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Increasing the order of convergence of multistep methods for solving systems of equations under weak conditions

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Dedicated to the memory of Professor Ştefan Măruşter

Abstract. In the present paper, we consider a construction proposed in Xiao and Yin (2016) to improve the order of convergence of the method from p to $p+2m$ under weaker assumptions. Using the idea of restricted convergence domains we extend the applicability of the method considered by Xiao and Yin (2016). Numerical examples where earlier results cannot apply to solve equations but our results can apply are also given in this study.

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1 Introduction

Consider the problem of approximating a locally unique solution x^* of the nonlinear equation

$$F(x) = 0, \tag{1.1}$$

where $F : D \subseteq X \longrightarrow Y$ is a Fréchet-differentiable operator defined on a convex subset D of X. Here X and Y are Banach spaces. Higher order multi-point methods are studied in the literature (see [1–4, 12, 13, 15]) for approximating the solution x^* of (1.1). But, very often the computational cost of these higher order method is very high [5–11, 14–16]. Numerous researchers have tried to obtain a general law to accelerate the convergence for all the iterative methods which use Newton iteration as a predictor [5–11]. Kou et al. [11] and Cordero et al. [5–7] introduced the following construction (when $X = Y = \mathbb{R}^n$),

$$\begin{aligned} z_k &= \phi(x_k, y_k), \\ x_{k+1} &= z_k - F'(y_k)^{-1}F(z_k), \end{aligned}$$

where $y_k = x_k - F'(x_k)^{-1}F(x_k)$ and ϕ is the iteration function. Using the above construction Kou et al. [11] and Cordero et al. [5–7] improved the order of the given iterative method from p to $p+2$. Further, for using extended Newton iteration as a predictor and accelerating the order of convergence the following construction is introduced in [16]:

$$\begin{aligned} y_k &= x_k - aF'(x_k)^{-1}F(x_k), \\ z_k &= \phi(x_k, y_k), \\ x_{k+1} &= z_k - \left\{ 2\left[\frac{1}{2a}F'(y_k) + \left(1 - \frac{1}{2a}\right)F'(x_k)\right]^{-1} - F'(x_k)^{-1} \right\} F(z_k). \end{aligned}$$

Using the above construction Xiao and Yin [16] improved the order of the given iterative method from p to $p + 2$.

In the present paper, we consider the following construction considered in [16] to improve the order of convergence of the method from p to $p + 2m$,

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ z_n &= \phi_1(x_n, y_n) \\ z_n^{(1)} &= z_n - \phi(x_n, y_n)F(z_n) \\ &\vdots \\ z_n^{(m-1)} &= z_n^{(m-2)} - \phi(x_n, y_n)F(z_n^{(m-2)}) \\ x_{n+1} &= z_n^{(m)} = z_n^{(m-1)} - \phi(x_n, y_n)F(z_n^{(m-1)}), \end{aligned} \tag{1.2}$$

where both ϕ functions are small as in method (1.2), $\phi_1 : D \times D \longrightarrow X$ is an iteration function with convergence order greater or equal to two, $\phi : D \times D \longrightarrow L(X)$ given by $\phi(x_n, y_n) = \frac{1}{3}\{4[3F'(y_n) - F'(x_n)]^{-1} + F'(x_n)\}$ and $m \geq 2$ is an integer. Our goal is to weaken the assumptions in [16], so that the applicability of the method (1.2) can be improved.

In earlier studies such as [5–11, 14–16], higher order methods are considered for approximating the solution x^* of (1.1). But the convergence analysis of these methods requires assumptions of the form (see e.g [16]):

$$\|F'''(x) - F'''(y)\| \leq L\|x - y\|, \quad x, y \in D, \quad L \geq 0 \tag{1.3}$$

or

$$\|F'''(x) - F'''(y)\| \leq w(\|x - y\|), \quad x, y \in D, \tag{1.4}$$

where $w(z)$ is a nondecreasing continuous function for $z > 0$ and $w(0) = 0$.

Notice that F'' and F''' do not appear in method (1.2). Hence, the results in the earlier studies using them cannot apply. A motivational example of (1.1) that does not satisfy (1.3) or (1.4) is the following:

$$F(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0 \\ 0, & x = 0, \end{cases} \tag{1.5}$$

where $F : [-\frac{3}{2}, \frac{1}{2}] \longrightarrow \mathbb{R}$. We have that

$$F'(x) = 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2$$

$$F''(x) = 6x \ln x^2 + 20x^3 - 12x^2 + 10x$$

and

$$F'''(x) = 6 \ln x^2 + 60x^2 - 24x + 22.$$

Then, obviously, function F''' is unbounded on D. In this paper, we extend the applicability of method (1.2) by using hypotheses only on the first derivative that appear on it.

The rest of the paper is organized as follows. In Section 2 we present the local convergence analysis. We also provide a radius of convergence, computable error bounds and uniqueness result, not provided in the earlier works on this method. Special cases and numerical examples are given in the last section.

2 Local convergence analysis

The local convergence analysis of method (1.2) is based on some scalar functions and parameters. Let $m \geq 2$ be an integer, $\lambda > 1$ be a parameter w_0, w, v be continuous, non-negative, non-decreasing functions defined on the interval $[0,+\infty)$.

Suppose equation

$$w_0(t) = 1 \tag{2.1}$$

has a minimal positive zero r_0. Define functions g and h_g, on the interval $[0,r_0)$ by,

$$g(t) = \frac{\int_0^1 w((1-\theta)t)d\theta}{1 - w_0(t)},$$

and

$$h_g(t) = g(t) - 1.$$

Next, we shall show the existence of zeros for certain scalar functions appearing in the proof of Theorem 2.1 relying on the intermediate value theorem. We have that $h_g(0) = -1 < 0$ and $h_g(t) \to +\infty$ as $t \to r_0^-$. By the intermediate value theorem, function h_g has zeros in the interval $(0, r_0)$. Denote by r_g the smallest such zero.

Let p_1 be a continuous, non-negative, non-decreasing function on the interval $[0, r_g)$ such that

$$p_1(\rho_0)\rho_0^{\lambda-1} > 1 \tag{2.2}$$

for some $\rho_0 \in (0, r_g)$. Let $h_{p_1}(t) = p_1(t)t^{\lambda-1}-1$. We have that $h_{p_1}(0) = -1 < 0$ and $h_{p_1}(\rho_0) > 0$. Denote by ρ_1 the smallest zero of function h_{p_1} on the interval $(0, \rho_0)$.

Define functions q and h_q on the interval $[0, r_0)$ by $q(t) = \frac{1}{2}(2w_0(g(t)t) + w_0(t))$ and

$$h_q(t) = q(t) - 1.$$

We have $h_q(0) = -1 < 0$ and $h_q(t) \to +\infty$ as $t \to r_0^-$. Denote by r_q the smallest zero of functions h_q on the interval $(0,r_0)$.

Case 1:

$$r_q \leq \rho_1. \tag{2.3}$$

Define function p on the interval $[0, r_q)$ by $p(t) = \frac{1}{3}(\frac{2}{1-q(t)} + \frac{1}{1-w_0(t)})$.
Notice that $p(0) = 1$ and $p(t) \to +\infty$ as $t \to r_0^-$.

Define functions ψ_1 and h_{ψ_1} on the interval $[0, r_q)$ by $\psi_1(t) = (1 + p_1(t)\int_0^1 v(\theta g(t)t^\lambda)d\theta)g(t)t^{\lambda-1}$ and $h_{\psi_1}(t) = \psi_1(t) - 1$. We have that $h_{\psi_1}(0) = -1 < 0$ and $h_{\psi_1}(t) \to +\infty$ as $t \to r_q^-$. Denote by r_{ψ_1} the smallest zero of function h_{ψ_1} on the interval $[0, \rho_q)$. Let $m \geq 1$ be an integer and $j = 2, 3, \cdots, m$. Define functions ψ_j and h_{ψ_j} on $[0, r_q)$, $\psi_j(t) = (1+p(t)\int_0^1 v(\theta\psi_{j-1}(t)t)dt)\psi_{j-1}(t)$ and $h\psi_j(t) = \psi_j(t) - 1$. We have that $h\psi_2(0) = -1 < 0$ and

$$\begin{aligned} h_{\psi_2}(r_{\psi_1}) &= (1 + p(r_{\psi_1})\int_0^1 v(\theta\psi_1(r\psi_1)r_{\psi_1})d\theta)\psi_1(r_{\psi_1}) - 1 \\ &= p(r_{\psi_1})\int_0^1 v(\theta r_{\psi_1})d\theta > 0, \end{aligned}$$

since $\psi_1(r_{\psi_1}) = 0$. Denote by r_{ψ_2} the smallest zero of function h_{ψ_2} on the interval $(0, r_{\psi_1})$. Similarly, we have that $h_{\psi_j}(0) = -1 < 0$ and

$$h_{\psi_j}(r_{\psi_{j-1}}) = p(r_{\psi_{j-1}}) \int_0^1 r(\theta r\psi_{j-1})d\theta > 0.$$

Denote by r_{ψ_j} the smallest zero of function $h\psi_j$ on the interval $(0, r_{\psi_{j-1}})$.

Define the radius of convergence r by

$$r = \min\{r_g, \rho_1, r_{\psi_i}\}, \ \ i = 1, 2, 3, \dots, m. \tag{2.4}$$

Then, we have that for each $t \in [0, r)$

$$0 \le g(t) < 1, \tag{2.5}$$

$$0 \le g_1(t)t^{\lambda-1} < 1, \tag{2.6}$$

$$0 \le \psi_j(t) < 1, \tag{2.7}$$

$$0 \le w_0(t) < 1, \tag{2.8}$$

$$0 \le q(t) < 1, \tag{2.9}$$

and

$$1 \le p(t). \tag{2.10}$$

Case 2:

$$\rho_1 < r_q.$$

We have that $h_{\psi_1}(0) = -1 < 0$ and

$$\begin{aligned} h_{\psi_1}(\rho_1) &= (1 + p(\rho_1) \textstyle\int_0^1 v(\theta g_1(\rho_1)\rho_1^{\lambda})d\theta) g_1(\rho_1)\rho_1^{\lambda-1} - 1 \\ &= p(\rho_1) \textstyle\int_0^1 v(\theta\rho_1)d\theta > 0, \end{aligned}$$

since $g_1(\rho_1)\rho_1^{\lambda-1} = 1$. Denote again by r_{ψ_1} the smallest zero of function h_{ψ_1} on the interval $(0, \rho_1^{\lambda-1})$. Also we have again that $h_{\psi_j}(0) = -1 < 0$ and

$$h_{\psi_j}(r_{\psi_{j-1}}) = p(r_{\psi_{j-1}}) \int_0^1 v(\theta r\psi_{j-1})d\theta > 0,$$

by (2.10). Hence the radius of convergence can again be defined by (2.4)

Let $U(a, \rho)$, $\overline{U}(a, \rho)$ stand respectively, for the open and closed balls in X, with center $a \in X$ and of radius $\rho > 0$. Next, we present the local convergence analysis of method (1.2) using the preceding notation.

Theorem 2.1. *Let $F : D \subset X \to Y$ be a Fréchet differentiable operator and $\lambda > 1$ be a parameter. Suppose that there exists $x^* \in D$, non-negative, non-decreasing functions p_1, w_0 such that (2.2) is satisfied, defined on the interval $[0, +\infty)$ such that for each $x \in D$,*

$$F(x^*) = 0, \quad F'(x^*)^{-1} \in L(Y, X), \tag{2.11}$$

$$\|z(x) - x^*\| \leq p_1(\|x - x^*\|)\|x - x^*\|^\lambda \tag{2.12}$$

and

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*)\| \leq w_0(\|x - x^*\|), \tag{2.13}$$

where $z(x) = \phi_1(x, x - F'(x)^{-1}F'(x))$. Moreover suppose there exist continuous, non-negative, non-decreasing functions w, v defined on the interval $[0, r_0)$ such that for each $x, y \in D_1 := D \cup U(x^*, r_0)$,

$$\|F'(x^*)^{-1}(F'(x) - F'(y)\| \leq w(\|x - y\|), \tag{2.14}$$

$$\|F'(x^*)^{-1}F'(x)\| \leq v(\|x - y\|), \tag{2.15}$$

and

$$U(x^*, r) \subseteq D, \tag{2.16}$$

where the radius of convergence is defined by (2.4). Then, the sequence $\{x_n\}$ generated for $x_0 \in U(x^*, r) - \{x^*\}$ by method (1.2) is well defined in $U(x^*, r)$, remains in $U(x^*, r)$ for each $n = 0, 1, 2, \ldots$ and converges to x^*. Moreover, the following estimates hold

$$\|y_n - x^*\| \leq g(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < r, \tag{2.17}$$

$$\|z_n - x^*\| \leq p_1(\|x_n - x^*\|)\|x_n - x^*\|^\lambda \leq \|x_n - x^*\|, \tag{2.18}$$

$$\|z_n^{(i)} - x^*\| \leq \psi_i(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|. \tag{2.19}$$

In particular, since $z_n^{(m)} = x_{n+1}$

$$\|x_{n+1} - x^*\| \leq \psi_m(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \tag{2.20}$$

where, the functions g, p and ψ_i are defined previously. Furthermore,for $\bar{r_0} \in [r, r_0)$, the limit point x^* is the only solution of equation $F(x) = 0$ in $D_2 = D \cap U(x^*, \bar{r})$ if $w_0(\bar{r_0}) < 1$.

Proof. We shall show using mathematical induction that sequence $\{x_n\}$ generated by (1.2) is well defined in $U(x^*, r)$, remains in $U(x^*, r)$ for each $n =$

$0, 1, 2, \cdots$ and converges to x^* so that estimates (2.17)-(2.20) are satisfied. By (2.1), (2.8), (2.13) and the hypotheses $x_0 \in U(x^*, r) - \{x^*\}$, we have that

$$\|F'(x^*)^{-1}(F'(x_0) - F'(x^*)\| \leq w_0(\|x_0 - x^*\|) \leq w_0(r) < 1. \tag{2.21}$$

It follows from (2.21) and the Banach Lemma on invertible operators [1, 11, 13] that $F'(x)^{-1} \in L(Y, X)$, y_0 is well defined by the first substep of method (1.2) and

$$\|F'(x_0)^{-1}F'(x^*)\| \leq \frac{1}{1 - w_0(\|x_0 - x^*\|)}. \tag{2.22}$$

We can write the identity,

$$y_0 - x^* = x_0 - x^* - F'(x_0)^{-1}F(x_0). \tag{2.23}$$

Using (2.5), (2.11)-(2.14), (2.22) and (2.23), we get in turn that

$$\begin{aligned} \|y_0 - x^*\| &\leq \|F'(x_0)^{-1}F'(x^*)\| \| \textstyle\int_0^1 F'(x^*)^{-1}[F'(x^* + \theta(x_0 - x^*)) \\ &\quad -F'(x_0)](x_0 - x^*)d\theta\| \\ &\leq \frac{\int_0^1 w((1-\theta)\|x_0-x^*\|)d\theta\|x_0-x^*\|}{1-w_0(\|x_0-x^*\|)}, \\ &= g(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < r, \end{aligned} \tag{2.24}$$

which shows (2.17) for $n = 0$ and $y_0 \in U(x^*, r)$.

We can write by (2.11) that

$$F(x_0) = F(x_0) - F(x^*) = \int_0^1 F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)d\theta. \tag{2.25}$$

Notice that $x^* + \theta(x_0 - x^*) \in U(x^*, r)$ for each $\theta \in [0, 1]$, since $\|x^* + \theta(x_0 - x^*) - x^*\| = \theta\|x_0 - x^*\| < r$. In view of (2.15) and (2.25), we get that

$$\|F'(x^*)^{-1}F(x_0) \leq \int_0^1 v(\theta\|x_0 - x^*\|)d\theta. \tag{2.26}$$

Notice that z_0 is well-defined by (2.22) and the second sub-step of method (1.2). By the second substep of method (1.2), for $n = 0$, (2.6) and (2.12), we obtain that

$$\begin{aligned} \|z_0 - x^*\| &= \|\phi_1(x_0, y_0) - x^*\| \\ &\leq p_1(\|x_0 - x^*\|)\|x_0 - x^*\|^\lambda \\ &\leq \|x_0 - x^*\| < 1, \end{aligned} \tag{2.27}$$

which shows (2.18) for $n = 0$ and $z_0 \in U(x^*, r)$. Next, we must show that linear operator $\phi(x_0, y_0)$ is well- defined. Using (2.4), (2.9), (2.13) and (2.24), we have in turn that

$$\begin{aligned} & \|(2F'(x^*))^{-1}((3F'(y_0) - F'(x_0)) - 2F'(x^*))\| \qquad (2.28)\\ \leq & \frac{1}{2}[2\|F'(x^*)^{-1}(F'(y_0) - F'(x^*))\| + \|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\|]\\ \leq & \frac{1}{2}[2w_0(\|y_0 - x^*\|) + w_0(\|x_0 - x^*\|)]\\ \leq & \frac{1}{2}[2w_0(g(\|x^* - x_0\|)(\|x^* - x_0\|)) + w_0(\|x^* - x_0\|)]\\ \leq & \frac{1}{2}[2w_0(g(r)r) + w_0(r)] = q(r) < 1. \end{aligned}$$

Hence we get that $z_0^{(i)}$ exist, $i = 0, 1, 2, \cdots, m$ by method (1.2), $\phi(x_0, y_0)$ exists and

$$\|\phi(x_0, y_0)F'(x^*)\| \leq \frac{1}{3}\left(\frac{2}{1 - q(\|x_0 - x^*\|)} + \frac{1}{1 - w_0(\|x_0 - x^*\|)}\right) = p(\|x_0 - x^*\|). \qquad (2.29)$$

Using (2.26) for $z_0^{(i)} = x_0$, (2.4), (2.7), (2.29), we have in turn that

$$\begin{aligned} \|z_0^{(1)} - x^*\| &= \|z_0 - x^*\| + \|\phi(x_0, y_0)F'(x^*)\|\|F'(x^*)^{-1}F(z_0)\|\\ &\leq (1 + \|\phi(x_0, y_0)F'(x^*)\| \textstyle\int_0^1 v(\theta\|z_0 - x^*\|)d\theta)\|z_0 - x^*\|\\ &\leq (1 + p(\|x_0 - x^*\|) \textstyle\int_0^1 v(\theta g_1(\|x_0 - x^*\|)\|x_0 - x^*\|^\lambda)d\theta)\\ &\quad \times g_1(\|x_0 - x^*\|)\|x_0 - x^*\|^\lambda\\ &= \psi_1(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < r, \end{aligned} \qquad (2.30)$$

and for $j = 2, 3, \cdots, m$

$$\begin{aligned} \|z_0^{(j)} - x^*\| &\leq (1 + p(\|x_0 - x^*\|) \textstyle\int_0^1 v(\theta\|z_0^{(j-1)} - x^*\|)d\theta)\|z_0^{(j-1)} - x^*\|\\ &\leq (1 + p(\|x_0 - x^*\|) \textstyle\int_0^1 v(\theta\psi_{j-1}(\|x_0 - x^*\|)\|x_0 - x^*\|)d\theta)\\ &\quad \times\psi_{j-1}(\|x_0 - x^*\|)\|x_0 - x^*\|\\ &= \psi_j(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < r, \end{aligned} \qquad (2.31)$$

Hence estimates (2.19)-(2.20) hold for $n = 0$. By simply replacing $x_0, y_0, z_0, z_0^{(i)}$ by $x_k, y_k, z_k, z_k^{(i)}$ in the preceding estimates, we arrive at estimates (2.17)-(2.20). Then, from the estimates

$$\|x_{n+1} - x^*\| \leq c\|x_k - x^*\| < r, \tag{2.32}$$

where $c = \psi_m(\|x_0 - x^*\|) \in [0, 1)$, we deduce that $\lim_{k\to\infty} x_k = x^*$ and $x_{k+1} \in U(x^*, r)$. Finally to show the uniqueness part, let $T = \int_0^1 F'(x^*+\theta(y^*-x^*))d\theta$ where $y^* \in D_2$ with $F(y^*) = 0$. Using (2.13), we obtain that

$$\begin{aligned} \|F'(x^*)^{-1}(T - F'(x^*))\| &\leq \int_0^1 w_0(\theta\|x^* - y^*\|)d\theta \\ &\leq w_0(\bar{r_0})d\theta < 1, \end{aligned} \tag{2.33}$$

Hence, we have that $T^{-1} \in L(Y, X)$. Then, from the identity $0 = F(y^*) - F(x^*) = T(y^* - x^*)$, we conclude that $x^* = y^*$. □

Remark 2.1. (a) In the case when $w_0(t) = L_0t, w(t) = Lt$, the radius $r_A = \frac{2}{2L_0+L}$ was obtained by Argyros in [1] as the convergence radius for Newton's method under condition (2.7)-(2.9). Notice that the convergence radius for Newton's method given independently by Rheinboldt [13] and Traub [15] is given by

$$\rho = \frac{2}{3L} < r_1.$$

As an example, let us consider the function $f(x) = e^x - 1$. Then $x^* = 0$. Set $\Omega = U(0, 1)$. Then, we have that $L_0 = e - 1 < l = e$, so $\rho = 0.24252961 < r_1 = 0.324947231$.

Moreover, the new error bounds [1] are:

$$\|x_{n+1} - x^*\| \leq \frac{L}{1 - L_0\|x_n - x^*\|}\|x_n - x^*\|^2,$$

whereas the old ones are

$$\|x_{n+1} - x^*\| \leq \frac{L}{1 - L\|x_n - x^*\|}\|x_n - x^*\|^2.$$

Clearly, the new error bounds are more precise, if $L_0 < L$. Clearly, we do not expect the radius of convergence of method (1.2) given by r to be larger than r_A.

(b) The local results can be used for projection methods such as Arnoldi's method, the generalized minimum residual method (GMREM), the generalized conjugate method (GCM) for combined Newton/finite projection methods and in connection to the mesh independence principle in order to develop the cheapest and most efficient mesh refinement strategy [1–4, 13].

(c) The results can be also be used to solve equations where the operator F' satisfies the autonomous differential equation [1–4, 13]:

$$F'(x) = P(F(x)),$$

where P is a known continuous operator. Since $F'(x^*) = P(F(x^*)) = P(0)$, we can apply the results without actually knowing the solution x^*. Let as an example $F(x) = e^x - 1$. Then, we can choose $P(x) = x+1$ and $x^* = 0$.

(d) It is worth noticing that method (1.2) are not changing if we use the new instead of the old conditions [16]. Moreover, for the error bounds in practice we can use the computational order of convergence (COC)

$$\xi = \frac{ln\frac{\|x_{n+2}-x_{n+1}\|}{\|x_{n+1}-x_n\|}}{ln\frac{\|x_{n+1}-x_n\|}{\|x_n-x_{n-1}\|}}, \qquad \text{for each } n = 1, 2, \ldots$$

or the approximate computational order of convergence (ACOC)

$$\xi^* = \frac{ln\frac{\|x_{n+2}-x^*\|}{\|x_{n+1}-x^*\|}}{ln\frac{\|x_{n+1}-x^*\|}{\|x_n-x^*\|}}, \qquad \text{for each} \quad n = 0, 1, 2, \ldots$$

instead of the error bounds obtained in Theorem 2.1.

(e) In view of (2.13) and the estimate

$$\begin{aligned} \|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}(F'(x) - F'(x^*)) + I\| \\ &\leq 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq 1 + w_0(\|x - x^*\|) \end{aligned}$$

condition (2.15) can be dropped and can be replaced by

$$v(t) = 1 + w_0(t)$$

or

$$v(t) = 1 + w_0(r_0),$$

since $t \in [0, r_0)$.

3 Numerical Examples

The numerical examples are presented in this section.

Example 3.1. Let $X = Y = \mathbb{R}^3, D = \bar{U}(0,1), x^* = (0,0,0)^T$. Define function F on D for $w = (x,y,z)^T$ by

$$F(w) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T.$$

Then, the Fréchet-derivative is given by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Choose $\psi_1(x_n, y_n) = y_n - F'(y_n)^{-1}F(y_n)$, $\lambda = 1$, $p_1(t) = \frac{Lt}{1-L_0t}$. Notice that using conditions (2.11)-(2.15), we get $w_0(t) = L_0t, w(t) = Lt, v(t) = L$, $L_0 = e-1, L = e^{\frac{1}{L_0}}$. The parameters are

$$r_g = 0.3827,\ \rho_1 = 0.2885,\ r_{\psi_1} = 0.2553,\ r_{\psi_2} = 0.1681 = r.$$

Example 3.2. Let $X = Y = C[0,1]$, the space of continuous functions defined on $[0,1]$ and be equipped with the max norm. Let $D = \overline{U}(0,1)$. Define function F on D by

$$F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3 d\theta. \tag{3.1}$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta,\ \text{for each}\ \xi \in D.$$

Then, we get that $x^* = 0$, $w_0(t) = L_0t, w(t) = Lt, v(t) = 2$, $L_0 = 7.5$, $L = 15$. The parameters for method are

$$r_g = 0.0667,\ \rho_1 = 0.0444,\ r_{\psi_1} = 0.0374,\ r_{\psi_2} = 0.0214 = r.$$

Example 3.3. Returning back to the motivational example at the introduction of this study, we have $w_0(t) = w(t) = 96.6629073t$, $v(t) = sup\|F(x^*)^{-1}F(x)\| = 0.7272$ and $v_1(t) = sup\|F(x^*)^{-1}F(x)\| = 0.3411$. Then the parameters are

$$r_g = 0.0096,\ \rho_1 = 0.0090 = r,\ r_{\psi_1} = 0.0091,\ r_{\psi_2} = 0.0199.$$

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