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Research Article

Indefinite Almost Paracontact Metric Manifolds

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We introduce the concept of (ε)-almost paracontact manifolds, and in particular, of (ε)-para-Sasakian manifolds. Several examples are presented. Some typical identities for curvature tensor and Ricci tensor of (ε)-para Sasakian manifolds are obtained. We prove that if a semi-Riemannian manifold is one of flat, proper recurrent or proper Ricci-recurrent, then it cannot admit an (ε)-para Sasakian structure. We show that, for an (ε)-para Sasakian manifold, the conditions of being symmetric, semi-symmetric, or of constant sectional curvature are all identical. It is shown that a symmetric spacelike (resp., timelike) (ε)-para Sasakian manifold M^n is locally isometric to a pseudohyperbolic space $H^n_\nu(1)$ (resp., pseudosphere $S^n_\nu(1)$). At last, it is proved that for an (ε)-para Sasakian manifold the conditions of being Ricci-semi-symmetric, Ricci-symmetric, and Einstein are all identical.

1. Introduction

In 1976, an almost paracontact structure (φ, ξ, η) satisfying $\varphi^2 = I - \eta \otimes \xi$ and $\eta(\xi) = 1$ on a differentiable manifold was introduced by Satō [1]. The structure is an analogue of the almost contact structure [2, 3] and is closely related to almost product structure (in contrast to almost contact structure, which is related to almost complex structure). An almost contact manifold is always odd dimensional but an almost paracontact manifold could be even dimensional as well. In 1969, Takahashi [4] introduced almost contact manifolds equipped with associated pseudo-Riemannian metrics. In particular, he studied Sasakian manifolds equipped with an associated pseudo-Riemannian metric. These indefinite almost contact metric manifolds and indefinite Sasakian manifolds are also known as (ε)-almost contact metric manifolds and (ε)-Sasakian manifolds, respectively [5–7]. Also, in 1989, Matsumoto [8] replaced the structure vector field ξ by $-\xi$ in an almost paracontact manifold and associated a Lorentzian metric with the resulting structure and called it a Lorentzian almost paracontact manifold.

An (ε)-Sasakian manifold is always odd dimensional. Recently, we have observed that there does not exist a lightlike surface in a 3-dimensional (ε)-Sasakian manifold. On the other hand, in a Lorentzian almost paracontact manifold given by Matsumoto, the semi-Riemannian metric has only index 1 and the structure vector field ξ is always timelike. These circumstances motivate us to associate a semi-Riemannian metric, not necessarily Lorentzian, with an almost paracontact structure, and we shall call this indefinite almost paracontact metric structure an (ε)-almost paracontact structure, where the structure vector field ξ will be spacelike or timelike according as $\varepsilon = 1$ or $\varepsilon = -1$.

In this paper we initiate study of (ε)-almost paracontact manifolds, and in particular, (ε)-para Sasakian manifolds. The paper is organized as follows. Section 2 contains basic definitions and some examples of (ε)-almost paracontact manifolds. In Section 3, some properties of normal almost paracontact structures are discussed. Section 4 contains definitions of an (ε)-paracontact structure and an (ε)-s-paracontact structure. A typical example of an (ε)-s-paracontact structure is also presented. In Section 5, we introduce the notion of an (ε)-para Sasakian structure and study some of its basic properties. We find some typical identities for curvature tensor and Ricci tensor. We prove that if a semi-Riemannian manifold is one of flat, proper recurrent, or proper Ricci-recurrent, then it cannot admit an (ε)-para Sasakian structure. We show that, for an (ε)-para Sasakian manifold, the conditions of being symmetric, semi-symmetric, or of constant sectional curvature are all identical. More specifically, it is shown that a symmetric spacelike (ε)-para Sasakian manifold M^n is locally isometric to a pseudohyperbolic space $H^n_\nu(1)$, and a symmetric timelike (ε)-para Sasakian manifold M^n is locally isometric to a pseudosphere $S^n_\nu(1)$. At last, it is proved that for an (ε)-para Sasakian manifold, the conditions of being Ricci-semi-symmetric, Ricci-symmetric, and Einstein are all identical. Unlike 3-dimensional (ε)-Sasakian manifold, which cannot possess a lightlike surface, the study of lightlike surfaces of 3-dimensional (ε)-para Sasakian manifolds will be presented in a forthcoming paper.

2. (ε)-Almost Paracontact Metric Manifolds

Let M be an almost paracontact manifold [1] equipped with an almost paracontact structure (φ, ξ, η) consisting of a tensor field φ of type $(1,1)$, a vector field ξ, and a 1-form η satisfying

$$\varphi^2 = I - \eta \otimes \xi, \tag{2.1}$$

$$\eta(\xi) = 1, \tag{2.2}$$

$$\varphi\xi = 0, \tag{2.3}$$

$$\eta \circ \varphi = 0. \tag{2.4}$$

It is easy to show that the relation (2.1) and one of the three relations (2.2), (2.3), and (2.4) imply the remaining two relations of (2.2), (2.3), and (2.4). On an n-dimensional almost paracontact manifold, one can easily obtain

$$\varphi^3 - \varphi = 0, \tag{2.5}$$

$$\operatorname{rank}(\varphi) = n - 1. \tag{2.6}$$

Equation (2.5) gives an $f(3\ ,-1)$-*structure* [9].

Throughout the paper, by a semi-Riemannian metric [10] on a manifold M, we understand a non-degenerate symmetric tensor field g of type $(0,2)$. In particular, if its index is 1, it becomes a Lorentzian metric [11]. A sufficient condition for the existence of a Riemannian metric on a differentiable manifold is paracompactness. The existence of Lorentzian or other semi-Riemannian metrics depends upon other topological properties. For example, on a differentiable manifold, the following statements are equivalent: (1) there exits a Lorentzian metric on M, (2) there exists a non-vanishing vector field on M, and (3) either M is non-compact, or M is compact and has Euler number $\chi(M) = 0$. Also, for instance, the only compact surfaces that can be made Lorentzian surfaces are the tori and Klein bottles, and a sphere S^n admits a Lorentzian metric if and only if n is odd ≥ 3.

Now, we give the following.

Definition 2.1. Let M be a manifold equipped with an almost paracontact structure (φ, ξ, η). Let g be a semi-Riemannian metric with $\operatorname{index}(g) = \nu$ such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \varepsilon\eta(X)\eta(Y), \quad X, Y \in TM, \tag{2.7}$$

where $\varepsilon = \pm 1$. Then we say that M is an (ε)-*almost paracontact metric manifold* equipped with an (ε)-*almost paracontact metric structure* $(\varphi, \xi, \eta, g, \varepsilon)$. In particular, if $\operatorname{index}(g) = 1$, then an (ε)-almost paracontact metric manifold will be called a *Lorentzian almost paracontact manifold*. In particular, if the metric g is positive definite, then an (ε)-almost paracontact metric manifold is the usual *almost paracontact metric manifold* [1].

Equation (2.7) is equivalent to

$$g(X, \varphi Y) = g(\varphi X, Y), \tag{2.8}$$

along with

$$g(X, \xi) = \varepsilon\eta(X), \tag{2.9}$$

for all $X, Y \in TM$. From (2.9) it follows that

$$g(\xi, \xi) = \varepsilon, \tag{2.10}$$

that is, the structure vector field ξ is never lightlike. Since g is non-degenerate metric on M and ξ is non-null, therefore the paracontact distribution

$$D = \{X \in TM : \eta(X) = 0\} \tag{2.11}$$

is non-degenerate on M.

Definition 2.2. Let $(M, \varphi, \xi, \eta, g, \varepsilon)$ be an (ε)-almost paracontact metric manifold (resp., a Lorentzian almost paracontact manifold). If $\varepsilon = 1$, then M will be said to be a *spacelike (ε)-almost paracontact metric manifold* (resp., a *spacelike Lorentzian almost paracontact manifold*). Similarly, if $\varepsilon = -1$, then M will be said to be a *timelike (ε)-almost paracontact metric manifold* (resp., a *timelike Lorentzian almost paracontact manifold*).

Note that a timelike Lorentzian almost paracontact structure is a Lorentzian almost paracontact structure in the sense of Mihai and Roşca [12], Matsumoto [13], which differs in the sign of the structure vector field of the Lorentzian almost paracontact structure given by Matsumoto [8].

Example 2.3. Let $\mathbb{R}^3$ be the 3-dimensional real number space with a coordinate system (x, y, z). We define

$$\begin{gathered}
\eta = dy, \qquad \xi = \frac{\partial}{\partial y}, \\
\varphi\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial z}, \qquad \varphi\left(\frac{\partial}{\partial y}\right) = 0, \qquad \varphi\left(\frac{\partial}{\partial z}\right) = \frac{\partial}{\partial x}, \\
g_1 = (dx)^2 - (dy)^2 + (dz)^2, \\
g_2 = -(dx)^2 + (dy)^2 - (dz)^2.
\end{gathered} \tag{2.12}$$

Then the set $(\varphi, \xi, \eta, g_1)$ is a timelike Lorentzian almost paracontact structure, while the set $(\varphi, \xi, \eta, g_2)$ is a spacelike (ε)-almost paracontact metric structure. We note that $\operatorname{index}(g_1) = 1$ and $\operatorname{index}(g_2) = 2$.

Example 2.4. Let $\mathbb{R}^3$ be the 3-dimensional real number space with a coordinate system (x, y, z). We define

$$\begin{gathered}
\eta = dz - y\,dx, \qquad \xi = \frac{\partial}{\partial z}, \\
\varphi\left(\frac{\partial}{\partial x}\right) = -\frac{\partial}{\partial x} - y\frac{\partial}{\partial z}, \qquad \varphi\left(\frac{\partial}{\partial y}\right) = -\frac{\partial}{\partial y}, \qquad \varphi\left(\frac{\partial}{\partial z}\right) = 0, \\
g_1 = (dx)^2 + (dy)^2 - \eta \otimes \eta, \\
g_2 = (dx)^2 + (dy)^2 + (dz)^2 - y(dx \otimes dz + dz \otimes dx), \\
g_3 = -(dx)^2 + (dy)^2 + (dz)^2 - y(dx \otimes dz + dz \otimes dx).
\end{gathered} \tag{2.13}$$

Then, the set (φ, ξ, η) is an almost paracontact structure in $\mathbb{R}^3$. The set $(\varphi, \xi, \eta, g_1)$ is a timelike Lorentzian almost paracontact structure. Moreover, the trajectories of the timelike structure vector ξ are geodesics. The set $(\varphi, \xi, \eta, g_2)$ is a spacelike Lorentzian almost paracontact structure. The set $(\varphi, \xi, \eta, g_3)$ is a spacelike (ε)-almost paracontact metric structure $(\varphi, \xi, \eta, g_3, \varepsilon)$ with $\operatorname{index}(g_3) = 2$.

Example 2.5. Let $\mathbb{R}^5$ be the 5-dimensional real number space with a coordinate system (x, y, z, t, s). Defining

$$\begin{gathered}
\eta = ds - y\,dx - t\,dz, \qquad \xi = \frac{\partial}{\partial s}, \\
\varphi\left(\frac{\partial}{\partial x}\right) = -\frac{\partial}{\partial x} - y\frac{\partial}{\partial s}, \qquad \varphi\left(\frac{\partial}{\partial y}\right) = -\frac{\partial}{\partial y}, \\
\varphi\left(\frac{\partial}{\partial z}\right) = -\frac{\partial}{\partial z} - t\frac{\partial}{\partial s}, \qquad \varphi\left(\frac{\partial}{\partial t}\right) = -\frac{\partial}{\partial t}, \qquad \varphi\left(\frac{\partial}{\partial s}\right) = 0, \\
g_1 = (dx)^2 + (dy)^2 + (dz)^2 + (dt)^2 - \eta \otimes \eta, \\
\begin{aligned}
g_2 = &- (dx)^2 - (dy)^2 + (dz)^2 + (dt)^2 + (ds)^2 - t(dz \otimes ds + ds \otimes dz) \\
&- y(dx \otimes ds + ds \otimes dx),
\end{aligned}
\end{gathered} \tag{2.14}$$

the set $(\varphi, \xi, \eta, g_1)$ becomes a timelike Lorentzian almost paracontact structure in $\mathbb{R}^5$, while the set $(\varphi, \xi, \eta, g_2)$ is a spacelike (ε)-almost paracontact structure. Note that $\operatorname{index}(g_2) = 3$.

The Nijenhuis tensor $[J, J]$ of a tensor field J of type $(1,1)$ on a manifold M is a tensor field of type $(1,2)$ defined by

$$[J, J](X, Y) \equiv J^2[X, Y] + [JX, JY] - J[JX, Y] - J[X, JY], \tag{2.15}$$

for all $X, Y \in TM$. If M admits a tensor field J of type $(1,1)$ satisfying

$$J^2 = I, \tag{2.16}$$

then it is said to be an *almost product manifold* equipped with an *almost product structure* J. An almost product structure is *integrable* if its Nijenhuis tensor vanishes. For more details we refer to [14].

Example 2.6. Let (M^n, J, G) be a semi-Riemannian almost product manifold such that

$$J^2 = I, \qquad G(JX, JY) = G(X, Y). \tag{2.17}$$

Consider the product manifold $M^n \times \mathbb{R}$. A vector field on $M^n \times \mathbb{R}$ can be represented by $(X, f(d/dt))$, where X is tangent to M, f a smooth function on $M^n \times \mathbb{R}$, and t the coordinates of $\mathbb{R}$. On $M^n \times \mathbb{R}$ we define

$$\begin{gathered}
\eta = dt, \qquad \xi = \frac{d}{dt}, \qquad \varphi\left(\left(X, f\frac{d}{dt}\right)\right) = JX, \\
g\left(\left(X, f\frac{d}{dt}\right), \left(Y, h\frac{d}{dt}\right)\right) = G(X, Y) + \varepsilon f h.
\end{gathered} \tag{2.18}$$

Then $(\varphi, \xi, \eta, g, \varepsilon)$ is an (ε)-almost paracontact metric structure on the product manifold $M^n \times \mathbb{R}$.

Example 2.7. Let $(M, \psi, \xi, \eta, g, \varepsilon)$ be an (ε)-almost contact metric manifold. If we put $\varphi = \psi^2$, then $(M, \varphi, \xi, \eta, g, \varepsilon)$ is an (ε)-almost paracontact metric manifold.

3. Normal Almost Paracontact Manifolds

Let M be an almost paracontact manifold with almost paracontact structure (φ, ξ, η) and consider the product manifold $M \times \mathbb{R}$, where $\mathbb{R}$ is the real line. A vector field on $M \times \mathbb{R}$ can be represented by $(X, f(d/dt))$, where X is tangent to M, f a smooth function on $M \times \mathbb{R}$, and t the coordinates of $\mathbb{R}$. For any two vector fields $(X, f(d/dt))$ and $(Y, h(d/dt))$, it is easy to verify the following:

$$\left[\left(X, f\frac{d}{dt}\right), \left(Y, h\frac{d}{dt}\right)\right] = \left([X, Y], (Xh - Yf)\frac{d}{dt}\right). \tag{3.1}$$

Definition 3.1. If the induced almost product structure J on $M \times \mathbb{R}$ defined by

$$J\left(X, f\frac{d}{dt}\right) \equiv \left(\varphi X + f\xi, \eta(X)\frac{d}{dt}\right) \tag{3.2}$$

is integrable, then we say that the almost paracontact structure (φ, ξ, η) is *normal*.

This definition is conformable with the definition of normality given in [15]. As the vanishing of the Nijenhuis tensor $[J, J]$ is a necessary and sufficient condition for the integrability of the almost product structure J, we seek to express the conditions of normality in terms of the Nijenhuis tensor $[\varphi, \varphi]$ of φ. In view of (2.15), (3.2), (3.1), and (2.1)–(2.4) we have

$$\begin{aligned}
[J, J]\left(\left(X, f\frac{d}{dt}\right), \left(Y, h\frac{d}{dt}\right)\right) = \Big(&[\varphi, \varphi](X, Y) - 2d\eta(X, Y)\xi - h(\pounds_\xi\varphi)X + f(\pounds_\xi\varphi)Y, \\
&\{(\pounds_{\varphi X}\eta)Y - (\pounds_{\varphi Y}\eta)X - h(\pounds_\xi\eta)X + f(\pounds_\xi\eta)Y\}\frac{d}{dt}\Big),
\end{aligned} \tag{3.3}$$

where $\pounds_X$ denotes the Lie derivative with respect to X. Since $[J, J]$ is skew symmetric tensor field of type $(1,2)$, it suffices to compute $[J, J]((X, 0), (Y, 0))$ and $[J, J]((X, 0), (0, (d/dt)))$. Thus we have

$$\begin{aligned}
[J, J]((X, 0), (Y, 0)) &= \left([\varphi, \varphi](X, Y) - 2d\eta(X, Y)\xi, ((\pounds_{\varphi X}\eta)Y - (\pounds_{\varphi Y}\eta)X)\frac{d}{dt}\right), \\
[J, J]\left((X, 0), \left(0, \frac{d}{dt}\right)\right) &= -\left((\pounds_\xi\varphi)X, ((\pounds_\xi\eta)X)\frac{d}{dt}\right).
\end{aligned} \tag{3.4}$$

We are thus led to define four types of tensors $\overset{1}{N}$, $\overset{2}{N}$, $\overset{3}{N}$, and $\overset{4}{N}$, respectively, by (see also [1])

$$\overset{1}{N} \equiv [\varphi, \varphi] - 2d\eta \otimes \xi, \tag{3.5}$$

$$\overset{2}{N} \equiv (\pounds_{\varphi X}\eta)Y - (\pounds_{\varphi Y}\eta)X, \tag{3.6}$$

$$\overset{3}{N} \equiv \pounds_{\xi}\varphi, \tag{3.7}$$

$$\overset{4}{N} \equiv \pounds_{\xi}\eta. \tag{3.8}$$

Thus the almost paracontact structure (φ, ξ, η) will be normal if and only if the tensors defined by (3.5)–(3.8) vanish identically.

Taking account of (2.1)–(2.5) and (3.5)–(3.8), it is easy to obtain the following.

Lemma 3.2. *Let M be an almost paracontact manifold with an almost paracontact structure (φ, ξ, η). Then*

$$\overset{4}{N}(X) = 2d\eta(\xi, X), \tag{3.9}$$

$$\overset{2}{N}(X, Y) = 2\big(d\eta(\varphi X, Y) + d\eta(X, \varphi Y)\big), \tag{3.10}$$

$$\overset{1}{N}(X, \xi) = -\overset{3}{N}(\varphi X) = -[\xi, X] + \varphi[\xi, \varphi X] + \xi(\eta(X))\xi, \tag{3.11}$$

$$\overset{1}{N}(\varphi X, Y) = -\varphi[\varphi, \varphi](X, Y) - \overset{2}{N}(X, Y)\xi - \eta(X)\overset{3}{N}(Y). \tag{3.12}$$

Consequently,

$$\overset{2}{N}(X, \varphi Y) = 2\big(d\eta(\varphi X, \varphi Y) + d\eta(X, Y)\big) + \eta(Y)\overset{4}{N}(X), \tag{3.13}$$

$$\overset{4}{N}(X) = \eta\Big(\overset{1}{N}(X, \xi)\Big) = \overset{2}{N}(\xi, \varphi X) = -\eta\Big(\overset{3}{N}(\varphi X)\Big), \tag{3.14}$$

$$\overset{4}{N}(\varphi X) = -\eta([\xi, \varphi X]) = -\eta\Big(\overset{3}{N}(X)\Big), \tag{3.15}$$

$$\varphi\Big(\overset{1}{N}(X, \xi)\Big) = \overset{3}{N}(X) + \overset{4}{N}(\varphi X)\xi, \tag{3.16}$$

$$\eta\Big(\overset{1}{N}(\varphi X, Y)\Big) = -\overset{2}{N}(X, Y) + \eta(X)\overset{4}{N}(\varphi Y). \tag{3.17}$$

From (3.14), it follows that if $\overset{2}{N}$ or $\overset{3}{N}$ vanishes then $\overset{4}{N}$ vanishes. In view of (3.14), (3.16), and (3.17), we can state the following.

Theorem 3.3. *If, in an almost paracontact manifold* M*,* $\overset{1}{N}$ *vanishes, then* $\overset{2}{N}$*,* $\overset{3}{N}$*, and* $\overset{4}{N}$ *vanish identically. Hence, the almost paracontact structure is normal if and only if* $\overset{1}{N} = 0$*.*

Some equations given in Lemma 3.2 are also in [1]. First part of Theorem 3.3 is given as Theorem 3.4 of [1]. Now, we find a necessary and sufficient condition for the vanishing of $\overset{2}{N}$ in the following.

Proposition 3.4. *The tensor* $\overset{2}{N}$ *vanishes if and only if*

$$d\eta(\varphi X, \varphi Y) = -\, d\eta(X, Y). \tag{3.18}$$

Proof. The necessary part follows from (3.13). Conversely, from (3.18) and (2.3), we have

$$0 = d\eta\left(\varphi^2 X, \varphi\xi\right) = -\, d\eta(\varphi X, \xi), \tag{3.19}$$

which along with (2.1), when used in (3.18), yields

$$d\eta(X, \varphi Y) = -\, d\eta\left(\varphi X, \varphi^2 Y\right) = -\, d\eta(\varphi X, Y), \tag{3.20}$$

which in view of (3.10) proves that $\overset{2}{N} = 0$. □

From the definition of $\overset{3}{N}$ and $\overset{4}{N}$, it follows that [1, Theorem 3.1] the tensor $\overset{3}{N}$ (resp., $\overset{4}{N}$) vanishes identically if and only if φ (resp., η) is invariant under the transformation generated by infinitesimal transformations ξ. Consequently, in a normal almost paracontact manifold, φ and η are invariant under the transformation generated by infinitesimal transformations ξ.

The tangent sphere bundle over a Riemannian manifold has naturally an almost paracontact structure in which $\overset{3}{N} = 0$ and $\overset{4}{N} = 0$ [16]. Also an almost paracontact structure (φ, ξ, η) is said to be *weak normal* [15] if the almost product structures $J_1 = \varphi + \eta \otimes \xi$ and $J_2 = \varphi - \eta \otimes \xi$ are integrable. Then an almost paracontact structure is normal if and only if it is weak normal and $\overset{4}{N} = 0$.

4. (ε)-s-Paracontact Metric Manifolds

The fundamental $(0, 2)$ symmetric tensor of the (ε)-almost paracontact metric structure is defined by

$$\Phi(X, Y) \equiv g(X, \varphi Y), \tag{4.1}$$

for all $X, Y \in TM$. Also, we get

$$(\nabla_X \Phi)(Y, Z) = g((\nabla_X \varphi)Y, Z) = (\nabla_X \Phi)(Z, Y),$$
$$(\nabla_X \Phi)(\varphi Y, \varphi Z) = -(\nabla_X \Phi)(Y, Z) + \eta(Y)(\nabla_X \Phi)(\xi, Z) + \eta(Z)(\nabla_X \Phi)(Y, \xi), \tag{4.2}$$

for all $X, Y, Z \in TM$.

Definition 4.1. We say that $(\varphi, \xi, \eta, g, \varepsilon)$ is an (ε)*-paracontact metric structure* if

$$2\Phi(X, Y) = (\nabla_X \eta)Y + (\nabla_Y \eta)X, \quad X, Y \in TM. \tag{4.3}$$

In this case M is an (ε)*-paracontact metric manifold*.

The condition (4.3) is equivalent to

$$2\Phi = \varepsilon £_\xi g, \tag{4.4}$$

where £ is the operator of Lie differentiation. For $\varepsilon = 1$ and g Riemannian, M is the usual paracontact metric manifold [17].

Definition 4.2. An (ε)-almost paracontact metric structure $(\varphi, \xi, \eta, g, \varepsilon)$ is called an (ε)*-s-paracontact metric structure* if

$$\nabla \xi = \varepsilon \varphi. \tag{4.5}$$

A manifold equipped with an (ε)-s-paracontact structure is said to be (ε)*-s-paracontact metric manifold*.

Equation (4.5) is equivalent to

$$\Phi(X, Y) = g(\varphi X, Y) = \varepsilon g(\nabla_X \xi, Y) = (\nabla_X \eta)Y, \quad X, Y \in TM. \tag{4.6}$$

We have the following.

Theorem 4.3. *An (ε)-almost paracontact metric manifold is an (ε)-s-paracontact metric manifold if and only if it is an (ε)-paracontact metric manifold such that the structure 1-form η is closed.*

Proof. Let M be an (ε)-s-paracontact metric manifold. Then in view of (4.6) we see that η is closed. Consequently, M is an (ε)-paracontact metric manifold.

Conversely, let us suppose that M is an (ε)-paracontact metric manifold and η is closed. Then

$$\Phi(X, Y) = \frac{1}{2}\{(\nabla_X \eta)Y + (\nabla_Y \eta)X\} = (\nabla_X \eta)Y, \tag{4.7}$$

which implies (4.6). □

Proposition 4.4. *If in an* (ε)*-almost paracontact metric manifold the structure* 1*-form* η *is closed, then*

$$\nabla_\xi \xi = 0. \tag{4.8}$$

Proof. First we note that $g(\nabla_X \xi, \xi) = 0$ and in particular

$$g(\nabla_\xi \xi, \xi) = 0. \tag{4.9}$$

If η is closed, then for any vector X orthogonal to ξ we get

$$0 = 2\varepsilon \, d\eta(\xi, X) = -\varepsilon\eta([\xi, X]) = -g(\xi, [\xi, X]) = -g(\xi, \nabla_\xi X) = g(\nabla_\xi \xi, X), \tag{4.10}$$

which completes the proof. □

Using techniques similar to those introduced in [18, Section 4], we give the following.

Example 4.5. Let us assume the following:

$$\begin{gathered} a, b, c, d \in \{1, \dots, p\}, \quad \lambda, \mu, \upsilon \in \{1, \dots, q\}, \\ i, j, k \in \{1, \dots, p+q\}, \qquad \lambda' = p + \lambda, \qquad n = p + q + 1. \end{gathered} \tag{4.11}$$

Let $\theta : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}$ be a smooth function. Define a function $\varphi : \mathbb{R}^n \to \mathbb{R}$ by

$$\varphi\left(x^1, \dots, x^n\right) \equiv \theta\left(x^1, \dots, x^{p+q}\right) + x^n. \tag{4.12}$$

Now, define a 1-form η on $\mathbb{R}^n$ by

$$\eta_i = \frac{\partial \theta}{\partial x^i} \equiv \theta_i \,, \qquad \eta_n = 1. \tag{4.13}$$

Next, define a vector field ξ on $\mathbb{R}^n$ by

$$\xi \equiv \frac{\partial}{\partial x^n}, \tag{4.14}$$

and a $(1,1)$ tensor field φ on $\mathbb{R}^n$ by

$$\varphi X \equiv X^a \frac{\partial}{\partial x^a} - X^{\lambda'} \frac{\partial}{\partial x^{\lambda'}} + \left(-\theta_a X^a + \theta_{\lambda'} X^{\lambda'} \right) \frac{\partial}{\partial x^n}, \tag{4.15}$$

for all vector fields

$$X = X^a \frac{\partial}{\partial x^a} + X^{\lambda'} \frac{\partial}{\partial x^{\lambda'}} + X^n \frac{\partial}{\partial x^n}. \tag{4.16}$$

Let $f_i : \mathbb{R}^n \to \mathbb{R}$ be $(p+q)$ smooth functions. We define a tensor field g of type $(0,2)$ by

$$g(X,Y) \equiv \left(f_i - (\theta_i)^2\right)X^iY^i - \theta_i\theta_j X^iY^j - \theta_i\left(X^iY^n + X^nY^i\right) - X^nY^n, \tag{4.17}$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$ are $(p+q)$ smooth functions such that

$$f_i - (\theta_i)^2 > 0, \quad i \in \{1,\ldots,p+q\}. \tag{4.18}$$

Then (φ,ξ,η,g) is a timelike Lorentzian almost paracontact structure on $\mathbb{R}^n$. Moreover, if the $(p+q)$ smooth functions $f_i : \mathbb{R}^n \to \mathbb{R}$ are given by

$$\begin{aligned} f_a &= F_a\left(x^1,\ldots,x^{p+q}\right)e^{-2x^n} + (\theta_a)^2, \quad a \in \{1,\ldots,p\}, \\ f_{\lambda'} &= F_{\lambda'}\left(x^1,\ldots,x^{p+q}\right)e^{2x^n} + (\theta_{\lambda'})^2, \quad \lambda \in \{1,\ldots,q\}, \end{aligned} \tag{4.19}$$

for some smooth functions $F_i > 0$, then we get a timelike Lorentzian s-paracontact manifold.

5. (ε)-Para Sasakian Manifolds

We begin with the following.

Definition 5.1. An (ε)-almost paracontact metric structure is called an *(ε)-para Sasakian structure* if

$$(\nabla_X\varphi)Y = -\,g(\varphi X,\varphi Y)\xi - \varepsilon\eta(Y)\varphi^2X, \qquad X,Y \in TM, \tag{5.1}$$

where ∇ is the Levi-Civita connection with respect to g. A manifold endowed with an (ε)-para Sasakian structure is called an *(ε)-para Sasakian manifold*.

For $\varepsilon = 1$ and g Riemannian, M is the usual para Sasakian manifold [17, 18]. For $\varepsilon = -1$, g Lorentzian, and ξ replaced by $-\xi$, M becomes a Lorentzian para Sasakian manifold [8].

Example 5.2. Let $\mathbb{R}^3$ be the 3-dimensional real number space with a coordinate system (x,y,z). We define

$$\begin{gathered} \eta = dz, \qquad \xi = \frac{\partial}{\partial z}, \\ \varphi\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial x}, \qquad \varphi\left(\frac{\partial}{\partial y}\right) = -\frac{\partial}{\partial y}, \qquad \varphi\left(\frac{\partial}{\partial z}\right) = 0, \\ g = e^{2\varepsilon x^3}(dx)^2 + e^{-2\varepsilon x^3}(dy)^2 + \varepsilon(dz)^2. \end{gathered} \tag{5.2}$$

Then $(\varphi,\xi,\eta,g,\varepsilon)$ is an (ε)-para Sasakian structure.

Theorem 5.3. *An (ε)-para Sasakian structure $(\varphi, \xi, \eta, g, \varepsilon)$ is always an (ε)- s-paracontact metric structure, and hence an (ε)-paracontact metric structure.*

Proof. Let M be an (ε)-para Sasakian manifold. Then from (5.1) we get

$$\varphi\nabla_X\xi = -(\nabla_X\varphi)\xi = \varepsilon\varphi^2 X, \quad X, Y \in TM. \tag{5.3}$$

Operating by φ to the above equation, we get (4.5). □

The converse of the above theorem is not true. Indeed, the (ε)-s-paracontact structure in the Example 4.5 need not be (ε)-para Sasakian.

Theorem 5.4. *An (ε)-para Sasakian structure is always normal.*

Proof. In an almost paracontact manifold M, we have

$$\overset{1}{N}(X, Y) = (\nabla_X\varphi)\varphi Y - (\nabla_Y\varphi)\varphi X + (\nabla_{\varphi X}\varphi)Y - (\nabla_{\varphi Y}\varphi)X - \eta(X)\nabla_Y\xi + \eta(Y)\nabla_X\xi, \tag{5.4}$$

for all vector fields X, Y in M. Now, let M be an (ε)-para Sasakian manifold. Then it is (ε)-s-paracontact, and therefore using (5.1) and (4.5) in (5.4), we get $\overset{1}{N} = 0$. □

Problem 1. Whether a normal (ε)-paracontact structure is (ε)-para Sasakian or not, consider the following.

Lemma 5.5. *Let M be an (ε)-para Sasakian manifold. Then the curvature tensor R satisfies*

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \quad X, Y \in TM. \tag{5.5}$$

Consequently,

$$R(X, Y, Z, \xi) = -\,\eta(X)g(Y, Z) + \eta(Y)g(X, Z), \tag{5.6}$$

$$\eta(R(X, Y)Z) = -\,\varepsilon\eta(X)g(Y, Z) + \varepsilon\eta(Y)g(X, Z), \tag{5.7}$$

$$R(\xi, X)Y = -\varepsilon g(X, Y)\xi + \eta(Y)X, \tag{5.8}$$

for all vector fields X, Y, Z in M.

Proof. Using (4.5), (5.1), and (2.1) in

$$R(X, Y)\xi = \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,Y]}\xi, \tag{5.9}$$

we obtain (5.5). □

If we put

$$R_0(X, Y)W = g(Y, W)X - g(X, W)Y, \quad X, Y, W \in TM, \tag{5.10}$$

then in an (ε)-para Sasakian manifold M (5.5) and (5.8) can be rewritten as

$$R(X,Y)\xi = -\varepsilon R_0(X,Y)\xi, \tag{5.11}$$

$$R(\xi,X) = -\varepsilon R_0(\xi,X), \tag{5.12}$$

respectively.

Lemma 5.6. *In an (ε)-para Sasakian manifold M, the curvature tensor satisfies*

$$\begin{aligned} R(X,Y,\varphi Z,W) - R(X,Y,Z,\varphi W) = {} & \varepsilon\Phi(Y,Z)g(\varphi X,\varphi W) - \varepsilon\Phi(X,Z)g(\varphi Y,\varphi W) \\ & + \varepsilon\Phi(Y,W)g(\varphi X,\varphi Z) - \varepsilon\Phi(X,W)g(\varphi Y,\varphi Z) \\ & + \eta(Y)\eta(Z)g(X,\varphi W) - \eta(X)\eta(Z)g(Y,\varphi W) \\ & + \eta(Y)\eta(W)g(X,\varphi Z) - \eta(X)\eta(W)g(Y,\varphi Z), \end{aligned} \tag{5.13}$$

$$\begin{aligned} R(X,Y,\varphi Z,\varphi W) - R(X,Y,Z,W) = {} & \varepsilon\Phi(Y,Z)\Phi(X,W) - \varepsilon\Phi(X,Z)\Phi(Y,W) \\ & + \varepsilon g(\varphi X,\varphi Z)g(\varphi Y,\varphi W) - \varepsilon g(\varphi Y,\varphi Z)g(\varphi X,\varphi W) \\ & + \eta(Z)\{\eta(Y)g(X,W) - \eta(X)g(Y,W)\} \\ & - \eta(W)\{\eta(Y)g(X,Z) - \eta(X)g(Y,Z)\}, \end{aligned} \tag{5.14}$$

$$R(X,Y,\varphi Z,\varphi W) = R(\varphi X,\varphi Y,Z,W), \tag{5.15}$$

$$\begin{aligned} R(\varphi X,\varphi Y,\varphi Z,\varphi W) = {} & R(X,Y,Z,W) + \eta(Z)\{\eta(Y)g(X,W) - \eta(X)g(Y,W)\} \\ & - \eta(W)\{\eta(Y)g(X,Z) - \eta(X)g(Y,Z)\}, \end{aligned} \tag{5.16}$$

for all vector fields X,Y,Z,W in M.

Proof. Writing (5.1) equivalently as

$$(\nabla_Y\Phi)(Z,W) = -\varepsilon\eta(Z)g(\varphi Y,\varphi W) - \varepsilon\eta(W)g(\varphi Y,\varphi Z), \quad Y,Z,W \in TM, \tag{5.17}$$

and differentiating covariantly with respect to X, we get

$$\begin{aligned} -\varepsilon(\nabla_X\nabla_Y\Phi)(Z,W) = {} & \Phi(X,Z)g(\varphi Y,\varphi W) + \eta(Z)(\nabla_X\Phi)(Y,\varphi W) \\ & + \eta(Z)g(\varphi(\nabla_X Y),\varphi W) + \eta(Z)(\nabla_X\Phi)(\varphi Y,W) \\ & + \Phi(X,W)g(\varphi Y,\varphi Z) + \eta(W)(\nabla_X\Phi)(Y,\varphi Z) \\ & + \eta(W)g(\varphi(\nabla_X Y),\varphi Z) + \eta(W)(\nabla_X\Phi)(\varphi Y,Z), \end{aligned} \tag{5.18}$$

for all $X, Y, Z, W \in TM$. Now using (5.18) in the Ricci identity

$$((\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})\Phi)(Z, W) = -\Phi(R(X, Y)Z, W) - \Phi(Z, R(X, Y)W), \tag{5.19}$$

we obtain (5.13). Equation (5.14) follows from (5.13) and (5.6). Equation (5.15) follows from (5.14). Finally, equation (5.16) follows from (5.14) and (5.15). □

Equation (5.5) may also be obtained by (5.16). Equations (5.13)–(5.16) are generalizations of the (3.2) and (3.3) in [19]. Now, we prove the following:

Theorem 5.7. *An* (ε)*-para Sasakian manifold cannot be flat.*

Proof. Let M be a flat (ε)-para Sasakian manifold. Then from (5.6) we get

$$\eta(X)g(Y, Z) = \eta(Y)g(X, Z), \tag{5.20}$$

from which we obtain

$$g(\varphi X, \varphi Z) = 0, \tag{5.21}$$

for all $X, Z \in TM$, a contradiction. □

A non-flat semi-Riemannian manifold M is said to be *recurrent* [20] if its Ricci tensor R satisfies the recurrence condition

$$(\nabla_W R)(X, Y, Z, V) = \alpha(W)R(X, Y, Z, V), \quad X, Y, Z, V \in TM, \tag{5.22}$$

where α is a 1-form. If $\alpha = 0$ in the above equation, then the manifold becomes *symmetric* in the sense of Cartan [21]. We say that M is proper recurrent if $\alpha \neq 0$.

Theorem 5.8. *An* (ε)*-para Sasakian manifold cannot be proper recurrent.*

Proof. Let M be a recurrent (ε)-para Sasakian manifold. Then from (5.22), (5.6), and (4.5) we obtain

$$\varepsilon R(X, Y, Z, \varphi W) = g(X, Z)\{\Phi(Y, W) - \alpha(W)\eta(Y)\} - g(Y, Z)\{\Phi(X, W) - \alpha(W)\eta(X)\} \tag{5.23}$$

for all $X, Y, Z, W \in TM$. Putting $Y = \xi$ in the above equation, we get

$$\alpha(W)g(\varphi X, \varphi Z) = 0, \quad X, Z, W \in TM, \tag{5.24}$$

a contradiction. □

Let $n \geq 2$ and $0 \leq \nu \leq n$. Then [10, Definition 23, page 110] the following are given.

(1) The *pseudosphere* of radius $r > 0$ in R^{n+1}_{ν} is the hyperquadric

$$S^n_{\nu}(r) = \left\{ p \in R^{n+1}_{\nu} : \langle p, p \rangle = r^2 \right\}, \tag{5.25}$$

with dimension n and index ν.

(2) The *pseudohyperbolic space* of radius $r > 0$ in $R^{n+1}_{\nu+1}$ is the hyperquadric

$$H^n_{\nu}(r) = \left\{ p \in R^{n+1}_{\nu+1} : \langle p, p \rangle = -\, r^2 \right\}, \tag{5.26}$$

with dimension n and index ν.

Theorem 5.9. *An (ε)-para Sasakian manifold is symmetric if and only if it is of constant curvature $-\varepsilon$. Consequently, a symmetric spacelike (ε)-para Sasakian manifold is locally isometric to a pseudohyperbolic space $H^n_{\nu}(1)$ and a symmetric timelike (ε)-para Sasakian manifold is locally isometric to a pseudosphere $S^n_{\nu}(1)$.*

Proof. Let M be a symmetric (ε)-para Sasakian manifold. Then putting $\alpha = 0$ in (5.23), we obtain

$$\varepsilon R(X, Y, Z, \varphi W) = g(X, Z)\Phi(Y, W) - g(Y, Z)\Phi(X, W), \tag{5.27}$$

for all $X, Y, Z, W \in TM$. Writing φW in place of W in the above equation and using (2.7) and (5.6), we get

$$R(X, Y, Z, W) = -\, \varepsilon \{ g(Y, Z)g(X, W) - g(X, Z)g(Y, W) \}, \tag{5.28}$$

which shows that M is a space of constant curvature $-\varepsilon$. The converse is trivial. □

Corollary 5.10. *If an (ε)-para Sasakian manifold is of constant curvature, then*

$$\Phi(Y, Z)\Phi(X, W) - \Phi(X, Z)\Phi(Y, W) = -\, g(\varphi Y, \varphi Z) g(\varphi X, \varphi W) + g(\varphi X, \varphi Z) g(\varphi Y, \varphi W) \tag{5.29}$$

for all $X, Y, Z, W \in TM$.

Proof. Obviously, if an (ε)-para Sasakian manifold is of constant curvature k, then $k = -\varepsilon$. Therefore, using (5.28) in (5.14), we get (5.29). □

Apart from recurrent spaces, semi-symmetric spaces are another well-known and important natural generalization of symmetric spaces. A semi-Riemannian manifold (M, g) is a *semi-symmetric space* if its curvature tensor R satisfies the condition

$$R(X, Y) \cdot R = 0, \tag{5.30}$$

for all vector fields X, Y on M, where $R(X, Y)$ acts as a derivation on R. Symmetric spaces are obviously semi-symmetric, but the converse need not be true. In fact, in dimension greater than two there always exist examples of semi-symmetric spaces which are not symmetric. For more details we refer to [22].

Given a class of semi-Riemannian manifolds, it is always interesting to know that whether, inside that class, semisymmetry implies symmetry or not. Here, we prove the following.

Theorem 5.11. *In an (ε)-para Sasakian manifold, the condition of semi-symmetry implies the condition of symmetry.*

Proof. Let M be a symmetric (ε)-para Sasakian manifold. Let the condition of being semi-symmetric be true, that is,

$$R(V, U) \cdot R = 0, \quad V, U \in TM. \tag{5.31}$$

In particular, from the condition $R(\xi, U) \cdot R = 0$, we get

$$0 = [R(\xi, U), R(X, Y)]\xi - R(R(\xi, U)X, Y)\xi - R(X, R(\xi, U)Y)\xi, \tag{5.32}$$

which in view of (5.12) gives

$$\begin{aligned} 0 = {} & g(U, R(X, Y)\xi)\xi - \eta(R(X, Y)\xi)U \\ & - g(U, X)R(\xi, Y)\xi + \eta(X)R(U, Y)\xi - g(U, Y)R(X, \xi)\xi \\ & + \eta(Y)R(X, U)\xi - \eta(U)R(X, Y)\xi + R(X, Y)U. \end{aligned} \tag{5.33}$$

Equation (5.11) then gives

$$R = -\varepsilon R_0. \tag{5.34}$$

Therefore M is of constant curvature $-\varepsilon$, and hence symmetric. □

In view of Theorems 5.9 and 5.11, we have the following.

Corollary 5.12. *Let M be an (ε)-para Sasakian manifold. Then the following statements are equivalent.*

(i) *M is symmetric.*

(ii) *M is of constant curvature $-\varepsilon$.*

(iii) *M is semi-symmetric.*

(iv) *M satisfies $R(\xi, U) \cdot R = 0$.*

Now, we need the following.

Lemma 5.13. *In an n-dimensional (ε)-para Sasakian manifold M the Ricci tensor S satisfies*

$$S(\varphi Y, \varphi Z) = S(Y, Z) + (n-1)\eta(Y)\eta(Z), \tag{5.35}$$

for all $Y, Z \in TM$. Consequently,

$$S(\varphi Y, Z) = S(Y, \varphi Z), \tag{5.36}$$

$$S(Y, \xi) = -(n-1)\eta(Y). \tag{5.37}$$

Proof. Contracting (5.16), we get (5.35). Replacing Z by φZ in (5.35), we get (5.36). Putting $Z = \xi$ in (5.35), we get (5.37).

A semi-Riemannian manifold M is said to be *Ricci-recurrent* [23] if its Ricci tensor S satisfies the condition

$$(\nabla_X S)(Y, Z) = \alpha(X)S(Y, Z), \quad X, Y, Z \in TM, \tag{5.38}$$

where α is a 1-form. If $\alpha = 0$ in the above equation, then the manifold becomes *Ricci-symmetric*. We say that M is proper Ricci-recurrent, if $\alpha \neq 0$. □

Theorem 5.14. *An (ε)-para Sasakian manifold cannot be proper Ricci-recurrent.*

Proof. Let M be an n-dimensional (ε)-para Sasakian manifold. If possible, let M be proper Ricci-recurrent. Then

$$(\nabla_X S)(Y, \xi) = \alpha(X)S(Y, \xi) = -\,(n-1)\alpha(X)\eta(Y). \tag{5.39}$$

But we have

$$(\nabla_X S)(Y, \xi) = (n-1)\left(\nabla_X \eta\right)Y - \varepsilon S(Y, \varphi X). \tag{5.40}$$

Using (5.40) in (5.39), we get

$$\varepsilon S(\varphi X, Y) + (n-1)\Phi(X, Y) = (n-1)\alpha(X)\eta(Y). \tag{5.41}$$

Putting $Y = \xi$ in the above equation, we get $\alpha(X) = 0$, a contradiction. □

A semi-Riemannian manifold M is said to be *Ricci-semi-symmetric* [24] if its Ricci tensor S satisfies the condition

$$R(X, Y) \cdot S = 0, \tag{5.42}$$

for all vector fields X, Y on M, where $R(X, Y)$ acts as a derivation on S.

at last, we prove the following.

Theorem 5.15. *For an n-dimensional (ε)-para Sasakian manifold M, the following three statements are equivalen.*

(a) *M is an Einstein manifold.*

(b) *M is Ricci-symmetric.*

(c) *M is Ricci-semi-symmetric.*

Proof. Obviously, the statement (a) implies each of the statements (b) and (c). Let (b) be true. Then putting $\alpha = 0$ in (5.41), we get

$$\varepsilon S(\varphi X, Y) + (n-1)\Phi(X, Y) = 0. \tag{5.43}$$

Replacing X by φX in the above equation, we get

$$S = -\ \varepsilon(n-1)g, \tag{5.44}$$

which shows that the statement (a) is true. At last, let (c) be true. In particular,

$$(R(\xi, X)\cdot S)(Y, \xi) = 0 \tag{5.45}$$

implies that

$$S(R(\xi, X)Y, \xi) + S(Y, R(\xi, X)\xi) = 0, \tag{5.46}$$

which in view of (5.8) and (5.37) again gives (5.44). This completes the proof. □

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