0$ and some $(x, y) \in A \setminus U_{\delta_1}$,

$$\begin{aligned} &L((0,0), (nx + n\varepsilon/2, ny + n\varepsilon/2)) \\ &\quad + L((nx - n\varepsilon/2, ny - n\varepsilon/2), (na, nb)) \\ &\quad \ge L((0,0), (na, nb)). \end{aligned} \tag{3.6}$$

By (3.5), the probability of (3.6) happening for fixed (x, y) vanishes as $n \to \infty$. The event in (1.7) is contained in a finite union of events (3.6), indexed by (x, y) ranging over the centers of a partitioning of $A \setminus U_\delta$ in $\varepsilon \times \varepsilon$ rectangles. This partitioning does not change with n, hence we get the limit in (1.7).

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