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INDECOMPOSABLE MODULES OVER NAGATA VALUATION DOMAINS

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ABSTRACT. For a discrete valuation ring R, let $\mathrm{fr}(R)$ be the supremum of the ranks of indecomposable finite rank torsion-free R-modules. Then $\mathrm{fr}(R) = 1, 2, 3$, or ∞. A complete list of indecomposables is given if $\mathrm{fr}(R) \leq 3$, in which case R is known to be a Nagata valuation domain.

Let R be a discrete valuation ring with prime p and quotient field Q, and let R^* be the p-adic completion of R with quotient field Q^*. Define $\mathrm{fr}(R) = \sup\{\mathrm{rank}\, X : X$ indecomposable torsion-free R-module of finite rank$\}$. In this paper, we show that $\mathrm{fr}(R) = 1, 2, 3$, or ∞. This resolves a conjecture by P. Va′mos that $\mathrm{fr}(R) = 1, 2$, or ∞.

It is well known that $\mathrm{fr}(R) = \infty$ in case $[Q^* : Q]$ is infinite and $\mathrm{fr}(R) = 1$ if $[Q^* : Q] = 1$. Call R a *Nagata valuation domain* if $2 \leq [Q^* : Q]$ is finite [Z]. In this case $\mathrm{char}\, Q^* = q > 0$; $Q^* = Q(u)$ for some unit u of R^* with $u^n = \lambda$, a unit of R; and $[Q^* : Q]$ is a power of q [V, R]. Examples of Nagata valuation domains are given in [N] and [V].

Zanardo [Z] shows that if $[Q^* : Q] = 2$, then $\mathrm{fr}(R) = 2$. Moreover, in this case there are, up to isomorphism, only three indecomposables: R, Q, and R^*. His example showing that $\mathrm{fr}(R) \geq 6$ for $[Q^* : Q] = 3$ is in error.

Henceforth, assume $[Q^* : Q] = n \geq 2$. Then Q^* is a *splitting field* for each finite rank R-module X; i.e., $R^* \otimes X$ is the direct sum of a free R^*-module and a Q^*-module. Thus, quasi-homomorphism results of Lady [L1, L3] for modules over a discrete valuation ring with a fixed splitting field are applicable.

As summarized in [L1, Theorem 1] and proved in [L3, Theorem 5.1], for:

$n = 2$, there are three strongly indecomposables: R, Q, and R^*;

$n = 3$, there are five strongly indecomposables: R, Q, R^*, C^-R (p-rank 1, rank 2), and C^+R^* (p-rank 2, rank 3);

$n = 4$, there are strongly indecomposables of arbitrarily large finite rank, but all strongly indecomposables are potentially describable (tame representation type);

$n \geq 5$, there are strongly indecomposables of arbitrarily large finite rank, but a description is generally regarded as hopeless (wild representation type).

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Since strongly indecomposables are indecomposable, Lady's theorem yields $\mathrm{fr}(R) = \infty$ for $n \geq 4$. We give an alternate proof by easily constructed examples in §3. This is sufficient for our purposes and avoids the deep arguments used in [L3].

The only unresolved case is $n = 3$. In this case, we show that $\mathrm{fr}(R) = 3$ and give a complete list of indecomposables up to isomorphism: R, Q, R^*, C^-R, and infinitely many of p-rank 2, rank 3 (all quasi-isomorphic to C^+R^*). The strongly indecomposable R-module C^+R^* is the quasi-homomorphism dual of R^* defined in [A1].

1. Preliminaries

The *p-rank* of an R-module X is the R/pR-dimension of X/pX. Fundamental properties of p-rank are given in [A1].

Lemma 1.1 [A1, Proposition 1.3, Lemma 1.5]. *Two finite rank R-modules G and H are quasi-isomorphic if and only if $p\text{-}\mathrm{rank}\, G = p\text{-}\mathrm{rank}\, H$, $\mathrm{rank}\, G = \mathrm{rank}\, H$, and there is a monomorphism $f\colon G \to H$. Moreover, quasi-isomorphism implies isomorphism for modules of p-rank 1.*

2. Indecomposables for $[Q^* : Q] = 3$

As noted in the introduction, $\mathrm{char}\, Q = 3$ and $Q^* = Q(u)$ for some unit u of R^* with $u^3 = \lambda$, a unit of R. This notation is maintained throughout the rest of this section.

Define $A[u]$ to be the pure R-submodule of R^* generated by $\{1, u\}$. Then $A[u] = (Q \oplus Qu) \cap R^*$ is strongly indecomposable with p-rank 1 and rank 2 and, hence, is quasi-isomorphic to C^-R by Lady's theorem. The following lemma is proved in [Z, Proposition 5] using Kurosch matrix-invariant arguments from [A1]. However, it can also be proved directly from the definition of $A[u]$ (a proof is not included).

Lemma 2.1 [Z, Corollary 12, Theorem 8]. *The module $A[u]$ is (strongly) indecomposable. Moreover, if X is an indecomposable R-module of rank ≤ 2, then X is isomorphic to R, Q, or $A[u]$.*

Next let $a, b \in R^* \backslash R$ and define $A[a, b]$ to be the pure R-submodule of $R^* \oplus R^*$ generated by $(1, 0)$, $(0, 1)$, and (a, b). In particular, $QA[a, b] = Q(1, 0) \oplus Q(0, 1) \oplus Q(a, b)$ and $A[a, b] = QA[a, b] \cap (R^* \oplus R^*)$. Up to isomorphism, this definition of $A[a, b]$ coincides with that of [Z]. Then $A[a, b]$ has p-rank 2 and rank 3. A routine argument shows that $A[a, b]$ is (strongly) indecomposable if and only if $\{1, a, b\}$ is a Q-independent set. In this case, $A[a, b]$ is quasi-isomorphic to C^+R^* by Lady's theorem. Moreover, $A[u, u^2]$ is the quasi-homomorphism dual of R^*, noting that R^* has p-rank 1 and rank 3.

Lemma 2.2. *Suppose that (a, b) and (c, d) are R^*-vectors.*

(a) *If $(c, d) = s(a, b)M + P$ for an invertible 2×2 R-matrix M, a Q-vector P, and $0 \neq s \in Q$, then $A[a, b] \approx A[c, d]$.*
(b) $A[u, p^i u^2] \approx A[p^i u, u^2]$.
(c) *If r is a unit of R and $j > i$, then $A[u + p^i r u^2, p^j u^2] \approx A[u, p^j u^2]$.*

Proof. (a) Define an R^*-automorphism ϕ of $R^* \oplus R^*$ by $\phi(x) = xM^{-1}$. Then ϕ induces a homomorphism $A[c, d] \to A[a, b]$ since $(Q \oplus Q)M^{-1}$ is contained in $Q \oplus Q$ and $(c, d)M^{-1} = s(a, b) + PM^{-1}$. In fact, this is an isomorphism since $A[a, b]$ and $\phi(A[c, d])$ are both pure rank-3 submodules of $R^* \oplus R^*$.

(b) Let $A = A[u, p^i u^2]$ and $B = A[p^i u, u^2]$ with $i \geq 1$. There is an R^*-endomorphism ϕ of $R^* \oplus R^*$ defined by

$$\phi(1, 0) = (1, 1) = (1, 0) + (0, 1) \in A,$$
$$\phi(0, 1) = (-u^{-2}, -p^i u^{-1} + p^{2i}) = -\lambda^{-1}(u, p^i u^2) + p^{2i}(0, 1) \in A,$$

recalling that $u^3 = \lambda$. Now ϕ is an automorphism as

$$\begin{pmatrix} 1 & 1 \\ -u^{-2} & -p^i u^{-1} + p^{2i} \end{pmatrix}$$

has determinant $d \equiv -u^{-2} \pmod{pR^*}$, a unit of R^*. Moreover, $\phi(B)$ is contained in A since

$$\begin{aligned}\phi(p^i u, u^2) &= p^i u(1, 1) + u^2(-u^{-2}, -p^i u^{-1} + p^{2i}) = (p^i u - 1, p^{2i} u^2) \\ &= p^i(u, p^i u^2) - (1, 0) \in A.\end{aligned}$$

It follows that $\phi\colon B \to A$ is an isomorphism.

(c) Let $A = A[u, p^j u^2]$ and $B = A[u + p^i r u^2, p^j u^2]$, and assume that either $i \geq 1$ or else $i = 0$ and ru is not congruent to 1 modulo pR^*.

Define an R^*-endomorphism ϕ of $R^* \oplus R^*$ by

$$\begin{aligned}\phi(1, 0) &= (1 - p^i r u, -p^{i+j} r u^2 + p^{2i+j} r^2 \lambda) \\ &= (1, 0) - p^i r(u, p^j u^2) + p^{2i+j} r^2 \lambda(0, 1) \in A, \\ \phi(0, 1) &= (0, 1 - p^{3i} r^3 \lambda) = (1 - p^{3i} r^3 \lambda)(0, 1) \in A.\end{aligned}$$

Then ϕ is an automorphism if $i \geq 1$, since the coefficient determinant $d = (1 - p^i r u)(1 - p^{3i} r^3 \lambda) \equiv 1 \pmod{pR^*}$. If $i = 0$, then $d = (1 - ru)(1 - \lambda r^3)$. Since $\operatorname{char} Q^* = 3$, $1 - \lambda r^3 = 1 - (ru)^3 = (1 - ru)^3$, whence $d = (1 - ru)^4$. Thus, ϕ is an automorphism, as ru is not congruent to 1 mod pR^*.

Now $\phi(B)$ is contained in A since

$$\begin{aligned}&\phi(u + p^i r u^2, p^j u^2) \\ &\quad = (u + p^i r u^2)\phi(1, 0) + p^j u^2 \phi(0, 1) \\ &\quad = (u + p^i r u^2)(1 - p^i r u, -p^{i+j} r u^2 + p^{2i+j} r^2 \lambda) + p^j u^2(0, 1 - p^{3i} r^3 \lambda) \\ &\quad = (u - p^{2i} r^2 \lambda, p^j u^2 - p^{i+j} r \lambda) \\ &\quad = (u, p^j u^2) - p^{2i} r^2 \lambda(1, 0) - p^{i+j} r \lambda(0, 1) \in A,\end{aligned}$$

recalling that $u^3 = \lambda$. As in the proof of (b), $B \approx \phi(B) = A$.

It remains to show that it is sufficient to assume that either $i \geq 1$ or else $i = 0$ and u is not congruent to 1 modulo pR^*. To see this, assume that $i = 0$ and $ru = 1 + ps$ for some $s = s_0 + s_1 u + s_2 u^2 \in R^*$. Then $u + ru^2 = (2 + ps_0)u + ps_1 u^2 + ps_2 \lambda$. Since $ps_2 \lambda \in Q$, it follows from (a) that $B = A[u + ru^2, p^j u^2] \approx A[(2 + ps_0)u + ps_1 u^2, p^j u^2]$. As $\operatorname{char} Q = 3$, $2 + ps_0 = -1 + ps_0$ is a unit of R^*. Thus, $B \approx A[u + ptu^2, p^j u^2]$ for $t = (2 + ps_0)^{-1} s_1$ by (a). If $i' = p\text{-height}(pt) \geq j$, an application of (a) shows that $B \approx A$. Otherwise, $j < i'$ and $i' \geq 1$, as desired.

Theorem 2.3. *If X is an indecomposable R-module of rank 3, then X is isomorphic to R^* or $A[u, p^j u^2]$ for some j.*

Proof. Note that p-rank $X \neq 0$ or 3, as X is reduced with no free summands (see [A1]). If p-rank $X = 1$, then X embeds in its completion which is isomorphic to R^*. Since R^* also has p-rank 1 and rank 3, $X \approx R^*$ by Lemma 1.1.

Now assume that X is indecomposable with p-rank 2 and rank 3. Then $X \approx A[a, b]$ with $(a, b) = (u, u^2)M$ for some 2×2 R-matrix M with $\det M \neq 0$ [Z]. We outline another proof that avoids matrix invariants. Let $Rx \oplus Ry$ be a basic submodule of X and extend to a maximal free submodule $Rx \oplus Ry \oplus Rz$ of X. Then X embeds as a pure submodule of $R^*x \oplus R^*y \approx (R^* \otimes X)/d(R^* \otimes X)$, where $d(R^* \otimes X)$ is the maximal divisible submodule. It follows that $X \approx A[a, b]$, where image $z = ax \oplus by$ for $a, b \in R^*$. Since $Q^* = Q(u) = Q \oplus Qu \oplus Qu^2$, we may write $(a, b) = (u, u^2)M + P$ for some R-matrix M and R-vector P. Apply Lemma 2.2(a) to see that, up to isomorphism, P may be chosen to be 0.

In view of Lemma 2.2(a), the isomorphism class of $A[a, b]$ is preserved by invertible R-column operations on M. In particular, it suffices to assume that M is of the form

$$\begin{pmatrix} p^k & 0 \\ p^r & p^j \end{pmatrix}$$

with $i < j$ and r either 0 or a unit of R. This follows from the observation that if an element in a row has least p-height, then the other entry in its row can be set to 0 using an invertible R-column operation. Moreover, column interchange and multiplication of a column by a unit are invertible R-operations.

We now have $X \approx A[a, b]$ with $(a, b) = (p^k u + p^i r u^2, p^j u^2)$, $j > i$ and r either 0 or a unit of R.

First, assume $k \leq i$. Then $X \approx A[u + p^{i-k} r u^2, p^{j-k} u^2]$ by Lemma 2.2(a). Moreover, $A[u + p^{i-k} r u^2, p^{j-k} u^2] \approx A[u, p^{j-k} u^2]$ via Lemma 2.2(c). Thus, $X \approx A[u, p^{j-k} u^2]$.

Now assume $k > i$. Factor out p^i and apply Lemma 2.2(a) to assume, up to isomorphism, that $[a, b] = [p^{k-i} u + r u^2, p^{j-i} u^2]$. If $r = 0$, then $X \approx A[a, b] \approx A[u, p^t u^2]$ for some t, obtained by factoring out $p^{\min\{k-i, j-i\}}$ and applying Lemma 2.2(b) in the case $k - i > j - i$.

Finally assume that r is a unit. Then $X \approx A[a, b] = A[r u^2 + p^{k-i} u, p^{j-i} u^2] \approx A[u^2 + p^{i'} r' u, p^{j'} u^2]$ for $i' = k - i$, $j' = j - i$, and $r' = r^{-1}$ (Lemma 2.2(a)). Since $(u^2)^2 = u\lambda$, substituting v for u^2 in the latter term and relabeling exponents and units gives $X \approx A[v + p^i r v^2, p^j v]$ for a unit $r = r'/\lambda$ of R. Invertible R-column operations on

$$\begin{pmatrix} 1 & p^j \\ p^i r & 0 \end{pmatrix}$$

reduce to the case that $X \approx A(v + p^i r v^2, p^{i+j} v^2)$. However, $Q^* = Q(u) = Q(v)$ with $v^3 = \lambda^2$, a unit of R. Thus, Lemma 2.2, with u replaced by v, is true. The argument of the first case then shows that $X \approx A[v, p^t v^2]$ for some t. Hence, by Lemma 2.2, $X \approx A[u^2, p^t \lambda u] \approx A[u^2, p^t u] \approx A[p^t u, u^2] \approx A[u, p^t u^2]$, as desired.

For finite rank torsion-free R-modules G and H, define $S_G(H)$ to be the image of the evaluation map $\mathrm{Hom}(G, H)\otimes_R G \to H$. Fundamental properties of $S_G(-)$ are given in [A2, Chapter 5] for torsion-free abelian groups of finite rank.

Proposition 2.4. (a) *If* $A[u, p^j u^2] \approx A[u, p^j u^2]$, *then* $i = j$.

(b) *There are embeddings* $A[u, p^i u^2] \to A[u, p^{i-1}u^2]$ *and* $A[u, p^{i-1}u^2] \to A[u, p^i u^2]$. *In each case the image has index* p.

(c) *If* G *and* H *are indecomposable with* p*-rank* 2 *and rank* 3, *then* $S_G(H) = H$.

Proof. (a) can be proven as in [Z, Proposition 16] for the case $i = 0$, $j = 1$. We outline an alternate proof that avoids matrix invariants. An R-isomorphism $\phi: A = A[u, p^i u^2] \to B = A[u, p^j u^2]$ lifts to an R^*-isomorphism of completions $\phi^*: A^* = R^* \oplus R^* \to B^* = R^* \oplus R^*$. Since $\phi(u, p^i u^2) \in B$ and $\phi^{-1}(u, p^j u^2) \in A$, it follows from a computation of p-heights that $i = j$.

(b) There is a monomorphism $f: A[u, p^{i-1}u^2] \to A[u, p^i u^2]$ induced by an R^*-endomorphism ϕ of $R^* \oplus R^*$ with $\phi(1, 0) = (1, 0)$ and $\phi(0, 1) = (0, p)$. Moreover, there is a monomorphism $f': A[u, p^i u^2] \to A[u, p^{i-1}u^2]$ induced by $\phi'(1, 0) = (p, 0)$ and $\phi'(0, 1) = (0, 1)$. Note that $ff' = p$ and $f'f = p$. Hence, if $H_i = A[u, p^i u^2]$, then pH_i is contained in image f. But p-rank $H_i = 2$ and H_i is not isomorphic to H_{i-1} by (b). It follows that $H_i/\text{image} f \approx R/pR$. Similarly, $H_{i-1}/\text{image} f' \approx R/pR$.

(c) For $i \geq 1$ and for ϕ' and ϕ defined as in the proof of (b), there is $g: A[p^{i-1}u, u^2] \to A[p^i u, u^2]$ induced by ϕ' and $g': A[p^i u, u^2] \to A[p^{i-1}u, u^2]$ induced by ϕ with $gg' = p$ and $g'g = p$. It now follows that if $G_i = A[p^i u, u^2]$, then $G_i/\text{image}\, g \approx R/pR \approx G_{i-1}/\text{image}\, g'$.

In view of Theorem 2.3, it is sufficient to show that

$$f \oplus \delta_i g \delta_{i-1}^{-1}: H_{i-1} \oplus H_{i-1} \to H_i \quad \text{and} \quad f' \oplus \delta_{i-1} g' \delta_i^{-1}: H_i \oplus H_i \to H_{i-1}$$

are onto, for δ_i the isomorphism $G_i = A[p^i u, u^2] \to A[u, p^i u^2] = H_i$ given in Lemma 2.2(b). Assume that $f \oplus \delta_i g \delta_{i-1}^{-1}$ is not onto. Since $H_i/pH_i \approx R/pR \oplus R/pR$ and pH_i is properly contained in both the image of f and the image of $\delta_i g \delta_{i-1}^{-1}$, it follows that image f = image $\delta_i g \delta_{i-1}^{-1}$. Hence, $f\delta_{i-1}(G_{i-1}) = \delta_i g(G_{i-1})$. But this is a contradiction, as can be seen by observing that f is a restriction of ϕ and g is a restriction of ϕ'. The proof that $f' \oplus \delta^{-1} g'$ is onto is analogous.

Lemma 2.5. *Assume that* X *is a finite rank* R*-module with submodule* K *such that* $A = X/K \approx A[u]$ *or* $A[u, p^i u^2]$ *for some* $i \geq 0$. *If* $S_A(X) = X$, *then* K *is a summand of* X.

Proof. It suffices to prove that $\mathrm{End}(A[u])$ and $\mathrm{End}(A[u, p^i u^2])$ are commutative. This is a consequence of [AR2, Theorems 5.6 and 5.8] as the abelian group proof therein carries over to modules over discrete valuation rings. Recall that $A[u]$ has p-rank 1 and is reduced. Hence its completion is isomorphic to R^*. In particular, $\mathrm{End}(A[u])$ is isomorphic to a subring of R^*. Moreover, $A[u, p^i u^2]$ is quasi-isomorphic to $A[u, u^2]$ which is the dual of R^*, as noted above. Thus, $Q\,\mathrm{End}(A[u, p^i u^2]) = Q\,\mathrm{End}(A[u, u^2]) = Q\,\mathrm{End}(R^*) = QR^*$. It follows that $\mathrm{End}(A[u, p^i u^2])$ is commutative.

Theorem 2.6. *If* X *is a finite rank* R*-module, then* X *is the direct sum of modules of rank* ≤ 3.

Proof. Choose pure strongly indecomposable submodules X_i of X with $X/(X_1 \oplus \cdots \oplus X_m)$ p^k-bounded. Each X_j is isomorphic to R, R^*, Q, $A[u]$, or $A[u, p^r u^2]$ for some $r \geq 0$ by Lady's theorem, Lemma 2.1, and Theorem 2.3. If X_i is isomorphic to the pure injective module R^* or Q, then X_i is a summand of X. Moreover, if $X_i \approx R$, then X has a cyclic summand, since X modulo the pure submodule generated by $\{X_j : j \neq i\}$ is isomorphic to R.

We may now assume that each X_j is isomorphic to $A[u]$ of some $A[u, p^i u^2]$. By induction on rank X and $|X/(X_1 \oplus \cdots \oplus X_m)|$, it suffices to further assume that $X/(X_1 \oplus \cdots \oplus X_m) \approx R/pR$ and prove that X has a summand of rank ≤ 3. Write $X = (X_1 \oplus \cdots \oplus X_m) + R(x_1 + \cdots + x_m)/p$. Let K be the pure submodule of X generated by $\{X_j : j \neq 1\}$ and $A = X/K$, quasi-isomorphic to X_1. Then A has p-rank 1, rank 2 or p-rank 2, rank 3 and has no free summands, being quasi-isomorphic to a strongly indecomposable X_1. Hence, A is indecomposable [A1, Proposition 4.1].

It is now sufficient to prove that $S_A(X) = X$, in which case X has a summand isomorphic to A of rank ≤ 3 by Lemma 2.5. There is some $Y = X_i$, say $i = 1$, with $S_Y(X_j) = X_j$ for each j. This follows from the natural exact sequence $A[u, p^r u^2] \to A[u] \to 0$, Proposition 2.4(c), and the assumption that each $X_j \approx A[u]$ or $A[u, p^r u^2]$. Moreover, for $A = X/K \approx X_1 + R(x_1/p)$, $S_A(X_j) = X_j$ for each j, again by Proposition 2.4(c) or Lemma 2.1 and the fact that A is indecomposable.

Write $X_i' = pX_i + Rx_i$, an indecomposable module for the same reason that A is. For each i, there is $y_i \in A$, a unit r_i of R, and $f_i : A \to X_i'$ with $f_i(y_i) \equiv r_i x_i \pmod{pX_i}$. This is because if $X_i' = S_A(X_i')$ is contained in pX_i, then $x_i \in pX_i$ and letting $r_i = 1$ will do. Note, for future reference, that we may as well assume that $f_i(y_i) \equiv x_i \pmod{pX_i}$. To see this, choose a unit s_i of R with $1 = r_i s_i + pt_i$, $t_i \in R$. Then $s_i f_i(y_i) \equiv x_i \pmod{pX_i}$, as desired.

We begin with the case $m = 2$ and find $x \in A$ and $g_i : A \to X_i'$ with $g_1(x) \equiv x_1 \pmod{pX_1}$ and $g_2(x) \equiv x_2 \pmod{pX_2}$. If either $f_1(y_2) \equiv s_1 x_1 \pmod{pX_1}$ or $f_2(y_1) \equiv s_2 x_2 \pmod{pX_2}$ for units s_1, s_2 of R, then let $x = y_2$, respectively, $x = y_1$. Otherwise, $f_1(y_2) \in pX_1$ and $f_2(y_1) \in pX_2$. In this case, let $x = y_1 + y_2$. In any case, there are units t_i of R with $f_i(x) \equiv t_i x_i \pmod{pX_i}$. As above, choose g_i to be an appropriate R-unit multiple of f_i.

Next let $A' = pA + Rx$, an indecomposable submodule of A for the same reason that A is indecomposable. Restriction induces a well-defined $\phi = g_1 \oplus g_2 : A' \to pX = pX_1 \oplus pX_2 + R(x_1 \oplus x_2)$ with $\phi(x) \in (x_1 \oplus x_2) + pX_1 \oplus pX_2$. Since $S_A(A') = A'$ by Proposition 2.4(c) and $S_A(X_i) = X_i$ for each i, it follows that $S_A(pX) = pX$ and so $S_A(X) = X$. This completes the proof for $m = 2$.

We illustrate an induction argument with $m = 3$. From the $m = 2$ case $S_A(X_{12'}) = X_{12'}$ for $X_{12'} = pX_1 \oplus pX_2 + R(x_1 \oplus x_2)$. Consequently, there is $x \in A$ and $g_{12} : A \to X_{12'}$ with $g_{12}(x) \equiv (x_1 \oplus x_2) \pmod{pX_1 \oplus pX_2}$. Otherwise, $x_1 \oplus x_2 \in S_A(X_{12'}) = X_{12'}$ is contained in $pX_1 \oplus pX_2$. Recall that there is $y_3 = A$ and $f_3 : A \to X_i'$ with $f_3(y_3) \equiv x_3 \pmod{pX_3}$. If $f_3(x) \equiv s_3 x_3 \pmod{pX_3}$ for some unit s_3 of R, then let $a = x$. If $g_{12}(y_3) \equiv s(x_1 \oplus x_2) \pmod{pX_1 \oplus pX_2}$ for some unit s of R, then let $a = y_3$. Otherwise, let $a = x + y_3$. It follows that $a \in A$ with $g_{12}(a) \equiv r(x_1 \oplus x_2) \pmod{pX_1 \oplus pX_2}$ and $f_3(a) \equiv r_3 x_3 \pmod{pX_3}$ for units r and r_3 of R. As in the $m = 2$ case, we may assume that $r = r_3 = 1$ and construct $\phi : A' = pA + Ra \to pX$ with $\phi(a) \equiv (x_1 \oplus x_2 \oplus x_3)$

(mod $pX_1 \oplus pX_2 \oplus pX_3$). It follows, as above, that $S_A(X) = X$.

The proof is concluded by an induction on m; the argument for passing from m to $m+1$ is analogous to that of the preceding paragraph.

3. Indecomposables for $[Q^*: Q] = n \geq 4$

The following are examples showing that $\mathrm{fr}(R) = \infty$ for $[Q^*: Q] = n \geq 4$. The detailed computations needed to verify that the modules are actually strongly indecomposable are omitted.

Example 3.1. Assume $n \geq 4$. Given $m \geq 2$, there is a strongly indecomposable R-module with p-rank m and rank $2m$.

Proof. Case I: $\mathrm{char}\, Q^* \geq 5$. Since Q^* is purely inseparable over Q, there is $u \in R^*$ such that $1, u, u^2, u^3$, and u^4 are Q-independent. Let M be an $m \times m$ simple Jordan block R-matrix, i.e., the diagonal elements of M are a fixed unit λ of R, the super diagonal elements are all 1's, and the remaining entries are 0. Define $X = A[\Gamma] = (R^*)^m \cap (Q^m \oplus Q^m\Gamma)$, where $\Gamma = uM + u^2 I_m$, and R-module with p-rank m and rank $2m$.

It can be shown that $\mathrm{End}(X)$ is represented by the set of $2m \times 2m$ R-matrices $\left(\begin{smallmatrix}\Pi & 0\\ 0 & \Pi\end{smallmatrix}\right)$ with $\Pi M = M\Pi$. This can be seen by equating Q-coefficients $1, u, u^2, u^3$, and u^4. Consequently, $Q\,\mathrm{End}(X) \approx Q[M] \approx Q[x]/\langle (x-\lambda)^m \rangle$ is a ring with no nontrivial idempotents, whence X is strongly indecomposable.

Case II: $\mathrm{char}\, Q^* = 3$. If there is $u \in R^*$ with $1, u, u^2, u^3$, and u^4 Q-independent, the construction of Case I suffices. Otherwise, there are $u, v \in R^*$ with $u^3, v^3 \in R$ and $1, u, v, u^2v, v^2u, u^2v^2, u^2$, and v^2 are Q-independent. Choose M as in Case I, and define $X = A[\Gamma]$ for $\Gamma = uM + vI$. An argument similar to that of Case I shows that $Q\,\mathrm{End}(X) \approx Q[X]/\langle (x-\lambda)^m \rangle$ and X is strongly indecomposable.

Case III: $\mathrm{char}\, Q^* = 2$. We are left with two possibilities not covered in Case I: there is $u \in R^*$ with $1, u, u^2$, and u^3 Q-independent and $u^4 \in R$, or there is $u, v \in R^*$ with $u^2, v^2 \in R$ and $\{1, u, v, uv\}$ Q-independent. In the first case, define $X = A[\Gamma]$ for $\Gamma = uM + u^3 I$. An argument similar to that of Case I shows that X is strongly indecomposable.

For the second case, define $X = A[\Gamma]$ for $\Gamma = uM + vI$. Once again, it can be shown that $Q\,\mathrm{End}(X)$ has no nontrivial idempotents, but the argument is slightly more complicated than the previous cases.

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