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Abstract

I. N. Bernstein, I. M. Gelfand and V. A. Ponomarev have recently shown that the bijection, first observed by P. Gabriel, between the indecomposable representations of graphs ("quivers") with a positive definite quadratic form and the positive roots of this form can be proved directly. Appropriate functors produce all indecomposable representations from the simple ones in the same way as the canonical generators of the Weyl group produce all positive roots from the simple ones.

This method is extended in two directions. In order to deal with all Dynkin diagrams rather than with those having single edges only, we consider valued graphs ("species"). In addition, we consider valued graphs with positive semi-definite quadratic form, i.e. extended Dynkin diagrams.

The main result of the paper describes all indecomposable representations up to the homogeneous ones, of a valued graph with positive semi-definite quadratic form. These indecomposable representations are of two types: those of discrete dimension type, and those of continuous dimension type. The indecomposable representations of discrete dimension type are determined by their dimension vectors: these are precisely the positive roots of the corresponding quadratic form. The continuous dimension vectors are the positive integral vectors in the radical space of the quadratic form and are thus the positive multiples of a fixed dimension vector. The full subcategory of all images of maps between direct sums of indecomposable representations of continuous dimension type is an abelian exact subcategory, which is called the category of all regular representations. It is the product of two categories U and H, where H is the largest direct factor containing only representations of continuous dimension type. The representations in H are called homogeneous and their behaviour depends very strongly on the particular modulation of the valued graph. One can reduce the study of the category H to the study of the homogeneous representations of a simpler valued graph, namely of a bimodule. On the other hand, the structure of the category U can be determined completely: it is the direct product of at most three indecomposable categories, each of which has only a finite number of simple objects, is serial, and has global dimension 1.

The indecomposable representations which are non-regular can be described in the following way: there are two endofunctors C^+ and C^- on the category of all representations, called the Coxeter functors, such that the list of all representations of the form $C^{-r}\mathbf{P}$ and $C^{+r}\mathbf{Q}$, where $\mathbf{P}$ is an indecomposable projective representation and $\mathbf{Q}$ is an indecomposable injective representation, is a complete list of all non-regular indecomposable representations. Also there is a numerical invariant, called the defect, which measures the behaviour of the indecomposable representations, and depends only on the dimension type. The defect of a representation is negative, zero, or positive, if and only if it is of the form $C^{-r}\mathbf{P}$, regular, or of the form $C^{+r}\mathbf{Q}$, respectively.

The paper concludes with tables of all valued graphs with positive semi-definite quadratic form. The tables provide, in condensed form, most of the information which is available about the representation theory of these valued graphs.

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Introduction

In a recent paper, I. N. Bernstein, I. M. Gelfand and V. A. Ponomarev [2] have shown that the bijection between the indecomposable representations of graphs ("quivers") with a positive definite quadratic form and the positive roots of this form observed by P. Gabriel [8] earlier, can be proved directly. They have introduced certain functors which allow to construct all the indecomposable representations from the one-dimensional ones in the same way as the canonic generators of the Weyl group produce all positive roots from the basic ones.

In this paper, we are going to extend this result in two directions. On one hand, we shall consider valued graphs (and therefore "species" of [9], [4]) instead of graphs in order to deal with all Dynkin diagrams rather than with those having single edges only; in this way, we recover previous results of ours [4]. And, on the other hand, we shall consider also valued graphs with positive semidefinite form (i.e. the extended Dynkin diagrams) and describe, up to the homogeneous ones, all their indecomposable representations. In the case of extended Dynkin diagrams with single edges, this yields the previous results of L. A. Nazarova [15] and P. Donovan and M. R. Freislich [7].

A *valued graph* $(\Gamma, \mathbf{d})$ is a finite set Γ (of vertices) together with non-negative integers d_{ij} for all pairs $i, j \in \Gamma$ such that $d_{ii} = 0$ and subject to the condition that there exist (non-zero) natural numbers f_i satisfying[1]

$$d_{ij}f_j = d_{ji}f_i \quad \text{for all} \quad i, j \in \Gamma.$$

In addition, we shall always assume that the valued graph $(\Gamma, \mathbf{d})$ is *connected* in the sense that, for every $k, l \in \Gamma$, there is a sequence $k, \ldots, i, j, \ldots, l$ of vertices of Γ such that $d_{ij} \neq 0$ for each pair of subsequent vertices i, j. Note that d_{ij} may differ from d_{ji}, but that $d_{ij} \neq 0$ if and only if $d_{ji} \neq 0$; let us call such pairs $\{i, j\}$ the edges of $(\Gamma, \mathbf{d})$, and the vertices i, j neighbours. In notation, we shall use the symbol $\underset{i}{\cdot} \xrightarrow{(d_{ij},\, d_{ji})} \underset{j}{\cdot}$ for the edges of $(\Gamma, \mathbf{d})$; if $d_{ij} = 1 = d_{ji}$, we write simply $\underset{i}{\cdot} — \underset{j}{\cdot}$. Let us remark that one can prove easily that every tree (graph without circuits) can be turned into a valued graph by choosing pairs (d_{ij}, d_{ji}) of arbitrary natural numbers ($\neq 0$) for all edges of that tree.

An *orientation* Ω of a valued graph $(\Gamma, \mathbf{d})$ is given by prescribing, for each edge $\{i, j\}$ of $(\Gamma, \mathbf{d})$, an order (indicated by an arrow: $\underset{i}{\cdot} \longrightarrow \underset{j}{\cdot}$). Given an orientation Ω and a vertex $k \in \Gamma$, define a new orientation $s_k\Omega$ of $(\Gamma, \mathbf{d})$ by reversing the direction of arrows along all edges containing k. A vertex $k \in \Gamma$ is said to be a *sink* (or a *source*) with respect to Ω if $\underset{k}{\cdot} \longleftarrow \underset{i}{\cdot}$ (or $\underset{k}{\cdot} \longrightarrow \underset{i}{\cdot}$) for all neighbours $i \in \Gamma$ of k. And, an orientation Ω of $(\Gamma, \mathbf{d})$ is said to be *admissible* if there is an ordering $k_1, k_2, \ldots, k_n$ of Γ such that each vertex k_t is a sink

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[1] Observe that there is a one-to-one correspondence between valued graphs and symmetrizable Cartan matrices (see [12]).

with respect to the orientation $s_{k_{t-1}} \cdots s_{k_2} s_{k_1} \Omega$ for all $1 \leqslant t \leqslant n$; such an ordering is called an admissible ordering for Ω. It is easy to see that an orientation Ω of the valued graph $(\Gamma, \mathbf{d})$ is admissible if and only if there is no circuit with orientation $\underset{i_1}{\cdot} \longrightarrow \underset{i_2}{\cdot} \longrightarrow \underset{i_3}{\cdot} \cdots \underset{i_{t-1}}{\cdot} \longrightarrow \underset{i_t = i_1}{\cdot}$; therefore, in particular, every orientation of a tree is admissible.

A *modulation* $\mathfrak{M}$ of a valued graph $(\Gamma, \mathbf{d})$ is a set of division rings F_i, $i \in \Gamma$, together with an F_i-F_j-bimodule ${}_iM_j$ and an F_j-F_i-bimodule ${}_jM_i$ for all edges $\{i, j\}$ of $(\Gamma, \mathbf{d})$ such that

(i) there are F_j-F_i-bimodule isomorphisms

$$ {}_jM_i \approx \operatorname{Hom}_{F_i}({}_iM_j, F_i) \approx \operatorname{Hom}_{F_j}({}_iM_j, F_j) $$

and

(ii) $\dim ({}_iM_j)_{F_j} = d_{ij}$.

A *realization* $(\mathfrak{M}, \Omega)$ of a valued graph $(\Gamma, \mathbf{d})$ is a modulation $\mathfrak{M}$ of $(\Gamma, \mathbf{d})$ together with an admissible orientation Ω. A *representation* $\mathbf{X} = (X_i, {}_j\varphi_i)$ of a realization $(\mathfrak{M}, \Omega)$ of $(\Gamma, \mathbf{d})$ is a set of finite-dimensional right F_i-spaces X_i, $i \in \Gamma$, together with F_j-linear mappings

$$ {}_j\varphi_i : X_i \otimes_{F_i} {}_iM_j \longrightarrow X_j $$

for all oriented edges[2] $\underset{i}{\cdot} \longrightarrow \underset{j}{\cdot}$. A morphism $\boldsymbol{\alpha} : \mathbf{X} \longrightarrow \mathbf{X}'$ from a representation $\mathbf{X} = (X_i, {}_j\varphi_i)$ to $\mathbf{X}' = (X'_i, {}_j\varphi'_i)$ is defined as a set $\boldsymbol{\alpha} = (\alpha_i)$ of F_i-linear mappings $\alpha_i : X_i \longrightarrow X'_i$, $i \in \Gamma$, satisfying

$$ {}_j\varphi'_i(\alpha_i \otimes 1) = \alpha_j\, {}_j\varphi_i \quad \text{for all edges } \underset{i}{\cdot} \longrightarrow \underset{j}{\cdot}. $$

One can see easily that the representations of $(\mathfrak{M}, \Omega)$ form an abelian category which we shall always denote by $L(\mathfrak{M}, \Omega)$.

Given a valued graph $(\Gamma, \mathbf{d})$, denote by $\mathbf{Q}^\Gamma$ the vector space of all $\mathbf{x} = (x_i)_{i \in \Gamma}$ over the rational numbers. In particular, for each $k \in \Gamma$, $\mathbf{k} \in \mathbf{Q}^\Gamma$ denotes the vector with $x_k = 1$ and $x_i = 0$ otherwise. Also, for each $k \in \Gamma$, define the linear transformation $s_k : \mathbf{Q}^\Gamma \longrightarrow \mathbf{Q}^\Gamma$ by $s_k\mathbf{x} = \mathbf{y}$, where $y_i = x_i$ for $i \neq k$ and

$$ y_k = -x_k + \sum_{i \in \Gamma} d_{ik} x_i . $$

The symbol $W = W_\Gamma$ will always denote the *Weyl group*, i.e. the group of all linear transformations of $\mathbf{Q}^\Gamma$ generated by the reflexions s_k, $k \in \Gamma$. A vector $\mathbf{x} \in \mathbf{Q}^\Gamma$ satisfying $w\mathbf{x} = \mathbf{x}$ for all $w \in W$ will be called *stable*. A vector $\mathbf{x} \in \mathbf{Q}^\Gamma$ is called a *root* (of $(\Gamma, \mathbf{d})$) if there exists $k \in \Gamma$ and $w \in W$ such that $\mathbf{x} = w\mathbf{k}$. A root $\mathbf{x}$ is said to be *positive* (or negative) if $x_i \geqslant 0$ (or $x_i \leqslant 0$) for all $i \in \Gamma$.

Given a representation $\mathbf{X} = (X_i, {}_j\varphi_i)$ of a realization $(\mathfrak{M}, \Omega)$, we may define the mapping

$$ \mathbf{dim} : L(\mathfrak{M}, \Omega) \longrightarrow \mathbf{Q}^\Gamma $$

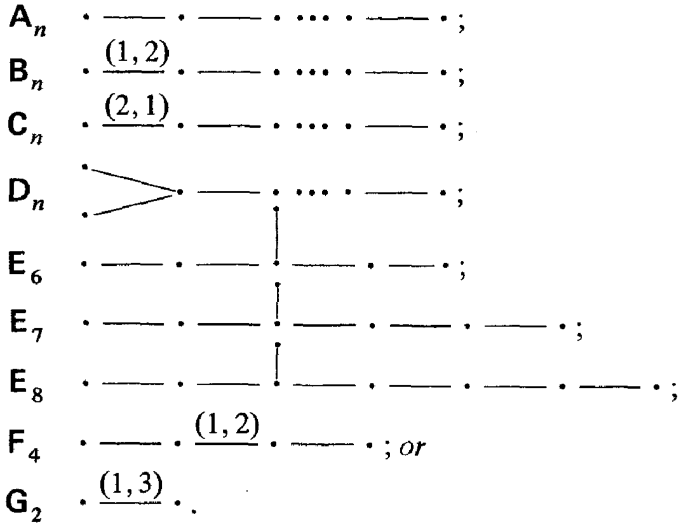
by $\mathbf{dim}\, \mathbf{X} = (x_i)$, where $x_i = \dim (X_i)_{F_i}$ for all $i \in \Gamma$. The vector $\mathbf{dim}\, \mathbf{X}$ is called the *dimension type* of the representation $\mathbf{X}$.

The main result of this paper is the following

THEOREM. *Let* $(\Gamma, \mathbf{d})$ *be a valued graph. Let* $(\mathfrak{M}, \Omega)$ *be a realization of* $(\Gamma, \mathbf{d})$.

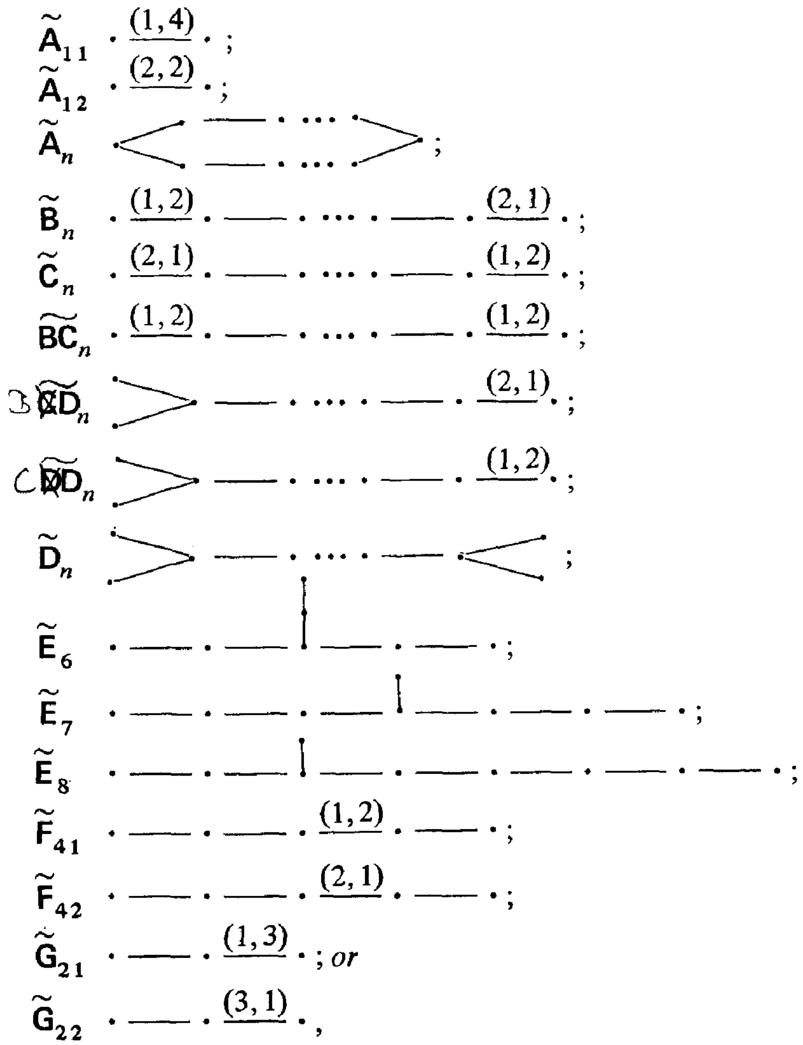
[2] We shall show later that ${}_j\varphi_i$ corresponds (bijectively) to an F_i-linear mapping ${}_j\bar{\varphi}_i : X_i \longrightarrow X_j \otimes_{F_j} {}_jM_i$, and thus, our definition of a representation coincides with that of P. Gabriel in [9]. Of course, realizations are called species in [9].

(a) *Then* $L(\mathfrak{M}, \Omega)$ *is of finite type if and only if* $(\Gamma, \mathbf{d})$ *is a Dynkin diagram, i. e. a valued graph of one of the forms*



Moreover, the mapping $\mathbf{dim}: L(\mathfrak{M}, \Omega) \longrightarrow \mathbf{Q}^{\Gamma}$ *induces a bijection between the isomorphism classes of indecomposable representations of* $(\mathfrak{M}, \Omega)$ *and the positive roots of* $(\Gamma, \mathbf{d})$.

(b) *If* $(\Gamma, \mathbf{d})$ *is an extended Dynkin diagram, i. e. a valued graph of one of the forms*



then the category $L(\mathfrak{M}, \Omega)$ *has two kinds of indecomposable representations: those of discrete dimension types and those of continuous dimension types. The mapping* **dim**: $L(\mathfrak{M}, \Omega) \longrightarrow \mathbf{Q}^\Gamma$ *induces a bijection between the isomorphism classes of indecomposable representations of* $(\mathfrak{M}, \Omega)$ *of discrete dimension types and the positive roots of* $(\Gamma, \mathbf{d})$. *The continuous dimension types are the positive integral multiples of the least stable positive integral vector of* $\mathbf{Q}^\Gamma$. *Moreover, the indecomposable representations of continuous dimension types can be derived from the indecomposable representation of continuous dimension type of a suitable realization of the graph* $\widetilde{\mathbf{A}}_{11}$ *or* $\widetilde{\mathbf{A}}_{12}$.

In fact, in the case of an extended Dynkin diagram, we can give a more detailed description of the category $L(\mathfrak{M}, \Omega)$ as follows:

The full subcategory of all direct sums of indecomposable representations of continuour dimension type is closed under images, kernels and cokernels only for diagrams $\widetilde{\mathbf{A}}_{11}$ and $\widetilde{\mathbf{A}}_{12}$. However, the full subcategory of all images of maps between direct sums of indecomposable representations of continuous dimension type is an abelian exact subcategory of $L(\mathfrak{M}, \Omega)$ which will be called the category $R(\mathfrak{M}, \Omega)$ of *regular* representations of $(\mathfrak{M}, \Omega)$. The largest direct factor of $R(\mathfrak{M}, \Omega)$ containing only representations of continuous dimension type will be called the category $H = H(\mathfrak{M}, \Omega)$ of *homogeneous* representations. The category U satisfying $R(\mathfrak{M}, \Omega) = H \times U$ can be described completely: It is an abelian category with a finite number of simple objects, and every indecomposable representation belonging to U is serial (i.e. has a unique composition series in U). In fact, U is the direct product of at most three indecomposable categories, and each of these has global dimension one. Furthermore, given $(\mathfrak{M}, \Omega)$, there exists a realization $(\mathfrak{M}', \Omega')$ of either $\widetilde{\mathbf{A}}_{11}$ or $\widetilde{\mathbf{A}}_{12}$ which determines all indecomposable representations of continuous dimension type of $L(\mathfrak{M}, \Omega)$ in the following sense: There exists a full exact embedding $T: L(\mathfrak{M}', \Omega') \longrightarrow L(\mathfrak{M}, \Omega)$ and (at most three) representations **X** in $L(\mathfrak{M}', \Omega')$ such that the full subcategory of $L(\mathfrak{M}', \Omega')$ of all regular representations without subobjects of the form **X** is equivalent to $H(\mathfrak{M}, \Omega)$.

The indecomposable representations of $(\mathfrak{M}, \Omega)$ which are non-regular can be described in the following way: There are two endofunctors C^+ and C^- on the category $L(\mathfrak{M}, \Omega)$ called the *Coxeter functors,* having the property that the list of all representations $C^{-m}\mathbf{P}$ and $C^{+m}\mathbf{Q}$ where $m \geqslant 0$, **P** is an indecomposable projective representation and **Q** is indecomposable injective representation, is just a complete list of all non-regular indecomposable representations. The dimension types of these representations correspond to $c^{-m}(\mathbf{dim}\,\mathbf{P})$ and $c^m(\mathbf{dim}\,\mathbf{Q})$, where c is the corresponding Coxeter transformation. Also, there is a numerical invariant, called the *defect,* characterizing the behaviour of indecomposable representations **X** in $L(\mathfrak{M}, \Omega)$: the defect of X is negative, or zero, or positive if and only if **X** is of the form $C^{-m}\mathbf{P}$, or **X** is regular, or **X** is of the form $C^{+m}\mathbf{Q}$, respectively.

Let us observe that a change in orientation of a valued graph results in a change of the indecomposable projective and injective representations, as well as the Coxeter functors. Thus, also the dimension types of the regular representations depend on the orientation. However, the categories $R(\mathfrak{M}, \Omega)$ and $R(\mathfrak{M}, \Omega')$ with an arbitrary modulation $\mathfrak{M}$ and orientations Ω

and Ω', are equivalent except in the case of the diagram $\widetilde{\mathbf{A}}_n$; in fact, the equivalences are given by functors similar to the Coxeter functors. For the diagram $\widetilde{\mathbf{A}}_n$, the category $\mathcal{R}(\widetilde{\mathbf{A}}_n, \Omega)$ is determined by the number of arrows pointing in one direction in the orientation Ω. Thus, in summary, we may consider in the tables of Chapter 6 a fixed orientation.

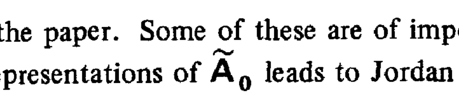
The results on the representations of valued graphs can be translated to the theory of representations of finite-dimensional associative algebras over a field or, more generally, to certain classes of artinian rings. Indeed, given an artinian ring R, we obtain a valued graph $\Gamma_R = (\Gamma, \mathbf{d})_R$ of R as follows: Assuming, without loss of generality, that R is basic, we have $R/\mathrm{Rad}\, R = \Pi_{1 \leqslant i \leqslant n} F_i$ and $\mathrm{Rad}\, R/(\mathrm{Rad}\, R)^2 = \Pi_{1 \leqslant i,j \leqslant n}\, {}_iN_j$ with uniquely determined division rings F_i and F_i-F_j-bimodules ${}_iN_j$; let $\{1, 2, \ldots, n\}$ be the set of vertices of the graph Γ_R and let, for each ${}_iN_j \neq 0$, $\underset{i}{\cdot} \xrightarrow{(d', d'')} \underset{j}{\cdot}$ with $d' = \dim({}_iN_j)_{F_j}$ and $d'' = \dim_{F_i}({}_iN_j)$ be a "valued" edge $\{i, j\}$. Now, if for each pair $\{i, j\}$, ${}_iN_j = 0$ or ${}_jN_i = 0$, this graph is a valued graph and $\mathfrak{M} = \{F_i, {}_iN_j\}_{1 \leqslant i,j \leqslant n}$ is a modulation of this graph provided that there exist numbers f_i with $d_{ij} f_j = d_{ji} f_i$ and that the dualization conditions $\mathrm{Hom}_{F_i}({}_iN_j, F_i) \approx \mathrm{Hom}_{F_j}({}_iN_j, F_j)$ hold. In particular, these are satisfied in the case that R is a finitely generated algebra over a central field K; then, also the numbers f_i can be interpreted as the indices $[F_i : K]$. Define the orientation Ω of Γ_R by $\underset{i}{\cdot} \longrightarrow \underset{j}{\cdot}$ if ${}_iN_j \neq 0$. In this way, one can assign to many artinian rings a modulation of a valued graph and an orientation. Conversely, given a modulation $\mathfrak{M}$ of a valued graph and an orientation Ω, the category $L(\mathfrak{M}, \Omega)$ is equivalent to the category of all right modules over the tensor ring $R = R_{(\mathfrak{M}, \Omega)}$ defined as follows: $R = \bigoplus_{t \geqslant 0} N^{(t)}$, where $S = N^{(0)} = \Pi_{1 \leqslant i \leqslant n} F_i$, $N^{(1)} = \Pi_{1 \leqslant i,j \leqslant n}\, {}_iN_j$, and $N^{(t)} = N^{(t-1)} \bigotimes_S N^{(1)}$ for $t \geqslant 2$, with the component-wise addition and the multiplication induced by taking tensor products. If Ω is, in addition, an admissible orientation, then $R_{(\mathfrak{M}, \Omega)}$ is always an artinian hereditary ring. From here, one can deduce the fact that a hereditary finite-dimensional K-algebra R if of finite representation type if and only if $(\Gamma, \mathbf{d})_R$ is a disjoint union of Dynkin diagrams. In fact, through the above translation, one can obtain in this case a complete description of the category of R-modules. In a similar way, a complete description can be also given for the category of all modules over a finite-dimensional K-algebra R with $(\mathrm{Rad}\, R)^2 = 0$ using the so-called separated diagram of R. For a more detailed account on these questions, we refer to [9] and [4].

The definition of a valued graph excludes graphs with loops and multiple edges. Also, we consider only orientations which are admissible. In particular, the graphs

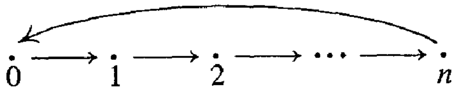
$\widetilde{\mathbf{A}}_0$;

$\widetilde{\mathbf{A}}_1 \quad \cdot = \cdot\,;$

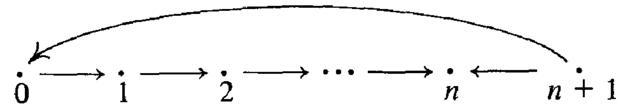
and $\widetilde{\mathbf{A}}_n$ $(n \geqslant 1)$ with the cyclic orientation



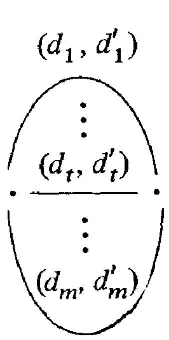
do not appear explicitly in the paper. Some of these are of importance in applications; indeed, the classification of representations of $\widetilde{\mathbf{A}}_0$ leads to Jordan normal forms of square matrices and the classification of representations of $\cdot \rightrightarrows \cdot$ leads to the Kronecker's classification of matrix pairs. However, it is easy to see that the representations of $\widetilde{\mathsf{A}}_n$ $(n \geq 0)$ with respect to the orientation



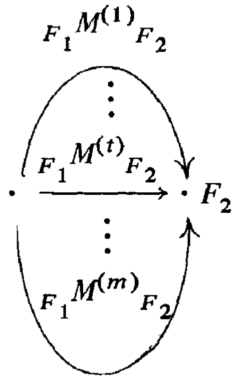
can be identified with the representations $\mathbf{X}$ of $\widetilde{\mathsf{A}}_{n+1}$ with respect to the orientation



such that the mapping $X_n \leftarrow X_{n+1}$ is the identity. Also, the category of all representations of the graph



with respect to the realization F_1



is precisely the category of all representations of the valued graph

$$\cdot \xrightarrow{(\sum_t d_t,\ \sum_t d_t')} \cdot \text{ with respect to the realization } F_1 \xrightarrow{\bigoplus_t {}_{F_1}M^{(t)}{}_{F_2}} F_2.$$

The above theorem improves the previous results of P. Gabriel [8, 9], M. M. Kleiner, L. A. Nazarova and A. V. Roiter [16, 11], the authors [4], W. Müller [13] and I. N. Bernstein, I. M. Gelfand, and V. A. Ponomarev [2] in the part (a) and those of L. A. Nazarova [14, 15], I. M. Gelfand and V. A. Ponomarev [10] and P. Donovan and M. R. Freislich [7] in the part (b). It has been announced at the "Workshop on Indecomposable Representations", Universität Bonn, November 1973 and a summary appeared in [5]. A preliminary version of this paper appeared as Carleton Mathematical Lecture Notes [6]; a major change is the addition of the fifth chapter. The idea of describing the category $\mathcal{R}(\mathfrak{M}, \Omega)$ by finding its simple objects and their extensions is due to P. Gabriel [unpublished]. In fact, he used this method to present the results of [10] (i.e. the representations of the extended Dynkin diagram $\widetilde{\mathsf{D}}_4$) in his course of lectures at Carleton University, Summer 1972. The authors wish to express their gratitude to him for introducing them to the subject.

The first chapter of the paper deals with the theory of valued graphs and provides a listing of all positive roots of valued graphs with a positive (definite or semidefinite) quadratic form by means of a given Coxeter transformation. In particular, in the semidefinite case, a general definition of the defect of a vector in $\mathbf{Q}^\Gamma$ is given. As counterpart to the Coxeter transformations, the Coxeter functors are defined and studied in the second chapter. The results of these chapters furnish, in particular, the arguments for the proof of part (a) and the description of all indecomposable representations of non-zero defect in part (b) of the Theorem. The third chapter deals with the theory of indecomposable representations of zero defect which is then applied to determine all simple representations of zero defect in the following fourth chapter and to advance the proof of part (b) of the Theorem. Moreover, the results of Chapters 3 and 4, provide a further refinement of the part (b) which is stated in Theorem 3.5. The description of the homogeneous representations in terms of representations of the graphs $\widetilde{\mathbf{A}}_{11}$ and $\widetilde{\mathbf{A}}_{12}$ is given in Chapter 5. And, finally, the results on extended Dynkin diagrams are tabulated in Chapter 6.

1. Valued Graphs: Coxeter Transformations, Defect and Listing of Roots

Let $(\Gamma, \mathbf{d})$ be a valued graph and $\mathbf{Q}^\Gamma$ the corresponding rational vector space. Define a symmetric bilinear form $B = B_\Gamma$ on $\mathbf{Q}^\Gamma$ as follows: For $\mathbf{x}, \mathbf{y} \in \mathbf{Q}^\Gamma$,

$$B(\mathbf{x}, \mathbf{y}) = \sum_i f_i x_i y_i - \frac{1}{2} \sum_{i,j} d_{ij} f_j x_i y_j.$$

Thus, the corresponding quadratic form $Q = Q_\Gamma$ can be expressed in the form

$$Q(\mathbf{x}) = B(\mathbf{x}, \mathbf{x}) = \sum_i f_i x_i^2 - \sum_{\{i,j\}} d_{ij} f_j x_i x_j.$$

Note that B and Q are defined up to a positive rational multiple.

One can easily verify the following

Lemma 1.1. *For every $k \in \Gamma$ and every $\mathbf{x} \in \mathbf{Q}^\Gamma$, the linear transformation s_k of $\mathbf{Q}^\Gamma$ defined in Introduction has the form*

$$s_k \mathbf{x} = \mathbf{x} - 2\frac{B(\mathbf{x}, \mathbf{k})}{B(\mathbf{k}, \mathbf{k})}\mathbf{k}.$$

Consequently, $s_k^2 = 1$ and $B(s_k\mathbf{x}, s_k\mathbf{y}) = B(\mathbf{x}, \mathbf{y})$ for all $\mathbf{x}, \mathbf{y}$ of $\mathbf{Q}^\Gamma$.

The following proposition can be easily derived from Theorems 1 and 4 of N. Bourbaki [3], VI, 4.

Proposition 1.2. (a) *$(\Gamma, \mathbf{d})$ is a Dynkin diagram if and only if its quadratic form is positive definite.*

(b) *$(\Gamma, \mathbf{d})$ is an extended Dynkin diagram if and only if its quadratic form is positive semidefinite.*

For the benefit of the reader, we give a brief proof of Proposition 1.2. First, if $(\Gamma, \mathbf{d})$ is a Dynkin, or an extended Dynkin diagram, then a direct calculation shows that Q_Γ is positive definite or semidefinite, respectively. (The quadratic forms Q_Γ of extended Dynkin diagrams are listed in Tables, Chapter 6.)

Thus, let Q_Γ be a positive quadratic form of a connected graph $(\Gamma, \mathbf{d})$. Clearly, every edge $\{j, k\}$ of $(\Gamma, \mathbf{d})$ satisfies $d_{jk} \times d_{kj} \leqslant 4$; for, taking $\mathbf{x} \in \mathbf{Q}^\Gamma$ with $x_j = \frac{1}{2} d_{kj}$, $x_k = 1$ and $x_i = 0$ otherwise, $Q(\mathbf{x}) = \frac{1}{2} f_k(4 - d_{jk} \times d_{kj})$. By a similar argument, every pair of edges $\{j, k\}$ and $\{k, l\}$ with $d_{jk} \times d_{kj} = 3$ satisfies $d_{kl} \times d_{lk} = 1$: $Q(\mathbf{x}) = f_l(1 - d_{kl} \times d_{lk})$ for $\mathbf{x} \in \mathbf{Q}^\Gamma$ with $x_j = d_{kj} \times d_{lk}$, $x_k = 2d_{lk}$, $x_l = 1$ and $x_i = 0$ otherwise. Also, every circuit contained in $(\Gamma, \mathbf{d})$ is of type $\widetilde{\mathbf{A}}_m$, $m \leqslant n$; for, if $i_0, i_1, \ldots, i_m = i_0$ are its vertices, then, for $\mathbf{x} \in \mathbf{Q}^\Gamma$ with $x_{i_r} = 1$ $(0 \leqslant r \leqslant m)$ and $x_i = 0$ otherwise, $Q(\mathbf{x}) = \Sigma_{r=0}^{m-1} f_{i_r} \cdot (1 - d_{i_{r+1}i_r})$. Therefore, all $d_{i_{r+1}i_r} = 1$ and subsequently also all $d_{i_ri_{r+1}} = 1$.

Now, in order to complete the proof, it is sufficient to show that the quadratic form $Q_{\Gamma'}$, of every (full) subgraph $(\Gamma', \mathbf{d}')$ of $(\Gamma, \mathbf{d})$ is necessarily positive definite. For, as a consequence, no extended Dynkin diagram can appear as a proper subgraph of $(\Gamma, \mathbf{d})$ and a straightforward combinatorial argument establishes both (a) and (b) of Proposition 1.2. Thus, assume that $(\Gamma', \mathbf{d}')$ be a minimal subgraph with the property that $Q_{\Gamma'}$ is not positive definite. Let $\{j, k\}$ be an edge of $(\Gamma, \mathbf{d})$ such that $j \in \Gamma'$ and $k \notin \Gamma'$. Let $0 \neq \mathbf{x}' \in \mathbf{Q}^{\Gamma'}$ satisfy $Q_{\Gamma'}(\mathbf{x}') = 0$. Then, in view of minimality of $(\Gamma', \mathbf{d}')$, $x_i' \neq 0$ for all $i \in \Gamma'$. In fact, since the vector $\mathbf{x}''$ whose components are $|x_i'|$, $i \in \Gamma'$, always satisfies $Q_{\Gamma'}(\mathbf{x}'') \leqslant Q_{\Gamma'}(\mathbf{x}')$, we deduce that all components of $\mathbf{x}'$ are positive or that all are negative. Assume that $x_i' > 0$ for all $i \in \Gamma'$, and consider the vector $\mathbf{x} \in \mathbf{Q}^\Gamma$ defined by $x_i = x_i'$ for all $i \in \Gamma'$, $x_k = ½ d_{jk} x_j'$ and $x_i = 0$ otherwise. Then

$$Q_\Gamma(\mathbf{x}) = Q_{\Gamma'}(\mathbf{x}') - \frac{1}{4} f_k d_{jk}^2 x_j'^2 - \frac{1}{2} \sum_i f_k d_{jk} d_{ik} x_j' x_i,$$

where the summation runs through all edges $\{i, k\}$ with $i \in \Gamma'$. Thus $Q_\Gamma(\mathbf{x}) < 0$ in contradiction to our assumption and the proof is completed.

Following N. Bourbaki [3], the elements of the Weyl group W which are of the form

$$c = s_{k_n} \cdots s_{k_2} s_{k_1},$$

where $k_1, k_2, \ldots, k_n$ are all vertices of Γ (in a certain order) will be called *Coxeter transformations.* Obviously if c is a Coxeter transformation, so is c^{-1}. Note that given $c = s_{k_n} \cdots s_{k_2} s_{k_1}$, we may prescribe a unique orientation Ω_c to $(\Gamma, \mathbf{d})$ such that the ordering $k_1, k_2, \ldots, k_n$ is admissible. On the other hand, to every admissible orientation Ω of $(\Gamma, \mathbf{d})$, there exists (in an obvious way) a Coxeter transformation c such that $\Omega_c = \Omega$, and c is again uniquely determined. For, if $k_1, k_2, \ldots, k_n$ and $k_1', k_2', \ldots, k_n'$ are two admissible orderings of Γ with respect to Ω, then $s_{k_n} \cdots s_{k_2} s_{k_1} = s_{k_n'} \cdots s_{k_2'} s_{k_1'}$, since s_i and s_j commute for any pair of vertices $i, j \in \Gamma$ which are not neighbours.

LEMMA 1.3. *Let $(\Gamma, \mathbf{d})$ be a valued graph and $\mathbf{z} \in \mathbf{Q}^\Gamma$. Then the following statements are equivalent:*

(i) $c\mathbf{z} = \mathbf{z}$ *for a Coxeter transformation* c;

(ii) $s_k\mathbf{z} = \mathbf{z}$ *for all* $k \in \Gamma$ (*i. e.* $w\mathbf{z} = \mathbf{z}$ *for all* $w \in W$);

(iii) $B(\mathbf{z}, \mathbf{y}) = 0$ *for all* $\mathbf{y} \in \mathbf{Q}^\Gamma$.

If, moreover, the corresponding quadratic form Q is positive, then the above statements are equivalent to

(iv) $Q(\mathbf{z}) = 0$.

Proof. Let $c = s_{k_n} \cdots s_{k_2} s_{k_1}$. The fact that (i) implies (ii) follows inductively from the equality

$$z_{k_t} = (s_{k_n} \cdots s_{k_{t+1}} s_{k_t} \mathbf{z})_{k_t} \quad \text{for all } 1 \leqslant t < n.$$

The latter yields immediately that $s_{k_1}\mathbf{z} = s_{k_2}\mathbf{z} = \cdots = s_{k_n}\mathbf{z} = \mathbf{z}$. Also, using Lemma 1.1, we see that $s_k \mathbf{z} = \mathbf{z}$ is equivalent to $B_\Gamma(\mathbf{z}, \mathbf{k}) = 0$. Consequently, since the vectors $\mathbf{k}$ form a basis of $\mathbf{Q}^\Gamma$, the statements $s_k\mathbf{z} = \mathbf{z}$ for all $k \in \Gamma$ and $B(\mathbf{z}, \mathbf{y}) = 0$ for all $\mathbf{y} \in \mathbf{Q}^\Gamma$ are equivalent. And, the equivalenee of (iii) and (iv) is obvious.

DEFINITION. *Given a valued graph* $(\Gamma, \mathbf{d})$, *define its radical subspace by*

$$N = N_\Gamma = \{\mathbf{x} \in \mathbf{Q}^\Gamma \mid w\mathbf{x} = \mathbf{x} \ \textit{for all} \ w \in W\}.$$

Thus, if $(\Gamma, \mathbf{d})$ is a Dynkin diagram, then $N = \{0\}$. And, if $(\Gamma, \mathbf{d})$ is an extended Dynkin diagram, then N is a one-dimensional subspace of $\mathbf{Q}^\Gamma$ generated by the "canonic" vector $\mathbf{n} = \mathbf{n}_\Gamma$; these vectors are listed in the tables of Chapter 6. Note that all components of the vectors $\mathbf{n}$ are positive and that, in each case, at least one component of $\mathbf{n}$ equals 1. Moreover, it has been shown in [1] that the existence of a positive vector in N implies that $(\Gamma, \mathbf{d})$ is an extended Dynkin diagram.

Notice that the set $R = \{\mathbf{x} \in \mathbf{Q}^\Gamma \mid \mathbf{x} = w\mathbf{k}$ for some $w \in W$ and $k \in \Gamma\}$ of all roots satisfies the defining properties (but the finiteness condition) for a reduced root system of [3]: R generates $\mathbf{Q}^\Gamma$; for $\mathbf{y} \in R$, the linear transformation

$$\mathbf{x} \longmapsto \mathbf{x} - \frac{B(\mathbf{x}, \mathbf{y})}{B(\mathbf{y}, \mathbf{y})}\mathbf{y}$$

maps R into R; for $\mathbf{x}, \mathbf{y} \in R$, $2B(\mathbf{x}, \mathbf{y})/B(\mathbf{y}, \mathbf{y})$ is an integer; and, if $\mathbf{x} \in R$, then $2\mathbf{x} \notin R$.

In particular, if $\mathbf{x} \in \mathbf{Q}^\Gamma$ is a root and $\mathbf{x} = w\mathbf{k}$, then $-\mathbf{x} = w(-\mathbf{k}) = (ws_k)\mathbf{k}$ is also a root. Furthermore, as an immediate consequence of the following more general result, one of the roots $\mathbf{x}$ or $-\mathbf{x}$ is always positive (i.e. $x_i \geqslant 0$ for all $i \in \Gamma$).

LEMMA 1.4. *Let* $(\Gamma, \mathbf{d})$ *be a valued graph such that the corresponding quadratic form* Q *is positive. If* $\mathbf{x}$ *is a positive root and* $k \in \Gamma$, *then either* $s_k\mathbf{x}$ *is positive or* $\mathbf{x} = \mathbf{k}$.

Proof. Let $\mathbf{x} = w\mathbf{j}$ for a certain $j \in \Gamma$. Since $0 \leqslant B(\mathbf{x} \pm \mathbf{k}, \mathbf{x} \pm \mathbf{k}) = B(\mathbf{x}, \mathbf{x}) + B(\mathbf{k}, \mathbf{k}) \pm 2B(\mathbf{x}, \mathbf{k})$, we get

$$|B(\mathbf{x}, \mathbf{k})| \leqslant \frac{1}{2}(B(\mathbf{k}, \mathbf{k}) + B(\mathbf{j}, \mathbf{j})) = \frac{1}{2}(f_k + f_j).$$

Thus, we have for the integer $2B(\mathbf{x}, \mathbf{k})$ the following inequality

$$-(f_k + f_j) \leqslant 2B(\mathbf{x}, \mathbf{k}) \leqslant f_k + f_j.$$

First, if $2B(\mathbf{x}, \mathbf{k}) = f_k + f_j$, then $B(\mathbf{x} - \mathbf{k}, \mathbf{x} - \mathbf{k}) = 0$, and thus $\mathbf{x} - \mathbf{k} \in N$. Thus, if $(\Gamma, \mathbf{d})$ is a Dynkin diagram, then $\mathbf{x} = \mathbf{k}$. Otherwise, $\mathbf{x} = \mathbf{k} + h\mathbf{n}$ with a suitable integer h. Therefore, in case that $h \neq 0$, $s_k\mathbf{x} = h\mathbf{n} - \mathbf{k} > 0$, because all components of $\mathbf{n}$ are $\geqslant 1$.

Second, if $2B(\mathbf{x}, \mathbf{k}) \leqslant 0$, then

$$s_k\mathbf{x} = \mathbf{x} - 2\frac{B(\mathbf{x}, \mathbf{k})}{B(\mathbf{k}, \mathbf{k})}\mathbf{k}$$

has all components greater or equal to those of $\mathbf{x}$, and is therefore positive..

Finally, let $1 \leqslant 2B(\mathbf{x}, \mathbf{k}) < f_k + f_j$. Then, calculating the kth component of

$$2B(\mathbf{x}, \mathbf{k})\mathbf{k} = (\mathbf{x} - s_k\mathbf{x})B(\mathbf{k}, \mathbf{k}),$$

we get

$$1 \leqslant 2B(\mathbf{x}, \mathbf{k}) = f_k(x_k + x_k - \sum_j d_{jk}x_j);$$

hence, $x_k > 0$, and therefore $x_k \geqslant 1$.

Now, $2B(\mathbf{x}, \mathbf{k})/B(\mathbf{k}, \mathbf{k}) < 1 + f_j/f_k$, and therefore, if $f_j \leqslant f_k$, $2B(\mathbf{x}, \mathbf{k})/B(\mathbf{k}, \mathbf{k}) \leqslant 1$. Consequently,

$$s_k\mathbf{x} = \mathbf{x} - 2\frac{B(\mathbf{x}, \mathbf{k})}{B(\mathbf{k}, \mathbf{k})}\mathbf{k}$$

is positive.

Thus, we have so far proved our Proposition in the case that $f_j \leqslant f_k$. If $f_j > f_k$, we shall reduce the problem to this case as follows:

Consider the valued graph $(\Gamma', \mathbf{d}')$ defined by $\Gamma' = \Gamma$ and $d'_{ij} = d_{ji}$. Note that we may choose $f'_i = 1/f_i$. Further, define the delation $\Delta : \mathbf{Q}^\Gamma \longrightarrow \mathbf{Q}^\Gamma$ by $\Delta\mathbf{x} = \mathbf{y}$, where

$$y_i = (\Delta\mathbf{x})_i = f_i x_i \text{ for each } i \in \Gamma.$$

We are going to show that

$$\Delta s_k = s'_k \Delta$$

(where s'_k denotes the involutions of $\mathbf{Q}^{\Gamma'}$). Indeed, for $i \neq k$,

$$(s'_k\Delta\mathbf{x})_i = (\Delta\mathbf{x})_i = f_i x_i = (\Delta s_k\mathbf{x})_i;$$

and,

$$\begin{aligned}(s'_k\Delta\mathbf{x})_k &= -f_k x_k + \sum_i d_{ik} f_i x_i \\ &= f_k(-x_k + \sum_i d_{ik} x_i) = (\Delta s_k \mathbf{x})_k.\end{aligned}$$

Now, $\mathbf{x} = w\mathbf{j}$, where w is a product of suitable involutions s_k; consider the vector $\mathbf{x}' = w'\mathbf{j}$, where w' is the corresponding product of the involutions s'_k. Thus

$$\mathbf{x}' = w'\frac{1}{f_j} f_j \mathbf{j} = \frac{1}{f_j} w'\Delta\mathbf{j} = \frac{1}{f_j}\Delta\mathbf{x}$$

is a positive root in $\mathbf{Q}^{\Gamma'}$. Since

$$Q_{\Gamma'}(\mathbf{x}') = Q_{\Gamma'}(\mathbf{j}) = \frac{1}{f_j} < \frac{1}{f_k} = Q_{\Gamma'}(\mathbf{k}),$$

we see immediately that, according to the first part of the proof,

either $\mathbf{x}' = \mathbf{k}$ or $s'_k\mathbf{x}'$ is positive.

However, if $\mathbf{x}' = \mathbf{k}$, then $\Delta\mathbf{x} = f_j\mathbf{k}$ implies that $\mathbf{x} = (f_j/f_k)\mathbf{k}$, a contradiction. If $s'_k\mathbf{x}'$ is positive, then

$$f_j s'_k \mathbf{x}' = f_j s'_k \frac{1}{f_j}\Delta\mathbf{x} = \Delta s_k \mathbf{x},$$

and therefore also $s_k\mathbf{x}$ is positive.

The proof is completed.

PROPOSITION 1.5. *Let $(\Gamma, \mathbf{d})$ be a valued graph with a ~~positive~~ non-negative quadratic form. Then the group $\overline{W}$ of all linear transformations of $\mathbf{Q}^\Gamma/N$ induced by the transformations of W is finite.*

Proof. Let f be a natural number with $f_i \leqslant f$ for all $i \in \Gamma$. Let M be the set of all integral vectors $\mathbf{x} \in \mathbf{Q}^\Gamma$ such that $Q(\mathbf{x}) \leqslant f$. Let P be a regular matrix which transforms Q into a diagonal form:

$$Q(\mathrm{x}) = \sum_{i=1}^{t} c_i y_i^2, \text{ where } \mathrm{y} = \mathrm{x}P \text{ and } 0 < c_i \in \mathbf{Q} \text{ for } 1 \leqslant i \leqslant t \leqslant n.$$

Thus, NP consists of all vectors $\mathrm{y} = (y_i)$ with $y_i = 0$ for $1 \leqslant i \leqslant t$. Let h be the common denominator of the entries in P; hence, hP is an integral matrix. Consequently, if $\mathrm{x} \in M$, then $h\mathrm{y}$ is an integral vector and, moreover $|y_i| \leqslant \sqrt{f/c_i}$ for $1 \leqslant i \leqslant t$. Therefore, under the transformation P, M is mapped into a set which is modulo NP finite. Thus, also the set $\bar{M} = M + N/N$ is finite.

Now, the group $\bar{W}$ of the automorphisms on $\mathbf{Q}^\Gamma/N$ induced by W transforms $\bar{M}$ into itself. And, since $\bar{M}$ contains a basis of $\mathbf{Q}^\Gamma/N$, $\bar{W}$ can be embedded into the symmetric group on $\bar{M}$ and thus is finite, as required.

Let $(\Gamma, \mathbf{d})$ be an extended Dynkin diagram. Proposition 1.5 allows us to introduce a very important concept, that of the *defect* $\partial_c\mathrm{x}$ of a vector $\mathrm{x} \in \mathbf{Q}^\Gamma$ with respect to a Coxeter transformation $c \in W$ (or, what is the same, with respect to an admissible orientation of $(\Gamma, \mathbf{d})$): If $\bar{c} \in \bar{W}$ is of order m, then, given $\mathrm{x} \in \mathbf{Q}^\Gamma$,

$$c^m\mathrm{x} = \mathrm{x} + (\partial_c\mathrm{x})\mathbf{n}, \text{ where } \partial_c\mathrm{x} \in \mathbf{Q}.$$

In this fashion, $\partial_c : \mathbf{Q}^\Gamma \longrightarrow \mathbf{Q}$ defines a linear form and thus, for the *defect vector* $\boldsymbol{\partial}_c = (\partial_c\mathrm{i}) \in \mathbf{Q}^\Gamma$, we have

$$\partial_c\mathrm{x} = \sum_i (\partial_c\mathrm{i})x_i.$$

Notice that, since one of the integral components of $\mathbf{n}$ equals 1, all $\partial_c\mathbf{i}$ are integral. Also, one can see easily that

$$\partial_c(c\mathrm{x}) = \partial_c\mathrm{x}.$$

LEMMA 1.6. *Let* $\mathrm{x} \geqslant 0$ *be a positive root of* $(\Gamma, \mathbf{d})$ *whose quadratic form is ~~positive~~ and let* $c = s_{k_n} \cdots s_{k_2}s_{k_1}$ *be a Coxeter transformation of* $\mathbf{Q}^\Gamma$. *Then*

(i) $c\mathrm{x} \not\geqslant \mathbf{0}$ *if and only if* $\mathrm{x} = \mathrm{p}_{k_t}$ *for a suitable* $1 \leqslant t \leqslant n$, *where*

$$\mathrm{p}_{k_t} = s_{k_1}s_{k_2} \cdots s_{k_{t-1}}\mathrm{k}_t.$$

(ii) $c^{-1}\mathrm{x} \not\geqslant \mathbf{0}$ *if and only if* $\mathrm{x} = \mathrm{q}_{k_t}$ *for a suitable* $1 \leqslant t \leqslant n$, *where*

$$\mathrm{q}_{k_t} = s_{k_n}s_{k_{n-1}} \cdots s_{k_{t+1}}\mathrm{k}_t.$$

Proof. This is an immediate consequence of Lemma 1.4.

LEMMA 1.7. *Let* x *be a positive root of* $(\Gamma, \mathbf{d})$ *whose quadratic form is ~~positive~~ and let* $c = s_{k_n} \cdots s_{k_2}s_{k_1}$ *be a Coxeter transformation of* $\mathbf{Q}^\Gamma$. *Then either there exists an integer r such that* $c^r\mathrm{x} \not\geqslant 0$, *or* $(\Gamma, \mathbf{d})$ *is an extended Dynkin diagram and* $\partial_c\mathrm{x} = 0$.

Proof. If $(\Gamma, \mathbf{d})$ is a Dynkin diagram, then c is of finite order, say m, and $\mathrm{y} = \Sigma_{h=1}^m c^h\mathrm{x}$ satisfies $c\mathrm{y} = \mathrm{y}$. Consequently, $\mathrm{y} = \mathbf{0}$ and thus there exists $1 \leqslant r \leqslant m - 1$ such that $c^r\mathrm{x} \not\geqslant \mathbf{0}$.

If $(\Gamma, \mathbf{d})$ is an extended Dynkin diagram, then by Proposition 1.5, the order of $\bar{c} \in \bar{W}$ is finite; denote it again by m. Now, if $\partial_c\mathrm{x} \neq 0$, then

$$c^{sm}\mathrm{x} = \mathrm{x} + s(\partial_c\mathrm{x})\mathbf{n} \text{ for every integer } s.$$

From here it follows that, for a suitable $r = sm$ (such that $s(\partial_c \mathbf{x})$ is negative and large in absolute value), $c^r\mathbf{x} \not\geqslant \mathbf{0}$.

The following more general results cover the remaining case of a root of defect zero (see R. V. Moody [12]).

LEMMA 1.8. *Let* $(\Gamma, \mathbf{d})$ *be an extended Dynkin diagram. Then there exists a natural number* g, $1 \leqslant g \leqslant 3$, *(called the tier number) such that the positive roots* $\mathbf{x}$ *of* $(\Gamma, \mathbf{d})$ *are just the vectors of the form*

$$\mathbf{x} = \mathbf{x}_0 + rg\mathbf{n}$$

with a non-negative integer r *and a positive root* $\mathbf{x}_0$ *of* $(\Gamma, \mathbf{d})$ *satisfying* $\mathbf{x}_0 \leqslant g\mathbf{n}$.

The tables in Chapter 6 provide the values of g and, for a particular admissible orientation which will be used throughout the paper, the value of ∂_c. For each oriented extended Dynkin diagram, there exist roots $\mathbf{x}_* \leqslant g\mathbf{n}$ with $\partial_c\mathbf{x}_* = 0$ such that the other roots $\mathbf{x}_0 \leqslant g\mathbf{n}$ with $\partial_c\mathbf{x}_0 = 0$ are just the vectors of the form

$$\sum_{r \leqslant t \leqslant r+s} c^t\mathbf{x}_*, \quad r, s \geqslant 0,$$

which do not belong to the radical space N (and which satisfy the inequality $\leqslant g\mathbf{n}$). These vectors $\mathbf{x}_*$ can be found in the tables (Chapter 6): They are the dimension types of the first indecomposable representation in each orbit listed there.

Thus, we may summarize the preceding results and list all positive roots of a valued graph with a positive quadratic form.

PROPOSITION 1.9. *Let* $(\Gamma, \mathbf{d})$ *be a valued graph and let* c *be a Coxeter transformation of* $\mathbb{Q}^\Gamma$.

(a) *If* $(\Gamma, \mathbf{d})$ *is a Dynkin diagram and* m *is the order of* c, *let, for each* $1 \leqslant t \leqslant n$, a_t *be the largest integers such that all* $c^{-r}\mathbf{p}_{k_t}$ *with* $0 \leqslant r \leqslant a_t$ *are positive. Then the vectors*

$$\mathbf{x} = c^{-r}\mathbf{p}_{k_t}, \quad 0 \leqslant r \leqslant a_t, \quad 1 \leqslant t \leqslant n$$

are just all positive roots of $(\Gamma, \mathbf{d})$. *Similarly, if* b_t *is the largest integer such that all* $c^r\mathbf{q}_{k_t}$ *with* $0 \leqslant r \leqslant b_t$ *are positive, then the vectors*

$$\mathbf{x} = c^r\mathbf{q}_{k_t}, \quad 0 \leqslant r \leqslant b_t, \quad 1 \leqslant t \leqslant n,$$

are just all positive roots of $(\Gamma, \mathbf{d})$.

(b) *If* $(\Gamma, \mathbf{d})$ *is an extended Dynkin diagram, then*

(1) *the vectors* $\mathbf{x} = c^{-r}\mathbf{p}_{k_t}$, $0 \leqslant r$, $1 \leqslant t \leqslant n$, *are just all positive roots of* $(\Gamma, \mathbf{d})$ *of negative defect with respect to* c;

(2) *the vectors* $\mathbf{x} = c^r\mathbf{q}_{k_t}$, $0 \leqslant r$, $1 \leqslant t \leqslant n$, *are just all positive roots of* $(\Gamma, \mathbf{d})$ *of positive defect with respect to* c;

(3) *the vectors* $\mathbf{x} = \mathbf{x}_0 + rg\mathbf{n}$, $r \geqslant 0$, *where* $\mathbf{x}_0 \leqslant g\mathbf{n}$ *with* $\partial_c\mathbf{x}_0 = 0$ *can be derived from the tables of Chapter 6, are just all positive roots of defect zero with respect to c.*

Proof. (a) follows immediately from Lemma 1.7 (with r chosen so that $|r|$ is minimal) and Lemma 1.6.

(b) is a consequence of Lemmas 1.6, 1.7 and 1.8.

REMARK. Using the obvious relation

$$c\mathbf{p}_{k_t} = -\mathbf{q}_{k_t} \quad \text{for all } 1 \leqslant t \leqslant n,$$

and the fact that the number of roots of a Dynkin diagram is mn, (see, e.g. [3]), we deduce that the set

$$\{\mathbf{x} = c^{-r}\mathbf{p}_{k_t} \mid 0 \leqslant r \leqslant m-1,\ 0 \leqslant t \leqslant n\} = \{\mathbf{x} = c^{r}\mathbf{q}_{k_t} \mid 0 \leqslant r \leqslant m-1,\ 0 \leqslant t \leqslant n\}$$

is the set of all roots and thus that the sets defined in Proposition 1.9 (a) have ½ mn elements.

Let us conclude this chapter with a remark giving another characterization of positive roots of an extended Dynkin diagram of defect zero.

REMARK. Given a valued graph $(\Gamma, \mathbf{d})$ with a positive form, and a Coxeter transformation c, define in $\mathbf{Q}^\Gamma$ a partial order $\leqslant_c$ as follows:

$$\mathbf{x} \leqslant_c \mathbf{y} \text{ if and only if } c^t\mathbf{x} \leqslant c^t\mathbf{y} \text{ for all integers } t.$$

Obviously, this order is trivial if and only if $(\Gamma, \mathbf{d})$ is a Dynkin diagram. In case of an extended Dynkin diagram, we can speak about *c-positive roots*: these are those roots $\mathbf{x}$ for which $\mathbf{x} \geqslant_c \mathbf{0}$. Minimal c-positive roots will be called *simple c-positive roots.*

(a) *For a root* $\mathbf{x}$ *of an extended Dynkin diagram* $(\Gamma, \mathbf{d})$, *the following assertions are equivalent*:

(i) $\mathbf{x}$ *is c-positive*;

(ii) $\partial_c\mathbf{x} = 0$;

(iii) *c-orbit of* $\mathbf{x}$ *is finite.*

This follows immediately from Lemma 1.7. From the tables in Chapter 6, one can see that in each case, there are *at most three orbits of simple c-positive roots.* The simple c-positive roots are labelled there $\mathbf{dim}\,\mathbf{E}_r^{(t)}$. They are precisely the roots denoted in the remark preceding Proposition 1.9 by $c^t\mathbf{x}_*$, $t \geqslant 0$. That remark can be reformulated as follows.

(b) *Every c-positive root is a (uniquely determined) sum of simple c-positive roots from the same orbit.*

2. REALIZATION OF VALUED GRAPHS: THE COXETER FUNCTORS

Let F_i, F_j, F_k be fields. An F_i-F_j-bimodule ${}_iM_j$ is said to have a *dual bimodule* if the F_j-F_i-bimodules

$$\mathrm{Hom}_{F_i}({}_iM_j, F_i) \quad \text{and} \quad \mathrm{Hom}_{F_j}({}_iM_j, F_j)$$

are isomorphic. For example, if K is a common central subfield of F_i and F_j such that K operates centrally on ${}_iM_j$, and if $\dim_K {}_iM_j$ is finite, then ${}_iM_j$ has a dual bimodule. Note that the F_j-F_i-bimodule ${}_jM_i$ is a dual bimodule to ${}_iM_j$ if and only if there exist nondegenerate bilinear forms

$$\epsilon_i^j : {}_iM_j \underset{F_j}{\otimes} {}_jM_i \longrightarrow F_i, \quad \epsilon_j^i : {}_jM_i \underset{F_i}{\otimes} {}_iM_j \longrightarrow F_j.$$

Given ${}_iM_j$, the dual bimodule ${}_jM_i$ (if it exists) is unique (up to an isomorphism), whereas the bilinear forms ϵ_i^j and ϵ_j^i may vary.[3]

[3] Although they are used in the construction of the Coxeter functors, the category $L(\mathfrak{M}, \Omega)$, as well as all results are, of course, independent of a particular choice of these forms.

If both bimodules ${}_{F_i}({}_iM_j)_{F_j}$ *and* ${}_{F_j}({}_jM_k)_{F_k}$ *have a dual bimodule, then so has the* F_i-F_k-*bimodule* ${}_{F_i}({}_iM_j)_{F_j} \otimes {}_{F_j}({}_jM_k)_{F_k}$.[4] For, given ${}_jM_i$, ϵ_i^j, ϵ_j^i and ${}_kM_j$, ϵ_j^k, ϵ_k^j, one considers ${}_kM_j \otimes {}_jM_i$ together with the mappings

$$ {}_iM_j \otimes {}_jM_k \otimes {}_kM_j \otimes {}_jM_i \xrightarrow{1 \otimes \epsilon_j^k \otimes 1} {}_iM_j \otimes F_j \otimes {}_jM_i \approx {}_iM_j \otimes {}_jM_i \xrightarrow{\epsilon_i^j} F_i $$

and $\epsilon_k^j(1 \otimes \epsilon_j^i \otimes 1)$, which are obviously nondegenerate.

Now, let ${}_iM_j$ be an F_i-F_j-bimodule with the dual bimodule ${}_jM_i$. If $(X_i)_{F_i}$ and $(X_j)_{F_j}$ are vector spaces, then *there is a natural isomorphism*

$$ \mathrm{Hom}_{F_j}(X_i \otimes {}_iM_j, X_j) \approx \mathrm{Hom}_{F_i}(X_i, X_j \otimes {}_jM_i). $$

For, there is the well-known isomorphism

$$ \mathrm{Hom}_{F_j}({}_iM_j, X_j) \approx X_j \underset{F_j}{\otimes} \mathrm{Hom}_{F_j}({}_iM_j, F_j) \approx X_j \otimes {}_jM_i ; $$

hence, the adjointness of $\otimes$ and Hom yields

$$ \mathrm{Hom}_{F_j}(X_i \otimes {}_iM_j, X_j) \approx \mathrm{Hom}_{F_i}(X_i, \mathrm{Hom}_{F_j}({}_iM_j, X_j)) \approx \mathrm{Hom}_{F_i}(X_i, X_j \otimes {}_jM_i). $$

Thus, for each F_j-linear mapping ${}_j\varphi_i : X_i \otimes {}_iM_j \longrightarrow X_j$, we have attached canonically an F_i-linear mapping ${}_j\overline{\varphi}_i : X_i \longrightarrow X_j \otimes {}_jM_i$; conversely, for ${}_j\psi_i : X_i \longrightarrow X_j \otimes {}_jM_i$, there correspond a unique ${}_j\underline{\psi}_i : X_i \otimes {}_iM_j \longrightarrow X_j$, and we have ${}_j\underline{\overline{\varphi}}_i = {}_j\varphi_i$ and ${}_j\overline{\underline{\psi}}_i = {}_j\psi_i$. This notation will be used throughout this paper.

REMARK. Let G be a subfield of F, and assume that the bimodule ${}_GF_F$ has a dual bimodule. Note that this dual bimodule has to be ${}_FF_G$. Denote by ${}_FK_F$ the kernel of the multiplication mapping ${}_FF_G \otimes {}_GF_F \xrightarrow{\mu} {}_FF_F$. We claim that *also* ${}_GF_G$, ${}_FF_G \otimes {}_GF_F$, ${}_G(F/G)_G$ *and* ${}_FK_F$ *have dual bimodules.* This is obvious for ${}_GF_G = {}_GF_F \otimes {}_FF_G$, and ${}_FF_G \otimes {}_GF_F$; both are self dual. There exists a nondegenerate bilinear form ${}_GF_F \otimes {}_FF_G \longrightarrow {}_GG_G$, but this is just a nonzero map $\epsilon : {}_GF_G \longrightarrow {}_GG_G$. Let ${}_GL_G$ be the kernel of ϵ. The nondegenerate bilinear form for ${}_GF_G$ is

$$ \epsilon\mu : {}_GF_G \otimes {}_GF_G \longrightarrow {}_GG_G, $$

and since $\epsilon\mu({}_GG_G \otimes {}_GL_G) = 0 = \epsilon\mu({}_GL_G \otimes {}_GG_G)$, we conclude that $\epsilon\mu$ induces a nondegenerate bilinear form on ${}_G(F/G)_G \otimes {}_GL_G$, and on ${}_GL_G \otimes {}_G(F/G)_G$. Now, for ${}_FK_F$, we consider the nondegenerate bilinear form

$$ \mu(1 \otimes \epsilon \otimes 1) : {}_FF_G \otimes {}_GF_F \otimes {}_FF_G \otimes {}_GF_F \longrightarrow {}_FF_F. $$

Let $\omega = \overline{1} : F_F \longrightarrow F_G \otimes {}_GF_F$ be the mapping canonically attached to the identity $1 : F_F \otimes {}_FF_G \longrightarrow F_G$, with respect to ϵ; thus, by construction, the map

$$ F_G \approx F_F \otimes {}_FF_G \xrightarrow{\omega \otimes 1} F_G \otimes {}_GF_F \otimes {}_FF_G \xrightarrow{1 \otimes \epsilon} F_G \otimes {}_GG_G \approx F_G $$

is the identity. Note that ω is an F-F-bimodule mapping. We claim that

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[4] In dealing with tensor products of the form

$$ {}_iM_j \underset{F_j}{\otimes} {}_jM_k, $$

we shall usually omit the letter F_j.

$$\mu(1 \otimes \epsilon \otimes 1)(\omega({}_FF_F) \otimes {}_FK_F) = 0 = \mu(1 \otimes \epsilon \otimes 1)({}_FK_F \otimes \omega({}_FF_F)).$$

For, let $\Sigma_i x_i \otimes y_i$ be in $K \subseteq {}_FF_G \otimes {}_GF_F$; then

$$\mu(1 \otimes \epsilon \otimes 1)(\omega(1) \otimes x_i \otimes y_i) = \mu[(1 \otimes \epsilon)(\omega \otimes 1)(1 \otimes x_i) \otimes y_i] = \mu(x_i \otimes y_i) = x_i y_i$$

and therefore

$$\sum_i \mu(1 \otimes \epsilon \otimes 1)(\omega(1) \otimes x_i \otimes y_i) = \sum_i x_i y_i = 0.$$

As a consequence, $\mu(1 \otimes \epsilon \otimes 1)$ defines a nondegenerate bilinear form on

$$({}_FF_G \otimes {}_GF_F)/\omega({}_FF_F) \otimes {}_FK_F,$$

as well as on ${}_FK_F \otimes ({}_FF_G \otimes {}_GF_F)/\omega({}_FF_F)$.

If $\dim_G F = \dim F_G = 2$, ${}_GF_F$ has a dual bimodule if and only if there exists $f \in F \backslash G$ such that either all the commutators $[f, g] = fg - gf$, $g \in G$, belong to G or all $[f, g]$, $g \in G$, belong to fG (i.e. $fG = Gf$). For, a nondegenerate bilinear form $\epsilon : {}_GF_F \otimes {}_FF_G \longrightarrow {}_GG_G$ is simply a nonzero bimodule mapping $\epsilon : {}_GF_G \longrightarrow {}_GG_G$. Now, either ${}_G(F/G)_G \approx {}_GG_G$ which is equivalent to saying that there is $f \in F \backslash G$ with $[f, g] \subseteq G$, or ${}_GF_G = {}_GG_G \oplus {}_GH_G$ for some complement H, and this means that $H = Gf = fG$ for some $f \in F \backslash G$.

Now let $(\mathfrak{M}, \Omega)$ be a realization of a given valued graph $(\Gamma, \mathbf{d})$. Let $\mathbf{X} = (X_i, {}_j\varphi_i)$ be a representation of $(\mathfrak{M}, \Omega)$. Given a sink, or a source, k of the realization $(\mathfrak{M}, \Omega)$, we are going to define functors

$$S_k^+ : L(\mathfrak{M}, \Omega) \longrightarrow L(\mathfrak{M}, s_k\Omega),$$

or

$$S_k^- : L(\mathfrak{M}, \Omega) \longrightarrow L(\mathfrak{M}, s_k\Omega),$$

respectively.

First, let k be a sink and $\mathbf{X} = (X_i, {}_j\varphi_i) \in L(\mathfrak{M}, \Omega)$. Define $S_k^+ \mathbf{X} = \mathbf{Y} = (Y_i, {}_j\psi_i)$ as follows:

$$Y_i = X_i \quad \text{for all} \quad i \neq k;$$

let Y_k be the kernel in the diagram [5]

$$0 \longrightarrow Y_k \xrightarrow{({}_j\kappa_k)_j} \bigoplus_{j \in \Gamma} X_j \otimes {}_jM_k \xrightarrow{({}_k\varphi_j)_j} X_k,$$

$${}_j\psi_i = {}_j\varphi_i \text{ for } i \neq k \text{ and } {}_j\psi_k = {}_j\bar{\kappa}_k : Y_k \otimes {}_kM_j \longrightarrow Y_j.$$

Also, if $\boldsymbol{\alpha} = (\alpha_i) : \mathbf{X} \longrightarrow \mathbf{X}'$ is a morphism in $L(\mathfrak{M}, \Omega)$, then $S_k^+ \boldsymbol{\alpha} = \boldsymbol{\beta} = (\beta_i)$ is defined by $\beta_i = \alpha_i$ for $i \neq k$ and $\beta_k : Y_k \longrightarrow Y'_k$ as the restriction of

$$\bigoplus_{j \in \Gamma} (\alpha_j \otimes 1) : \bigoplus_{j \in \Gamma} X_j \otimes {}_jM_k \longrightarrow \bigoplus_{j \in \Gamma} X'_j \otimes {}_jM_k.$$

Note that $S_k^+ \mathbf{X} \in L(\mathfrak{M}, s_k\Omega)$ with changed (!) orientation.

Similarly, if k is a source of $(\mathfrak{M}, \Omega)$ and $\mathbf{X} = (X_i, {}_j\varphi_i) \in L(\mathfrak{M}, \Omega)$, define $S_k^- \mathbf{X} = \mathbf{Y} = (Y_i, {}_j\psi_i)$ as follows:

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[5] Here, and elsewhere, put ${}_k\varphi_j = 0$ for all $j \in \Gamma$ which are not neighbours of k.

$$Y_i = X_i \quad \text{for all} \quad i \neq k;$$

let Y_k be the cokernel in the diagram

$$X_k \xrightarrow{({}_i\bar{\varphi}_k)_i} \bigoplus_{i \in \Gamma} X_i \otimes {}_iM_k \xrightarrow{({}_k\pi_i)_i} Y_k \longrightarrow 0;$$

$${}_j\psi_i = {}_j\varphi_i \text{ for } j \neq k$$

and

$${}_k\psi_i = {}_k\pi_i : X_i \otimes {}_iM_k \longrightarrow Y_k.$$

In this way, we get $S_k^- \mathbf{X} \in L(\mathfrak{M}, s_k\Omega)$, again with a changed orientation. And, if $\boldsymbol{\alpha} : \mathbf{X} \longrightarrow \mathbf{X}'$ is a morphism in $L(\mathfrak{M}, \Omega)$, then $S_k^- \boldsymbol{\alpha} = \boldsymbol{\beta} = (\beta_i)$, where $\beta_i = \alpha_i$ for $i \neq k$ and β_k is the map induced by $\oplus_j \alpha_j \otimes 1$.

It can be easily seen that S_k^- is left adjoint to S_k^+; however, this fact will not be used in the paper.

Now, for each $k \in \Gamma$, define the representation $\mathbf{F}_k \in L(\mathfrak{M}, \Omega)$ by $\mathbf{F}_k = (X_i, {}_j\varphi_i)$, where

$$X_i = 0 \text{ for } i \neq k, X_k = F_k \text{ and all } {}_j\varphi_i = 0.$$

Observe that the representations $\mathbf{F}_k$ are just the simple objects of $L(\mathfrak{M}, \Omega)$.

PROPOSITION 2.1. *Let $(\mathfrak{M}, \Omega)$ be a realization of $(\Gamma, \mathbf{d})$. Let $\mathbf{X} \in L(\mathfrak{M}, \Omega)$.*

(i) *Let $k \in \Gamma$ be a sink. Then there is a canonical monomorphism*

$$\mu : S_k^- S_k^+ \mathbf{X} \longrightarrow \mathbf{X}.$$

In fact, $\mu(S_k^- S_k^+ \mathbf{X})$ has a complement in $\mathbf{X}$ which is a direct sum of copies of $\mathbf{F}_k$. Thus, if $\mathbf{X}$ is indecomposable, then either

$$\mathbf{X} \approx \mathbf{F}_k \text{ and } S_k^+ \mathbf{X} = \mathbf{0},$$

or μ is an isomorphism in which case

$$\operatorname{End}(S_k^+ \mathbf{X}) \approx \operatorname{End} \mathbf{X} \quad \text{and} \quad \mathbf{dim}(S_k^+ \mathbf{X}) = s_k(\mathbf{dim}\, \mathbf{X}).$$

(ii) *Let $k \in \Gamma$ be a source. Then there is a canonical epimorphism*

$$\epsilon : \mathbf{X} \longrightarrow S_k^+ S_k^- \mathbf{X}$$

Again, ϵ has a section $\epsilon' : S_k^+ S_k^- \mathbf{X} \longrightarrow \mathbf{X}$ and $\mathbf{X}$ is a direct sum of $\epsilon'(S_k^+ S_k^- \mathbf{X})$ and of copies of $\mathbf{F}_k$. Thus, if $\mathbf{X}$ is indecomposable, then either

$$\mathbf{X} = \mathbf{F}_k \text{ and } S_k^- \mathbf{X} = \mathbf{0},$$

or ϵ is an isomorphism in which case

$$\operatorname{End}(S_k^- \mathbf{X}) \approx \operatorname{End} \mathbf{X} \text{ and } \mathbf{dim}(S_k^- \mathbf{X}) = s_k(\mathbf{dim}\, \mathbf{X}).$$

Proof. (i) In the diagram

$$\begin{array}{ccccccc} 0 \longrightarrow (S_k^+ \mathbf{X})_k & \longrightarrow & \bigoplus_{j \in \Gamma} X_j \otimes {}_jM_k & \longrightarrow & (S_k^- S_k^+ \mathbf{X})_k & \longrightarrow & 0, \\ & & \Big\downarrow ({}_k\varphi_j)_j & \swarrow \mu_k & & & \\ & & X_k & & & & \end{array}$$

μ_k is obviously always a monomorphism. Now, if $({}_k\varphi_j)_j$ is not surjective, then $\mathbf{X}$ is a direct sum of the representation $\mathbf{Y} = (Y_i, {}_j\psi_i)$, where $Y_i = X_i$ for $i \neq k$, $Y_k = \operatorname{Im}[({}_k\varphi_j)_j]$ and all ${}_j\psi_i = {}_j\varphi_i$, and of copies of $\mathbf{F}_k$. Thus, if $\mathbf{X}$ is indecomposable, then $\mathbf{X} \approx \mathbf{F}_k$ (and then $S_k^+\mathbf{X} = \mathbf{0}$) or $({}_k\varphi_j)_j$ is surjective. In the latter case, μ_k is an isomorphism, and we have $S_k^- S_k^+ \mathbf{X} \approx \mathbf{X}$. Consequently, also $S_k^+ S_k^- S_k^+ \mathbf{X} \approx S_k^+ \mathbf{X}$, and therefore the composition of any two maps in

$$\operatorname{End}\mathbf{X} \xrightarrow{S_k^+} \operatorname{End}(S_k^+\mathbf{X}) \xrightarrow{S_k^-} \operatorname{End}(S_k^- S_k^+\mathbf{X}) \xrightarrow{S_k^+} \operatorname{End}(S_k^+ S_k^- S_k^+\mathbf{X})$$

is a bijection. Hence, $\operatorname{End}(S_k^+\mathbf{X})_k \approx \operatorname{End}\mathbf{X}$.

Finally, the exact sequence

$$0 \longrightarrow (S_k^+\mathbf{X})_k \longrightarrow \bigoplus_{i\in\Gamma} X_i \otimes {}_iM_k \longrightarrow X_k \longrightarrow 0$$

yields the formula

$$\begin{aligned} \dim\,[(S_k^+\mathbf{X})_k\,]_{F_k} &= \sum_{i\in\Gamma} \dim(X_i \otimes {}_iM_k)_{F_k} - \dim(X_k)_{F_k} \\ &= -\dim(X_k)_{F_k} + \sum_{i\in\Gamma} \dim({}_iM_k)_{F_k} \cdot \dim(X_i)_{F_i}; \end{aligned}$$

thus, since $\dim({}_iM_k)_{F_k} = d_{ik}$ and $(S_k^+\mathbf{X})_i = X_i$ for $i \neq k$, we get $\mathbf{dim}(S_k^+\mathbf{X}) = s_k(\mathbf{dim}\,\mathbf{X})$.

(ii) The second part of Proposition 2.1 can be proved in a similar way as (i) considering the diagram

$$\begin{array}{ccccccccc} & & & & X_k & & & & \\ & & & \overset{\epsilon_k}{\swarrow} & \Big\downarrow ({}_i\bar{\varphi}_k)_i & & & & \\ 0 \longrightarrow & (S_k^+ S_k^- \mathbf{X})_k & \longrightarrow & & \bigoplus_{i\in\Gamma} X_i \otimes {}_iM_k & \longrightarrow & (S_k^-\mathbf{X})_k & \longrightarrow 0. \end{array}$$

Let us illustrate the use of Proposition 2.1 on the following example.

EXAMPLE. Let G be a subfield of F with $\dim_G F = \dim F_G = d$ such that there exists a nontrivial bimodule mapping $\epsilon : {}_GF_G \longrightarrow {}_GG_G$. Then $\mathfrak{M} = (F_1 = G, F_2 = F, {}_1M_2 = {}_GF_F, {}_2M_1 = {}_FF_G)$ is a modulation of $(\Gamma, \mathbf{d}) = \underset{1}{\cdot} \xrightarrow{(1,d)} \underset{2}{\cdot}$. Let Ω be the orientation $\underset{1}{\cdot} \longrightarrow \underset{2}{\cdot}$. Now, ϵ defines a bimodule embedding $\omega : {}_FF_F \longrightarrow {}_FF_G \otimes {}_GF_F$ (such that $(1 \otimes \epsilon)(\omega \otimes 1)$ is the identity of $F_G = F_F \otimes {}_FF_G = F_G \otimes {}_GG_G$). If U_G is a G-subspace of a vector F-space V_F, then the mapping $U_G \hookrightarrow V_F$ determines a representation of $(\mathfrak{M}, \Omega)$. We claim that *there exists a unique indecomposable representation* $\mathbf{X} = (U_G, V_F, \varphi)$ *of* $(\mathfrak{M}, \Omega)$ *of dimension type* $(d, d-1)$ *given by the canonical mapping*

$$U_G = F_G \otimes {}_GG_G \longrightarrow (F_G \otimes {}_GF_F)/\omega(F) = V_F,$$

and that this mapping is an inclusion. Moreover, *the endomorphism ring of* $\mathbf{X}$ *is* F, which operates on ${}_FF_G \otimes {}_GG_G$ and $({}_FF_G \otimes {}_GF_F)/(\omega({}_FF_F)$ from the left canonically.

For, consider the obvious representation $\mathbf{Y} = (F_G, F_F, F_G \xleftarrow{1} F_F \otimes {}_FF_G)$ of $(\mathfrak{M}, s_2\Omega)$ and apply the functor S_1^-. Thus, we consider the mapping $\omega = \bar{1} : F_F \longrightarrow F_G \otimes {}_GF_F$, and form the cokernel $F_G \otimes {}_GF_F \longrightarrow (F_G \otimes {}_GF_F)/\omega(F)$. This shows that $\mathbf{X} = S_1^- \mathbf{Y}$. Since we know that $S_1^-\mathbf{Y}$ is indecomposable, the mapping $U_G \longrightarrow V_F$ has to be an inclusion. Since End $\mathbf{Y} = F$ with the canonical operation from the left, the same is true for $\mathbf{X}$. Also the representation $\mathbf{X}$ is uniquely determined because of the fact that $\mathbf{X} = S_1^- S_1^+ \mathbf{X} = S_1^- \mathbf{Y}$. Here, we make use of the fact that the only indecomposable representation of dimension type $\dim S_1^+ \mathbf{X}$ is $\mathbf{Y}$.

Later, in Proposition 2.4, we shall determine the indecomposable objects of $L(\mathfrak{M}, \Omega)$ which are projective or injective. The first step in this direction is the following rather obvious

LEMMA 2.2. *Let* $(\mathfrak{M}, \Omega)$ *be a realization of* $(\Gamma, \mathbf{d})$.

(i) *If* $k \in \Gamma$ *is a sink, then* $\mathbf{F}_k \in L(\mathfrak{M}, \Omega)$ *is projective.*

(ii) *If* $k \in \Gamma$ *is a source, then* $\mathbf{F}_k \in L(\mathfrak{M}, \Omega)$ *is injective.*

PROPOSITION 2.3. *Let* $\mathbf{X}$ *and* $\mathbf{X}'$ *be indecomposable representations in* $L(\mathfrak{M}, \Omega)$.

(i) *If* $k \in \Gamma$ *is a sink and* $S_k^+ \mathbf{X}' \neq \mathbf{0}$, *then* S_k^+ *induces an isomorphism* $\operatorname{Ext}^1(\mathbf{X}, \mathbf{X}') \longrightarrow \operatorname{Ext}^1(S_k^+ \mathbf{X}, S_k^+ \mathbf{X}')$.

(ii) *If* $k \in \Gamma$ *is a source and* $S_k^- \mathbf{X} \neq \mathbf{0}$, *then* S_k^- *induces an isomorphism* $\operatorname{Ext}^1(\mathbf{X}, \mathbf{X}') \longrightarrow \operatorname{Ext}^1(S_k^- \mathbf{X}, S_k^- \mathbf{X}')$.

Proof. Let $k \in \Gamma$ be a sink. If $S_k^+ \mathbf{X} = \mathbf{0}$, then $\mathbf{X} \approx \mathbf{F}_k$ is projective and therefore $\operatorname{Ext}^1(\mathbf{X}, \mathbf{X}') = 0$. Thus, we may assume that $S_k^+ \mathbf{X} \neq \mathbf{0}$.

We want to show that S_k^+ carries exact sequences $\mathbf{0} \longrightarrow \mathbf{X}' \xrightarrow{\mu} \mathbf{E} \xrightarrow{\epsilon} \mathbf{X} \longrightarrow \mathbf{0}$ into exact sequences. Obviously, we need only to verify that the kth component of

$$\mathbf{0} \xrightarrow{\mu} S_k^+ \mathbf{X}' \xrightarrow{S_k^+ \mu} S_k^+ \mathbf{E} \xrightarrow{S_k^+ \epsilon} S_k^+ \mathbf{X} \longrightarrow \mathbf{0}$$

is exact. To this end, consider the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (S_k^+ \mathbf{X}')_k & \xrightarrow{(S_k^+ \mu)_k} & (S_k^+ \mathbf{E})_k & \xrightarrow{(S_k^+ \epsilon)_k} & (S_k^+ \mathbf{X})_k & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigoplus_j X_j' \otimes {}_jM_k & \xrightarrow{\bigoplus \mu_k \otimes 1} & \bigoplus_j E_j \otimes {}_jM_k & \xrightarrow{\bigoplus \epsilon_k \otimes 1} & \bigoplus_j X_j \otimes {}_jM_k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_k' & \xrightarrow{\mu_k} & E_k & \xrightarrow{\epsilon_k} & X_k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

By assumption, the middle and the lower rows are exact. The first and the third columns are exact, because $\mathbf{X}$ and $\mathbf{X}'$ are indecomposable and $S_k^+ \mathbf{X} \neq \mathbf{0} \neq S^+ \mathbf{X}'$; therefore, also the second column is exact (the extension of two epimorphisms is an epimorphism). Finally, the upper row is exact because the formation of kernels is left exact, and a dimension argument implies that $(S_k^+ \epsilon)_k$ is surjective.

In this way, S_k^+ defines a map $\operatorname{Ext}^1(\mathbf{X}, \mathbf{X}') \longrightarrow \operatorname{Ext}^1(S_k^+ \mathbf{X}, S_k^+ \mathbf{X}')$, and obviously S_k^- defines the inverse map. This proves (i). The second part of Proposition 2.3 follows in a similar fashion.

Now, let $k_1, k_2, \ldots, k_n$ be an admissible ordering of Γ with respect to the orientation Ω. Then

$$C^+ = S^+_{k_n} \cdots S^+_{k_2} S^+_{k_1} \quad \text{and} \quad C^- = S^-_{k_1} S^-_{k_2} \cdots S^-_{k_n}$$

are functors from $L(\mathfrak{M}, \Omega)$ to $L(\mathfrak{M}, \Omega)$ (the same orientation!) and are called the *Coxeter functors* of $L(\mathfrak{M}, \Omega)$. Note that both C^+ and C^- depend only on the orientation Ω and not on a particular admissible ordering of Γ. Obviously, they depend on a particular choice of the bimodule mappings

$${}_jM_i \approx \mathrm{Hom}_{F_i}({}_iM_j, F_i) \approx \mathrm{Hom}_{F_j}({}_iM_j, F_j);$$

however, the choice of these mappings is irrelevant for the theory.

Also, for $1 \leqslant t \leqslant n$, we introduce the representations $\mathbf{P}_{k_t}$ and $\mathbf{Q}_{k_t}$ from $L(\mathfrak{M}, \Omega)$ by

$$\mathbf{P}_{k_t} = S^-_{k_1} S^-_{k_2} \cdots S^-_{k_{t-1}} \mathbf{F}_{k_t}, \quad \text{where } \mathbf{F}_{k_t} \in L(\mathfrak{M}, s_{k_t} s_{k_{t+1}} \cdots s_{k_n} \Omega),$$

and

$$\mathbf{Q}_{k_t} = S^+_{k_n} S^+_{k_{n-1}} \cdots S^+_{k_{t+1}} \mathbf{F}_{k_t}, \quad \text{where } \mathbf{F}_{k_t} \in L(\mathfrak{M}, s_{k_t} s_{k_{t-1}} \cdots s_{k_1} \Omega).$$

PROPOSITION 2.4. *Let C^+ and C^- be the Coxeter functors for $L(\mathfrak{M}, \Omega)$ and c the corresponding Coxeter transformation. Let $\mathbf{X}$ be an indecomposable representation from $L(\mathfrak{M}, \Omega)$. Then the four statements, both in* (i) *and* (ii), *are respectively equivalent.*

(i) (1) $\mathbf{X}$ *is projective*;

(2) $\mathbf{X} \approx \mathbf{P}_k$ *for some* $k \in \Gamma$;

(3) $C^+\mathbf{X} = \mathbf{0}$;

(4) $c(\dim \mathbf{X}) \ngtr 0$.

(ii) (1) $\mathbf{X}$ *is injective*;

(2) $\mathbf{X} \approx \mathbf{Q}_k$ *for some* $k \in \Gamma$;

(3) $C^-\mathbf{X} = \mathbf{0}$;

(4) $c^{-1}(\dim \mathbf{X}) \ngtr 0$.

Proof. Let $k_1, k_2, \ldots, k_n$ be an admissible ordering of Γ with respect to Ω. Using Proposition 2.1, we get

$$C^+\mathbf{P}_{k_t} = S^+_{k_n} \cdots S^+_{k_2} S^+_{k_1} S^-_{k_1} S^-_{k_2} \cdots S^-_{k_{t-1}} \mathbf{F}_{k_t} \approx S^+_{k_n} \cdots S^+_{k_t} \mathbf{F}_{k_t} = \mathbf{0}.$$

Conversely, if $C^+\mathbf{X} = \mathbf{0}$ and if t is the smallest index such that $S^+_{k_t} \cdots S^+_{k_2} S^+_{k_1} \mathbf{X} = \mathbf{0}$, then $S^+_{k_{t-1}} \cdots S^+_{k_2} S^+_{k_1} \mathbf{X} \approx \mathbf{F}_{k_t}$, and therefore $\mathbf{X} \approx S^-_{k_1} S^-_{k_2} \cdots S^-_{k_{t-1}} \mathbf{F}_{k_t} = \mathbf{P}_{k_t}$. The relations between the corresponding dimension types result immediately in the equivalence of (3) and (4).

Now, if we show that all $\mathbf{P}_k$, $k \in \Gamma$, are projective, then these n pair-wise nonisomorphic indecomposable representations must be just all indecomposable projective objects of $L(\mathfrak{M}, \Omega)$. Indeed, $L(\mathfrak{M}, \Omega)$ is equivalent to the category of all right R-modules, where $R = R_{(\mathfrak{M}, \Omega)}$ is the tensor algebra constructed from the corresponding realization $(\mathfrak{M}, \Omega)$ (see [9] or [4]), and since there are no circuits with cyclic orientation, R is semiprimary. Therefore, the indecomposable projective representations are precisely the projective covers of the simple representations.

Thus, let $\mathbf{X}'$ be an indecomposable representation of $(\mathfrak{M}, \Omega)$. First, assume that

$S^+_{k_{t-1}} \cdots S^+_{k_2} S^+_{k_1} \mathbf{X}' \neq \mathbf{0}$. Then, by Proposition 2.3,

$$\mathrm{Ext}^1(\mathbf{P}_{k_t}, \mathbf{X}') = \mathrm{Ext}^1(S^-_{k_1} S^-_{k_2} \cdots S^-_{k_{t-1}} \mathbf{F}_{k_t}, \mathbf{X}') \approx \mathrm{Ext}^1(\mathbf{F}_{k_t}, S^+_{k_{t-1}} \cdots S^+_{k_2} S^+_{k_1} \mathbf{X}'),$$

and this is zero because $\mathbf{F}_{k_t} \in L(\mathfrak{M}, s_{k_t} s_{k_{t+1}} \cdots s_{k_n} \Omega)$ is projective according to Lemma 2.2 (k_t is a sink with respect to $s_{k_t} s_{k_{t+1}} \cdots s_{k_n} \Omega$). But, if $S^+_{k_{t-1}} \cdots S^+_{k_2} S^+_{k_1} \mathbf{X}' = \mathbf{0}$, then $\mathbf{X}' = \mathbf{P}_{k_s}$ with $s < t$, and making use of Proposition 2.3 again, we get

$$\mathrm{Ext}^1(\mathbf{P}_{k_t}, \mathbf{X}') = \mathrm{Ext}^1(\mathbf{P}_{k_t}, \mathbf{P}_{k_s}) \approx \mathrm{Ext}^1(S^-_{k_s} \cdots S^-_{k_{t-2}} S^-_{k_{t-1}} \mathbf{F}_{k_t}, \mathbf{F}_{k_s}).$$

Thus, assume (without loss of generality) that $s = 1$ and consider an extension $\mathbf{0} \longrightarrow \mathbf{F}_{k_1} \longrightarrow \mathbf{E} \longrightarrow \mathbf{P}_{k_t} \longrightarrow \mathbf{0}$ with $\mathbf{E} = (E_i, {}_j\varphi_i)$. We claim that the mapping $\bigoplus_{j \in \Gamma} E_j \otimes {}_jM_{k_1} \longrightarrow E_{k_1}$ is not surjective; this follows from the fact that it is an extension of the corresponding map in $\mathbf{F}_{k_1}$, which is zero, by the corresponding map in $\mathbf{P}_{k_t}$, which is an isomorphism. This shows that $\mathbf{E}$ has a direct summand of the form $\mathbf{F}_{k_1}$, and, in this way, we get a splitting of the given sequence, as required. The second part of Proposition 2.4 can be proved by dual arguments.

PROPOSITION 2.5. *Let* $(\mathfrak{M}, \Omega)$ *be a realization of* $(\Gamma, \mathbf{d})$. *Let* C^+, C^- *be the Coxeter functors and c the Coxeter transformation with respect to* Ω. *Let* $\mathbf{X} \in L(\mathfrak{M}, \Omega)$ *be indecomposable. Then*

(i) *either*

$$\mathbf{X} \approx \mathbf{P}_k \text{ for some } k \in \Gamma \text{ and } C^+\mathbf{X} = \mathbf{0}$$

or

$$\mathbf{X} \approx C^- C^+ \mathbf{X}, \ \mathrm{End}(C^+\mathbf{X}) \approx \mathrm{End}\,\mathbf{X} \text{ and } \mathbf{dim}(C^+\mathbf{X}) = c(\mathbf{dim}\,\mathbf{X});$$

(ii) *either*

$$\mathbf{X} \approx \mathbf{Q}_k \text{ for some } k \in \Gamma \text{ and } C^-\mathbf{X} = \mathbf{0},$$

or

$$\mathbf{X} \approx C^+ C^- \mathbf{X}, \ \mathrm{End}(C^-\mathbf{X}) \approx \mathrm{End}\,\mathbf{X} \text{ and } \mathbf{dim}(C^-\mathbf{X}) = c^{-1}(\mathbf{dim}\,\mathbf{X}).$$

Proof. Both statements follow immediately from Proposition 2.1 and 2.4.

The preceding Proposition 2.5 together with Proposition 1.9 imply further

PROPOSITION 2.6. *Let* $(\mathfrak{M}, \Omega)$ *be a realization of a valued graph* $(\Gamma, \mathbf{d})$.

(a) *If* $(\Gamma, \mathbf{d})$ *is a Dynkin diagram, then the mapping* $\mathbf{dim}: L(\mathfrak{M}, \Omega) \longrightarrow \mathbf{Q}^\Gamma$ *provides a one-to-one correspondence between all positive roots of* $(\Gamma, \mathbf{d})$ *and all indecomposable representations in* $L(\mathfrak{M}, \Omega)$.

(b) *If* $(\Gamma, \mathbf{d})$ *is an extended Dynkin diagram, then the mapping* $\mathbf{dim}$ *provides a one-to-one correspondence between all positive roots of* $(\Gamma, \mathbf{d})$ *of nonzero defect and all indecomposable representations in* $L(\mathfrak{M}, \Omega)$ *of nonzero defect.*

In the final part of this chapter, we want to study in greater detail the $\mathrm{End}\,\mathbf{X}' - \mathrm{End}\,\mathbf{X}$-bimodule structure of $\mathrm{Ext}^1(\mathbf{X}, \mathbf{X}')$ for certain representations $\mathbf{X}$ and $\mathbf{X}'$.

PROPOSITION 2.7. *Let* $(\mathfrak{M}, \Omega)$ *be a realization of* $(\Gamma, \mathbf{d})$ *and* C^+, C^- *the corresponding Coxeter functors.*

(i) *If* $k \in \Gamma$ *is a sink, then* $\mathrm{Ext}^1(C^-\mathbf{F}_k, \mathbf{F}_k) \approx {}_{F_k}(F_k)_{F_k}$.

(ii) *If* $k \in \Gamma$ *is a source, then* $\mathrm{Ext}^1(\mathbf{F}_k, C^+\mathbf{F}_k) \approx {}_{F_k}(F_k)_{F_k}$.

Proof. (i) Let k be a sink. We shall construct an exact sequence

$$0 \longrightarrow \mathbf{F}_k \longrightarrow \mathbf{T} \longrightarrow C^-\mathbf{F}_k \longrightarrow 0$$

with a projective representation $\mathbf{T}$ which does not possess any direct summand of the form $\mathbf{F}_k$. Then, obviously $\mathrm{Hom}(\mathbf{T}, \mathbf{F}_k) = 0$ and $\mathrm{Ext}^1(\mathbf{T}, \mathbf{F}_k) = 0$, and therefore the exact sequence

$$\mathrm{Hom}(\mathbf{T}, \mathbf{F}_k) \longrightarrow \mathrm{Hom}(\mathbf{F}_k, \mathbf{F}_k) \longrightarrow \mathrm{Ext}^1(C^-\mathbf{F}_k, \mathbf{F}_k) \longrightarrow \mathrm{Ext}^1(\mathbf{T}, \mathbf{F}_k)$$

yields the required isomorphism

$$\mathrm{Ext}^1(C^-\mathbf{F}_k, \mathbf{F}_k) \approx \mathrm{Hom}(\mathbf{F}_k, \mathbf{F}_k) = {}_{F_k}(F_k)_{F_k}.$$

Now, in order to construct $\mathbf{T}$ and the above exact sequence, consider an admissible ordering $k = k_1, k_2, \ldots, k_n$ of Γ with respect to Ω. Let $\mathbf{W}_{k_t}$ be the representation of $(\mathfrak{M}, s_{k_t}s_{k_{t+1}} \cdots s_{k_n}\Omega)$ such that

$$(\mathbf{W}_{k_t})_{k_t} = ({}_{k_1}M_{k_t})_{F_{k_t}} \text{ and } (\mathbf{W}_{k_t})_i = 0 \text{ for } i \neq k_t.$$

Put

$$\mathbf{V}_{k_t} = S^-_{k_1}S^-_{k_2} \cdots S^-_{k_{t-1}}\mathbf{W}_{k_t}, \quad 1 \leqslant t \leqslant n.$$

Since $\mathbf{W}_{k_t}$ is the direct sum of $d_{k_1k_t}$ copies of $\mathbf{F}_{k_t}$, it follows readily that $\mathbf{V}_{k_t}$ is the direct sum of the same number of copies of $\mathbf{P}_{k_t}$; therefore, $\mathbf{V}_{k_t}$ is projective. Also, $\mathbf{V}_{k_1} = \mathbf{0}$. Observe that, for all $s \neq t$,

$$(\mathbf{V}_{k_t})_{k_s} = \bigoplus_{k_r \to k_s} (\mathbf{V}_{k_t})_{k_r} \otimes {}_{k_r}M_{k_s},$$

where the summation runs over all neighbours k_r of k_s such that $k_r \longrightarrow k_s$ in the orientation Ω (and therefore, a fortiori, $r > s$). On the other hand, the components of $C^-\mathbf{F}_k$ satisfy, for $s \neq 1$,

$$(C^-\mathbf{F}_k)_{k_s} = \left[\bigoplus_{k_r \to k_s} (C^-\mathbf{F}_k)_{k_r} \otimes {}_{k_r}M_{k_s}\right] \oplus ({}_{k_1}M_{k_s})_{F_{k_s}}.$$

Finally, we consider the exact sequence

$$0 \longrightarrow \mathbf{F}_k \longrightarrow \mathbf{T} \xrightarrow{\epsilon} C^-\mathbf{F}_k \longrightarrow 0,$$

where $\mathbf{T} = \bigoplus_t \mathbf{V}_{k_t}$ and $\epsilon = (\epsilon_{k_s})$ is defined by induction on $s = n, n-1, \ldots, 1$ as follows: For $s > 1$,

$$\epsilon_{k_s} = \left[\bigoplus \epsilon_{k_r} \otimes 1_{({}_{k_r}M_{k_s})}\right] \oplus 1_{({}_{k_1}M_{k_s})},$$

which means that ϵ_{k_s} identifies T_{k_s} with $(C^-\mathbf{F}_k)_{k_s}$. And, for $s = 1$, we have the exact sequence

$$0 \longrightarrow F_{k_1} \longrightarrow \bigoplus_{k_r} (C^-\mathbf{F}_k)_{k_r} \otimes {}_{k_r}M_{k_1} \xrightarrow{\pi} (C^-\mathbf{F}_k)_{k_1} \longrightarrow 0;$$

here, $\bigoplus_{k_r}(C^-\mathbf{F}_k)_{k_r} \otimes {}_{k_r}M_{k_1} = T_{k_1}$ and we put $\epsilon_{k_1} = \pi$. Thus, we have defined ϵ whose kernel is just $\mathbf{F}_k$, as required. This completes the proof of (i).

The assertion (ii) can be established by a similar argument.

PROPOSITION 2.8. *Let $(\mathfrak{M}, \Omega)$ be a realization of $(\Gamma, \mathbf{d})$ and C^+, C^- the corresponding Coxeter functors.*

(i) *For $\mathbf{X} = C^{-r}\mathbf{P}_k$ with $C^-\mathbf{X} \neq 0$,*

$$\mathrm{Ext}^1(C^-\mathbf{X}, \mathbf{X}) \approx {}_{F_k}(F_k)_{F_k}.$$

(ii) *For* $\mathbf{X} = C^{+r}\mathbf{Q}_k$ *with* $C^+\mathbf{X} \neq 0$,

$$\mathrm{Ext}^1(\mathbf{X}, C^+\mathbf{X}) \approx {}_{F_k}(F_k)_{F_k}.$$

Proof. Both statements are straightforward consequences of Propositions 2.7 and 2.3.

3. Representation of Defect zero: General Theory

Let $(\mathfrak{M}, \Omega)$ be a (fixed) realization of an extended Dynkin diagram $(\Gamma, \mathbf{d})$. The *defect* $\partial_c(\mathbf{dim}\ \mathbf{X})$ of a representation $\mathbf{X} \in L(\mathfrak{M}, \Omega)$ will be denoted simply by $\partial\mathbf{X}$. The general theory of representations of zero defect follows closely the theory of quadruples by I. M. Gelfand and V. A. Ponomarev. First, we have the following equivalence.

Lemma 3.1. *The following properties are equivalent for* $\mathbf{X} \in L(\mathfrak{M}, \Omega)$.

(i) $\mathbf{X}$ *is a direct sum of indecomposable representations of zero defect in* $L(\mathfrak{M}, \Omega)$;

(ii) $\partial\mathbf{X} = 0$ *and* $\partial\mathbf{X}' \leqslant 0$ *for every subobject* $\mathbf{X}'$ *of* $\mathbf{X}$ *in* $L(\mathfrak{M}, \Omega)$;

(iii) $\partial\mathbf{X} = 0$ *and* $\partial\mathbf{X}'' \geqslant 0$ *for every quotient* $\mathbf{X}''$ *of* $\mathbf{X}$ *in* $L(\mathfrak{M}, \Omega)$.

Proof (cf. [10]). Assume (i) and let $\mathbf{X}'$ be a subobject of $\mathbf{X}$ such that $\partial\mathbf{X}' > 0$. Denoting the order of the induced Coxeter transformation $\bar{c}$ on $\mathbf{Q}^\Gamma/N$ by m, we have, for arbitrary r,

$$\mathbf{dim}(C^{+mr}\mathbf{X}') = c^{mr}(\mathbf{dim}\ \mathbf{X}') = \mathbf{dim}\ \mathbf{X}' + r(\partial\mathbf{X}')\mathbf{n},$$

which can be arbitrarily large. On the other hand, since C^+ preserves monomorphisms (for, S_k^+ involves only construction of a certain kernel), we have

$$\mathbf{dim}(C^{+mr}\mathbf{X}') \leqslant \mathbf{dim}(C^{+mr}\mathbf{X}) = \mathbf{dim}\ \mathbf{X} \text{ for all } r,$$

a contradiction. Thus (i) implies (ii).

Conversely, (ii) implies (i). For, all summands in a direct decomposition of $\mathbf{X}$ have defect $\leqslant 0$, and the total sum of their defects is 0. Hence, all are of zero defect.

A dual argument yields the equivalence of (i) and (iii).

Definition. *The representations in* $L(\mathfrak{M}, \Omega)$ *satisfying the properties described in Lemma* 3.1 *will be called regular.*

Thus, we have

Proposition 3.2. *Let* $(\mathfrak{M}, \Omega)$ *be a realization of an extended Dynkin diagram. Let* $R(\mathfrak{M}, \Omega)$ *be the full subcategory of all regular representations in* $L(\mathfrak{M}, \Omega)$. *Then* $R(\mathfrak{M}, \Omega)$ *is an abelian exact subcategory of* $L(\mathfrak{M}, \Omega)$, *closed under extensions.*

Proof. Let $\mathbf{V}, \mathbf{W} \in R(\mathfrak{M}, \Omega)$ and let $\alpha : \mathbf{V} \longrightarrow \mathbf{W}$. Let $\alpha = \alpha''\alpha'$ with an epimorphism $\alpha' : \mathbf{V} \longrightarrow \mathbf{X}$ and a monomorphism $\alpha'' : \mathbf{X} \longrightarrow \mathbf{W}$. Then, by Lemma 3.1 (ii), $\partial\mathbf{X} \leqslant 0$, and by (iii), $\partial\mathbf{X} \geqslant 0$; hence $\partial\mathbf{X} = 0$. Since subobjects of $\mathbf{X}$ are also subobjects of $\mathbf{W}$, $\mathbf{X}$ is, in view of (ii), regular. Thus, images of morphisms between regular representations are regular. On the other hand, if

$$\mathbf{0} \longrightarrow \mathbf{V} \longrightarrow \mathbf{X} \longrightarrow \mathbf{W} \longrightarrow \mathbf{0}$$

is exact, then $\partial\mathbf{X} = \partial\mathbf{V} + \partial\mathbf{W}$ and therefore, if any two of the representations are of zero defect, so is the third one. In fact, if any two of the representations in the above sequence are regular, so is the third one; for, subobjects of $\mathbf{V}$ are subobjects of $\mathbf{X}$, quotients of $\mathbf{W}$ are quotients of $\mathbf{X}$ and subobjects of $\mathbf{X}$ are extensions of subobjects of $\mathbf{V}$ by subobjects of $\mathbf{W}$. This completes the proof.

A simple object $\mathbf{X}$ of $R(\mathfrak{M}, \Omega)$ is said to be *homogeneous* if $\mathbf{dim}\, \mathbf{X} \in N$. An arbitrary regular representation is called homogeneous if all its simple composition factors (in the category $R(\mathfrak{M}, \Omega)$) are homogeneous. The full subcategory of $R(\mathfrak{M}, \Omega)$ of all homogeneous representations will be denoted by $H(\mathfrak{M}, \Omega)$.

Now, consider the dual vector space $\mathbf{Q}^{\Gamma *}$ of all linear forms $\chi : \mathbf{Q}^\Gamma \longrightarrow \mathbf{Q}$. Write $\chi = (\chi_i)$ with respect to the dual basis, i.e. $\chi \mathbf{x} = \Sigma_{i \in \Gamma} \chi_i x_i$ for $\mathbf{x} = (x_i) \in \mathbf{Q}^\Gamma$. For a representation $\mathbf{X} = (X_i, {}_j\varphi_i) \in L(\mathfrak{M}, \Omega)$, we write $\chi \mathbf{X} = \chi(\mathbf{dim}\, \mathbf{X})$.

For each $w \in W$, define $w^* : \mathbf{Q}^{\Gamma *} \longrightarrow \mathbf{Q}^{\Gamma *}$ by

$$(\alpha w^*)(\mathbf{x}) = \alpha(w\mathbf{x}),\ \mathbf{x} \in \mathbf{Q}^\Gamma.$$

For $k \in \Gamma$, s_k^* is an involution and, writing $\chi s_k^* = \xi$, we obtain

$$\xi_k = -\chi_k \quad \text{and} \quad \xi_i = \chi_i + d_{ij}\chi_k \quad \text{for} \quad i \neq k.$$

For $c = s_{k_n} \cdots s_{k_2} s_{k_1}$, we have $c^* = s_{k_1}^* s_{k_2}^* \cdots s_{k_n}^*$, and

$$\{\chi \mid \chi c^* = \chi\} = \Delta_c$$

is a one-dimensional space generated by the defect vector ∂_c. (One can verify that it is a complement of the image of $\mathbf{Q}^\Gamma$ under the mapping $\mathbf{Q}^\Gamma \longrightarrow \mathbf{Q}^{\Gamma *}$ defined by $\mathbf{x} \longmapsto B_\Gamma(\mathbf{x}, -)$, whose kernel is obviously N.) Also

$$\{\mathbf{x} \mid \chi c^*(\mathbf{x}) = \chi(\mathbf{x}) \text{ for all } \chi \in \mathbf{Q}^{\Gamma *}\} = \{\mathbf{x} \mid c\mathbf{x} = \mathbf{x}\} = N.$$

DEFINITION. *A representation* $\mathbf{E} \in L(\mathfrak{M}, \Omega)$ *is said to possess an equation if there exists* $\eta \in \mathbf{Q}^{\Gamma *}$ *such that*

(i) $\eta \mathbf{E} > 0$;

(ii) *if* $\eta \mathbf{X} > 0$ *for some regular representation* $\mathbf{X}$, *then* $\mathbf{E} \hookrightarrow \mathbf{X}$;

(iii) *if* $\eta \mathbf{X} < 0$ *for some regular representation* $\mathbf{X}$, *then* $\mathbf{X} \twoheadrightarrow C^+\mathbf{E}$.

In what follows, $\mathbf{E}_r = C^{+r}\mathbf{E}$ (with $\mathbf{E}_0 = \mathbf{E}$) for a given $\mathbf{E} \in L(\mathfrak{M}, \Omega)$.

LEMMA 3.3. *Let* $\mathbf{E} \in L(\mathfrak{M}, \Omega)$ *possess an equation* η *and assume that, under the action of the Coxeter transformation* c, $\mathbf{dim}\, \mathbf{E}$ *has a finite orbit containing* $l \geqslant 2$ *elements. Then*

(1) $C^{+l}\mathbf{E} \approx \mathbf{E}$ *and all*

$$\mathbf{E}_r, \quad 0 \leqslant r < l,$$

are (mutually nonisomorphic) simple regular nonhomogeneous representations.

(2) $\eta \mathbf{X} = 0$ *for all simple regular representations* $\mathbf{X} \not\approx \mathbf{E}_0$, $\mathbf{X} \not\approx \mathbf{E}_1$.

(3) $\mathrm{Ext}^1(\mathbf{E}_r, \mathbf{X}) = 0$ *for all simple regular representations* $\mathbf{X} \not\approx \mathbf{E}_r$ *and* $\mathbf{X} \not\approx \mathbf{E}_{r+1}$, $0 \leqslant r < l$.

(4) $\mathrm{Ext}^1(\mathbf{X}, \mathbf{E}_r) = 0$ *for all simple regular representations* $\mathbf{X} \not\approx \mathbf{E}_r$ *and* $\mathbf{X} \not\approx \mathbf{E}_{r-1}$, $0 < r \leqslant l$.

Proof. (1) For arbitrary natural r, let $\mathbf{E}_r = \bigoplus \mathbf{X}_t$ be a direct decomposition into indecomposable representations. Since the orbit of $\mathbf{dim}\, \mathbf{E}_r$ is finite, all orbits of $\mathbf{dim}\, \mathbf{X}_t$ are finite and thus all $\partial \mathbf{X}_t = 0$. Hence, all $\mathbf{E}_r$ are regular representations. Furthermore, since $\eta(C^{+l}\mathbf{E}) = \eta\mathbf{E} > 0$, $\mathbf{E}$ can be embedded into $C^{+l}\mathbf{E}$ by (ii) of Definition; hence $C^{+l}\mathbf{E} = \mathbf{E}$. Moreover, all $\mathbf{E}_r$, $0 \leqslant r < l$, are simple in $R(\mathfrak{M}, \Omega)$; it is sufficient to show this for $\mathbf{E}$ because $\mathbf{E} \approx C^{+(l-r)}\mathbf{E}_r$. But η is additive on extensions and therefore, in view of $\eta\mathbf{E} > 0$, there is a composition factor $\mathbf{X}$ of $\mathbf{E}$ in $R(\mathfrak{M}, \Omega)$ with $\eta\mathbf{X} > 0$: by (ii), $\mathbf{E}$ can be embedded into $\mathbf{X}$ and thus $\mathbf{E} = \mathbf{X}$ is simple. Also, $\mathbf{E}$ is nonhomogeneous, because $\mathbf{dim}\, \mathbf{E} \notin N$.

(2) Now, let $\mathbf{X}$ be a simple regular representation; then by (ii) or (iii) of Definition $\eta\mathbf{X} = 0$ unless $\mathbf{X} \approx \mathbf{E}$ or $\mathbf{X} \approx \mathbf{E}_1$. In fact $\eta\mathbf{E}_1 = -\eta\mathbf{E}$ because $\Sigma_{0 \leqslant r < l}\eta\mathbf{E}_r = 0$; the latter follows from the fact that $\Sigma_{0 \leqslant r < l}\mathbf{dim}\, \mathbf{E}_r \in N$.

(3) Thus assume that $\mathbf{X} \not\approx \mathbf{E}$ and $\mathbf{X} \not\approx \mathbf{E}_1$ and consider an extension

$$\mathbf{0} \longrightarrow \mathbf{X} \longrightarrow \mathbf{V} \longrightarrow \mathbf{E} \longrightarrow \mathbf{0}.$$

Since $\eta\mathbf{V} = \eta\mathbf{X} + \eta\mathbf{E} > 0$, the regular representation $\mathbf{V}$ contains a representation $\mathbf{Y}$ isomorphic to $\mathbf{E}$. In view of $\eta\mathbf{X} = 0$ and $\eta\mathbf{E} > 0$, $\mathbf{X}$ and $\mathbf{Y}$ are nonisomorphic. Consequently, $\mathbf{X} \cap \mathbf{Y} = \mathbf{0}$, and $\mathbf{E} \approx \mathbf{Y} \hookrightarrow \mathbf{V} \twoheadrightarrow \mathbf{E}$ is the identity of $\mathbf{E}$, i.e. the sequence splits. According to Proposition 2.3,

$$\mathrm{Ext}^1(\mathbf{E}_r, \mathbf{X}) = \mathrm{Ext}^1(\mathbf{E}, C^{-r}\mathbf{X})$$

and the statement (2) follows.

(4) To prove the last statement, consider the extension

$$\mathbf{0} \longrightarrow \mathbf{E}_1 \longrightarrow \mathbf{V} \longrightarrow C^{-(r-1)}\mathbf{X} \longrightarrow \mathbf{0}, \quad 1 \leqslant r \leqslant l,$$

with a simple regular representation $\mathbf{X}$ such that $C^{-(r-1)}\mathbf{X} \not\approx \mathbf{E}$ and $C^{-(r-1)}\mathbf{X} \approx \mathbf{E}_1$. Then $\eta\mathbf{V} = \eta\mathbf{E}_1 < 0$ and $\mathbf{E}_1$ is a quotient of $\mathbf{V}$. Hence the sequence splits and, again by Proposition 2.3,

$$\mathrm{Ext}^1(\mathbf{X}, \mathbf{E}_r) = \mathrm{Ext}^1(C^{-(r-1)}\mathbf{X}, \mathbf{E}_1) = 0, \quad 0 < r \leqslant l,$$

for all $\mathbf{X} \not\approx \mathbf{E}_{r-1}$ and $\mathbf{X} \approx \mathbf{E}_r$.

The following lemma provides the final argument in the proof of Theorem 3.5.

LEMMA 3.4. *Let* $(\mathfrak{M}, \Omega)$ *be a realization of an extended Dynkin diagram* $(\Gamma, \mathbf{d})$ *and* C^+ *be the corresponding Coxeter functor. Let* i *be a source with respect to* Ω. *Let* $\mathbf{E}$ *be a representation of* $(\mathfrak{M}, \Omega)$ *such that* $C^+\mathbf{E} \neq \mathbf{0}$ *and* $E_i = 0$. *Then there exists* $k \in \Gamma$ *such that*

$$\mathrm{End}\, \mathbf{E} = F_k,\ \mathrm{Ext}^1(\mathbf{E}, \mathbf{E}) = 0 \text{ and } \mathrm{Ext}^1(\mathbf{E}, C^+\mathbf{E}) = {}_{F_k}(F_k)_{F_k}.$$

Proof. Denote by $L_i(\mathfrak{M}, \Omega)$ the full subcategory of $L(\mathfrak{M}, \Omega)$ of all representations $\mathbf{X}$ of $(\mathfrak{M}, \Omega)$ such that $X_i = 0$. Since i is a source, there is an admissible ordering $k_1, k_2, \ldots, k_n$ of Γ such that $k_n = i$. Thus

$$\mathbf{Y} = S^+_{k_{n-1}} \cdots S^+_{k_2} S^+_{k_1}\mathbf{E} \in L_i(\mathfrak{M}, \Omega)$$

and $C^+\mathbf{E}$ is an extension of $\mathbf{Y}$ by a direct sum $\mathbf{Z}$ of copies of $\mathbf{F}_i$:

$$\mathbf{0} \longrightarrow \mathbf{Y} \longrightarrow C^+\mathbf{E} \longrightarrow \mathbf{Z} \longrightarrow \mathbf{0}.$$

From here, we get the exact sequence

$$\mathrm{Hom}(\mathbf{E}, \mathbf{Z}) \longrightarrow \mathrm{Ext}^1(\mathbf{E}, \mathbf{Y}) \longrightarrow \mathrm{Ext}^1(\mathbf{E}, C^+\mathbf{E}) \longrightarrow \mathrm{Ext}^1(\mathbf{E}, \mathbf{Z}),$$

in which both the first and last terms vanish: the first one trivially and the last one because of injectivity of **Z**. Thus

$$\mathrm{Ext}^1(\mathbf{E}, C^+\mathbf{E}) \approx \mathrm{Ext}^1(\mathbf{E}, \mathbf{Y}).$$

Now, there is a canonic isomorphism

$$T\colon L_i(\mathfrak{M}, \Omega) \longrightarrow L(\mathfrak{M}', \Omega'),$$

where $(\mathfrak{M}', \Omega')$ is the restriction of $(\mathfrak{M}, \Omega)$ to the graph $(\Gamma', \mathbf{d}')$, where $\Gamma' = \Gamma \backslash \{i\}$ and $\mathbf{d}'$ is the induced valuation. Observe that $(\Gamma', \mathbf{d}')$ is a disjoint union of Dynkin diagrams. It turns out that

$$T(\mathbf{Y}) = C'^+ T(\mathbf{E}),$$

where $C'^+ = S^+_{k_{n-1}} \cdots S^+_{k_2} S^+_{k_1}$ is the Coxeter functor on $L(\mathfrak{M}', \Omega')$. Now, by Propositions 1.9 and 2.6,

$$T(\mathbf{E}) = C'^{+r}\mathbf{Q}_k \quad \text{for some } k \in \Gamma',$$

and by Proposition 2.8 (ii)

$$\mathrm{Ext}^1(T(\mathbf{E}), C'^+ T(\mathbf{E})) \approx {}_{F_k}(F_k)_{F_k}.$$

Also, by Propositions 2.5 and 2.3,

$$\mathrm{End}\, T(\mathbf{E}) = F_k \quad \text{and} \quad \mathrm{Ext}^1(T(\mathbf{E}), T(\mathbf{E})) = 0.$$

Lemma 3.4 follows.[6]

Now, given **E** which possesses an equation η, observe that $c^{*r}\eta$ is an equation of $\mathbf{E}_r$. Put

$$\bigcap_{0 \leqslant r < l} \mathrm{Ker}\, c^{*r}\eta = K_\eta.$$

By Lemma 3.3 (2), always $K_\eta \supseteq N$.

DEFINITION. *A finite set* $\{\mathbf{E}^t \mid 1 \leqslant t \leqslant h\}$ *of representations of* $(\mathfrak{M}, \Omega)$ *is said to be a generating set if*

(i) *the orbits of* $\mathbf{dim}\, \mathbf{E}^{(t)}$ *under the action of c are nontrivial and finite*;

(ii) *any two* $\mathbf{dim}\, \mathbf{E}^{(t)}$ *and* $\mathbf{dim}\, \mathbf{E}^{(t')}$ *for* $t \neq t'$ *belong to distinct orbits of c*;

(iii) *each* $\mathbf{E}^{(t)}$ *possesses an equation* $\eta^{(t)}$;

(iv) *there exists a source i with respect to* Ω *such that each* $\mathbf{E}^{(t)}$ *has the property that* $\mathbf{E}_i^{(t)} = 0$;

(v) $\bigcap_{1 \leqslant t \leqslant h} \mathrm{K}_{\eta^{(t)}} = N$.

Now, Lemmas 3.3 and 3.4 in combination with Proposition 2.3 yield the following results which refines the second part of Theorem in the Introduction.

THEOREM 3.5. *Let* $(\mathfrak{M}, \Omega)$ *be a realization of an extended Dynkin diagram and* C^+ *the corresponding Coxeter functor. Let* $\mathrm{E} = \{\mathbf{E}^{(t)} \mid 1 \leqslant t \leqslant h\}$ *be a generating set of regular representations of* $(\mathfrak{M}, \Omega)$. *Then the category* $R(\mathfrak{M}, \Omega)$ *is the product of* $H(\mathfrak{M}, \Omega)$ *and h*

[6] Here, $T(\mathbf{E})$ is a representation of a disjoint union of Dynkin diagrams. However, since it is indecomposable, it is, in fact, a representation of one connected component and so we may apply the previous results.

subcategories $\mathcal{R}^{(t)}$ *corresponding to the orbits* $\mathcal{O}^{(t)} = \{\mathbf{E}^{(t)}, C^+\mathbf{E}^{(t)}, \ldots, C^{+l_n}\mathbf{E}^{(t)}\}$, $1 \leqslant t \leqslant h$. *The indecomposable objects in* $\mathcal{R}^{(t)}$ *are serial with composition factors from* $\mathcal{O}^{(t)}$ *ordered in a (going down) sequence corresponding to the action of* C^+ : *each of them is fully determined by its (simple) socle and its length which both can be arbitrary.*

The last statement follows from a well-known characterization of serial abelian categories which (in the case of a module category) is due to T. Nakayama (see e.g. [9]). Since $L(\mathfrak{M}, \Omega)$ has global dimension $\leqslant 1$, the same is true for each $\mathcal{R}^{(t)}$. Since each simple object has a nontrivial extension by a suitable simple object, there are indecomposable objects of arbitrary length with a prescribed socle. As an immediate consequence, we get the following statement which will be used in Chapter 5: *If* $\mathbf{X}$ *is an indecomposable representation of continuous dimension type belonging to some* $\mathcal{R}^{(t)}$, *then* End $\mathbf{X}$ *is a division ring if and only if* $\mathbf{X}$ *has no proper subobject of continuous dimension type. This happens if and only if every simple representation of* $\mathcal{R}^{(t)}$ *appears precisely once as a composition factor of* $\mathbf{X}$. *Also, if* $\mathbf{Y}$ *is another indecomposable representation of continuous dimension type in* $\mathcal{R}^{(t)}$, *then* $\mathrm{Hom}(\mathbf{X}, \mathbf{Y}) \neq 0$.

The existence of a generating set $\mathcal{E}$ of Theorem 3.5 is proved, for each of the extended Dynkin diagrams, in the next chapter.

4. Simple Regular Nonhomogeneous Representations

In this chapter, we shall exhibit a generating set of regular representations for each of the extended Dynkin diagrams with a suitable orientation. The results will then be incorporated in the tables of Chapter 6; in particular, all simple regular nonhomogeneous representations will be listed there.

As a consequence of these results, we can formulate the following theorem.

THEOREM 4.1. *Let* $(\Gamma, \mathbf{d})$ *be an extended Dynkin diagram and* Ω *an admissible orientation. Let* $n + 1$ *be the number of its vertices and* h *the number of elements in a generating set of regular representations. Then* $0 \leqslant h \leqslant 3$ *and the number* l *of all simple regular nonhomogeneous representations is given by the formula*

$$l = n + h - 1.$$

The number h *is independent of* Ω *except in the case of the diagram* $\widetilde{\mathbf{A}}_n$ $(n \geqslant 2)$ *when* $h = 1$ *or* 2.

The formula in the theorem can be reinterpreted as follows: If $\mathcal{O}_t$ is an orbit of a simple regular nonhomogeneous representation under the action of C^+, denote by l_t the length of $\mathcal{O}_t$; then

$$\sum_t (l_t - 1) = n - 1.$$

We shall use letters $a, b, z, \ldots$, possibly with some indices, to denote the vertices of the diagrams; the vector spaces of a given representation attached to these vertices will be denoted by the corresponding capital letters $A, B, Z \ldots$ (with the respective indices). For convenience, $a_t^{(m)}, \ldots$ will denote also $\dim A_t^{(m)}, \ldots$. For further notation, such as using the letters H and G to denote subfields of F, we refer to the tables.

In order to show that a given set is a generating set, we have to verify conditions (i)–(v) of the definition. However, the conditions (i), (ii) and (iv), (v) can be checked in a routine manner, and thus we shall be concerned only with the condition (iii): In each case, we shall prove that the linear form $\eta^{(m)}$ listed in the tables is an equation for $\mathbf{E}_0^{(m)}$. Throughout the proofs, $\mathbf{X}$ stands always for a regular representation, $\mathbf{X}'$ for a (not necessarily regular) subobject and $\mathbf{X}''$ for the quotient of $\mathbf{X}$ by $\mathbf{X}'$. We shall use frequently the fact that, $\partial\mathbf{X}' \leqslant 0$ and $\partial\mathbf{X}'' \geqslant 0$.

Before we proceed to establish the existence of a generating set for each individual extended Dynkin diagram, we are going to give an outline of the proof that $\eta^{(t)}$ is an equation for $\mathbf{E}^{(t)}$ which will be systematically followed in each case. It is trivial to verify that $\eta^{(t)}\mathbf{E}^{(t)} > 0$. Hence, we need only to show that the other two conditions concerning regular representations $\mathbf{X}$ with $\eta^{(t)}\mathbf{X} \neq 0$ are satisfied. This will be achieved by the method of contraction to a compatible realization of a Dynkin diagram in the following sense. We shall outline the method of contraction in general, for arbitrary valued graphs.

DEFINITION. *Let* $(\mathfrak{M}, \Omega)$ *be a realization of a valued graph* $(\Gamma, \mathbf{d})$. *A realization* $(\mathfrak{M}', \Omega')$ *of a valued graph* $(\Gamma', \mathbf{d}')$ *is said to be a contraction of* $(\mathfrak{M}, \Omega)$ *if* $\Gamma' \subseteq \Gamma$ *and*

(i) *for every two* $k, l \in \Gamma'$, *for which there is an oriented sequence of edges* $k = i_0 \longrightarrow i_1 \longrightarrow \cdots \longrightarrow i_q = l$ *in* $(\Gamma, \mathbf{d})$ (*in the orientation* Ω), *there is an edge* $k — l$ *in* $(\Gamma', \mathbf{d}')$;

(ii) *for every oriented* (*with respect to* Ω') *edge* $k \longrightarrow l$ *in* $(\Gamma', \mathbf{d}')$, *there is a unique sequence of edges* $L_{kl} = (k = i_0 — \cdots — i_p — i_{p+1} — \cdots — i_q = l)$, *and it is oriented in* Ω *by* $i_p \longrightarrow i_{p+1}$ *for all* $0 \leqslant p \leqslant q-1$;

(iii) $L_{kl} \cap L_{k'l'} = \varnothing$ *for* $\{k, l\} \cap \{k', l'\} = \varnothing$;

(iv) $F'_k = F_k$ *for every* $k \in \Gamma'$; *and*

(v) ${}_kM'_l = {}_kM_{i_1} \otimes \cdots \otimes {}_{i_p}M_{i_{p+1}} \otimes \cdots \otimes {}_{i_{q-1}}M_l$, *where* $\{i_p, i_{p+1}\}$ *runs through* L_{kl}, *for every edge* $k - l$ *of* $(\Gamma', \mathbf{d}')$.

Now, for each $k \in \Gamma'$, define

$$\downarrow_k = \{i \in \Gamma \mid i = i_0 \longrightarrow i_1 \longrightarrow \cdots \longrightarrow i_q = k \text{ with } i_p \notin \Gamma' \text{ for all } 0 \leqslant p < q\}$$

and

$$\uparrow_k = \{i \in \Gamma \mid k = i_0 \longrightarrow i_1 \longrightarrow \cdots \longrightarrow i_q = i \text{ with } i_p \notin \Gamma' \text{ for all } 0 < p \leqslant q\}$$

Thus, k belongs both to $\downarrow_k$ and $\uparrow_k$.

Let $\mathbf{X} = (X_i, {}_j\varphi_i)$ be an arbitrary representation of $(\mathfrak{M}, \Omega)$ and $(\mathfrak{M}', \Omega')$ a contraction of $(\mathfrak{M}, \Omega)$. Define the representation $\mathbf{U} = R(\mathbf{X})$ of $(\mathfrak{M}', \Omega')$ as follows: $\mathbf{U} = (U_k, {}_l\psi_k)$, where

$$U_k = X_k \text{ for all } k \in \Gamma'$$

and

$${}_l\psi_k : U_k \otimes {}_kM'_l \longrightarrow U_l \text{ for all } k \longrightarrow l \text{ (in } \Omega')$$

with ${}_l\psi_k = {}_{i_q}\varphi'_{i_{q-1}} \cdots {}_{i_{p+1}}\varphi'_{i_p} \cdots {}_{i_2}\varphi'_{i_2}\,{}_{i_1}\varphi'_{i_0}$ and

$$\begin{aligned}{}_{i_{p+1}}\varphi'_{i_p} = {}_{i_{p+1}}\varphi_{i_p} \otimes 1 \otimes \cdots \otimes 1 : X_{i_p} \otimes {}_{i_p}M_{i_{p+1}} \otimes {}_{i_{p+1}}M_{i_{p+2}} \\ \otimes \cdots \otimes {}_{i_{q-1}}M_{i_q} \longrightarrow X_{i_{p+1}} \otimes {}_{i_{p+1}}M_{i_{p+2}} \otimes \cdots \otimes {}_{i_{q-1}}M_{i_q} \text{ for all } 0 \leqslant p < q-1.\end{aligned}$$

On the other hand, if $\mathbf{V}$ is a representation of $(\mathfrak{M}', \Omega')$ which is a subobject of $\mathbf{U} = R(\mathbf{X})$, we can define canonically two subobjects of $\mathbf{X}: T(\mathbf{V})$ and $T'(\mathbf{V})$. But first, given $\mathbf{V} = (V_k, {}_l\psi_k)$, introduce the following notation:

If $i \in \Gamma$ belongs to $\downarrow_k$, and $L_k = (i = i_1 \longrightarrow i_1 \longrightarrow \cdots \longrightarrow i_q = k)$ is a path satisfying $i_p \notin \Gamma'$ for all $0 \leqslant p \leqslant q$, define V_{iL_k} as the inverse image of V_k under the mapping

$$ {}_{i_q}\overline{\varphi}'_{i_{q-1}} \cdots {}_{i_p}\overline{\varphi}'_{i_{p-1}} \cdots {}_{i_1}\overline{\varphi}'_{i_0} : X_{i_0} \longrightarrow X_{i_q} \otimes {}_{i_q}M_{i_{q-1}} \otimes \cdots \otimes {}_{i_1}M_{i_0}, $$

where

$$ {}_{i_{p+1}}\overline{\varphi}'_{i_p} = {}_{i_{p+1}}\overline{\varphi}_{i_p} \otimes 1 \otimes \cdots \otimes 1 : X_{i_p} \otimes {}_{i_p}M_{i_{p-1}} \otimes \cdots \otimes {}_{i_1}M_{i_0} $$
$$ \longrightarrow X_{i_{p+1}} \otimes {}_{i_{p+1}}M_{i_p} \otimes {}_{i_p}M_{i_{p-1}} \otimes \cdots \otimes {}_{i_1}M_{i_0} \qquad \text{for all } 0 \leqslant p \leqslant q-1, $$

and put

$$ V_{ik}^{\downarrow} = \bigcap_{L_k} V_{iL_k} \subseteq X_i, $$

where the indices run through all possible paths L_k described above.

In a similar fashion, if $i \in \Gamma$ belongs to $\uparrow_k$, and $L^k = (k = i_0 \longrightarrow i_1 \longrightarrow \cdots \longrightarrow i_q = i)$ is a path satisfying $i_p \notin \Gamma'$ for all $0 \leqslant p \leqslant q$, define V_{iL^k} as the image of V_k under the mapping ${}_{i_q}\overline{\varphi}_{i_0}$ defined above. And again, put

$$ V_{ik}^{\uparrow} = \sum_{L^k} V_{iL^k} \subseteq X_i, $$

where the indices run through all possible paths L^k.

Now, we define the representation $T(\mathbf{V})$ of $(\mathfrak{M}, \Omega)$ as follows: $T(\mathbf{V}) = (V_i, {}_j\varphi_i)$, where

$$ V_i = \bigcap_k V_{ik}^{\downarrow}, $$

where the indices run through all k such that $i \in \downarrow_k$ (and thus, if no such k exists, $V_i = X_i$) and ${}_j\varphi_i$ are induced by $\mathbf{X}$.

Similarly, we define the representation $T'(\mathbf{V})$ of $(\mathfrak{M}, \Omega)$ by $T'(\mathbf{V}) = (V'_i, {}_j\varphi'_i)$, where

$$ V'_i = \sum_k V_{ik}^{\downarrow}, $$

where the indices run through all k such that $i \in \uparrow_k$ (and thus, if no such k exists, $V_i = 0$) and the mappings ${}_j\varphi'_i$ are again induced by $\mathbf{X}$.

It is easy to verify that both $T(\mathbf{U})$ and $T'(\mathbf{U})$ are subobjects of $\mathbf{X}$. The following consequence will be repeatedly used.

LEMMA 4.2. *Let $(\mathfrak{M}', \Omega')$ be a contraction of $(\mathfrak{M}, \Omega)$. Let $\mathbf{X}$ be a representation of $(\mathfrak{M}, \Omega)$ and $\mathbf{U} = R(\mathbf{X})$ the corresponding representation of $(\mathfrak{M}', \Omega')$. If $\mathbf{V}$ is an indecomposable direct summand of $\mathbf{U}$, then $\mathbf{V}$ determines a subobject $T'(\mathbf{V})$ of $\mathbf{X}$ and a quotient $T''(\mathbf{V}) = \mathbf{X}/T(\mathbf{W})$ of $\mathbf{X}$, where $\mathbf{W}$ is a complement of $\mathbf{V}$ in $\mathbf{U}$.*

Now, our proof will consist of considering, for each extended Dynkin diagram and for each $\eta^{(t)}$, a contraction $(\mathfrak{M}', \Omega')$ of $(\mathfrak{M}, \Omega)$ defined by the nonzero components of $\eta^{(t)}$, and in decomposing $R(\mathbf{X})$ into indecomposable representations of $(\mathfrak{M}', \Omega')$. Since $(\mathfrak{M}', \Omega')$ is a realization of a Dynkin diagram, the direct summands are determined by their dimension type, and these are the positive roots of the diagram. From the condition $\eta^{(t)}\mathbf{X} > 0$, we deduce that one particular summand $\mathbf{V}_0$ must occur in the decomposition and show that the subobject $T'(\mathbf{V}_0)$ of $\mathbf{X}$ is isomorphic to $\mathbf{E}^{(t)} = \mathbf{E}_0^{(t)}$. Similarly, using the condition $\eta^{(t)}\mathbf{X} < 0$, we single out another particular direct summand $\mathbf{V}_1$ of $R(\mathbf{X})$, and show that the quotient $T''(\mathbf{V}_1)$ of $\mathbf{X}$ is isomorphic to $\mathbf{E}_1^{(t)} = C^+\mathbf{E}_0^{(t)}$. These conclusions are obtained in each case as a result of a very simple elimination process of those direct summands $\mathbf{V}$ which would lead to subobjects of $\mathbf{X}$ of positive defect or to quotients of $\mathbf{X}$ of negative defect. In the case of diagrams E, F and G we shall make use of an additional information that the orientation is chosen in such a way that all mappings ${}_j\overline{\varphi}_i : X_j \longrightarrow X_j \otimes {}_jM_i$ are monomorphisms; for, otherwise we would get a subobject of $\mathbf{X}$ of a positive defect.

Although we shall deal in the proofs with the dimension types only, the actual description of the simple regular nonhomogeneous representation given in the tables follows immediately from the form of the corresponding representation $\mathbf{V}_0$ or $\mathbf{V}_1$ of the contracted Dynkin diagram.

$\widetilde{\mathsf{A}}_{11}$ and $\widetilde{\mathsf{A}}_{12}$. Obviously, *there are no regular nonhomogeneous representations.*

$\widetilde{\mathsf{A}}_n$, $n > 1$. $\{\mathbf{E}_0 = \mathbf{E}_{c_p}, \mathbf{E}_0' = \mathbf{F}_{d_q}\}$ *is a generating set* (of course, the set contains only $\mathbf{E}_0$ if $q = 0$).

First, η is an equation for $\mathbf{E}$. Consider the contraction to the Dynkin diagram A_2 : $c_p \longrightarrow b$ (more precisely to the realization $F \longrightarrow F$ of the Dynkin diagram c_p — b), and decompose $R(\mathbf{X})$ into the direct sum of indecomposable representations. The dimension types of the direct summands are $(0, 1)$, $(1, 0)$ or $(1, 1)$. Now, if $\eta\mathbf{X} > 0$, then there must be a direct summand $\mathbf{V}_0$ of dimension type $(1, 0)$; for, η is additive and $\eta\mathbf{V} < 0$ for the summands $\mathbf{V}$ of type $(0, 1)$ and $\eta\mathbf{V} = 0$ for those of type $(1, 1)$. And, evidently $\mathbf{X}' = T'(\mathbf{V}_0) \approx \mathbf{E}_0$. If $\eta\mathbf{X} < 0$, there must be a direct summand $\mathbf{V}_1$ of dimension type $(0, 1)$. And, we get $\mathbf{X}'' = T''(\mathbf{V}_1) \approx \mathbf{E}_1$.

The same arguments show that η' is an equation for $\mathbf{E}'$.

$\widetilde{\mathsf{B}}_n$. $\{\mathbf{E} = \mathbf{F}_{z_{n-1}}\}$ *is a generating set.* Consider the contraction to the diagram B_2 : $z_{n-1} \xrightarrow{(2,1)} b$. The dimension types of the direct summands of $R(\mathbf{X})$ are $(0, 1)$, $(1, 0)$, $(1, 1)$ and $(1, 2)$. If $\eta\mathbf{X} > 0$, a summand $\mathbf{V}_0$ of type $(1, 0)$ must occur and $T'(\mathbf{V}_0) \approx \mathbf{E}_0$. If $\eta\mathbf{X} < 0$, $R(\mathbf{X})$ has a direct summand of type $(0, 1)$ or $(1, 2)$. But, there is no direct summand $\mathbf{V}$ of type $(0, 1)$, because it would determine a quotient $T''(\mathbf{V})$ of $\mathbf{X}$ of negative defect -1. Hence, there is a summand $\mathbf{V}_1$ of type $(1, 2)$ and the quotient $T''(\mathbf{V}_1)$ is isomorphic to $\mathbf{E}_1$.

$\widetilde{\mathsf{C}}_n$. $\{\mathbf{E} = \mathbf{F}_{z_{n-1}}\}$ *is a generating set.* Consider the contraction to the diagram B_2 : $z_{n-1} \xrightarrow{(1,2)} b$. If $\eta\mathbf{X} > 0$, there is a direct summand $\mathbf{V}_0$ of $R(\mathbf{X})$ of dimension type $(1, 0)$ and $T'(\mathbf{V}_0) \approx \mathbf{E}_0$. If $\eta\mathbf{X} < 0$, then, for the same reason as in $\widetilde{\mathsf{B}}_n$, there is a summand $\mathbf{V}_1$ of type $(1, 1)$ and $T''(\mathbf{V}_1) \approx \mathbf{E}_1$.

$\widetilde{\mathsf{BC}}_n$. $\{\mathbf{E} = \mathbf{F}_{z_{n-1}}\}$ *is a generating set.* Again, consider the contraction to B_2 : $z_{n-1} \xrightarrow{(1,2)} b$. As before, if $\eta\mathbf{X} > 0$, there is a summand $\mathbf{V}_0$ of $R(\mathbf{X})$ of type $(1, 0)$ and $T'(\mathbf{V}_0) = \mathbf{E}_0$; and, if $\eta\mathbf{X} < 0$, there is a summand $\mathbf{V}_1$ of type $(1, 1)$ and $T''(\mathbf{V}_1) \approx \mathbf{E}_1$.

$\widetilde{\mathsf{BD}}_n$. $\{\mathbf{E} = \mathbf{F}_{z_{n-2}}, \mathbf{E}' = {}^{\;0}_{F}F\,F \cdots F\,F\,G\}$ *is a generating set.* First, consider the contraction to $\mathsf{B}_2 : z_{n-2} \xrightarrow{(2,1)} b$. If $\eta\mathbf{X} > 0$, there is a direct summand $\mathbf{V}_0$ of $R(\mathbf{X})$ of dimension type $(1, 0)$ and $T'(\mathbf{V}_0) \approx \mathbf{E}_0$. If $\eta\mathbf{X} < 0$, then a summand $\mathbf{V}_1$ of type $(1, 2)$ occurs and $T''(\mathbf{V}_1) \approx \mathbf{E}_1$.

Second, consider the contraction to the diagram $a_2 \xrightarrow{(2,1)} b$. If $\eta'\mathbf{X} > 0$, then a direct summand $\mathbf{V}_0'$ of $R(\mathbf{X})$ of dimension type $(1, 1)$ occurs and $T'(\mathbf{V}_0') \approx \mathbf{E}_0'$. If $\eta'\mathbf{X} < 0$, then there is a summand $\mathbf{V}_1'$ of type $(0, 1)$ in the decomposition and we get $T''(\mathbf{V}_1') \approx \mathbf{E}_1'$.

$\widetilde{\mathsf{CD}}_n$. $\{\mathbf{E} = \mathbf{F}_{z_{n-2}}, \mathbf{E}' = {}^{F}_{0}F\,F \cdots F\,F\,F\}$ *is a generating set.* First, consider the contraction to the diagram $z_{n-2} \xrightarrow{(1,2)} b$. As before, if $\eta\mathbf{X} > 0$, there is a direct summand $\mathbf{V}_0$ of $R(\mathbf{X})$ of type $(1, 0)$ and $T''(\mathbf{V}_0) \approx \mathbf{E}_0$. And, $\eta\mathbf{X} < 0$ implies the existence of $\mathbf{V}_1$ of type $(1, 1)$ and $T''(\mathbf{V}_1) \approx \mathbf{E}_1$.

Second, consider the contraction to $a_2 \xrightarrow{(1,2)} b$. If $\eta'\mathbf{X} > 0$, there is a direct summand $\mathbf{V}_0'$ of type $(2, 1)$ and $T'(\mathbf{V}_0') \approx \mathbf{E}_0'$. And, if $\eta'\mathbf{X} < 0$, then a summand $\mathbf{V}_1'$ of type $(0, 1)$ yields $T''(\mathbf{V}_1') \approx \mathbf{E}_1'$.

$\widetilde{\mathsf{D}}_n$. $\left\{\mathbf{E} = \mathbf{F}_{z_{n-3}}, \mathbf{E}' = {}^{0}_{F}F\,F \cdots F\,F^{0}_{F}, \mathbf{E}'' = {}^{0}_{F}F\,F \cdots F\,F^{F}_{0}\right\}$ *is a generating set.*
First, consider the contraction to the diagram

$$\mathsf{A}_3 : z_{n-3} \begin{smallmatrix} \nearrow b_1 \\ \searrow b_2 \end{smallmatrix}.$$

In the decomposition of $R(\mathbf{X})$, the dimension types of the direct summands are $0\,{}^{1}_{0}$, $1\,{}^{0}_{0}$, $0\,{}^{0}_{1}$, $1\,{}^{1}_{0}$, $1\,{}^{0}_{1}$ and $1\,{}^{1}_{1}$. Now, if $\eta\mathbf{X} > 0$ there must be a direct summand $\mathbf{V}_0$ of type $1\,{}^{0}_{0}$; and $T'(\mathbf{V}_0) \approx \mathbf{E}_0$. If $\eta\mathbf{X} < 0$, there must be a direct summand of type $0\,{}^{1}_{0}$, or $0\,{}^{0}_{1}$, or $1\,{}^{1}_{1}$. But the first two cases lead to a quotient of $\mathbf{X}$ of negative defect -1. Hence, there is a summand $\mathbf{V}_1$ of type $1\,{}^{1}_{1}$, and $T''(\mathbf{V}_1) \approx \mathbf{E}_1$.

Second, consider the contraction to $a_2 \longrightarrow b_1$. Then a summand $\mathbf{V}_0'$ of type $(1, 0)$ yields the subobject $T'(\mathbf{V}_0') \approx \mathbf{E}_0'$ in case that $\eta'\mathbf{X} > 0$ and a summand $\mathbf{V}_1'$ of type $(0, 1)$ yields the quotient $T'(\mathbf{V}_1') \approx \mathbf{E}_1'$ if $\eta'\mathbf{X} < 0$.

Finally, η'' is an equation for $\mathbf{E}_0''$ by the same arguments as above.

$\widetilde{\mathsf{E}}_6$. $\left\{\mathbf{E} = \begin{smallmatrix} & & 0 & & \\ & & F & & \\ 0 & F & F & F & 0\end{smallmatrix}, \mathbf{E}' = \begin{smallmatrix} & & 0 & & \\ & & 0 & & \\ 0 & F & F & F & F\end{smallmatrix}, \mathbf{E}'' = \begin{smallmatrix} & & 0 & & \\ & & F & & \\ 0 & 0 & F & F & F\end{smallmatrix}\right\}$ *is a generating set.*

First, consider the contraction to the diagram

$$\mathsf{D}_n : a_2 \longrightarrow \begin{smallmatrix} c_2 \\ \downarrow \\ z \end{smallmatrix} \longleftarrow b_2.$$

If $\eta\mathbf{X} > 0$, then in a decomposition of $R(\mathbf{X})$, there must be a summand of dimension type $\begin{smallmatrix} & 1 & \\ 0 & 0 & 0\end{smallmatrix}$, or $\begin{smallmatrix} & 1 & \\ 1 & 0 & 0\end{smallmatrix}$, or $\begin{smallmatrix} & 0 & \\ 0 & 0 & 1\end{smallmatrix}$, or $\begin{smallmatrix} & 1 & \\ 1 & 1 & 1\end{smallmatrix}$. But all the first three summands will determine a subobject of $\mathbf{X}$ of positive defect. Hence, there is $\mathbf{V}_0$ of type $\begin{smallmatrix} & 1 & \\ 1 & 1 & 1\end{smallmatrix}$ which determines the subobjects $T'(\mathbf{V}_0) \approx \mathbf{E}_0$. If $\eta\mathbf{X} < 0$, then there must be a summand of type $\begin{smallmatrix} & 0 & \\ 0 & 1 & 0\end{smallmatrix}$, or $\begin{smallmatrix} & 1 & \\ 0 & 1 & 0\end{smallmatrix}$, or $\begin{smallmatrix} & 1 & \\ 1 & 1 & 0\end{smallmatrix}$, or $\begin{smallmatrix} & 0 & \\ 0 & 1 & 1\end{smallmatrix}$, or $\begin{smallmatrix} & 1 & \\ 1 & 2 & 1\end{smallmatrix}$, But again, with the exception of a summand $\mathbf{V}_1$ of type $\begin{smallmatrix} & 1 & \\ 1 & 2 & 1\end{smallmatrix}$, all other would lead to a quotient of $\mathbf{X}$ of negative defect. And $\mathbf{V}_1$ determines the quotient $T''(\mathbf{V}_1) \approx \mathbf{E}_1$.

Second, consider the contraction to the diagram $\mathbf{A}_3 : a_2 \longrightarrow z \longleftarrow b_1$. If $\eta'\mathbf{X} > 0$, then a direct summand $\mathbf{V}_0'$ of $R(\mathbf{X})$ of dimension type (1, 1, 1) must occur, which determines the subobject $T'(\mathbf{V}_0') \approx \mathbf{E}_0'$. And a summand $\mathbf{V}_1'$ of type (0, 1, 0) determines the quotient $T''(\mathbf{V}_1') \approx \mathbf{E}_1'$.

Finally the proof that η'' is an equation for $\mathbf{E}_0''$ follows the same lines.

$\widetilde{\mathbf{E}}_7$. $\{\mathbf{E} = 0\ F\ F\ \overset{F}{F}\ F\ F\ 0,\ \mathbf{E}' = 0\ 0\ 0\ \overset{F}{F}\ F\ F\ 0,\ \mathbf{E}'' = 0\ \ F \times 0\ \ F \times 0\ \ \overset{(1,\,1)F}{F \times F}\ \ F \times F\ \ 0 \times F\ \ 0 \times F\}$ *is a generating set.*

First, consider the contraction to

$$\mathbf{D}_4 : a_3 \longrightarrow \overset{\displaystyle c \atop \downarrow}{z} \longleftarrow b_3.$$

If $\eta\mathbf{X} > 0$, then there must be a direct summand of $R(\mathbf{X})$ of dimension type $0\ \overset{1}{0}\ 0$, or $1\ \overset{0}{0}\ 0$, or $0\ \overset{0}{0}\ 1$, or $1\ \overset{1}{1}\ 1$. Using a by now standard argument, a summand $\mathbf{V}_0$ of type $1\ \overset{1}{1}\ 1$ must occur and $T'(\mathbf{V}_0) \approx \mathbf{E}_0$. If $\eta\mathbf{X} < 0$, then a summand of type $0\ \overset{0}{1}\ 0$, or $0\ \overset{1}{1}\ 0$, or $0\ \overset{0}{1}\ 1$, or $1\ \overset{1}{2}\ 1$ must occur. All but the last one lead to a quotient of $\mathbf{X}$ of negative defect. And the summand $\mathbf{V}_1$ of type $1\ \overset{1}{2}\ 1$ determines $T''(\mathbf{V}_1) \approx \mathbf{E}_1$.

Second, consider the contraction to

$$\mathbf{A}_3 : \overset{\displaystyle c \atop \downarrow}{z} \longleftarrow b_2.$$

If $\eta'\mathbf{X} > 0$, a summand $\mathbf{V}_0'$ of type $\overset{1}{1}\ 1$ must occur and $T'(\mathbf{V}_0') \approx \mathbf{E}_0'$. If $\eta'\mathbf{X} < 0$, $\mathbf{V}_1'$ of type $\overset{0}{1}\ 0$ must occur as a summand and $T''(\mathbf{V}_1') \approx \mathbf{E}_1'$.

Finally, consider the contraction to

$$\mathbf{D}_5 : a_2 \longrightarrow \overset{\displaystyle c \atop \downarrow}{z} \longleftarrow b_3 \longleftarrow b_1.$$

If $\eta''\mathbf{X} > 0$, then there must be a direct summand of $R(\mathbf{X})$ of dimension type $1\ \overset{0}{0}\ 0\ 0$, or $0\ \overset{0}{0}\ 1\ 0$, or $0\ \overset{0}{0}\ 0\ 1$, or $0\ \overset{1}{0}\ 0\ 0$, or $1\ \overset{0}{1}\ 1\ 1$, or $1\ \overset{1}{1}\ 1\ 0$, or $0\ \overset{1}{1}\ 1\ 1$, or $1\ \overset{1}{1}\ 1\ 1$, or $1\ \overset{1}{2}\ 2\ 1$. Again, by routine elimination of all types with the exception of the last one, we deduce existence of $\mathbf{V}_0''$ of dimension type $1\ \overset{1}{2}\ 2\ 1$ which determines $T'(\mathbf{V}_0'') \approx \mathbf{E}_0''$. Similarly, if $\eta''\mathbf{X} < 0$, then all but the last one of the dimension types $0\ \overset{0}{1}\ 0\ 0$, $1\ \overset{0}{1}\ 0\ 0$, $0\ \overset{0}{1}\ 1\ 0$, $0\ \overset{0}{0}\ 1\ 1$, $0\ \overset{1}{1}\ 0\ 0$ and $1\ \overset{1}{2}\ 1\ 0$ can be eliminated. .nd the direct summand $\mathbf{V}_1''$ of dimension $1\ \overset{1}{2}\ 1\ 0$ determines the quotient $T''(\mathbf{V}_1'') \approx \mathbf{E}_1''$.

$\widetilde{\mathbf{E}}_8$. $\{\mathbf{E} = 0\ 0\ 0\ 0\ F\ \overset{F}{F}\ F\ 0,\ \mathbf{E}_1 = 0\ 0\ \ F \times 0\ \ F \times 0\ \ F \times 0\ \ \overset{(1,\,1)F}{F \times F}\ \ F \times F\ \ 0 \times F,$

$\mathbf{E}'' = 0\ \ F \times 0 \times 0\ \ F \times 0 \times 0\ \ F \times F \times 0\ \ F \times F \times 0\ \ \overset{(1,\,1,\,0)F + (0,\,1,\,1)F}{F \times F \times F}\ \ 0 \times F \times F\ \ 0 \times 0 \times F\}$ *is a generating set.*

Consider the contraction to

$$\mathsf{D}_4 : a_5 \longrightarrow \overset{\displaystyle c \atop \downarrow}{z} \longleftarrow b_2,$$

and proceed as in the case of η in $\tilde{\mathsf{E}}_6$ or $\tilde{\mathsf{E}}_7$.

Second, consider the contraction to

$$\mathsf{D}_5 : a_3 \longrightarrow \overset{\displaystyle c \atop \downarrow}{z} \longleftarrow b_2 \longleftarrow b_1,$$

and proceed as in the case of η'' in $\tilde{\mathsf{E}}_7$.

Finally, consider the contraction to

$$\mathsf{E}_6 : a_2 \longrightarrow a_4 \longrightarrow \overset{\displaystyle c \atop \downarrow}{z} \longleftarrow b_2 \longleftarrow b_1.$$

If $\eta''\mathbf{X} > 0$, there are 18 possible dimension types of direct summands of $R(\mathbf{X})$, for which the η-value is positive. However, all, with the exception of the type $1\ 2\ \overset{2}{3}\ 2\ 1$ determine a subobject of negative defect. And, the direct summand $\mathbf{V}_0''$ of the mentioned type yields the subobject $T'(\mathbf{V}_0'') \approx \mathbf{E}_0''$. If $\eta''\mathbf{X} < 0$, there are 10 possible dimension types of direct summands of $R(\mathbf{X})$ having η-value negative. Again, by simple elimination, we conclude that a summand $\mathbf{V}_1''$ of dimension type $1\ 2\ \overset{1}{3}\ 2\ 1$ must occur and $T''(\mathbf{V}_1'') \approx \mathbf{E}_1''$.

$\tilde{\mathsf{F}}_{41}$. *$\{\mathbf{E} = 0\ 0\ F\ F\ F,\ \mathbf{E}' = 0\ G\ G\ F\ F\}$ is a generating set.* First, consider the contraction to $\mathsf{B}_3 : a_3 \xrightarrow{(1,2)} z \longleftarrow b$. If $\eta\mathbf{X} > 0$, there must be a summand $\mathbf{V}_0$ of type $(2, 1, 1)$ in the decomposition of $R(\mathbf{X})$ and $T'(\mathbf{V}_0) \approx \mathbf{E}_0$. If $\eta\mathbf{X} < 0$, there is a summand $\mathbf{V}_1$ of type $(2, 2, 1)$ and $T''(\mathbf{V}_1) \approx \mathbf{E}_1$.

Second, consider the contraction to $a_2 \xrightarrow{(1,2)} z \longleftarrow b$. If $\eta'\mathbf{X} > 0$ there is a summand $\mathbf{V}_0'$ of type $(1, 1, 1) : T'(\mathbf{V}_0') \approx \mathbf{E}_0'$. If $\eta'\mathbf{X} < 0$, then a summand $\mathbf{V}_1'$ of type $(1, 1, 0)$ occurs and $T''(\mathbf{V}_1') \approx \mathbf{E}_1'$.

$\tilde{\mathsf{F}}_{42}$. *$\{\mathbf{E} = 0\ F\ F\ G\ G,\ \mathbf{E}' = 0\ F\ F\ F\ 0\}$ is a generating set.* First, consider the contraction to $\mathsf{B}_3 : a_2 \longrightarrow z \xleftarrow{(2,1)} b_1$. If $\eta\mathbf{X} > 0$, then there is a direct summand $\mathbf{V}_0$ of $R(\mathbf{X})$ of dimension type $(1, 1, 1)$ and $T'(\mathbf{V}_0) \approx \mathbf{E}_0$. If $\eta\mathbf{X} < 0$, then the existence of a summand $\mathbf{V}_1$ of type $(0, 1, 1)$ yields the quotient $T''(\mathbf{V}_1) \approx \mathbf{E}_1$.

Second, consider the contraction to $a_2 \longrightarrow z \xleftarrow{(2,1)} b_2$. If $\eta'\mathbf{X} > 0$, we establish easily the existence of a direct summand $\mathbf{V}_0'$ of $R(\mathbf{X})$ of type $(1, 1, 2) : T'(\mathbf{V}_0') \approx \mathbf{E}_0'$. And, if $\eta'\mathbf{X} < 0$, there must be a summand $\mathbf{V}_1'$ of type $(1, 2, 2)$: $T''(\mathbf{V}_0') \approx \mathbf{E}_1'$.

$\tilde{\mathsf{G}}_{21}$. *$\{\mathbf{E} = 0\ F\ F\}$ is a generating set.* Consider the contraction to the Dynkin diagram $\mathsf{G}_2 : a_2 \xrightarrow{(1,3)} z$. If $\eta\mathbf{X} > 0$, then there must be a direct summand $\mathbf{V}_0$ of $R(\mathbf{X})$ of dimension type $(3, 1)$ and $T'(\mathbf{V}_0) \approx \mathbf{E}_0$. If $\eta\mathbf{X} < 0$, then there is a summand $\mathbf{V}_1$ of type $(3, 2)$ and $T''(\mathbf{V}_1) \approx \mathbf{E}_1$.

$\tilde{\mathsf{G}}_{22}$. *$\{\mathbf{E} = 0\ F\ \ G + fG$ with $f \in F \backslash G\}$ is a generating set.* Consider the contraction to $\mathsf{G}_2 : z \xleftarrow{(3,1)} b$. If $\eta\mathbf{X} > 0$, then there is a direct summand $\mathbf{V}_0$ of $R(\mathbf{X})$ of dimension type $(1, 2)$ determining $T'(\mathbf{V}_0) \approx \mathbf{E}_0$. And, $\eta\mathbf{X} < 0$ implies the existence of a summand $\mathbf{V}_1$ of type $(1, 1)$. Thus there is a quotient $T''(\mathbf{V}_1) \approx \mathbf{E}_1$ of $\mathbf{X}$.

5. Homogeneous Representations

In this section we assume again that $(\mathfrak{M}, \Omega)$ is a realization of an extended Dynkin diagram $(\Gamma, \mathbf{d})$. We want to show that the study of the homogeneous representations can be reduced to the study of the homogeneous representations of a realization of a diagram of type $\widetilde{\mathbf{A}}_{11}$ or $\widetilde{\mathbf{A}}_{12}$.

The realization of a diagram of type $\widetilde{\mathbf{A}}_{11}$ or $\widetilde{\mathbf{A}}_{12}$ will simply be called a bimodule, and denoted by ${}_F M_G$; more precisely, if $(\Gamma, \mathbf{d}) = \underset{1}{\cdot} \overset{(d_{12}, d_{21})}{\longrightarrow} \underset{2}{\cdot}$ is such a diagram with the orientation Ω defined by $\underset{1}{\cdot} \longrightarrow \underset{2}{\cdot}$, and the modulation $\mathfrak{M} = (F_1, F_2, {}_1M_2, {}_2M_1)$, we write simply $F = F_1$, $G = F_2$ and ${}_F M_G = {}_{F_1}({}_1M_2)_{F_2}$. Then ${}_F M_G$ determines completely the realization $(\mathfrak{M}, \Omega)$ and, in this way, we will consider ${}_F M_G$ as the realization of $(\Gamma, \mathbf{d})$. Thus, a representation (U_F, V_G, φ) of ${}_F M_G$ consists of two vector spaces U_F, V_G, and a G-linear mapping $\varphi: U_F \otimes {}_F M_G \longrightarrow V_G$. Note that, for every ${}_F M_G$, $\mathcal{R}({}_F M_G) = \mathcal{H}({}_F M_G)$.

Theorem 5.1. *Let $(\mathfrak{M}, \Omega)$ be a realization of an extended Dynkin diagram, and let $\mathcal{R}(\mathfrak{M}, \Omega) = \mathcal{H}(\mathfrak{M}, \Omega) \times \mathcal{R}^{(1)} \times \cdots \times \mathcal{R}^{(h)}$. Then there exists a bimodule ${}_F M_G$ of type $\widetilde{\mathbf{A}}_{11}$ or $\widetilde{\mathbf{A}}_{12}$, a full exact embedding $T: \mathcal{H}({}_F M_G) \longrightarrow \mathcal{R}(\mathfrak{M}, \Omega)$ and h simple objects $\mathbf{R}^{(1)}$, $\mathbf{R}^{(2)}, \ldots, \mathbf{R}^{(h)}$ in $\mathcal{H}({}_F M_G)$ such that*

(i) *the image of objects of $\mathcal{H}({}_F M_G)$ under T have continuous dimension types;*

(ii) *for all t, $T(\mathbf{R}^{(t)}) \in \mathcal{R}^{(t)}$;*

(iii) *the full subcategory of $\mathcal{H}({}_F M_G)$ of all objects without subobject of the form* $\mathbf{R}$ $\mathbf{R}^{(t)}$, $1 \leqslant t \leqslant h$, *is equivalent to $\mathcal{H}(\mathfrak{M}, \Omega)$ under T.*

As a consequence, we get also some information about the category $\mathcal{H}({}_F M_G)$. Namely, $\mathcal{H}({}_F M_G)$ is the product of $h + 1$ categories; h of these are uniserial categories with a unique simple object $\mathbf{R}^{(t)}$ and the remaining one is described in (iii); the objects have no composition factors of the form $\mathbf{R}^{(t)}$.

The proof of the theorem will consist in a case by case inspection. In the tables of Chapter 6, there are listed a bimodule ${}_F M_G$, a functor $T: \mathcal{L}({}_F M_G) \longrightarrow \mathcal{L}(\mathfrak{M}, \Omega)$, and the representations $\mathbf{R}^{(t)}$ of ${}_F M_G$. As we will show, these data satisfy the following conditions:

(0) *T is a full and exact embedding* (or, at least, the restriction of T to the category $\mathcal{L}e({}_F M_G)$ of all representations (U_F, V_G, φ) with a surjective mapping φ is a full and exact embedding).

(i) *If $\mathbf{X}$ is a representation in $\mathcal{L}e({}_F M_G)$, then $T(\mathbf{X})$ has continuous dimension type if and only if $\mathbf{X}$ has continuous dimension type.*

(ii)′ *$T(\mathbf{R}^{(t)})$ contains a simple object of $\mathcal{R}^{(t)}$ as a subobject, and* End $\mathbf{R}^{(t)}$ *is a division ring.*

(iii)′ *Every homogeneous representation of $(\mathfrak{M}, \Omega)$ is an image under T.*

We claim that these conditions imply the assertions of the theorem. For, since $T: \mathcal{L}e({}_F M_G) \longrightarrow \mathcal{L}(\mathfrak{M}, \Omega)$ is a full embedding, any representation $\mathbf{X}$ in $\mathcal{L}e({}_F M_G)$ is indecomposable if and only if $T(\mathbf{X})$ is indecomposable. Now, by (ii)′, $T(\mathbf{R}^{(t)})$ is indecomposable, and has no subobject in $\mathcal{R}(\mathfrak{M}, \Omega)$ of continuous dimension type. Therefore $\mathbf{R}^{(t)}$ has to be simple, because T is exact and satisfies (i). Since $T(\mathbf{R}^{(t)})$ is indecomposable and contains a simple subobject of $\mathcal{R}^{(t)}$, it belongs to $\mathcal{R}^{(t)}$. Let $\mathcal{C}$ be the full subcategory of all objects of $H({}_FM_G)$ without subobjects of the form $\mathbf{R}^{(t)}$, $1 \leqslant t \leqslant h$. If $\mathbf{Y} \neq 0$ belongs to C, then $\mathrm{Hom}(\mathbf{R}^{(t)}, \mathbf{Y}) = 0$ (since $\mathbf{R}^{(t)}$ is simple) for all t; thus, also $\mathrm{Hom}(T(\mathbf{R}^{(t)}, T(\mathbf{Y})) = 0$, and therefore $T(\mathbf{Y})$ is homogeneous (using the remark at the end of Chapter 3). This shows that the restriction of T to C gives a full and exact embedding of C into $H(\mathfrak{M}, \Omega)$. But every indecomposable homogeneous representation of $(\mathfrak{M}, \Omega)$ is of the form $T(\mathbf{Y})$ with an indecomposable representation $\mathbf{Y}$ and of continuous dimension type. Also, no $\mathbf{R}^{(t)}$ can be embedded into $\mathbf{Y}$; for, otherwise, $T(\mathbf{R}^{(t)}) \hookrightarrow T(\mathbf{Y})$, and $T(\mathbf{Y})$ would belong to $R^{(t)}$. This implies that the functor $T\colon C \longrightarrow H(\mathfrak{M}, \Omega)$ is dense and therefore T is an equivalence.

Now, we are going to consider the individual extended Dynkin diagrams. In all cases, condition (i) is satisfied trivially and, in most cases, it is also very easy to see that the condition (iii)$'$ is satisfied: one uses the properties of a simple regular representation $\mathbf{X}$ with $\eta(\mathbf{X}) = 0$ which are listed in the last column of the tables. These properties are satisfied for every simple homogeneous object, and therefore for every homogeneous representation at all. As a result, we are mainly concerned with conditions (0) and (ii)$'$.

$\widetilde{\mathbf{A}}_n$. It is obvious that $T(\mathbf{R})$ contains

$$0 \begin{smallmatrix} \nearrow & 0 \to 0 \cdots \to 0 \to N_F & \searrow \\ \searrow & 0 \to 0 \cdots \to 0 \to \ 0 & \nearrow \end{smallmatrix} 0,$$

whereas $T(\mathbf{R}')$ contains

$$0 \begin{smallmatrix} \nearrow & 0 \to 0 \to \cdots 0 \to 0 & \searrow \\ \searrow & 0 \to 0 \to \cdots 0 \to F_F & \nearrow \end{smallmatrix} 0.$$

$\widetilde{\mathbf{B}}_n$. Since $F_{G_1} \otimes {}_{G_1}F_{G_2} \xrightarrow{\text{mult}} F_{G_2}$ comes, in fact, from an F-linear mapping $F_{G_1} \otimes {}_{G_1}F_F \xrightarrow{\text{mult}} F_F$, its kernel K is an F-subspace of $F_{G_1} \otimes {}_{G_1}F_F$, and $\dim K_F = 1$. Thus $T(\mathbf{R})$ contains

$$0 \to 0 \to \cdots 0 \to K_F \to 0$$

as a subobject.

$\widetilde{\mathbf{C}}_n$. Obviously, the kernel of $\varphi\colon (F_1)_G \otimes {}_G(F_2)_{F_2} \twoheadrightarrow (F_1/G)_G \otimes {}_G(F_2)_{F_2} \approx (F_2)_{F_2}$ is $K_{F_2} = G_G \otimes {}_G(F_2)_{F_2}$; thus $T(\mathbf{R})$ contains

$$0 \to 0 \to \cdots 0 \to G_G \to 0$$

as a subobject.

$\widetilde{\mathbf{BC}}_n$. The kernel of the mapping $G_H \otimes {}_HG_G \xrightarrow{\text{mult}} G_G$ is a nonzero G-subspace K_G of $G_H \otimes {}_HG_G$; therefore

$$0 \to 0 \to \cdots 0 \to K_G \to 0$$

is contained in $T(\mathbf{R})$.

$\widetilde{\mathbf{BD}}_n$. The kernel of the mapping $F_F \otimes {}_FF_G \otimes {}_GF_F \xrightarrow{\text{mult}} F_F$ is a nonzero F-subspace K_F of $F_F \otimes {}_FF_G \otimes {}_GF_F$, and thus $K_F \oplus 0 \subseteq \Gamma_\varphi$; therefore,

$$0 \to 0 \to \cdots 0 \to K_F \oplus 0 \to 0$$

is contained in $T(\mathbf{R})$.

On the other hand, let $\varphi\colon F_F \otimes {}_FF_G \otimes {}_GF_F \approx F_G \otimes {}_GF_F \twoheadrightarrow (F/G)_G \otimes {}_GF_F \approx F_F$. Then $\Gamma_\varphi \cap F_F \oplus 0 = G_G \oplus 0$, and therefore

$$\begin{array}{l} 0 \searrow \\ \qquad F_F \oplus 0 \longrightarrow \cdots \longrightarrow F_F \oplus 0 \longrightarrow (F/G)_G \oplus 0 \\ F_F \nearrow \end{array}$$

is contained in $T(\mathbf{R}')$.

$\widetilde{\mathbf{CD}}_n$. Since the kernel K of $F_G \otimes {}_G F_G \xrightarrow{\text{mult}} F_G$ is an F-subspace of $F_G \otimes {}_G F_F$, it follows that $K \oplus 0 \subseteq \Gamma_\varphi$ is an F-subspace; thus,

$$\begin{array}{l} 0 \searrow \\ \qquad 0 \longrightarrow 0 \cdots 0 \longrightarrow K \oplus 0 \longrightarrow 0 \\ 0 \nearrow \end{array}$$

is contained in $T(\mathbf{R})$.

For the projection mapping $F_G \otimes {}_G F_G \twoheadrightarrow (F/G)_G \otimes {}_G F_G \approx F_G$, we see that $\Gamma_\varphi \cap (F_G \oplus 0) \otimes {}_G F_F = (G_G \oplus 0) \otimes {}_G F_F$ and hence $T(\mathbf{R}')$ contains

$$\begin{array}{l} 0 \searrow \\ \qquad (G_G \oplus 0) \otimes {}_G F_F \longrightarrow \cdots \longrightarrow (G_G \oplus 0) \otimes {}_G F_F \longrightarrow (F/G) \otimes {}_G F_G. \\ G_G \otimes {}_G F_F \nearrow \end{array}$$

$\widetilde{\mathbf{D}}_n$. Obviously, $T(\mathbf{R})$ contains

$$\begin{array}{l} 0 \searrow \qquad\qquad\qquad\qquad\quad \nearrow 0 \\ \qquad 0 \longrightarrow 0 \cdots 0 \longrightarrow \Gamma_{\text{id}} \qquad , \\ 0 \nearrow \qquad\qquad\qquad\qquad\quad \searrow 0 \end{array}$$

where $\Gamma_{\text{id}} = \{(f, f) \mid f \in F\}$ is the graph of the identity mapping. If $\varphi_1 = 0$, then $\Gamma_1 = U \oplus 0$; thus

$$\begin{array}{l} 0 \searrow \qquad\qquad\qquad\qquad\qquad\qquad\quad \nearrow 0 \\ \qquad F \oplus 0 \longrightarrow F \oplus 0 \longrightarrow \cdots \longrightarrow F \oplus 0 \\ F \nearrow \qquad\qquad\qquad\qquad\qquad\qquad\quad \searrow F \end{array}$$

is contained in $T(\mathbf{R}')$. Similarly, $\mathbf{E}_0'' \subseteq T(\mathbf{R}'')$.

$\widetilde{\mathbf{E}}_6$. First, we show that $T\colon L({}_F M_F) \longrightarrow L(\mathfrak{M}, \Omega)$ is a full embedding. Let

$$\begin{array}{c} \qquad\qquad\qquad\qquad\qquad\qquad C_1 \\ \qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\ \qquad\qquad\qquad\qquad\qquad\qquad C_2 \\ \qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\ T(U_F, V_F, (\varphi_1, \varphi_2)) = A_1 \hookrightarrow A_2 \hookrightarrow Z \hookleftarrow B_2 \hookleftarrow B_1. \end{array}$$

Then $A_2 \cap B_2 = 0\ U\ 0$; thus, all three direct components of Z are determined. Furthermore, $(A_1 + C_1) \cap B_2$ identifies $0\ U\ 0$ with $0\ 0\ U$, $(C_1 + B_1) \cap A_2$ defines the graph of φ_2, whereas $C_2 \cap A_2$ is the graph of φ_1.

Note that $T(\mathbf{R})$ contains a copy of $\mathbf{E}_0$ determined by the central vector space $0\ U\ 0$, $T(\mathbf{R}')$ contains a copy of $\mathbf{E}_0'$, with the central vector space $\{(0, u, u) \mid u \in U\}$, and $T(\mathbf{R}'')$ contains $\mathbf{E}_0''$ with the central vector space $0\ 0\ U$.

$\widetilde{\mathbf{E}}_7$. Again, we show that T is a full embedding. Obviously, the direct components of Z in $T(U_F, V_F, (\varphi_1, \varphi_2))$ are determined, and $(A_1 + C) \cap B_3$ determines the diagonals between $0\ U\ 0\ 0$, $0\ 0\ U\ 0$ and $0\ 0\ 0\ U$. Also φ_1 is determined by $(C + 0\ U\ 0\ 0) \cap V\ 0\ U\ 0$, and φ_2 is determined by $(C + 0\ 0\ 0\ U) \cap V\ 0\ U\ 0$.

$T(\mathbf{R})$ contains a copy of $\mathbf{E}_0$, with the central vector space generated by the elements $(0, u, u, 0)$, $u \in U$. $T(\mathbf{R}')$ contains a copy of $\mathbf{E}_0'$, with the central vector space generated by the elements $(0, 0, u, u)$, $u \in U$. Finally, $T(\mathbf{R}'')$ contains $\mathbf{E}_0''$, namely

$$\{(0, u, 0, -u) \mid u \in U\}$$
$$\downarrow$$
$$0\,0\,0\,0 \hookrightarrow 0\,U\,0\,0 \hookrightarrow 0\,U\,0\,0 \hookrightarrow 0\,U\,0\,U \hookleftarrow 0\,U\,0\,U \hookleftarrow 0\,0\,0\,U \hookleftarrow 0\,0\,0\,U$$

$\widetilde{\mathbf{E}}_8$. To show that T is a full embedding, one notes that the components $V\,0\,0\,0\,0\,0$, $0\,0\,U\,0\,0\,0$, and $0\,0\,0\,0\,U\,0$ of the central vector space Z in $T(U_F, V_F, (\varphi_1, \varphi_2))$ are determined using only the subspaces A_i and B_j. Also, $(C + A_1) \cap A_4 \cap B_2 = 0\,0\,0\,U\,0\,0$, and then $(C + 0\,0\,0\,U\,0\,0) \cap A_2 = 0\,U\,0\,0\,0\,0$, and $(C + V\,0\,0\,U\,0\,0) \cap B_2 = 0\,0\,0\,0\,0\,U$. This shows that all components are determined. Obviously, C provides the identifications of the different copies of U, as well as the definition of the graphs of φ_1 and φ_2.

Now, $T(\mathbf{R})$ contains a copy of $\mathbf{E}_0$, since $(0, 0, 1, 1, 1, 0)$ belongs to $A_5 \cap B_2 \cap C$. Similarly, $T(\mathbf{R}')$ contains a copy of $\mathbf{E}_0'$, since $(0, 0, 1, 0, 1, 1) = (0, 0, 1, 0, 0, 0) + (0, 0, 0, 0, 1, 1)$ belongs both to C and to $(A_3 \cap B_2) + B_1$. Finally, $T(\mathbf{R}'')$ contains a copy of $\mathbf{E}_0''$ whose central vector space is $0\,U\,0\,U\,0\,U$, and whose vector space C is generated by the two elements $(0, 0, 0, 1, 0, -1)$ and $(0, 1, 0, 1, 0, 0)$.

In the remaining four cases, we shall use the concept of an interior and that of a closure of a G-subspace in a vector F-space (where G is a subfield of F) defined in the introduction to the tables in Chapter 6 (for details, see [4]).

$\widetilde{\mathbf{F}}_{41}$. The functor T is a full embedding: If $T(U_G, V_G, \varphi) = (A_1 \to A_2 \to A_3 \to Z \leftarrow B)$, then $U_G = A_1$, $V_G = A_2 \cap \underline{A}_3$, and B is just the graph of φ.

We claim that $T(\mathbf{R})$ contains a copy of $\mathbf{E}_0$. The kernel of the mapping $F_G \otimes {}_G F_F \twoheadrightarrow (F/G)_G \otimes {}_G F_F$ is the F-subspace $K_F = G_G \otimes {}_G F_F$. Thus $0 \times K$ is an F-subspace of $F_G \otimes {}_G F_F \to F_G \otimes {}_G F_F$, and $T(\mathbf{R})$ contains

$$0 \to 0 \to 0 \times K \to 0 \times K \leftarrow 0 \times K.$$

On the other hand, let φ be the identity mapping $\varphi: G_G \otimes {}_G F_F \to G_G \otimes {}_G F_F$. Then $\Gamma_\varphi \cap G_G \times G_G = (1, 1)G$ and therefore $T(\mathbf{R}')$ contains a copy of $\mathbf{E}_0'$, namely

$$0 \to (1, 1)G \to (1, 1)G \to \Gamma_\varphi \leftarrow \Gamma_\varphi.$$

Perhaps we should point out that the category of all images under T is given by all representations which satisfy the conditions $\underline{A}_2 = 0$, $\overline{A}_2 = Z$, $\underline{A}_3 \oplus B = Z$, $\overline{A}_1 \oplus \underline{A}_3 = W$. The conditions $\overline{A}_1 \oplus B = Z$ and $A_2 \oplus B = Z$ are not satisfied for all images under T, but rather for the homogeneous ones.

$\widetilde{\mathbf{F}}_{42}$. First, we show that the functor T is a full embedding. Since $\underline{U_F \otimes_F F_G \otimes_G G_G} = 0$, we see that $\underline{B}_2 = 0 \times V_F$. Also $U_F \otimes {}_F N_F = \overline{B}_1 \cap (A_1 + \underline{B}_2)$, and $U_F = U_F \otimes {}_F N_F \otimes {}_F N_F^*$. Here we use the fact that $\dim_F N = \dim N_F = 1$, so that N^* exists with ${}_F N_F \otimes {}_F N_F^* = {}_F F_F$. Finally, φ is determined by A_2.

Second, $T(\mathbf{R})$ contains $\mathbf{E}_0$, because $(1 \otimes 1) \times 0$ belongs to $A_2 \cap B_1$ in this case. On the other hand, $T(\mathbf{R}')$ contains a copy of $\mathbf{E}_1'$, because the kernel of the multiplication mapping $F_G \otimes {}_G F_F \xrightarrow{\text{mult}} F_F$ is just N_F, so that $A_1 \cap \overline{B}_1 \neq 0$.

$\widetilde{\mathbf{G}}_{21}$. The functor T: $L({}_G(F/G)_G) \to L(\mathfrak{M}, \Omega)$ defines an equivalence of categories between the category $L\varrho({}_G(F/G)_G)$ of all representations $\varphi: U_G \otimes {}_G(F/G)_G \to V_G$ with a surjective φ, and the category $\mathcal{C}$ of all representations $(A_1 \hookrightarrow A_2 \hookrightarrow Z)$ of $(\mathfrak{M}, \Omega)$ which have the property that $(A_1 \hookrightarrow Z)$, considered as a representation of $\cdot \xrightarrow{(1,3)} \cdot$, is a direct sum of copies of $(G_G \hookrightarrow F_F)$. It is obvious that $H({}_G(F/G)_G) \subseteq \mathcal{L}e({}_G(F/G)_G)$.

Also, using the fact that a homogeneous representation $\mathbf{X}$ does not allow a nonzero homomorphism from $\mathbf{E}_1$ into $\mathbf{X}$, we get $H(\mathfrak{M}, \Omega) \subseteq \mathcal{C}$. Thus $(A_1 \hookrightarrow Z)$ has to be a direct sum of copies of $(0 \hookrightarrow F_F)$ and $(G_G \hookrightarrow F_F)$. However, if there is a copy of $(0 \hookrightarrow F_F)$, then $\dim(A_1)_G < \dim Z_F$, and, using the defect argument, $\dim(A_2)_G > 2\dim Z_F$, which would imply the existence of a subobject of the form $\mathbf{E}_0$.

Now, consider $T(\mathbf{R})$, where $\mathbf{R} = (F_F, (F_G \otimes {}_G F_G)/(F_G \otimes {}_G G_G + \omega(F_F)), \pi)$. Since $(F_G \otimes G_G) \cap \omega(F) = 0$ according to the example following Proposition 2.1, $\dim T(\mathbf{R}) = (3, 6, 9)$. Obviously, $T(\mathbf{R})$ contains a copy of $\mathbf{E}_0$, namely

$$0 \longrightarrow \omega(F_F) \longrightarrow \omega(F_F),$$

and the quotient is $\mathbf{E}_1$. But $T(\mathbf{R})$ is not the direct sum of $\mathbf{E}_0$ and $\mathbf{E}_1$; for, in $T(\mathbf{R})$, the closure $\overline{A}_1$ of $A_1 = F_G \otimes {}_G G_G$ is $Z = F_G \otimes {}_G F_F$. Thus $\operatorname{End}\mathbf{R} \approx \operatorname{End} T(\mathbf{R})$ is a division ring.

$\widetilde{\mathbf{G}}_{22}$. Let ${}_F M_F = ({}_F F_G \otimes {}_G F_F)/\omega(F)$ and ${}_F N_G = ({}_F F_G \otimes {}_G G_G + \omega F)/\omega(F)$. According to the example following Proposition 2.1, the endomorphism ring of the pair $(M_F \longleftarrow N_G)$, considered as a representation of $F \cdot \xleftarrow{{}_G F_F} \cdot\, G$, is F. Now, the direct sums of copies of $(M_F \supseteq N_G)$ form a category $\mathcal{C}$ which is equivalent to the category of all vector F-spaces, under the functor which maps an F-space U_F to $(U_F \otimes {}_F M_F \supseteq U_F \otimes {}_F N_G)$. [In order to show that it is an equivalence, one easily checks that it is a dense full embedding.] Thus, there is a reversed functor which associates to a direct sum $(Z_F \supseteq B_G)$ of copies of $(M_F \supseteq N_G)$ a vector space U_F such that

$$U_F \otimes {}_F(M_F \supseteq N_G) \approx (Z_F \supseteq B_G).$$

We claim that T provides an equivalence between the category $\mathcal{L}e({}_F M_F)$ and the category $\mathcal{C}$ of all representations $(A_F \hookrightarrow Z_F \hookleftarrow B_G)$ such that $(Z \hookleftarrow B)$ is a direct sum of indecomposable representations of dimension type $(2, 3)$. Of course, T maps $\mathcal{L}e({}_F M_F)$ into $\mathcal{C}$. But, obviously, $T\colon \mathcal{L}e({}_F M_F) \longrightarrow \mathcal{C}$ is also dense. Moreover, T is a full embedding, since $(A_F \longrightarrow Z_F \longleftarrow B_G)$ determines U_G and the kernel of φ, and therefore also φ.

It is easy to see that $T(\mathbf{R})$ contains a copy of $\mathbf{E}_0$.

6. Tables

The following tables summarize the results of Chapters 4 and 5 on the indecomposable regular representations. Each of the tables provide the following information on the respective extended Dynkin diagram:

1. The type of the valued graph including the notation for the vertices.
2. The quadratic form of the graph.
3. The vector **n** stable under the action of Weyl group, and the tier number g.
4. The defect vector ∂_c with respect to an orientation which is indicated here.
5. A modulation $\mathfrak{M} = (F_i, {}_iM_j)$. As a rule, if ${}_iM_j$ is one-dimensional on both sides, we assume $F_i = F_j$ and ${}_{F_i}({}_iM_j)_{F_j} = {}_{F_i}(F_i)_{F_i}$. This is not, in general, possible for the diagram $\widetilde{\mathbf{A}}_n$, because it is not a tree; here, we assume that $F_i = F$ for all $i \in \Gamma$ and that ${}_iM_j = {}_FF_F$ for all but one edge. Also, if ${}_iM_j$ is one-dimensional as a left F_i-module, we may assume that F_j is a subfield of F_i and ${}_iM_j = {}_{F_i}(F_i)_{F_j}$; and, similarly for the case when ${}_iM_j$ is a one-dimensional right F_j-module. In the tables we omit to display these bimodules explicitly.
6. The simple nonhomogeneous regular representations. The list is divided into the orbits of C^+. Besides the representations, we include their dimension type, their image under C^+, the equation $\eta = \eta_0$ for $\mathbf{E} = \mathbf{E}_0$ and the derived equations η_r for other representations $\mathbf{E}_r$. The last column in the tables points out the consequences of the equality $\eta_r^{(t)}\mathbf{X} = 0$ for a simple regular representation $\mathbf{X}$. In particular, a simple regular representation is homogeneous if and only if all conditions given in the last column are satisfied simultaneously. [The consequences follow immediately from the proofs of the tables. However, they can be easily derived from the tables alone: that is to say, the assumption that a particular consequence, for a regular $\mathbf{X}$, is not satisfied implies that, for the respective $\mathbf{E}_r^{(t)}$, either $\mathbf{E}_r^{(t)} \hookrightarrow \mathbf{X}$ or $\mathbf{X} \twoheadrightarrow \mathbf{E}_r^{(t)}$.] Here, the capital letters $A, B, Z, \ldots$ (possibly with some indices) denote the vector spaces of the representation $\mathbf{X}$ in the corresponding vertices $a, b, z, \ldots$ of Γ. In case that G is a subfield of F and the mapping $A_G \longrightarrow Z_F \otimes {}_FF_G$ is a monomorphism, we shall assume that A_G is a G-subspace of the F-space Z_F. In such a situation, we shall denote by $\underline{A}$ the maximal F-subspace $\{a \in A \mid aF \subseteq A\}$ contained in A (the "F-interior" of A) and by $\overline{A}$ the F-subspace $\Sigma_{a\in A}\, aF$ (of Z_F) generated by A (the "F-closure" of A). Observe also that the source i needed in the proofs of Chapter 4 is always a or a_1.
7. A bimodule ${}_FM_G$ whose representations determine the homogeneous representations of the corresponding extended Dynkin diagram.
8. A functor $T\colon L({}_FM_G) \longrightarrow L(\mathfrak{M}, \Omega)$ which defines the correspondence.
9. The simple regular representations $\mathbf{R}$ of ${}_FM_G$ such that $T(\mathbf{R})$ is not simple. Always, $\mathbf{E}_0^{(t)} \subseteq T(\mathbf{R}^{(t)})$ with the exception of the diagram $\widetilde{\mathbf{F}}_{42}$, where $T(\mathbf{R}')$ contains $(\mathbf{E}_1')$.

Finally, let us make the following convention: If ${}_FM_G = M_1 \oplus M_2$, we shall decompose $\varphi\colon U_F \otimes {}_FM_G \longrightarrow V_G$ is an obvious manner into (φ_1, φ_2). Also, writing down a representation, we shall usually specify the vector spaces and the mappings by $U_{F_1} \xrightarrow{\varphi} U_{F_2}$ (although these mappings are defined between $U_{F_1} \otimes {}_{F_1}M_{F_2}$ and U_{F_2}).

$\widetilde{A}_{11}$

$a \xrightarrow{(1,\,4)} b;$

$Q(x) = (x_a - 2x_b)^2 = (a - 2b)^2;$

$n = 2 \text{ —— } 1; g = 2;$

$\partial_c = 1 \longrightarrow -2;$

$\mathfrak{M} = G \text{ —— } F$ with $[F:G] = 4;$

All regular representations are homogeneous.

$\widetilde{A}_{12}$

$a \xrightarrow{(2,\,2)} b;$

$Q(x) = (x_a - x_b)^2 = (a - b)^2;$

$n = 1 \text{ —— } 1; g = 1;$

$\partial_c = 2 \longrightarrow -2;$

$\mathfrak{M} = F_1 \xrightarrow{{}_{F_1}M_{F_2}} F_2$ with $\dim_{F_1} M = \dim M_{F_2} = 2;$

All regular representations are homogeneous.

$\widetilde{\mathbf{A}}_n$

$$a\begin{smallmatrix}\nearrow c_1 - c_2 - \cdots - c_p \searrow \\ \searrow d_1 - d_2 - \cdots - d_q \nearrow\end{smallmatrix} b, \text{ where } p \geqslant q \text{ and } p+q = n-1;$$

$Q(\mathbf{x}) = \sum (x_i - x_j)^2$, where the summation runs through all edges $i - j$;

$$\mathbf{n} = 1\begin{smallmatrix}\nearrow 1 - 1 - \cdots - 1 \searrow \\ \searrow 1 - 1 - \cdots - 1 \nearrow\end{smallmatrix} 1; \quad g = 1; \qquad \Big|\, \partial_c = n+1 \begin{smallmatrix}\nearrow 0 \to 0 \to \cdots \to 0 \searrow \\ \searrow 0 \to 0 \to \cdots \to 0 \nearrow\end{smallmatrix} -(n+1);$$

$$\mathfrak{M} = F\begin{smallmatrix}\nearrow F - F - \cdots - F \searrow \\ \searrow F - F - \cdots - F \nearrow_N\end{smallmatrix} F,$$ where ${}_FN = {}_FF$ and the right action of F on N is given by a field automorphism of F;

	$\mathbf{E}_r^{(t)}$	$\dim \mathbf{E}_r^{(t)}$	$C^+\mathbf{E}_r^{(t)}$	$\eta_r^{(t)}$	isomorphisms for simple $\mathbf{X}$ with $\eta_r^{(t)}\mathbf{X} = 0$
for $p \geqslant 1$	$\mathbf{E}_0 = 0\begin{smallmatrix}0\,0\cdots 0\,F\\0\,0\cdots 0\,0\end{smallmatrix}0$	$0\begin{smallmatrix}0\,0\cdots 0\,1\\0\,0\cdots 0\,0\end{smallmatrix}0$	$\mathbf{E}_1$	$0\begin{smallmatrix}0\,0\cdots 0\,1\\0\,0\cdots 0\,0\end{smallmatrix}-1$	$C_p \to B$
	$\mathbf{E}_1 = F\begin{smallmatrix}0\,0\cdots 0\,0\\F\,F\cdots F\,F\end{smallmatrix}F$	$1\begin{smallmatrix}0\,0\cdots 0\,0\\1\,1\cdots 1\,1\end{smallmatrix}1$	$\mathbf{E}_2$	$1\begin{smallmatrix}-1\,0\cdots 0\,0\\0\,0\cdots 0\,0\end{smallmatrix}0$	$A \to C_1$
	$\mathbf{E}_2 = 0\begin{smallmatrix}F\,0\cdots 0\,0\\0\,0\cdots 0\,0\end{smallmatrix}0$	$0\begin{smallmatrix}1\,0\cdots 0\,0\\0\,0\cdots 0\,0\end{smallmatrix}0$	$\mathbf{E}_3$	$0\begin{smallmatrix}1\,-1\cdots 0\,0\\0\;\;0\cdots 0\,0\end{smallmatrix}0$	$C_1 \to C_2$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\mathbf{E}_p = 0\begin{smallmatrix}0\,0\cdots F\,0\\0\,0\cdots 0\,0\end{smallmatrix}0$	$0\begin{smallmatrix}0\,0\cdots 1\,0\\0\,0\cdots 0\,0\end{smallmatrix}0$	$\mathbf{E}_0$	$0\begin{smallmatrix}0\,0\cdots 1\,-1\\0\,0\cdots 0\;\;0\end{smallmatrix}0$	$C_{p-1} \to C_p$
for $q \geqslant 1$	$\mathbf{E}'_0 = 0\begin{smallmatrix}0\,0\cdots 0\,0\\0\,0\cdots 0\,F\end{smallmatrix}0$	$0\begin{smallmatrix}0\,0\cdots 0\,0\\0\,0\cdots 0\,1\end{smallmatrix}0$	$\mathbf{E}'_1$	$0\begin{smallmatrix}0\,0\cdots 0\,0\\0\,0\cdots 0\,1\end{smallmatrix}-1$	$D_q \to B$
	$\mathbf{E}'_1 = F\begin{smallmatrix}F\,F\cdots F\,F\\0\,0\cdots 0\,0\end{smallmatrix}F$	$1\begin{smallmatrix}1\,1\cdots 1\,1\\0\,0\cdots 0\,0\end{smallmatrix}1$	$\mathbf{E}'_2$	$1\begin{smallmatrix}0\,0\cdots 0\,0\\-1\,0\cdots 0\,0\end{smallmatrix}0$	$A \to D_1$
	$\mathbf{E}'_2 = 0\begin{smallmatrix}0\,0\cdots 0\,0\\F\,0\cdots 0\,0\end{smallmatrix}0$	$0\begin{smallmatrix}0\,0\cdots 0\,0\\1\,0\cdots 0\,0\end{smallmatrix}0$	$\mathbf{E}'_3$	$0\begin{smallmatrix}0\;\;0\cdots 0\,0\\1\,-1\cdots 0\,0\end{smallmatrix}0$	$D_1 \to D_2$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\mathbf{E}'_q = 0\begin{smallmatrix}0\,0\cdots 0\,0\\0\,0\cdots F\,0\end{smallmatrix}0$	$0\begin{smallmatrix}0\,0\cdots 0\,0\\0\,0\cdots 1\,0\end{smallmatrix}0$	$\mathbf{E}'_0$	$0\begin{smallmatrix}0\,0\cdots 0\;\;0\\0\,0\cdots 1\,-1\end{smallmatrix}0$	$D_{q-1} \to D_q$

$M = {}_FF_F \oplus {}_FN_F$;

$$T(U_F, V_F, (\varphi_1, \varphi_2)) = U_F \begin{smallmatrix} \overset{1}{\nearrow} U_F \overset{1}{\longrightarrow} U_F \overset{1}{\longrightarrow} \cdots \overset{1}{\longrightarrow} U_F \overset{\varphi_1}{\searrow} \\ \underset{1}{\searrow} U_F \underset{1}{\longrightarrow} U_F \underset{1}{\longrightarrow} \cdots \underset{1}{\longrightarrow} U_F \underset{\varphi_2}{\nearrow} \end{smallmatrix} V_F;$$

$\mathbf{R} = (F_F, N_F, (0, 1))$ satisfies $\mathbf{E}_0 \subseteq T(\mathbf{R})$ for $p \geqslant 1$;

$\mathbf{R}' = (F_F, F_F, (1, 0))$ satisfies $\mathbf{E}'_0 \subseteq T(\mathbf{R}')$ for $q \geqslant 1$.

$$\widetilde{\mathbf{B}}_n$$

$$a \xrightarrow{(1,2)} z_1 - z_2 - \cdots - z_{n-1} \xrightarrow{(2,1)} b;$$

$$Q(\mathrm{x}) = (a - z_1)^2 + \sum_{1 \leqslant t \leqslant n-2} (z_t - z_{t+1})^2 + (z_{n-1} - b)^2;$$

$\mathbf{n} = 1 - 1 - 1 - \cdots - 1 - 1$; $g = 2$;

$\partial_c = 2 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow -2$;

$\mathfrak{M} = G_1 - F - F - \cdots - F - G_2$ with $[F:G_1] = [F:G_2] = 2$;

$\mathbf{E}_r$	$\dim \mathbf{E}_r$	$C^+\mathbf{E}_r$	η_r	isomorphisms for simple $\mathbf{X}$ with $\eta_r\mathbf{X} = 0$
$\mathbf{E}_0 = 000\cdots 0F0$	$000\cdots 010$	$\mathbf{E}_1$	$000\cdots 01{-1}$	$Z_{n-1} \longrightarrow \mathrm{Hom}({}_FF_{G_2}, B_{G_2})$
$\mathbf{E}_1 = FFF\cdots FFF$	$211\cdots 112$	$\mathbf{E}_2$	$1{-1}0\cdots 000$	$A \otimes_{G_1} F_F \longrightarrow Z_1$
$\mathbf{E}_2 = 0F0\cdots 000$	$010\cdots 000$	$\mathbf{E}_3$	$01{-1}\cdots 000$	$Z_1 \longrightarrow Z_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathbf{E}_{n-1} = 000\cdots F00$	$000\cdots 100$	$\mathbf{E}_0$	$000\cdots 1{-1}0$	$Z_{n-2} \longrightarrow Z_{n-1}$

$M = {}_{G_1}F_{G_2}$;

$T(U_{G_1}, V_{G_2}, \varphi) = U_{G_1} \xrightarrow{1} U_{G_1} \otimes {}_{G_1}F_F \xrightarrow{1} \cdots \xrightarrow{1} U_{G_1} \otimes {}_{G_1}F_F \xrightarrow{\varphi} V_{G_2}$;

$\mathbf{R} = (F_{G_1}, F_{G_2}, F_{G_1} \otimes {}_{G_1}F_{G_2} \xrightarrow{\text{mult}} F_{G_2})$ satisfies $\mathbf{E}_0 \subseteq T(\mathbf{R})$;

$\widetilde{\mathbf{C}}_n$

$a \overset{(2,1)}{\text{———}} z_1 — z_2 — \cdots — z_{n-1} \overset{(1,2)}{\text{———}} b$

$Q(\mathbf{x}) = (2a - z_1)^2 + \sum_{1 \leqslant t \leqslant n-2} (z_t - z_{t+1})^2 + (z_{n-1} - 2b)^2;$

$\mathbf{n} = 1 — 2 — 2 — \cdots — 2 — 1;\ g = 1;$

$\partial_c = 2 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow -2;$

$\mathfrak{M} = F_1 — G — G — \cdots — G — F_2$ with $[F_1 : G] = [F_2 : G] = 2$;

$\mathbf{E}_r$	$\dim \mathbf{E}_r$	$C^+\mathbf{E}_r$	η_r	isomorphisms for simple **X** with $\eta_r\mathbf{X} = 0$
$\mathbf{E}_0 = 0\ 0\ 0 \cdots 0\ G\ 0$	$0\ 0\ 0 \cdots 0\ 1\ 0$	$\mathbf{E}_1$	$0\ 0\ 0 \cdots 0\ 1\ {-2}$	$Z_{n-1} \longrightarrow B_G$
$\mathbf{E}_1 = F_1\ G\ G \cdots G\ G\ F_2$	$1\ 1\ 1 \cdots 1\ 1\ 1$	$\mathbf{E}_2$	$2\ {-1}\ 0 \cdots 0\ 0\ 0$	$A_G \longrightarrow Z_1$
$\mathbf{E}_2 = 0\ G\ 0 \cdots 0\ 0\ 0$	$0\ 1\ 0 \cdots 0\ 0\ 0$	$\mathbf{E}_3$	$0\ 1\ {-1} \cdots 0\ 0\ 0$	$Z_1 \longrightarrow Z_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathbf{E}_{n-1} = 0\ 0\ 0 \cdots G\ 0\ 0$	$0\ 0\ 0 \cdots 1\ 0\ 0$	$\mathbf{E}_0$	$0\ 0\ 0 \cdots 1\ {-1}\ 0$	$Z_{n-2} \longrightarrow Z_{n-1}$

$M = {}_{F_1}(F_1)_G \otimes {}_G(F_2)_{F_2};$

$T(U_{F_1}, U_{F_2}, \varphi) = (U_{F_1} \xrightarrow{1} U_{F_1} \otimes {}_{F_1}(F_1)_G \xrightarrow{1} \cdots \xrightarrow{1} U_{F_1} \otimes {}_{F_1}(F_1)_G \xrightarrow{\varphi} V_{F_2};$

$\mathbf{R} = ((F_1)_{F_1}, (F_2)_{F_2}, (F_1)_G \otimes {}_G(F_2)_{F_2} \twoheadrightarrow (F_1/G)_G \otimes {}_G(F_2)_{F_2} \approx (F_2)_{F_2})$ satisfies $\mathbf{E}_0 \subseteq T(\mathbf{R});$

$\widetilde{\mathbf{BC}}_n$

$a \overset{(1,2)}{\text{——}} z_1 — z_2 — \cdots — z_{n-1} \overset{(1,2)}{\text{——}} b$;

$Q(\mathbf{x}) = (a - z_1)^2 + \sum_{1 \leqslant t \leqslant n-2} (z_t - z_{t+1})^2 + (z_{n-1} - 2b)^2$;

$\mathbf{n} = 2 — 2 — 2 — \cdots — 2 — 1$; $g = 2$;

$\partial_c = 1 \to 0 \to 0 \to \cdots \to 0 \to -2$;

$\mathfrak{M} = H — G — G — \cdots — G — F$ with $H \subseteq G \subseteq F$ and $[F:G] = [G:H] = 2$;

$\mathbf{E}_r$	$\dim \mathbf{E}_r$	$C^+\mathbf{E}_r$	η_r	isomorphisms for simple $\mathbf{X}$ with $\eta_r\mathbf{X} = 0$
$\mathbf{E}_0 = 0\,0\,0 \cdots 0\,G\,0$	$0\,0\,0 \cdots 0\,1\,0$	$\mathbf{E}_1$	$0\,0\,0 \cdots 0\,1\,{-2}$	$Z_{n-1} \to B_G$
$\mathbf{E}_1 = G\,G\,G \cdots G\,G\,F$	$2\,1\,1 \cdots 1\,1\,1$	$\mathbf{E}_2$	$1\,{-1}\,0 \cdots 0\,0\,0$	$A_H \otimes {}_HG_G \to Z_1$
$\mathbf{E}_2 = 0\,G\,0 \cdots 0\,0\,0$	$0\,1\,0 \cdots 0\,0\,0$	$\mathbf{E}_3$	$0\,1\,{-1} \cdots 0\,0\,0$	$Z_1 \to Z_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathbf{E}_{n-1} = 0\,0\,0 \cdots G\,0\,0$	$0\,0\,0 \cdots 1\,0\,0$	$\mathbf{E}_0$	$0\,0\,0 \cdots 1\,{-1}\,0$	$Z_{n-2} \to Z_{n-1}$

$M = {}_HF_F$;

$T(U_H, V_F, \varphi) = U_H \xrightarrow{1} U_H \otimes {}_HG_G \xrightarrow{1} \cdots \to U_H \otimes {}_HG_G \xrightarrow{\varphi} V_F$;

$\mathbf{R} = (G_H, F_F, G_H \otimes {}_HF_F \xrightarrow{\text{mult}} F_F)$ satisfies $\mathbf{E}_0 \subseteq T(\mathbf{R})$;

$\widetilde{\mathbf{BD}}_n$

$$\begin{matrix} a_1 \\ a_2 \end{matrix} \!\!>\! z_1 - z_2 - \cdots - z_{n-2} \xrightarrow{(2,\,1)} b;$$

$$Q(\mathbf{x}) = \sum_{t=1,2} (2a_t - a_t)^2 + 2 \sum_{1 \leqslant t \leqslant n-3} (z_t - z_{t+1})^2 + 2(z_{n-2} - b)^2;$$

$$\mathbf{n} = \begin{matrix} 1 \\ 1 \end{matrix} \!\!>\! 2 - 2 - \cdots - 2 - 2; \quad g = 1;$$

$$\partial_c = \begin{matrix} 1 \\ 1 \end{matrix} \!\!\rightrightarrows\! 0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow -1;$$

$$\mathfrak{M} = \begin{matrix} F \\ F \end{matrix} \!\!>\! F - F - \cdots - F - G \text{ with } [F:G] = 2;$$

$\mathbf{E}_r^{(t)}$	$\dim \mathbf{E}_r^{(t)}$	$\mathbf{C}^+\mathbf{E}_r^{(t)}$	$\eta_r^{(t)}$	isomorphisms for simple $\mathbf{X}$ with $\eta_r^{(t)}\mathbf{X} = 0$
$\mathbf{E}_0 = {}^{0}_{0}\,0\,0 \cdots 0\,F\,0$	${}^{0}_{0}\,0\,0 \cdots 0\,1\,0$	$\mathbf{E}_1$	${}^{0}_{0}\,0\,0 \cdots 0\,1\,{-1}$	$Z_{n-2} \longrightarrow \mathrm{Hom}({}_FF_G, B_G)$
$\mathbf{E}_1 = {}^{F}_{F}\,F\,F \cdots F\,F\,F$	${}^{1}_{1}\,1\,1 \cdots 1\,1\,2$	$\mathbf{E}_2$	${}^{1}_{1}\,{-1}\,0 \cdots 0\,0\,0$	$A_1 \oplus A_2 \longrightarrow Z_1$
$\mathbf{E}_2 = {}^{0}_{0}\,F\,0 \cdots 0\,0\,0$	${}^{0}_{0}\,1\,0 \cdots 0\,0\,0$	$\mathbf{E}_3$	${}^{0}_{0}\,1\,{-1} \cdots 0\,0\,0$	$Z_1 \longrightarrow Z_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathbf{E}_{n-2} = {}^{0}_{0}\,0\,0 \cdots F\,0\,0$	${}^{0}_{0}\,0\,0 \cdots 1\,0\,0$	$\mathbf{E}_0$	${}^{0}_{0}\,0\,0 \cdots 1\,{-1}\,0$	$Z_{n-3} \longrightarrow Z_{n-2}$
$\mathbf{E}'_0 = {}^{0}_{F}\,F\,F \cdots F\,F\,G$	${}^{0}_{1}\,1\,1 \cdots 1\,1\,1$	$\mathbf{E}'_1$	${}^{0}_{2}\,0\,0 \cdots 0\,0\,{-1}$	$A_1 \approx A_2$
$\mathbf{E}'_1 = {}^{F}_{0}\,F\,F \cdots F\,F\,G$	${}^{1}_{0}\,1\,1 \cdots 1\,1\,1$	$\mathbf{E}'_0$	${}^{2}_{0}\,0\,0 \cdots 0\,0\,{-1}$	$A_1 \approx A_2$

$M = {}_FF_G \otimes {}_GF_F$;

$$T(U_F, V_F, \varphi) = \begin{matrix} V_F \\ U_F \end{matrix} \!\!\rightrightarrows\! U_F \oplus V_F \xrightarrow{1} U_F \oplus V_F \xrightarrow{1} \cdots \xrightarrow{1} U_F \oplus V_F \longrightarrow (U_G \oplus V_G)/\Gamma_\varphi,$$

where $\Gamma_\varphi = \{(u, \varphi(u \otimes 1 \otimes 1)) \mid u \in U\}$ is a G-subspace of $U_G \oplus V_G$;

$\mathbf{R} = (F_F, F_F, F_F \otimes {}_FF_G \otimes {}_GF_F \xrightarrow{\text{mult}} F_F)$ satisfies $\mathbf{E}_0 \subseteq T(\mathbf{R})$;

$\mathbf{R}' = (F_F, F_F, F_F \otimes {}_FF_G \otimes {}_GF_F \approx F_G \otimes {}_GF_F \twoheadrightarrow (F/G)_G \otimes {}_GF_F \approx F_F)$ satisfies $\mathbf{E}'_0 \subseteq T(\mathbf{R}')$;

$\widetilde{\mathbf{CD}}_n$

$$\begin{matrix} a_1 \searrow \\ \quad z_1 - z_2 - \cdots - z_{n-2} \overset{(1,2)}{\longrightarrow} b; \\ a_2 \nearrow \end{matrix}$$

$$Q(\mathrm{x}) = \sum_{t=1,2} (2a_t - z_1)^2 + 2 \sum_{1 \leqslant t \leqslant n-3} (z_t - z_{t+1})^2 + 2(z_{n-2} - 2b)^2;$$

$$\mathrm{n} = {}^{1}_{1}\!\!> 2 - 2 - \cdots - 2 - 1; \quad g = 2;$$

$$\partial_c = {}^{1}_{1}\!\!\rightrightarrows 0 \to 0 \to \cdots \to 0 \to -2;$$

$$\mathfrak{M} = {}^{G}_{G}\!\!> G - G - \cdots - G - F \text{ with } [F:G] = 2;$$

$\mathbf{E}_r^{(t)}$	$\dim \mathbf{E}_r^{(t)}$	$C^+\mathbf{E}_r^{(t)}$	$\eta_r^{(t)}$	isomorphisms for simple X with $\eta_r^{(t)}\mathbf{X} = 0$
$\mathbf{E}_0 = {}^{0}_{0}\,0\,0 \cdots 0\,G\,0$	${}^{0}_{0}\,0\,0 \cdots 0\,1\,0$	$\mathbf{E}_1$	${}^{0}_{0}\,0\,0 \cdots 0\,1\,{-2}$	$Z_{n-2} \to B_G$
$\mathbf{E}_1 = {}^{G}_{G}\,G\,G \cdots G\,G\,F$	${}^{1}_{1}\,1\,1 \cdots 1\,1\,1$	$\mathbf{E}_2$	${}^{1}_{1}\,{-1}\,0 \cdots 0\,0\,0$	$A_1 \oplus A_2 \to Z_1$
$\mathbf{E}_2 = {}^{0}_{0}\,G\,0 \cdots 0\,0\,0$	${}^{0}_{0}\,1\,0 \cdots 0\,0\,0$	$\mathbf{E}_3$	${}^{0}_{0}\,1\,{-1} \cdots 0\,0\,0$	$Z_1 \to Z_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathbf{E}_{n-2} = {}^{0}_{0}\,0\,0 \cdots G\,0\,0$	${}^{0}_{0}\,0\,0 \cdots 1\,0\,0$	$\mathbf{E}_0$	${}^{0}_{0}\,0\,0 \cdots 1\,{-1}\,0$	$Z_{n-3} \to Z_{n-2}$
$\mathbf{E}'_0 = {}^{0}_{F}\,F\,F \cdots F\,F\,F$	${}^{0}_{2}\,2\,2 \cdots 2\,2\,1$	$\mathbf{E}'_1$	${}^{0}_{1}\,0\,0 \cdots 0\,0\,{-1}$	$A_1 \approx A_2$
$\mathbf{E}'_1 = {}^{F}_{0}\,F\,F \cdots F\,F\,F$	${}^{2}_{0}\,2\,2 \cdots 2\,2\,1$	$\mathbf{E}'_0$	${}^{1}_{0}\,0\,0 \cdots 0\,0\,{-1}$	$A_1 \approx A_2$

$M = {}_GF_G$; representations of ${}_GF_G$ are considered both as mappings $U_G \otimes {}_GF_G \longrightarrow V_G$ and as mappings $U_G \otimes {}_GF_F \longrightarrow V_G \otimes {}_GF_F$;

$$T(U_G, V_G, \varphi) = \begin{matrix} V_G \otimes {}_GF_F \searrow \\ \qquad (U_G \oplus V_G) \otimes {}_GF_F \longrightarrow \cdots \longrightarrow (U_G \oplus V_G) \otimes {}_GF_F \\ U_G \otimes {}_GF_F \nearrow \end{matrix}$$
$$\longrightarrow (U_G \oplus V_G) \otimes {}_GF_F/\Gamma_\varphi,$$

where Γ_φ = graph of $U_G \otimes {}_GF_F \longrightarrow V_G \otimes {}_GF_F$;

$\mathbf{R} = (F_G, F_G, F_G \otimes {}_GF_G \xrightarrow{\text{mult}} F_G)$ satisfies $\mathbf{E}_0 \subseteq T(\mathbf{R})$;

$\mathbf{R}' = (F_G, F_G, F_G \otimes {}_GF_G \twoheadrightarrow (F/G)_G \otimes {}_GF_G \approx F_G)$ satisfies $\mathbf{E}'_0 \subseteq T(\mathbf{R}')$;

$\tilde{\mathbf{D}}_n$

$$\begin{smallmatrix} a_1 & & & & & & & & b_1 \\ & \searrow & & & & & & \nearrow & \\ & & z_1 & - & z_2 & - \cdots - & z_{n-3} & & \\ & \nearrow & & & & & & \searrow & \\ a_2 & & & & & & & & b_2 \end{smallmatrix};$$

$$Q(\mathbf{x}) = \sum_{t=1,2} (2a_t - z_1)^2 + 2 \sum_{1 \leqslant t \leqslant n-4} (z_t - z_{t+1})^2 + \sum_{t=1,2} (z_{n-3} - 2b_t)^2;$$

$$\mathbf{n} = {}^{1}_{1}\!\!>\! 2 - 2 - \cdots - 2 \!<\!{}^{1}_{1}; \quad g = 1;$$

$$\partial_c = {}^{1}_{1}\!\!\rightrightarrows 0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \rightrightarrows {}^{-1}_{-1};$$

$$\mathfrak{M} = {}^{F}_{F}\!\!>\! F - F - \cdots - F \!<\!{}^{F}_{F};$$

$\mathbf{E}_r^{(t)}$	$\dim \mathbf{E}_r^{(t)}$	$C^+\mathbf{E}_r^{(t)}$	$\eta_r^{(t)}$	isomorphisms for simple $\mathbf{X}$ with $\eta_r^{(t)}\mathbf{X} = 0$
$\mathbf{E}_0 = {}^{0}_{0}\,0\;0\cdots 0\;F\,{}^{0}_{0}$	${}^{0}_{0}\,0\;0\cdots 0\;1\,{}^{0}_{0}$	$\mathbf{E}_1$	${}^{0}_{0}\,0\;0\cdots 0\;1\,{}^{-1}_{-1}$	$Z_{n-3} \to B_1 \oplus B_2$
$\mathbf{E}_1 = {}^{F}_{F}\,F\;F\cdots F\;F\,{}^{F}_{F}$	${}^{1}_{1}\,1\;1\cdots 1\;1\,{}^{1}_{1}$	$\mathbf{E}_2$	${}^{1}_{1}\,{-1}\;0\cdots 0\;0\,{}^{0}_{0}$	$A_1 \oplus A_2 \to Z_1$
$\mathbf{E}_2 = {}^{0}_{0}\,F\;0\cdots 0\;0\,{}^{0}_{0}$	${}^{0}_{0}\,1\;0\cdots 0\;0\,{}^{0}_{0}$	$\mathbf{E}_3$	${}^{0}_{0}\,1\;{-1}\cdots 0\;0\,{}^{0}_{0}$	$Z_1 \to Z_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathbf{E}_{n-3} = {}^{0}_{0}\,0\;0\cdots F\;0\,{}^{0}_{0}$	${}^{0}_{0}\,0\;0\cdots 1\;0\,{}^{0}_{0}$	$\mathbf{E}_0$	${}^{0}_{0}\,0\;0\cdots 1\;{-1}\,{}^{0}_{0}$	$Z_{n-4} \to Z_{n-3}$
$\mathbf{E}'_0 = {}^{0}_{F}\,F\;F\cdots F\;F\,{}^{0}_{F}$	${}^{0}_{1}\,1\;1\cdots 1\;1\,{}^{0}_{1}$	$\mathbf{E}'_1$	${}^{0}_{1}\,0\;0\cdots 0\;0\,{}^{-1}_{0}$	$A_2 \to B_1$
$\mathbf{E}'_1 = {}^{F}_{0}\,F\;F\cdots F\;F\,{}^{F}_{0}$	${}^{1}_{0}\,1\;1\cdots 1\;1\,{}^{1}_{0}$	$\mathbf{E}'_0$	${}^{1}_{0}\,0\;0\cdots 0\;0\,{}^{0}_{-1}$	$A_1 \to B_2$
$\mathbf{E}''_0 = {}^{0}_{F}\,F\;F\cdots F\;F\,{}^{F}_{0}$	${}^{0}_{1}\,1\;1\cdots 1\;1\,{}^{1}_{0}$	$\mathbf{E}''_1$	${}^{0}_{1}\,0\;0\cdots 0\;0\,{}^{0}_{-1}$	$A_2 \to B_2$
$\mathbf{E}''_1 = {}^{F}_{0}\,F\;F\cdots F\;F\,{}^{0}_{F}$	${}^{1}_{0}\,1\;1\cdots 1\;1\,{}^{0}_{1}$	$\mathbf{E}''_0$	${}^{1}_{0}\,0\;0\cdots 0\;0\,{}^{-1}_{0}$	$A_1 \to B_1$

$M = {}_F(F \times F)_F$;

$$T(U_F, V_F, (\varphi_1, \varphi_2)) = {}^{V_F}_{U_F}\!\!\rightrightarrows U_F \oplus V_F \longrightarrow U_F \oplus V_F \longrightarrow \cdots \longrightarrow U_F \oplus V_F \rightrightarrows {}^{(U_F \oplus V_F)/\Gamma_1}_{(U_F \oplus V_F)/\Gamma_2},$$

where Γ_i is the graph of φ_i $(i = 1, 2)$;

$\mathbf{R} = (F_F, F_F, (1, 1))$ satisfies $\mathbf{E}_0 \subseteq T(\mathbf{R})$;

$\mathbf{R}' = (F_F, F_F, (0, 1))$ satisifes $\mathbf{E}'_0 \subseteq T(\mathbf{R}')$;

$\mathbf{R}'' = (F_F, F_F, (1, 0))$ satisfies $\mathbf{E}''_0 \subseteq T(\mathbf{R}'')$;

$\tilde{\mathrm{E}}_6$

$$\begin{array}{ccccccccc} & & & & c_1 & & & & \\ & & & & | & & & & \\ & & & & c_2 & & & & \\ & & & & | & & & & \\ a_1 & — & a_2 & — & z & — & b_2 & — & b_1 \end{array}$$

$Q(x) = (6a_1 - 3a_2)^2 + (6b_1 - 3b_2)^2 + (6c_1 - 3c_2)^2 + 3[(3a_2 - 2z)^2 + (3b_2 - 2z)^2 + (3c_2 - 2z)^2]$;

$$\mathbf{n} = \begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & & | & & & & \\ & & & & 2 & & & & \\ & & & & | & & & & \\ 1 & — & 2 & — & 3 & — & 2 & — & 1 \end{array}; \quad g = 1; \quad \partial_c = \begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & & \downarrow & & & & \\ & & & & 1 & & & & \\ & & & & \downarrow & & & & \\ 1 & \longrightarrow & 1 & \longrightarrow & -3 & \longleftarrow & 1 & \longleftarrow & 1 \end{array};$$

$$\mathfrak{M} = \begin{array}{ccccccccc} & & & & F & & & & \\ & & & & | & & & & \\ & & & & F & & & & \\ & & & & | & & & & \\ F & — & F & — & F & — & F & — & F \end{array};$$

	$\mathbf{E}_r^{(t)}$	$\dim \mathbf{E}_r^{(t)}$	$C^+\mathbf{E}_r^{(t)}$	$\eta_r^{(r)}$	consequences for simple $\mathbf{X}$ with $\eta_r^{(t)}\mathbf{X} = 0$
$\mathbf{E}_0 =$	$\begin{smallmatrix} & & 0 & & \\ & & F & & \\ 0 & F & F & F & 0 \end{smallmatrix}$	$\begin{smallmatrix} & & 0 & & \\ & & 1 & & \\ 0 & 1 & 1 & 1 & 0 \end{smallmatrix}$	$\mathbf{E}_1$	$\begin{smallmatrix} & & 0 & & \\ & & 1 & & \\ 0 & 1 & -2 & 1 & 0 \end{smallmatrix}$	$Z/A_2 \oplus Z/B_2 \oplus Z/C_2 = Z$
$\mathbf{E}_1 =$	$\begin{smallmatrix} & & (1,1)F & & \\ & & (1,1)F & & \\ F\times 0 & F\times 0 & F\times F & 0\times F & 0\times F \end{smallmatrix}$	$\begin{smallmatrix} & & 1 & & \\ & & 1 & & \\ 1 & 1 & 2 & 1 & 1 \end{smallmatrix}$	$\mathbf{E}_0$	$\begin{smallmatrix} & & 1 & & \\ & & 0 & & \\ 1 & 0 & -1 & 0 & 1 \end{smallmatrix}$	$A_1 \oplus B_1 \oplus C_1 = Z$
$\mathbf{E}'_0 =$	$\begin{smallmatrix} & & 0 & & \\ & & 0 & & \\ 0 & F & F & F & F \end{smallmatrix}$	$\begin{smallmatrix} & & 0 & & \\ & & 0 & & \\ 0 & 1 & 1 & 1 & 1 \end{smallmatrix}$	$\mathbf{E}'_1$	$\begin{smallmatrix} & & 0 & & \\ & & 0 & & \\ 0 & 1 & -1 & 0 & 1 \end{smallmatrix}$	$A_2 \oplus B_1 = Z$
$\mathbf{E}'_1 =$	$\begin{smallmatrix} & & F & & \\ & & F & & \\ 0 & 0 & F & F & 0 \end{smallmatrix}$	$\begin{smallmatrix} & & 1 & & \\ & & 1 & & \\ 0 & 0 & 1 & 1 & 0 \end{smallmatrix}$	$\mathbf{E}'_2$	$\begin{smallmatrix} & & 1 & & \\ & & 0 & & \\ 0 & 0 & -1 & 1 & 0 \end{smallmatrix}$	$B_2 \oplus C_1 = Z$
$\mathbf{E}'_2 =$	$\begin{smallmatrix} & & 0 & & \\ & & F & & \\ F & F & F & 0 & 0 \end{smallmatrix}$	$\begin{smallmatrix} & & 0 & & \\ & & 1 & & \\ 1 & 1 & 1 & 0 & 0 \end{smallmatrix}$	$\mathbf{E}'_0$	$\begin{smallmatrix} & & 0 & & \\ & & 1 & & \\ 1 & 0 & -1 & 0 & 0 \end{smallmatrix}$	$C_2 \oplus A_1 = Z$
$\mathbf{E}''_0 =$	$\begin{smallmatrix} & & 0 & & \\ & & F & & \\ 0 & 0 & F & F & F \end{smallmatrix}$	$\begin{smallmatrix} & & 0 & & \\ & & 1 & & \\ 0 & 0 & 1 & 1 & 1 \end{smallmatrix}$	$\mathbf{E}''_1$	$\begin{smallmatrix} & & 0 & & \\ & & 1 & & \\ 0 & 0 & -1 & 0 & 0 \end{smallmatrix}$	$B_1 \oplus C_2 = Z$
$\mathbf{E}''_1 =$	$\begin{smallmatrix} & & 0 & & \\ & & 0 & & \\ F & F & F & F & 0 \end{smallmatrix}$	$\begin{smallmatrix} & & 0 & & \\ & & 0 & & \\ 1 & 1 & 1 & 1 & 0 \end{smallmatrix}$	$\mathbf{E}''_2$	$\begin{smallmatrix} & & 0 & & \\ & & 0 & & \\ 1 & 0 & -1 & 1 & 0 \end{smallmatrix}$	$A_1 \oplus B_2 = Z$
$\mathbf{E}''_2 =$	$\begin{smallmatrix} & & F & & \\ & & F & & \\ 0 & F & F & 0 & 0 \end{smallmatrix}$	$\begin{smallmatrix} & & 1 & & \\ & & 1 & & \\ 0 & 1 & 1 & 0 & 0 \end{smallmatrix}$	$\mathbf{E}''_0$	$\begin{smallmatrix} & & 1 & & \\ & & 0 & & \\ 0 & 1 & -1 & 0 & 0 \end{smallmatrix}$	$C_1 \oplus A_2 = Z$

$M = {}_F(F \times F)_F$;

$\mathbf{R} = (F_F, F_F, (0, 1))$;
$\mathbf{R}' = (F_F, F_F, (1, 0))$;
$\mathbf{R}'' = (F_F, F_F, (1, 1))$;

$$T(U_F, V_F, (\varphi_1, \varphi_2)) = \begin{array}{ccccccccc} & & & & C_1 & & & & \\ & & & & \downarrow & & & & \\ & & & & C_2 & & & & \\ & & & & \downarrow & & & & \\ V\,0\,0 & \longrightarrow & V\,U\,0 & \longrightarrow & V\,U\,U & \longleftarrow & 0\,U\,U & \longleftarrow & 0\,0\,U, \end{array}$$

where $C_1 = \{(\varphi_2(u), u, u) \mid u \in U\}$ and $C_2 = C_1 + \{(\varphi_1(u), u, 0) \mid u \in U\}$;

$\tilde{\mathbf{E}}_7$

$$\begin{array}{ccccccc} & & & c & & & \\ & & & | & & & \\ a_1 - & a_2 - & a_3 - & z & - b_3 & - b_2 & - b_1; \end{array}$$

$$Q(\mathbf{x}) = 6[(2a_1 - a_2)^2 + (2b_1 - b_2)^2 + 2[(3a_2 - 2a_3)^2 + (3b_2 - 2b_3)^2] + (4a_3 - 3z)^2 + (4b_3 - 3z)^2 + 6(2c - z)^2;$$

$$\mathbf{n} = \begin{array}{ccccccc} & & & 2 & & & \\ & & & | & & & \\ 1 - & 2 - & 3 - & 4 & - 3 & - 2 & - 1; \end{array} \quad g = 1;$$

$$\partial_c = \begin{array}{ccccccc} & & & 2 & & & \\ & & & \downarrow & & & \\ 1 \longrightarrow & 1 \longrightarrow & 1 \longrightarrow & -4 & \longleftarrow 1 & \longleftarrow 1 & \longleftarrow 1; \end{array}$$

$$\mathfrak{M} = \begin{array}{ccccccc} & & & F & & & \\ & & & | & & & \\ F - & F - & F - & F & - F & - F & - F \end{array}$$

	$\mathbf{E}_r^{(t)}$							dim $\mathbf{E}_r^{(t)}$	$C^+\mathbf{E}_r^{(t)}$	$\eta_r^{(t)}$	consequences for simple **X** with $\eta_r^{(t)}\mathbf{X} = 0$
				F				1		1	
$\mathbf{E}_0 =$	0	0	F	F	F	0	0	0 0 1 1 1 0 0	$\mathbf{E}_1$	0 0 1 −2 1 0 0	$(A_3 \cap B_3) \oplus C = Z/A_3 \oplus Z/B_3 \oplus Z/C = Z$
				$(1,1)F$				1		1	
$\mathbf{E}_1 =$	$F\times 0$	$F\times 0$	$F\times 0$	$F\times F$	$0\times F$	$0\times F$	$0\times F$	1 1 1 2 1 1 1	$\mathbf{E}_2$	1 0 0 −1 0 0 1	$A_1 \oplus B_1 \oplus C = Z$
				0				0		0	
$\mathbf{E}_2 =$	0	F	F	F	F	F	0	0 1 1 1 1 1 0	$\mathbf{E}_3$	0 1 0 −1 0 1 0	$A_2 \oplus B_2 = Z$
				F				1		1	
$\mathbf{E}'_0 =$	0	0	0	F	F	F	0	0 0 0 1 1 1 0	$\mathbf{E}'_1$	0 0 0 −1 0 1 0	$B_2 \oplus C = Z$
				F				Φ		0	
$\mathbf{E}'_1 =$	F	F	F	F	F	0	0	1 1 1 1 1 0 0	$\mathbf{E}'_2$	1 0 0 −1 1 0 0	$A_1 \oplus B_3 = Z$
				F				1		1	
$\mathbf{E}'_2 =$	0	F	F	F	0	0	0	0 1 1 1 0 0 0	$\mathbf{E}'_3$	0 1 0 −1 0 0 0	$A_2 \oplus C = Z$
				F				Φ		0	
$\mathbf{E}'_3 =$	0	0	F	F	F	F	F	0 0 1 1 1 1 1	$\mathbf{E}'_0$	0 0 1 −1 0 0 1	$(A_3 \oplus B_1 = Z$
				$(1,1)F$				1		1	
$\mathbf{E}''_0 =$	0	$F\times 0$	$F\times 0$	$F\times F$	$F\times F$	$0\times F$	$0\times F$	0 1 1 2 2 1 1	$\mathbf{E}''_1$	0 1 0 −2 1 0 1	$(A_2 \cap B_3) \oplus (B_3 \cap C) \oplus B_1 = Z$
				$(1,1)F$				1		1	
$\mathbf{E}''_1 =$	$F\times 0$	$F\times 0$	$F\times F$	$F\times F$	$0\times F$	$0\times F$	0	1 1 2 2 1 1 0	$\mathbf{E}''_0$	1 0 1 −2 0 1 0	$(A_3 \cap B_2) \oplus (A_3 \cap C) \oplus A_1 = Z$

$M = {}_F(F \times F)_F$;

$$T(U_F, V_F, (\varphi_1, \varphi_2)) = V000 \hookrightarrow VU00 \hookrightarrow VUU0 \hookrightarrow \overset{\displaystyle C}{\overset{\downarrow}{VUUU}} \hookleftarrow 0UUU \hookleftarrow 00UU \hookleftarrow 000U,$$

where C is generated by the elements $(\varphi_1(u), u, u, 0)$ and $(\varphi_2(u), 0, u, u)$ with $u \in U$;

$\mathbf{R} = (F_F, F_F, (0, 1))$;

$\mathbf{R}' = (F_F, F_F, (1, 0))$;

$\mathbf{R}'' = (F_F, F_F, (1, 1))$;

$\tilde{\mathbf{E}}_8$

$$a_1 - a_2 - a_3 - a_4 - a_5 - \overset{c}{\overset{|}{z}} - b_2 - b_1;$$

$$Q(\mathbf{x}) = 30(2a_1 - a_2)^2 + 10(3a_2 - 2a_3)^2 + 5(4a_3 - 3a_4)^2 + 3(5a_4 - 4a_5)^2 + 30(2b_1 - b_2)^2 + 2(6a_5 - 5z)^2 + 10(3b_2 - 2z)^2 + 30(2c - z)^2;$$

$$\mathbf{n} = 1 - 2 - 3 - 4 - 5 - \overset{3}{\overset{|}{6}} - 4 - 2; \quad g = 1; \; \partial_c = 1 \to 1 \to 1 \to 1 \to 1 \to \overset{3}{\overset{|}{-6}} \leftarrow 2 \leftarrow 2; \qquad \mathfrak{M} = F - F - F - F - F - \overset{F}{\overset{|}{F}} - F - F;$$

	$\mathbf{E}_r^{(t)}$								$\dim \mathbf{E}_r^{(t)}$	$C^+\mathbf{E}_r^{(t)}$	$\eta_r^{(t)}$	consequences for simple $\mathbf{X}$ with $\eta_r^{(t)}\mathbf{X} = 0$
$\mathbf{E}_0 =$	0	0	0	0	F	$\overset{F}{F}$	F	0	$0\,0\,0\,0\,1\,\overset{1}{1}\,1\,0$	$\mathbf{E}_1$	$0\,0\,0\,0\,1\,\overset{1}{-2}\,1\,0$	$(A_5 \cap B_2) \oplus C = Z/A_5 \oplus Z/B_2 \oplus Z/C = Z$
$\mathbf{E}_1 =$	$F\times 0$	$F\times 0$	$F\times 0$	$F\times 0$	$F\times 0$	$\overset{(1,1)F}{F\times F}$	$0\times F$	$0\times F$	$1\,1\,1\,1\,1\,\overset{1}{2}\,1\,1$	$\mathbf{E}_2$	$1\,0\,0\,0\,0\,\overset{1}{-1}\,0\,1$	$A_1 \oplus B_1 \oplus C = Z$
$\mathbf{E}_2 =$	0	F	F	F	F	$\overset{0}{F}$	F	0	$0\,1\,1\,1\,1\,\overset{0}{1}\,1\,0$	$\mathbf{E}_3$	$0\,1\,0\,0\,0\,\overset{0}{-1}\,1\,0$	$A_2 \oplus B_2 = Z$
$\mathbf{E}_3 =$	0	0	F	F	F	$\overset{F}{F}$	0	0	$0\,0\,1\,1\,1\,\overset{1}{1}\,0\,0$	$\mathbf{E}_4$	$0\,0\,1\,0\,0\,\overset{1}{-1}\,0\,0$	$A_3 \oplus C = Z$
$\mathbf{E}_4 =$	0	0	0	F	F	$\overset{0}{F}$	F	F	$0\,0\,0\,1\,1\,\overset{0}{1}\,1\,1$	$\mathbf{E}_0$	$0\,0\,0\,1\,0\,\overset{0}{-1}\,0\,1$	$A_4 \oplus B_1 = Z$
$\mathbf{E}'_0 =$	0	0	$F\times 0$	$F\times 0$	$F\times 0$	$\overset{(1,1)F}{F\times F}$	$F\times F$	$0\times F$	$0\,0\,1\,1\,1\,\overset{1}{2}\,2\,1$	$\mathbf{E}'_1$	$0\,0\,1\,0\,0\,\overset{1}{-2}\,1\,1$	$B_1 \oplus (A_3 \cap B_2) \oplus (B_2 \cap C) = Z$
$\mathbf{E}'_1 =$	$F\times 0$	$F\times 0$	$F\times 0$	$F\times F$	$F\times F$	$\overset{(1,1)F}{F\times F}$	$0\times F$	0	$1\,1\,1\,2\,2\,\overset{1}{2}\,1\,0$	$\mathbf{E}'_2$	$1\,0\,0\,1\,0\,\overset{1}{-2}\,1\,0$	$A_1 \oplus (A_4 \cap B_2) \oplus (A_4 \cap C) = Z$
$\mathbf{E}'_2 =$	0	$F\times 0$	$F\times 0$	$F\times 0$	$F\times F$	$\overset{(1,1)F}{F\times F}$	$0\times F$	$0\times F$	$0\,1\,1\,1\,2\,\overset{1}{2}\,1\,1$	$\mathbf{E}'_0$	$0\,1\,0\,0\,1\,\overset{1}{-2}\,0\,1$	$A_2 \oplus (A_5 \cap B_1) \oplus (A_5 \cap C) = Z$
$\mathbf{E}''_0 =$	0	$F\times 0\times 0$	$F\times 0\times 0$	$F\times F\times 0$	$F\times F\times 0$	$\overset{(1,1,0)F + (0,1,1)F}{F\times F\times F}$	$0\times F\times F$	$0\times 0\times F$	$0\,1\,1\,2\,2\,\overset{2}{3}\,2\,1$	$\mathbf{E}''_1$	$0\,1\,0\,1\,0\,\overset{2}{-3}\,1\,1$	$A_2 \oplus B_1 \oplus (B_2 \cap C) \oplus (A_4 \cap C) = Z$
$\mathbf{E}''_1 =$	$F\times 0\times 0$	$F\times 0\times 0$	$F\times F\times 0$	$F\times F\times 0$	$F\times F\times F$	$\overset{(1,1,1)F}{F\times F\times F}$	$0\times F\times F$	$0\times 0\times F$	$1\,1\,2\,2\,3\,\overset{1}{3}\,2\,1$	$\mathbf{E}''_0$	$1\,0\,1\,0\,1\,\overset{1}{-3}\,1\,1$	$A_1 \oplus C \oplus (B_1 \cap A_5) \oplus (B_2 \cap A_3) = Z$

$M = {}_F(F \times F)_F$;

$\mathbf{R} = (F_F, F_F, (0, 1))$;

$\mathbf{R}' = (F_F, F_F, (1, 0))$;

$\mathbf{R}'' = (F_F, F_F, (1, 1))$;

$$T(U_F, V_F, (\varphi_1, \varphi_2)) = V00000 \hookrightarrow VU0000 \hookrightarrow VUU000 \hookrightarrow VUUU00 \hookrightarrow VUUUU0 \hookrightarrow \overset{C}{\overset{\cap}{VUUUUU}} \hookleftarrow 00UUUU \hookleftarrow 0000UU,$$

where C is generated by the elements $(\varphi_1(u), 0, u, u, u, 0)$, $(\varphi_2(u), 0, u, 0, u, u)$ and $(0, u, 0, u, 0, 0)$ with $u \in U$;

$\widetilde{\mathsf{F}}_{41}$

$a_1 — a_2 — a_3 \overset{(1,2)}{\text{———}} z — b;$

$Q(\mathrm{x}) = 3(2a_1 - a_2)^2 + (3a_2 - 2a_3)^2 + 2(2a_3 - 3z)^2 + 6(2b - z)^2;$

$\mathbf{n} = 1 — 2 — 3 — 2 — 1; \quad g = 2;$

$\mathfrak{d}_c = 1 \longrightarrow 1 \longrightarrow 1 \longrightarrow -4 \longleftarrow 2;$

$\mathfrak{M} = G — G — G — F — F$ with $[F:G] = 2;$

$\mathbf{E}_r^{(t)}$						$\dim \mathbf{E}_r^{(t)}$	$C^+\mathbf{E}_r^{(t)}$	$\eta_r^{(t)}$	consequences for simple $\mathbf{X}$ with $\eta_r^{(t)}\mathbf{X}=0$
$\mathbf{E}_0 =$	0	0	F	F	F	0 0 2 1 1	$\mathbf{E}_1$	0 0 1 −2 1	$\underline{A}_3 \oplus B = Z$
$\mathbf{E}_1 =$	$G\times G$	$G\times G$	$G\times G$	$F\times F$	$(1,1)F$	2 2 2 2 1	$\mathbf{E}_2$	1 0 0 −1 1	$\bar{A}_1 \oplus B = Z$
$\mathbf{E}_2 =$	0	F	F	F	0	0 2 2 1 0	$\mathbf{E}_0$	0 1 0 −1 0	$\bar{A}_2 = 0, \bar{A}_2 = Z$
$\mathbf{E}'_0 =$	0	G	G	F	F	0 1 1 1 1	$\mathbf{E}'_1$	0 1 0 −2 2	$A_2 \oplus B = Z$
$\mathbf{E}'_1 =$	G	G	F	F	0	1 1 2 1 0	$\mathbf{E}'_0$	1 0 1 −2 0	$\bar{A}_1 \oplus \underline{A}_3 = Z$

$M = {}_GF_G;$

$T(U_G, V_G, \varphi) = V_G \times 0 \hookrightarrow V_G \times U_G \hookrightarrow V_G \times (U_G \otimes {}_GF_G) \hookrightarrow (V_G \otimes {}_GF_F) \times (U_G \otimes {}_GF_F) \hookleftarrow \Gamma_\varphi$, where Γ_φ is the graph of $\varphi: U_G \otimes {}_GF_F \longrightarrow V_G \otimes {}_GF_F$;

$\mathbf{R} = (F_G, F_G, F_G \otimes {}_GF_G \twoheadrightarrow (F/G)_G \otimes {}_GF_G \approx F_G);$

$\mathbf{R}' = (G_G, G_G, G_G \otimes {}_GF_F \xrightarrow{\mathrm{id}} G_G \otimes {}_GF_F);$

$\widetilde{\mathbf{F}}_{42}$

$a_1 - a_2 - z \overset{(2,1)}{\longrightarrow} b_2 - b_1$;

$Q(\mathrm{x}) = 6(2a_1 - a_2)^2 + 3(2b_1 - b_2)^2 + 2(3a_2 - 2z)^2 + (3b_2 - 4z)^2$;

$\mathrm{n} = 1 - 2 - 3 - 4 - 2$; $g = 1$;

$\partial_c = 1 \longrightarrow 1 \longrightarrow -3 \longleftarrow 1 \longleftarrow 1$;

$\mathfrak{M} = F - F - F - G - G$ with $[F:G] = 2$;

$\mathbf{E}_r^{(t)}$						$\dim \mathbf{E}_r^{(t)}$	$C^+\mathbf{E}_r^{(t)}$	$\eta_t^{(t)}$	consequences for simple $\mathbf{X}$ with $\eta_r^{(t)}\mathbf{X}=0$
$\mathbf{E}_0 =$	0	F	F	G	G	0 1 1 1 1	$\mathbf{E}_1$	0 2 −2 0 1	$A_2 \oplus B_1 = Z$
$\mathbf{E}_1 =$	0	0	F	F	G	0 0 1 2 1	$\mathbf{E}_2$	0 0 −2 1 1	$\bar{B}_1 \oplus \underline{B}_2 = Z$
$\mathbf{E}_2 =$	0	F	F	G	0	1 1 1 1 0	$\mathbf{E}_0$	2 0 −2 1 0	$A_1 \oplus B_2 = Z$
$\mathbf{E}'_0 =$	F	F	F	F	0	0 1 1 2 0	$\mathbf{E}'_1$	0 1 −2 1 0	$A_2 \oplus \underline{B}_2 = Z$
$\mathbf{E}'_1 =$	$(1,f)F$ with $f \in F \setminus G$	$(1,f)F$	$F \times F$	$G \times G$	$G \times G$	1 1 2 2 2	$\mathbf{E}'_0$	1 0 −1 0 1	$A_1 \oplus \bar{B}_1 = Z$

$M = {}_FF_G \otimes {}_GF_F$;

$T(U_F, V_F, \varphi) = A_1 \longrightarrow A_2 \longrightarrow (U_F \otimes {}_FM_F) \times V_F \longleftarrow (U_F \otimes {}_FF_G \otimes {}_GG_G) \times V_F$
$\longleftarrow (U_F \otimes {}_FF_G \otimes {}_GG_G) \times 0$,

where A_2 is the graph of φ and $A_1 = A_2 \cap (U_F \otimes {}_FK_F \times V_F)$,
with ${}_FK_F = \ker({}_FF_G \otimes {}_GF_F \longrightarrow {}_FF_F)$;

$\mathbf{R} = (F_F, F_F, F_F \otimes {}_FF_G \otimes {}_GF_F \approx F_G \otimes {}_GF_F \twoheadrightarrow (F/G)_G \otimes {}_GF_F \approx F_F)$;

$\mathbf{R}' = (F_F, F_F, F_F \otimes {}_FF_G \otimes {}_GF_F \xrightarrow{\text{mult}} F_F)$;

$\tilde{\mathbf{G}}_{21}$

$a_1 - a_2 \overset{(1,3)}{\text{———}} z$;

$Q(\mathbf{x}) = (2a_1 - a_2)^2 + 3(a_2 - 2z)^2$;

$\mathbf{n} = 1 - 2 - 1$; $g = 3$;

$\partial_c = 1 \longrightarrow 1 \longrightarrow -3$;

$\mathfrak{M} = G - G - F$ with $[F:G] = 3$;

$\mathbf{E}_r$			$\dim \mathbf{E}_r$	$C^+\mathbf{E}_r$	η_r	consequences for simple $\mathbf{X}$ with $\eta_r\mathbf{X} = 0$
$\mathbf{E}_0 =$ 0	F	F	0 3 1	$\mathbf{E}_1$	0 1 −2	$\bar{A}_1 = Z$, $\underline{A}_2 = 0$
$\mathbf{E}_1 = G \times G +$ $(e,f)G$ such that $\{1, e, f\}$ is a basis of F_G	$G \times G +$ $(e,f)G$	$F \times F$	3 3 2	$\mathbf{E}_0$	1 0 −1	$A_1 \otimes {}_GF_F \longrightarrow Z$ is an isomorphism

$M = {}_G(F/G)_G$; representations of ${}_G(F/G)_G$ are considered as the mappings $\varphi: U_G \otimes {}_GF_G \longrightarrow V_G$ with $\varphi(U_G \otimes {}_GG_G) = 0$;

$T(U_G, V_G, \varphi: U_G \otimes {}_GF_G \longrightarrow V_G) = U_G \otimes {}_GG_G \hookrightarrow \ker\varphi \hookrightarrow U_G \otimes {}_GF_F$;

$\mathbf{R} = (F_G, (F_G \otimes {}_GF_G)/(F_G \otimes {}_GG_G + \omega(F_F)), \pi)$, where π is the projection, and $\omega = \bar{1}: F_F \longrightarrow F_G \otimes {}_GF_F$ is the mapping canonically attached to the identity $F_F \otimes {}_FF_G \longrightarrow F_G$ with respect to some $\epsilon: {}_GF_G \longrightarrow {}_GG_G$ (see the example after Proposition 2.1);

$\widetilde{\mathbf{G}}_{22}$

$z \xrightarrow{(3,1)} b$;

$Q(x) = 3(2a - z)^2 + (2b - 3z)^2$;

$\mathbf{n} = 1 — 2 — 3$; $g = 1$;

$\partial_c = 1 \longrightarrow -2 \longleftarrow 1$;

$\mathfrak{M} = F — F — G$ with $[F:G] = 3$;

$\mathbf{E}_r$			$\dim \mathbf{E}_r$	$C^+\mathbf{E}_r$	η_r	consequences for simple $\mathbf{X}$ with $\eta_r\mathbf{X} = 0$
$\mathbf{E}_0 = 0$	F	$G + fG$ with $f \in F\backslash G$	0 1 2	$\mathbf{E}_1$	0 −3 2	$B \oplus Bf = Z$ as abelian groups
$\mathbf{E}_1 = F$	F	G	1 1 1	$\mathbf{E}_0$	3 −3 1	$A \oplus B = Z$

$M = ({}_FF_G \otimes {}_GF_F) / \omega({}_FF_F)$, where $\omega = \overline{1} : F_F \longrightarrow F_G \otimes {}_GF_F$ is the mapping canonically attached to the identity $F_F \otimes {}_FF_G \longrightarrow F_G$ with respect to some $\epsilon : {}_GF_G \longrightarrow {}_GG_G$ (see the example after Proposition 2.1);

$T(U_F, V_F, \varphi) = \operatorname{Ker} \varphi \hookrightarrow U_F \otimes {}_FM_F \hookleftarrow U_F \otimes {}_FN_G$, where ${}_FN_G = ({}_FF_G \otimes {}_GG_G + \omega({}_FF_F)) / \omega({}_FF_F)$;

$\mathbf{R} = (F_F, M_F/xF, F_F \otimes {}_FM_F \approx M_F \twoheadrightarrow M_F/xF)$ with $0 \neq x \in N$;

References

1. Berman, S., Moody, R. and Wonenberger, Maria: *Certain matrices with null roots and finite Cartan matrices.* Indiana Univ. Math. J. **21**, 1091–1099 (1971/72).

2. Bernstein, I. N., Gelfand, I. M. and Ponomarev, V. A.: *Coxeter functors and Gabriel's theorem.* Uspechi Mat. Nauk **28**, 19–33 (1973); translated in Russian Math. Surveys **28**, 17–32 (1973).

3. Bourbaki, N.: *Groupe et algèbres de Lie*, Ch. 4, 5 et 6. Paris: Hermann 1968.

4. Dlab, V. and Ringel, C. M.: *On algebras of finite representation type.* J. Algebra **33**, 306–394 (1975).

5. ———, *Représentations des graphes valués.* C. R. Acad. Sc. Paris **278**, 537–540 (1974).

6. ———, *Representations of graphs and algebras*, Carleton Mathematical Lecture Notes No. 8, 1974.

7. Donovan, P. and Freislich, M. R.: *The representation theory of finite graphs and associated algebras.* Carleton Mathematical Lecture Notes No. 5, 1973.

8. Gabriel, P.: *Unzerlegbare Darstellungen* I. Man. Math. **6**, 71–103 (1972).

9. ———, *Indecomposable representations* II. Symposia Math. Ist. Naz. Alta Mat., Vol. XI, 81–104 (1973).

10. Gelfand, I. M. and Ponomarev, V. A.: *Problems of linear algebra and classification of quadruples of subspaces in a finite-dimensional vector space.* Coll. Math. Soc. Bolyai 5, Tihany (Hungary), 163–237 (1970).

11. Kleiner, M. M.: *Partially ordered sets of finite type.* Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **28**, 32–41 (1972).

12. Moody, R. V.: *Euclidean Lie algebras.* Can. J. Math. **21**, 1432–1454 (1969).

13. Müller, W.: *Unzerlegbare Moduln über artinschen Ringen.* Math. Z. **137**, 197–226 (1974).

14. Nazarova, L. A.: *Representation of quadruples.* Izv. Akad. Nauk SSSR, ser. Mat. **31**, 1361–1377 (1967).

15. ———, *Representation of quivers of infinite type.* Izv. Akad. Nauk SSSR, ser. mat. **37**, 752–791 (1973).

16. Nazarova, L. A. and Roiter, A. V.: *Representations of partially ordered sets.* Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **28**, 5–31 (1972).

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Addendum

We should like to use this opportunity and point out some of the developments in the theory of representations of graphs which took place after this paper was prepared for publication.

In order to complete the description of the category $L(\mathfrak{M}, \Omega)$ for a realization $(\mathfrak{M}, \Omega)$ of an extended Dynkin diagram, one would need the classification of the homogeneous indecomposable modules. In general situation, this seems to be very difficult. However, the subcategory $H(\mathfrak{M}, \Omega)$ can be described in the case of a K-realization (or "K-species"), that is in the case of a realization $(\mathfrak{M}, \Omega)$ such that all F_i contain a common central field K with $[F_i : K] < \infty$, which operates on each ${}_iM_j$ centrally. For these, $H(\mathfrak{M}, \Omega)$ is always a direct sum of uniserial subcategories of global dimension one with a single simple object (C. M. Ringel, Representations of K-species and bimodules, to appear in J. Algebra.) Thus in order to complete the classification of all indecomposable representations of K-realizations of extended Dynkin diagrams it only remains to describe all *simple* homogeneous representations of all K-realizations of the diagrams $\tilde{\mathbf{A}}_{11}$ and $\tilde{\mathbf{A}}_{12}$.

This can be done when the field K is the field $\mathbf{R}$ of the real numbers. For, obviously, there are only six different $\mathbf{R}$-realizations of this kind ($\mathbf{C}$ and $\mathbf{H}$ denote the fields of complex numbers and quaternions, respectively):

1) $\mathbf{R} \xrightarrow{{}_{\mathbf{R}}\mathbf{H}_{\mathbf{H}}} \mathbf{H}$ and $\mathbf{H} \xrightarrow{{}_{\mathbf{H}}\mathbf{H}_{\mathbf{R}}} \mathbf{R}$;

2) $F \xrightarrow{{}_FF_F \oplus {}_FF_F} F$ with $F = \mathbf{R}, \mathbf{C}$ or $\mathbf{H}$; and

3) $\mathbf{C} \xrightarrow{{}_{\mathbf{C}}\mathbf{C}_{\mathbf{C}} \oplus {}_{\mathbf{C}}\mathbf{C}_{\bar{\mathbf{C}}}} \mathbf{C}$, where the right action of $\mathbf{C}$ on ${}_{\mathbf{C}}\mathbf{C}_{\bar{\mathbf{C}}}$ is given by conjugation.

This follows from the fact that a bimodule ${}_FM_G$ can be considered as a left $F \otimes_K G^{op}$-module; now, $\mathbf{H} \otimes_{\mathbf{R}} \mathbf{H}^{op}$ is a simple algebra and $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C}$ is the product of two copies of $\mathbf{C}$.

In 1), the complete classification is given in V. Dlab and C. M. Ringel, Real subspaces of a vector space over the quaternions, to appear: The simple homogeneous representations are of the form $U_{\mathbf{R}} \hookrightarrow \mathbf{H}_{\mathbf{R}}$, where U is a two-dimensional $\mathbf{R}$-subspace of $\mathbf{H}$ containing the subfield $\mathbf{R}$, and thus the set of such representations is the two-dimensional real projective space. The endomorphism ring of every such representation is the field $\mathbf{C}$.

In 2), $H(\mathfrak{M}, \Omega)$ is the product of the module category of the corresponding polynomial ring $F[x]$ and a uniserial category with a single simple object whose dimension type is $(1, 1)$. Here, if $F = \mathbf{R}$, the set of all simple homogeneous representations is a compact real 2-hemisphere K; the endomorphism ring of a representation corresponding to a point on the boundary is the field $\mathbf{R}$, and that of a point in the interior is $\mathbf{C}$. The correspondence can be described as follows: Consider the hemisphere K as the one-point compactification of the closed upper real plane. For the points (a, b) with $b > 0$, the corresponding representation is

$$\mathbf{R} \times \mathbf{R} \underset{\begin{pmatrix} a & b \\ -b & a \end{pmatrix}}{\overset{1}{\rightrightarrows}} \mathbf{R} \times \mathbf{R};$$

for $(a, 0)$,

$$\mathbf{R} \underset{a}{\overset{1}{\rightrightarrows}} \mathbf{R};$$

and for the point ∞,

$$\mathbf{R} \underset{1}{\overset{0}{\rightrightarrows}} \mathbf{R}.$$

If $F = \mathbf{C}$, the set of simple homogeneous representations is the real 2-sphere and the endomorphism ring of every such representation is $\mathbf{C}$. We consider the real 2-sphere as a one-dimensional projective $\mathbf{C}$-space and then a correspondence is given, for $c \in \mathbf{C}$, by

$$\mathbf{C} \underset{c}{\overset{1}{\rightrightarrows}} \mathbf{C}$$

and for ∞, by

$$\mathbf{C} \underset{1}{\overset{0}{\rightrightarrows}} \mathbf{C}.$$

If $F = \mathbf{H}$, then the set of all simple homogeneous representations is again the compact real hemisphere K and the endomorphism rings of such representations are either $\mathbf{H}$ or $\mathbf{C}$, depending whether the corresponding point lies on the boundary or in the interior of K. Here, the correspondence between the points of K and the representations is given as follows: For (a, b) in the real upper half-plane $(b \geqslant 0)$,

$$\mathbf{H} \underset{a+bi}{\overset{1}{\rightrightarrows}} \mathbf{H},$$

and for the point ∞,

$$\mathbf{H} \underset{1}{\overset{0}{\rightrightarrows}} \mathbf{H}.$$

Finally, in 3), $H(\mathfrak{M}, \Omega)$ is the product of the category of all modules over the twisted polynomial ring $\mathbf{C}(x, -]$ with respect to the conjugation, and a uniserial category with a single simple object, again of dimension type $(1, 1)$ (V. Dlab and C. M. Ringel, Normal forms of real matrices with respect to complex similarity, to appear in Linear Algebra and Appl. Again, the set of all simple homogeneous representations is K; the correspondence is given as follows: for $c = (a, b)$ with $b > 0$ or $b = 0$ and $a < 0$,

$$\mathbf{C} \times \mathbf{C} \underset{\begin{pmatrix} 0 & c \\ 1 & 0 \end{pmatrix}}{\overset{1}{\rightrightarrows}} \mathbf{C} \times \mathbf{C},$$

for $b = 0$ and $a \geqslant 0$,

$$\mathbf{C} \underset{a}{\overset{1}{\rightrightarrows}} \mathbf{C},$$

and for the point ∞, again

$$\mathbf{C} \underset{1}{\overset{0}{\rightrightarrows}} \mathbf{C}.$$

The endomorphism ring of a representation corresponding to (a, b) with $b > 0$ is $\mathbf{C}$, to $(a, 0)$ with $a < 0$ is **H**, to $(a, 0)$ with $a > 0$ is **R** and to the points $(0, 0)$ and ∞, it is **C**.

From the above paper of C. M. Ringel, it follows easily that, given a K-realization $(\mathfrak{M}, \Omega)$ of a connected valued graph $(\Gamma, \mathbf{d})$, $L(\mathfrak{M}, \Omega)$ is of tame representation type if and only if $(\Gamma, \mathbf{d})$ is an extended Dynkin diagram. Namely, for all valued graphs with an indefinite quadratic form and for all K-realizations $(\mathfrak{M}, \Omega)$, the category $L(\mathfrak{M}, \Omega)$ is of wild representation type in the sense that there is a full exact embedding of the category of all modules over a free associative algebra with two generators over a commutative field.

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