Indefinite-Metric Quantum Field Theory

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Some recent progress in the quantum field theory based on the indefinite-metric Hilbert space is reviewed in a systematic way. Various problems (as shown in the table of contents below) are discussed extensively. The selection of topics is, however, dependent on the present author's interest; undescribed topics do not necessarily mean to be unimportant. An almost complete exposition is given concerning the finite-dimensional indefinite-metric space. On the indefinite-metric quantum field theory, the problem of interpretation and the unitarity of the physical S-matrix are discussed in detail. Also included are a number of new results and remarks on various existing theories. Some of them are construction of a field theory having only one Feynman integral, critical remarks on the quantization of a purely-imaginary-mass field and a criticism of the renormalization procedure of the Gupta-Bleuler quantum electrodynamics.

Contents

Chapter 1

Introduction

§ 1. Outline

An indefinite-metric space is a vector space in which an inner product $\langle l|k\rangle$ is defined for any two vectors $|k\rangle$ and $|l\rangle$, but $\langle k|k\rangle$ for $|k\rangle\neq 0$ is not necessarily positive. An indefinite-metric Hilbert space (though it is a self-contradictory nomenclature) is roughly an infinite-dimensional indefinite-metric space. For rigorous definition, we have to introduce topology, but unfortunately, as will be discussed in § 8, there is no very convenient way of introducing topology into this space.

The present paper is a review of some recent progress in the quantum field theory based on the indefinite-metric Hilbert space. Since there is an extensive review article by Nagy[N4)] on the indefinite-metric quantum field theory, which was published in 1966, we do not intend to present a complete review on this subject. We shall avoid reproducing the results which are already described in detail in Nagy's book, except for some basic points.

The use of the indefinite-metric Hilbert space in quantum field theory has been motivated for various purposes. As is well known, it is inevitable to introduce an indefinite-metric Hilbert space if we wish to formulate quantum electrodynamics in a manifestly covariant way. Since the Minkowski space is a finite-dimensional indefinite-metric space, the requirement of manifest covariance for higher-spin fields naturally leads us to the introduction of the indefinite-metric Hilbert space.

There is another interesting application of the indefinite-metric Hilbert space; it is to remove the ultraviolet-divergence difficulty of the conventional quantum field theory. Though it is almost trivial to remove all ultraviolet divergences from the theory by means of indefinite metric, we encounter a new difficulty due to the use of indefinite metric, because it conflicts with the usual probabilistic interpretation of quantum theory. The most natural way of avoiding this difficulty is to restrict physical states (i. e., observable states) to those which belong to a subspace having positive-definite metric. This idea is consistent if and only if the S-matrix satisfies the condition that if the initial state is a physical state then the final state is always a physical one. This condition is satisfied in quantum electrodynamics, but its satisfaction is extremely difficult in the theories in which the ultraviolet-divergence difficulty is removed. On the possibility of constructing such a consistent theory, Nagy's book was rather optimistic, but recent closer investigations have revealed an important mistake involved in the previous work (see § 10). On the other hand, it has been recently found that there is another possibility, which was overlooked previously, in constructing a divergence-free field theory which has the S-matrix satisfying the above condition (see § 16). To describe those topics is the main motivation of the present paper.

Concerning the description of indefinite metric, we make a remark on the so-called η-formalism. In the old literature, it was customary to represent the indefinite-metric Hilbert space in terms of the (positive-metric) Hilbert space by introducing an indefinite-metric operator η, that is, the inner product in the former was defined by the corresponding matrix element of η in the latter. It is known, however, that it is not only fruitless but also misleading to write η explicitly, because η is an operator quite different from any other physical operators and because the η-formalism violates the elegant manifest-covariance property of the indefinite-metric quantum field theory. Therefore we do not employ the η-formalism.

Finally, for later convenience, we explain terminology in the framework of the indefinite-metric quantum field theory. A non-zero vector in the indefinite-metric (Hilbert) space is called a state vector or simply a state. It should be noted that a state is not necessarily a normalized vector. For a state $|k\rangle$, $\langle k|k\rangle$ is, in abuse of language, called the norm of $|k\rangle$ (instead of the *squared* norm). A state having zero or negative norm is called a ghost. Let H be the Hamiltonian of a system. If a state $|k\rangle$ satisfies

$$(H-E_k)|k\rangle=0, \tag{1·1}$$

then $|k\rangle$ and E_k are called an eigenstate and an eigenvalue of H, respectively, as usual. Though H is assumed to be hermitian, E_k is not necessarily real; for E_k non-real, $|k\rangle$ is called a complex ghost, which necessarily has zero norm. The totality of the eigenstates of H is not complete in general. If

$$(H-E_k)^2|k, D\rangle=0 \quad \text{but} \quad (H-E_k)|k, D\rangle\neq 0, \tag{1·2}$$

then $|k, D\rangle$ is called a dipole ghost. More generally, if

$$(H-E_k)^n|k, M_n\rangle=0 \quad \text{but} \quad (H-E_k)^{n-1}|k, M_n\rangle\neq 0 \tag{1·3}$$

for $n\geq 2$, then $|k, M_n\rangle$ is called a multipole ghost. All the above states are called generalized eigenstates. It can be proved that a complete set is formed by generalized eigenstates of H at least in the *finite-dimensional* indefinite-metric space (see § 6).

The Minkowski-space metric tensor $g_{\mu\nu}$ is defined by $g_{00}=-g_{ll}=1$ $(l=1, 2, 3)$ and $g_{\mu\nu}=0$ for $\mu\neq\nu$. The usual tensor-analysis convention is employed throughout; for instance, $px=p_\mu x^\mu=g^{\mu\nu}p_\mu x_\nu=p_0x_0-\boldsymbol{px}$ and $\boldsymbol{px}=\sum_{l=1}^3 p_l x_l$.

§ 2. Historical review

Indefinite metric was first introduced into quantum field theory by Dirac (1942)[D3] and then reviewed by Pauli (1943).[P3] An attempt at removing ultraviolet divergences was made by Pauli and Villars (1949),[P5] whose regulator method was, in effect, the introduction of indefinite-metric auxiliary fields. This idea was more realistically formulated in the multimass theory, to which the most prominent contribution was made by Pais and Uhlenbeck (1950)[P1] (see § 14). On the other hand, the manifestly covariant quantum electrodynamics in the Feynman gauge was proposed by Gupta (1950)[G6] and made precise by Bleuler (1950).[B4] Subsequently, indefinite metric was applied to massive vector fields, a weak gravitational field, etc.

Lee (1954)[L3] proposed a solvable field-theory model, called the Lee model (see § 12), and found that the renormalization constant Z_2 becomes negative if no cutoff function is introduced in the interaction Hamiltonian. Källén and Pauli (1955)[K2] noted that it is necessary to introduce indefinite metric in order to renormalize the no-cutoff Lee model consistently, but then the appearance of a ghost in the final state violates the unitarity of the physical S-matrix.

Heisenberg (1957)[H7] first introduced a dipole ghost in the Lee model in order to justify the use of a propagator having a double pole in his non-linear spinor field theory.[H8] According to him, the unitarity trouble would not occur in the dipole-ghost theory. After Heisenberg's work, many authors (Pauli (1958),[P4] Froissart (1959),[F6] etc.) investigated various possibilities, such as complex ghosts, relativistic dipole ghosts, etc. in the framework of the indefinite-metric quantum field theory. Ascoli and Minardi (1958)[A3),A4] discussed its general features and clarified the condition in which the unitarity of the physical S-matrix is guaranteed (see § 10). In particular, their conclusion supported Heisenberg's one on the dipole-ghost theory. Pandit (1959)[P2] and Nagy (1960)[N3] summarized the results obtained in those days. Shimodaira (1960)[S7] (see § 10), Yokoyama (1961)[Y5)~Y7] (see § 15) and Tanaka (1963)[T3] (see § 16) proposed some interesting theories, hoping to obtain a divergence-free, unitary indefinite-metric field theory.

Though this is not a genuine field-theoretical problem, it was found by Nakanishi (1965)[N9] that the Bethe-Salpeter equation necessarily has the solutions corresponding to ghosts. Soon later, Nakanishi (1965)[N10] showed that at certain energies of degeneracy the existence of multipole ghosts is deduced in the Bethe-Salpeter formalism (for review, see Ref. N 14)). As its Reggeized version, the daughter trajectories found by Freedman and Wang (1967)[F5] involve the Reggeized ghosts[N12] and multipole ghosts. The Reggeized ghosts are further inherited by the generalized Veneziano amplitudes.[M1),*)]

Hinted by the multipole ghosts in the Bethe-Salpeter formalism, Nakanishi (1966)[N11)] proposed a manifestly covariant quantization of the electromagnetic field in the Landau gauge by using a dipole ghost. Independently, Lautrup (1967)[L2),**)] more thoroughly formulated a quantization of the electromagnetic field in the general covariant gauge without noting the necessity of a dipole ghost (see § 18).

Lee and Wick (1969, 1970)[L6)~L8)] and Lee (1970)[L5)] proposed a new complex-ghost quantum field theory (see § 16). They found that it is possible to obtain a divergence-free, unitary physical S-matrix if *relativistic* complex ghosts are used. It was pointed out by Nakanishi (1971),[N16)] however, that the physical S-matrix of this theory is not Lorentz-invariant in the second-order self-energy part (see § 16).

Nagy (1970)[N5)] constructed a dipole-ghost field-theory model whose physical S-matrix is not unitary, contrary to the conclusion of Heisenberg and Ascoli and Minardi. The reason for this result was clarified by Nakanishi (1971)[N17)] (see § 10). He showed that the use of dipole ghosts does not guarantee the unitarity of the physical S-matrix at all.

The present author conjectures that it is impossible to formulate satisfactorily a non-trivial, divergence-free quantum field theory whose physical S-matrix is unitary, macro-causal and Lorentz-invariant, without introducing a drastic change of the notion of space-time. This belief is based on the failure of all past attempts to obtain such a theory since Heisenberg and Pauli's proposal of quantum field theory. Of course, a number of people will not agree to the above conjecture. Indeed, several attempts at formulating satisfactory finite theories have been proposed recently.[***)]

§ 3. Representations of commutation relations

In this section, we show how the abnormal commutation relations lead us to the introduction of indefinite metric. For simplicity, we consider only one mode, avoiding the difficulty due to infinite degrees of freedom.

(A) *Anticommutation relations*

Let α and $\bar{\alpha}$ be two operators satisfying

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*) If the loop-graph contributions are summed up, multipole ghosts will also appear in the dual resonance model.

**) Part of his work was referred to in the Schladming Conference held in 1965 (private communication).

***) Some interesting examples are Kita's non-Lagrangian formulation of non-local field theory,[K4)~K6)] H. Yamamoto's non-hermitian Hamiltonian theory of complex ghosts,[Y1)~Y4)] Sudarshan and his collaborators' "shadow-state" theory[S10),G2),N21),N22)] and Taguchi and K. Yamamoto's regularized Hamiltonian theory.[T1)]

$$\alpha^2=\bar{\alpha}^2=0, \tag{3·1}$$

$$\alpha\bar{\alpha}+\bar{\alpha}\alpha=\pm 1. \tag{3·2}$$

In (3·2), the upper sign corresponds to the normal anticommutation relation, while the lower sign corresponds to the abnormal one. Let

$$N\equiv\pm\bar{\alpha}\alpha; \tag{3·3}$$

then with the aid of (3·2) and (3·1), we have

$$N^2=N, \tag{3·4}$$

that is, N is a projection operator. Let $\mathcal{V}$ be the operand space, and write $\mathcal{V}_1\equiv N\mathcal{V}$ and $\mathcal{V}_0\equiv(1-N)\mathcal{V}$. Then, any vector of $\mathcal{V}$ can be uniquely decomposed into a vector of $\mathcal{V}_0$ and that of $\mathcal{V}_1$, that is, $\mathcal{V}=\mathcal{V}_0\oplus\mathcal{V}_1$. As is easily seen by using (3·1)～(3·3), the operator α maps $\mathcal{V}_0$ to 0 and $\mathcal{V}_1$ to $\mathcal{V}_0$, while $\bar{\alpha}$ does $\mathcal{V}_0$ to $\mathcal{V}_1$ and $\mathcal{V}_1$ to 0. Furthermore, for the mapping $\mathcal{V}_1\to\mathcal{V}_0$ induced by α, there is an inverse mapping $\mathcal{V}_0\to\mathcal{V}_1$ induced by $\pm\bar{\alpha}$, that is, the correspondence between $\mathcal{V}_0$ and $\mathcal{V}_1$ is one to one. Hence we can represent any vector in $\mathcal{V}_0$ by a column-vector whose lower half consists of zeros only and any vector in $\mathcal{V}_1$ by a column-vector whose upper half consists of zeros only; then we have

$$\alpha=\begin{pmatrix}0 & \xi\\ 0 & 0\end{pmatrix},\quad \bar{\alpha}=\begin{pmatrix}0 & 0\\ \zeta & 0\end{pmatrix}, \tag{3·5}$$

where ξ and ζ are square matrices satisfying

$$\xi\zeta=\zeta\xi=\pm\delta, \tag{3·6}$$

where δ denotes the unit matrix (δ_{mn}). Since ξ and ζ are commutative, they can be diagonalized simultaneously. Changing their normalizations appropriately, we can set $\xi=\pm\delta$ and $\zeta=\delta$. In particular, if we consider an irreducible representation alone, we find

$$\alpha=\begin{pmatrix}0 & \pm 1\\ 0 & 0\end{pmatrix},\quad \bar{\alpha}=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}. \tag{3·7}$$

Let $|0\rangle$ and $|1\rangle$ be a basis vector in $\mathcal{V}_0$ and that in $\mathcal{V}_1$, respectively. Then (3·7) is rewritten as

$$\begin{aligned}\alpha|0\rangle&=0, & \bar{\alpha}|0\rangle&=|1\rangle,\\ \alpha|1\rangle&=\pm|0\rangle, & \bar{\alpha}|1\rangle&=0.\end{aligned} \tag{3·8}$$

Now, if we *require* any vectors in $\mathcal{V}$ to satisfy

$$\begin{aligned}\langle l|k\rangle^*&=\langle k|l\rangle,\\ \langle l|\alpha|k\rangle^*&=\langle k|\bar{\alpha}|l\rangle,\end{aligned} \tag{3·9}$$

where an asterisk denotes complex conjugation, then with the aid of (3·8) we find that

$$\langle 1|0\rangle = \pm\langle 1|\alpha|1\rangle = \pm\langle 1|\bar{\alpha}|1\rangle^* = 0,$$
$$\langle 0|1\rangle = \langle 1|0\rangle^* = 0,$$
$$\langle 1|1\rangle = \langle 1|\bar{\alpha}|0\rangle = \langle 0|\alpha|1\rangle^* = \pm\langle 0|0\rangle^* = \pm\langle 0|0\rangle. \quad (3\cdot 10)$$

Thus under the assumption (3·9), $\langle 0|0\rangle$ and $\langle 1|1\rangle$ cannot be simultaneously positive in the case of the abnormal anticommutation relation, that is, in this case the introduction of indefinite metric is inevitable.

(B) *Commutation relations*

We consider two operators α and $\bar{\alpha}$ satisfying

$$\alpha\bar{\alpha} - \bar{\alpha}\alpha = \pm 1. \quad (3\cdot 11)$$

It should be noted that there is no relation corresponding to (3·1). Since the abnormal commutation relation (lower sign) is transformed into the normal one (upper sign) by interchanging α and $\bar{\alpha}$, they are mutually equivalent without an additional requirement. Hence, from the physical consideration, we require that there exists a non-zero vector $|0\rangle$ such that

$$\alpha|0\rangle = 0. \quad (3\cdot 12)$$

By using (3·11), we see that the smallest invariant subspace involving $|0\rangle$ is the space $\mathcal{V}$ spanned by

$$|n\rangle \equiv (n!)^{-1/2}\bar{\alpha}^n|0\rangle. \quad (n=0, 1, 2, \cdots) \quad (3\cdot 13)$$

The vectors listed in (3·13) are linearly independent, because if there exists a polynomial f such that $f(\bar{\alpha})|0\rangle = 0$ then we have $f(\bar{\alpha})|n\rangle = 0$ for all n, that is, $f(\bar{\alpha}) = 0$ in $\mathcal{V}$; if g is a polynomial of the minimum degree such that $g(\bar{\alpha}) = 0$, then we have $0 = \alpha g(\bar{\alpha}) - g(\bar{\alpha})\alpha = \pm g'(\bar{\alpha})$, a result which is inconsistent with the assumption. Thus $\mathcal{V}$ is an infinite-dimensional space.

Now, we require (3·9) to hold. Then it is straightforward to show that

$$\langle n|n\rangle = (\pm 1)^n\langle 0|0\rangle. \quad (3\cdot 14)$$

Thus for the abnormal commutation relation we need an indefinite-metric Hilbert space.

As remarked above, if we take

$$\bar{\alpha}|0\rangle = 0 \quad (3\cdot 15)$$

instead of (3·12), then the abnormal case reduces to the normal one, so that the use of indefinite metric is avoided. For practical applications, however, the energy operator H will symbolically (i. e., suppressing irrelevant degrees of freedom) have the form

$$H = E\bar{\alpha}\alpha + E'\bar{\beta}\beta \tag{3·16}$$

with $E>0$ and $E'>0$, where β and $\bar{\beta}$ satisfy the normal commutation relation. Hence H has negative eigenvalues as well as positive ones. This "negative-energy quantization" method could be regarded as a substitute for the indefinite-metric theory, but we note that the former has the following serious defects:

(1) The vacuum becomes unstable.

(2) The divergence difficulty gets worse. Hence this method cannot be used for obtaining a divergence-free field theory.

(3) A transformation between α and β destroys the Fock space. Hence this method is completely unsuitable for formulating a manifestly covariant field theory.

(4) As mentioned in (A), this method cannot be applied to the abnormal anticommutation relation.

Thus the negative-energy quantization is almost useless.

Chapter 2

Finite-Dimensional Indefinite-Metric Space

The theory of a finite-dimensional vector space is mathematically well-established, and its formulation can be found in the mathematical literature.[H9)] But the description in mathematical books is usually inconvenient to physicists. The present chapter is an exposition of the theory of a finite-dimensional vector space in the terminology of indefinite metric.

§ 4. Definition of an *N*-dimensional indefinite-metric space

We employ the following notation: a, b, c, etc. stand for complex numbers, while vectors are denoted by $|k\rangle$, $|l\rangle$, etc. except for the zero vector, which is denoted by 0. The complex conjugate is indicated by affixing an asterisk. The symbol $\in$ means "belong to".

A set $\mathcal{V}$ of vectors is called a vector space if the following conditions are satisfied for any vectors $|k\rangle$, $|l\rangle$, $|m\rangle \in \mathcal{V}$:

1° $|k\rangle + |l\rangle \in \mathcal{V}$,

2° $|k\rangle + |l\rangle = |l\rangle + |k\rangle$,
$(|k\rangle + |l\rangle) + |m\rangle = |k\rangle + (|l\rangle + |m\rangle)$,

3° $0 \in \mathcal{V}$, $|k\rangle + 0 = |k\rangle$,
$-|k\rangle \in \mathcal{V}$, $|k\rangle + (-|k\rangle) = 0$,

4° $a|k\rangle \in \mathcal{V}$, $1|k\rangle = |k\rangle$,

5° $(a+b)|k\rangle = a|k\rangle + b|k\rangle$,
$a(b|k\rangle) = (ab)|k\rangle$,
$a(|k\rangle + |l\rangle) = a|k\rangle + a|l\rangle$.

Furthermore, $\mathcal{V}$ is a finite-dimensional vector space if

6° The number of linearly independent vectors is finite. [It is denoted by N.]

A vector space $\mathcal{V}$ is an indefinite-metric space if for any two vectors $|k\rangle$ and $|l\rangle$ of $\mathcal{V}$ an inner product (*or* a sesquilinear form) is defined in such a way that

7° $\langle l|k\rangle$ is a complex number,

8° $\langle l|k\rangle^* = \langle k|l\rangle$,

9° $\langle l|(a|k\rangle + b|k'\rangle) = a\langle l|k\rangle + b\langle l|k'\rangle$.

It is often convenient to define a "bra-vector" $\langle k|$, which is in one-to-one correspondence to a "ket-vector" $|k\rangle$. Then $a^*\langle k|$ corresponds to $a|k\rangle$ because of 8°.

An indefinite-metric space is more general than the Euclidean space, which, in addition to the above, has the properties

$$\langle k|k\rangle \geq 0 \tag{4·1}$$

and

$$\langle k|k\rangle = 0 \text{ if and only if } |k\rangle = 0. \tag{4·2}$$

Because of the lack of those properties, the norm of a vector $|k\rangle$ cannot be defined *mathematically* in terms of $\langle k|k\rangle$; nevertheless we call $\langle k|k\rangle$ the norm of $|k\rangle$. Since $\langle k|k\rangle$ is real because of 8°, we have three cases: positive norm $\langle k|k\rangle > 0$, zero norm $\langle k|k\rangle = 0$ and negative norm $\langle k|k\rangle < 0$.

Let $|1\rangle$, $|2\rangle, \cdots, |N\rangle$ be N linearly independent vectors of $\mathcal{V}$. Such a set is called a base. For any vector $|k\rangle \in \mathcal{V}$, we can expand it as

$$|k\rangle = \sum_{n=1}^{N} a_n |n\rangle. \tag{4·3}$$

We define

$$\eta_{mn} \equiv \langle m|n\rangle, \tag{4·4}$$

and call the $N \times N$ matrix $\eta = (\eta_{mn})$ an indefinite-metric matrix (*not an operator*). It is a hermitian matrix and of course base-dependent. It is an important property, characteristic to the finite-dimensional indefinite-metric space, that given (η_{mn}), the structure of $\mathcal{V}$ is uniquely determined.

Let $|\tilde{1}\rangle$, $|\tilde{2}\rangle, \cdots, |\tilde{N}\rangle$ form another base. We write

$$\tilde{\eta}_{mn} \equiv \langle \tilde{m}|\tilde{n}\rangle. \tag{4·5}$$

According to (4·3), we have

$$|\tilde{m}\rangle = \sum_n u_{nm} |n\rangle, \tag{4·6}$$

where $u = (u_{nm})$ is a non-singular matrix. Hence

$$\begin{aligned} \tilde{\eta}_{mn} &= \sum_{k,l} u_{km}^* \; u_{ln} \langle k|l\rangle \\ &= \sum_{k,l} u_{km}^* \; \eta_{kl} u_{ln} . \end{aligned} \tag{4·7}$$

In the matrix notation, (4·7) is rewritten as

$$\tilde{\eta} = u^{\dagger} \eta u, \tag{4·8}$$

where a dagger denotes hermitian conjugation. Therefore, by choosing an appropriate base, we can make the indefinite-metric matrix diagonal in such a way that its diagonal elements are $+1, -1$ and 0 only. This matrix form may be called the normal form.

We call $\mathcal{V}$ degenerate (*or* singular) if there exists a non-zero vector $|k\rangle$ in $\mathcal{V}$ such that $\langle l|k\rangle = 0$ for any vector $|l\rangle \in \mathcal{V}$. Since the existence of such a vector $|k\rangle$ has no effect on any quantity experssed in terms of inner products, it is convenient to consider a quotient space $\mathcal{V}/\mathcal{O}$ instead of $\mathcal{V}$ itself, where $\mathcal{O}$ denotes the totality of the vectors $|k\rangle$ such that $\langle l|k\rangle=0$ for any $|l\rangle$. Equivalently, we introduce an additional postulate:

10° $\mathcal{V}$ is non-degenerate, that is, $|k\rangle=0$ if and only if $\langle l|k\rangle=0$ for any $|l\rangle\in\mathcal{V}$.

THEOREM The space $\mathcal{V}$ is non-degenerate if and only if η is non-singular.

This is because all diagonal elements in the normal form of η are non-zero if and only if η is non-singular.

§ 5. Linear operators

If $T|k\rangle\in\mathcal{V}$ for any $|k\rangle\in\mathcal{V}$, then T is called an operator on $\mathcal{V}$. An operator T is linear if for any $|k\rangle,|l\rangle\in\mathcal{V}$

$$T(a|k\rangle+b|l\rangle)=aT|k\rangle+bT|l\rangle. \tag{5·1}$$

Given a base $\{|1\rangle,\cdots,|N\rangle\}$, a linear operator T is uniquely determined by an $N\times N$ matrix $t=(t_{nm})$, where

$$T|m\rangle=\sum_n t_{nm}|n\rangle. \tag{5·2}$$

It is evident that $T=0$ if and only if $t=0$. Let s be the matrix corresponding to another linear operator S. Then

$$\begin{aligned}ST|m\rangle&=\sum_l t_{lm}S|l\rangle=\sum_{l,n}t_{lm}s_{nl}|n\rangle\\&=\sum_n(st)_{nm}|n\rangle.\end{aligned} \tag{5·3}$$

Thus, ST corresponds to the product matrix st, that is, linear operators are faithfully represented by $N\times N$ matrices.

The matrix representation is of course base-dependent. Let $\tilde{t}$ be the matrix representation of T on a different base $\{|\tilde{1}\rangle,\cdots,|\tilde{N}\rangle\}$. Then by using (4·6), it is easy to show that

$$\tilde{t}=u^{-1}tu. \tag{5·4}$$

It should be noted that the transformation law of t is different from that of η, (4·8).

An operator $T^\dagger$ is called a hermitian conjugate of T if

$$\langle k|T^\dagger|l\rangle=\langle l|T|k\rangle^* \tag{5·5}$$

for any $|k\rangle$, $|l\rangle\in\mathcal{V}$. For any linear operator T, $T^\dagger$ always exists and is unique. This can be proved as follows.

Let

$$|k\rangle=\sum_{m=1}^{N}a_m|m\rangle,$$

$$|l\rangle=\sum_{m=1}^{N}b_m|m\rangle. \tag{5·6}$$

Then

$$\langle l|T|k\rangle=\sum_{p,m,n}b_p^* a_m t_{nm}\langle p|n\rangle$$
$$=\sum_{p,m,n}b_p^*\eta_{pn}t_{nm}a_m. \tag{5·7}$$

Likewise,

$$\langle k|S|l\rangle=\sum_{p,n,m}a_m^*\eta_{mn}s_{np}b_p. \tag{5·8}$$

By definition, in order that $S=T^\dagger$, it is necessary and sufficient that

$$\sum_n\eta_{mn}s_{np}=\sum_n t_{nm}^*\eta_{pn}^*, \tag{5·9}$$

that is,

$$\eta s=t^\dagger\eta^\dagger=t^\dagger\eta. \tag{5·10}$$

Since η is non-singular, the matrix $\eta^{-1}t^\dagger\eta$ exists and it defines $T^\dagger$ uniquely.

From (5·5), it is evident that a relation $T|k\rangle=|k'\rangle$ is equivalent to $\langle k|T^\dagger=\langle k'|$.

A linear operator H is called hermitian if $H^\dagger=H$. H is hermitian if and only if ηh is a hermitian matrix, where h is a matrix representation of H. We also define the following operators:

unitary operator U, $U^\dagger=U^{-1}$;

projection operator P, $P^2=P$; *)

nilpotent operator K, $K^n=0$ for some integer n.

We do not use the nomenclatures "pseudo-hermitian" or "self-adjoint" and "pseudo-unitary".

§ 6. Generalized eigenstates

In the physical problems, we suppose the existence of a Hamiltonian H, which is a hermitian operator. In the usual (positive-metric) space, any vector can be expressed as a superposition of the eigenvectors of H, that is, they form a base. In the indefinite-metric space, however, this statement is no longer true even in the finite-dimensional space. Hereafter, a non-zero vector in $\mathcal{V}$ is called a state.

*) P is not necessarily assumed to be hermitian.

Let T be a linear operator. If

$$T|k\rangle = c|k\rangle, \tag{6·1}$$

then a state $|k\rangle$ is called an eigenstate of T and a complex number c is called an eigenvalue of T. Let $\mathcal{W}$ be a subspace of $\mathcal{V}$. If $T|k\rangle \in \mathcal{W}$ for any $|k\rangle \in \mathcal{W}$, then $\mathcal{W}$ is called an invariant subspace of T. The space generated by an eigenstate of T is a one-dimensional invariant subspace of T. As mentioned above, $\mathcal{V}$ cannot in general be decomposed into a direct sum of one-dimensional invariant subspaces of T.

Let t be a matrix representation of T on an arbitrary base, and define a polynomial in x of degree N by

$$f(x) \equiv \det(x\delta - t), \tag{6·2}$$

where δ denotes the unit matrix (δ_{nm}). The polynomial $f(x)$ is *base-independent*, as is easily seen by using (5·4). We call $f(x)$ the *characteristic polynomial* of T.

CAYLEY-HAMILTON THEOREM

$$f(T) = 0. \tag{6·3}$$

Proof: We first rewrite

$$T|m\rangle = \sum_n t_{nm}|n\rangle \tag{6·4}$$

as

$$\sum_n s_{nm}(T)|n\rangle = 0, \tag{6·5}$$

where

$$s_{nm}(x) \equiv x\delta_{nm} - t_{nm}. \tag{6·6}$$

Let $(\tilde{s}_{nm}(x))$ be the adjoint matrix of the matrix $s = (s_{nm}(x))$. Then from (6·5) we have

$$\sum_{m,n} \tilde{s}_{ml}(T) s_{nm}(T)|n\rangle = 0 \quad \text{for} \quad l = 1, 2, \cdots, N, \tag{6·7}$$

that is,

$$\sum_n \Big(\sum_m s_{nm}(T)\tilde{s}_{ml}(T)\Big)|n\rangle = 0. \tag{6·8}$$

Since

$$(s\tilde{s})_{nl} = \det s \cdot \delta_{nl} = f\delta_{nl}, \tag{6·9}$$

(6·8) reduces to

$$f(T)|n\rangle = 0 \quad \text{for} \quad n = 1, \cdots, N. \tag{6·10}$$

Thus we establish (6·3).

The Cayley-Hamilton theorem is nothing but the compatibility condition of N simultaneous homogeneous equations (6·4).

According to the fundamental theorem of algebra, the characteristic polynomial is uniquely decomposed into

$$f(x)=\prod_{j=1}^{r}(x-c_j)^{N_j},\qquad \sum_{j=1}^{r}N_j=N, \tag{6·11}$$

where $c_j \neq c_i$ if $j\neq i$.

The minimal polynomial $g(x)$ (which exists) is defined to be a monic polynomial of the minimum degree such that $g(T)=0$, where "monic" means that the coefficient of the highest-degree term is 1. Because of (6·3), $f(x)$ must be divisible by $g(x)$. Hence, we can write

$$g(x)=\prod_{j=1}^{r}(x-c_j)^{n_j},\quad 0\leq n_j\leq N_j. \tag{6·12}$$

Of course, $g(x)$ does not necessarily coincide with $f(x)$.

PRIMARY DECOMPOSITION THEOREM The vector space $\mathcal{V}$ is decomposed into a direct sum of r invariant subspaces $\mathcal{V}_j$:

$$\mathcal{V}=\mathcal{V}_1\oplus\cdots\oplus\mathcal{V}_r. \tag{6·13}$$

Here $\mathcal{V}_j$ is the null space of $(T-c_j)^{n_j}$, namely, the set of all $|k\rangle$ such that $(T-c_j)^{n_j}|k\rangle=0$. [More precisely, $(T_j-c_j)^{n_j}$ is the minimal polynomial on $\mathcal{V}_j$, where T_j denotes the restriction of T to $\mathcal{V}_j$.]

Proof: We set

$$g_j(x)\equiv g(x)/(x-c_j)^{n_j}=\prod_{i\neq j}(x-c_i)^{n_i}. \tag{6·14}$$

Since $c_1,\cdots,c_r$ are all distinct, $g_1(x),\cdots,g_r(x)$ are relatively prime. Hence, as is well known, there are polynomials $h_1(x),\cdots,h_r(x)$ such that

$$\sum_{j=1}^{r}h_j(x)g_j(x)=1. \tag{6·15}$$

We define

$$P_j\equiv h_j(T)g_j(T). \tag{6·16}$$

Then (6·15) implies that

$$\sum_{j=1}^{r}P_j=1. \tag{6·17}$$

Furthermore,

$$P_iP_j=0\quad\text{for}\quad i\neq j, \tag{6·18}$$

because $g_i(x)g_j(x)$ is divisible by $g(x)$. Hence by multiplying (6·17) by P_j, we find

$$P_j^2=P_j. \tag{6·19}$$

Thus P_j is a projection operator.

We define $\mathcal{V}_j$ by the range of P_j, namely the totality of the states of the form $P_j|k\rangle$. Then from (6·17) it is evident that for any $|k\rangle\in\mathcal{V}$ we have

$$|k\rangle=\sum_{j=1}^{r}P_j|k\rangle, \quad P_j|k\rangle\in\mathcal{V}_j. \tag{6·20}$$

(1) The decomposition (6·20) is unique. In fact, if

$$|k\rangle=\sum_{j=1}^{r}|j\rangle, \quad |j\rangle\in\mathcal{V}_j, \tag{6·21}$$

then

$$\sum_j|j\rangle=\sum_j P_j|k\rangle. \tag{6·22}$$

Multiplying (6·22) by P_j, we have

$$P_j|j\rangle=P_j|k\rangle. \tag{6·23}$$

Since

$$P_j|j\rangle=\sum_{i=1}^{r}P_i|j\rangle=|j\rangle, \tag{6·24}$$

we find

$$|j\rangle=P_j|k\rangle. \tag{6·25}$$

(2) $\mathcal{V}_j$ is an invariant subspace of $\mathcal{V}$. This is because

$$TP_j=P_jT. \tag{6·26}$$

(3) $(T-c_j)^{n_j}|k\rangle=0$ for any $|k\rangle\in\mathcal{V}_j$. This follows from the fact that $(x-c_j)^{n_j}g_j(x)$ is equal to $g(x)$.

(4) $\mathcal{V}_j$ is the null space of $(T-c_j)^{n_j}$. This is shown as follows. Suppose that

$$(T-c_j)^{n_j}|k\rangle=0. \tag{6·27}$$

Since $P_i(i\neq j)$ contains a factor $(T-c_j)^{n_j}$, we have

$$P_i|k\rangle=0 \quad \text{for} \quad i\neq j. \tag{6·28}$$

Hence

$$|k\rangle=(1-\sum_{i\neq j}P_i)|k\rangle=P_j|k\rangle\in\mathcal{V}_j. \tag{6·29}$$

(5) $(T-c_j)^{n_j-1}|k\rangle\neq0$ for some $|k\rangle\in\mathcal{V}_j$. Otherwise, any state $|k\rangle=\sum_j P_j|k\rangle$ in $\mathcal{V}$ would go to zero if $g(T)/(T-c_j)$ acts on it.

Thus the theorem is established.

If we have

$$(T-c)^n|k\rangle=0, \tag{6·30}$$

but

$$(T-c)^{n-1}|k\rangle \neq 0 \tag{6·31}$$

for some positive integer n, $|k\rangle$ and c are called a generalized eigenstate (of order n) and a generalized eigenvalue, respectively. The primary decomposition theorem implies that *any state in a finite-dimensional vector space is always expressed as a superposition of generalized eigenstates of any given linear operator.*

Finally, we note that

$$N_j = \dim \mathcal{V}_j. \qquad (j=1, \cdots, r) \tag{6·32}$$

This relation can be proved by substituting the Jordan form of T in (6·2). [For the Jordan form, see the next section.]

§ 7. Standard matrix representation

Throughout this section, we consider a hermitian operator H (Hamiltonian) instead of a general linear operator T. The characteristic polynomial $f(x)$ of H is of real coefficients, because

$$\det(x\delta - h) = (\det \eta)^{-1} \det(x\eta - \eta h) \tag{7·1}$$

and η and ηh are hermitian. Hence the non-real generalized eigenvalues of H, if any, must appear in complex-conjugate pairs.

According to the primary decomposition theorem, a base is formed by generalized eigenstates of H, but then the indefinite-metric matrix η, in general, takes a complicated form. To simplify the expression for η can be achieved by transforming the matrix representation of the linear operator into its Jordan form. As stated in the mathematical literature,[H9)] the Jordan form is a standard matrix representation of an arbitrary linear operator, and it is a concept which is *independent* of the inner product. In our case, the transformation into the Jordan form is carried out in a simpler way by making use of the inner product. We follow the line of thought of Belinfante and Winternitz.[B1)]

Let E_j be a generalized eigenvalue of H and $\mathcal{V}(E_j)$ be the totality of the generalized eigenstates belonging to E_j. Then the primary decomposition theorem implies that

$$\mathcal{V} = \sum_j \oplus \mathcal{V}(E_j). \tag{7·2}$$

The following theorem holds even for an infinite-dimensional indefinite-metric space.

THEOREM For any $|k\rangle \in \mathcal{V}(E_j)$ and any $|l\rangle \in \mathcal{V}(E_i)$, we have an orthogonality relation

$$\langle l|k\rangle=0 \tag{7·3}$$

if $E_i^* \neq E_j$.

Proof: By definition, there exist some integers n_j and n_i such that

$$(H-E_j)^{n_j}|k\rangle=0,$$
$$\langle l|(H-E_i^*)^{n_i}=0. \tag{7·4}$$

Since $E_i^* \neq E_j$, two polynomials $(x-E_j)^{n_j}$ and $(x-E_i^*)^{n_i}$ are relatively prime; hence there exist two polynomials $\varphi_1(x)$ and $\varphi_2(x)$ such that

$$\varphi_1(x)(x-E_j)^{n_j}+\varphi_2(x)(x-E_i^*)^{n_i}=1. \tag{7·5}$$

Therefore

$$\langle l|k\rangle=\langle l|[\varphi_1(H)(H-E_j)^{n_j}+(H-E_i^*)^{n_i}\ \varphi_2(H)]|k\rangle$$
$$=0 \tag{7·6}$$

with the aid of (7·4). This completes the proof.

The above theorem implies that if E_j is non-real then

(1) all states of $\mathcal{V}(E_j)$ have zero norm;

(2) $\mathcal{V}(E_j^*)$ is non-zero if and only if $\mathcal{V}(E_j)$ is non-zero.

The latter proposition follows from the assumption that $\mathcal{V}$ is non-degenerate.

According to the above theorem, the $\mathcal{V}(E_i)\times\mathcal{V}(E_j)$ block of the matrix η vanishes if $E_i^* \neq E_j$. Hence the problem of simplifying the expression for η reduces to that in a subspace $\mathcal{V}(E_j)$ for E_j real and in a direct sum, $\mathcal{V}(E_j)\oplus\mathcal{V}(E_j^*)$, of two subspaces for E_j non-real. For simplicity, we omit the subscript j of E_j and n_j for a moment.

First, we discuss the case of E real. Let K be the restriction of $H-E$ to $\mathcal{V}(E)$, that is, $K=H-E$ in $\mathcal{V}(E)$ and K is undefined outside $\mathcal{V}(E)$. Of course, we have $K^\dagger=K$ and

$$K^n=0, \qquad K^{n-1}\neq 0. \quad (n\geq 1) \tag{7·7}$$

[Of course, $K^0\equiv 1$.] Thus K is a nilpotent hermitian operator. There exists a state $|k\rangle\in\mathcal{V}(E)$ such that

$$\langle k|K^{n-1}|k\rangle\neq 0, \tag{7·8}$$

because if $\langle k|K^{n-1}|k\rangle=0$ for any $|k\rangle\in\mathcal{V}(E)$, then considering two cases $|k\rangle=|l\rangle+|m\rangle$ and $|k\rangle=|l\rangle+i|m\rangle$ we immediately see $\langle m|K^{n-1}|l\rangle=0$ for any $|l\rangle$, $|m\rangle\in\mathcal{V}(E)$.

Let $\mathcal{W}_1$ be the n-dimensional subspace spanned by n states $|k\rangle, K|k\rangle, \cdots, K^{n-1}|k\rangle$, whose linear independence can easily be verified by multiplying K^m $(m=n-1, n-2, \cdots, 1)$ and using (7·7). Evidently, $\mathcal{W}_1$ is a cyclic invariant subspace of $\mathcal{V}(E)$, where "cyclic" means that this space can be generated by an operator K from a single state. Let $\mathcal{W}_1^\perp$ be the totality of the states $|l\rangle$ such that

$$\langle k|K^j|l\rangle=0 \quad \text{for } j=0, 1, \cdots, n-1. \tag{7·9}$$

Then we have the following results.

(1) $\mathcal{W}_1^\perp$ is an invariant subspace of $\mathcal{V}(E)$. This is because from (7·9) and (7·7) it follows that

$$\langle k|K^j\cdot K|l\rangle=0 \quad \text{for } j=0, 1, \cdots, n-1 \tag{7·10}$$

for any $|l\rangle\in\mathcal{W}_1^\perp$.

(2) $\mathcal{V}(E)$ is decomposed into a direct sum of $\mathcal{W}_1$ and $\mathcal{W}_1^\perp$:

$$\mathcal{V}(E)=\mathcal{W}_1\oplus\mathcal{W}_1^\perp. \tag{7·11}$$

This can be shown as follows. For any $|m\rangle\in\mathcal{V}(E)$, we write

$$|m\rangle=\sum_{j=0}^{n-1}c_jK^j|k\rangle+|l\rangle. \tag{7·12}$$

In order that $|l\rangle\in\mathcal{W}_1^\perp$, it is necessary and sufficient that

$$\langle k|K^p|m\rangle-\sum_{j=0}^{n-1}c_j\langle k|K^{p+j}|k\rangle=0$$
$$\text{for } p=0, 1, \cdots n-1. \tag{7·13}$$

Because of (7·8), c_0, c_1, $\cdots$, c_{n-1} are uniquely determined by (7·13) by setting $p=n-1, n-2, \cdots, 0$, successively. Thus the decomposition (7·12) with $|l\rangle\in\mathcal{W}_1^\perp$ exists and is unique.

Since the invariant subspace $\mathcal{W}_1$ has a similar structure as that of $\mathcal{V}(E)$, we can repeat the above procedure in $\mathcal{W}_1^\perp$ instead of $\mathcal{V}(E)$. Since the dimension of $\mathcal{V}(E)$ is finite, we can proceed by mathematical induction to obtain finally that

$$\mathcal{V}(E)=\sum_q\oplus\mathcal{W}_q, \tag{7·14}$$

where $\mathcal{W}_q$ is an n_q-dimensional, cyclic invariant subspace ($1\leq n_q\leq n$). The submatrix of η in $\mathcal{W}_q$ is a triangle matrix in which all elements below the non-principal diagonal line are zero. By a linear transformation of the base, it obviously reduces to a matrix whose non-zero elements are only on the non-principal diagonal line. More precisely, we can find a state $|k_q\rangle$ such that

$$\langle k_q|K^p|k_q\rangle=\sigma_q\delta_{p,n_q-1} \tag{7·15}$$

with $\sigma_q=\pm1$ (independently of p). Then, in $\mathcal{W}_q$, we choose a base $\{|k_q\rangle, K|k_q\rangle, \cdots, K^{n_q-1}|k_q\rangle\}$. Thus the submatrix of η in $\mathcal{V}(E)$ is a direct sum of the matrices which have the form

$$\begin{pmatrix} 0 & & +1 \\ & \cdot^{\cdot^{\cdot}} +1 & \\ +1 & & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & & -1 \\ & \cdot^{\cdot^{\cdot}} -1 & \\ -1 & & 0 \end{pmatrix}. \tag{7·16}$$

The matrix representation of K in $\mathcal{W}_q$ is

$$\begin{pmatrix} 0 & & & & 0 \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ 0 & & & 1 & 0 \end{pmatrix}, \tag{7·17}$$

because $K(K^p|k_q\rangle)=K^{p+1}|k_q\rangle$ $(p=0, 1, \cdots, n_q-1)$. Hence the matrix representation of $H=K+E$ in $\mathcal{V}(E)$ is a direct sum of the matrices of the form

$$\begin{pmatrix} E & & & & 0 \\ 1 & E & & & \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ 0 & & & 1 & E \end{pmatrix}. \tag{7·18}$$

Such a matrix representation is called the Jordan form of H.

Next, we consider the case of E non-real. As before, let K be the restriction of $H-E$ to $\mathcal{V}(E)$. Then we again have (7·7) but $K^\dagger \neq K$. Let $|k\rangle$ be a state such that

$$K^{n-1}|k\rangle \neq 0. \tag{7·19}$$

Then, according to the assumption that $\mathcal{V}$ is non-degenerate, there exists a state $|l\rangle \in \mathcal{V}(E^*)$ such that

$$\langle l|K^{n-1}|k\rangle \neq 0; \tag{7·20}$$

in particular,

$$(K^\dagger)^{n-1}|l\rangle \neq 0. \tag{7·21}$$

Let $\mathcal{U}_1$ be the space spanned by $|k\rangle, K|k\rangle, \cdots, K^{n-1}|k\rangle$, and $\mathcal{U}_1^*$ be the one spanned by $|l\rangle$, $K^\dagger|l\rangle, \cdots, (K^\dagger)^{n-1}|l\rangle$. Of course, $\mathcal{U}_1$ and $\mathcal{U}_1^*$ are n-dimensional, cyclic invariant subspaces of $\mathcal{V}(E)$ and $\mathcal{V}(E^*)$, respectively. As before, we define the orthogonal spaces $(\mathcal{U}_1^*)^\perp$ and $\mathcal{U}_1^\perp$, which are invariant subspaces of $\mathcal{V}(E)$ and $\mathcal{V}(E^*)$, respectively. Then

$$\mathcal{V}(E)=\mathcal{U}_1 \oplus (\mathcal{U}_1^*)^\perp,$$
$$\mathcal{V}(E^*)=\mathcal{U}_1^* \oplus \mathcal{U}_1^\perp. \tag{7·22}$$

Since $(\mathcal{U}_1^*)^\perp$ and $\mathcal{U}_1^\perp$ have similar structures as those of $\mathcal{V}(E)$ and $\mathcal{V}(E^*)$, respectively, we can repeat the above consideration. By mathematical induction, we obtain that

$$\mathcal{V}(E)=\sum_{q=1}^{s}\oplus\mathcal{U}_q,$$

$$\mathcal{V}(E^*)=\sum_{q=1}^{s}\oplus\mathcal{U}_q^* \tag{7·23}$$

with

$$1\leq\dim\mathcal{U}_q=\dim\mathcal{U}_q^*\equiv n_q\leq n. \tag{7·24}$$

In $\mathcal{U}_q\oplus\mathcal{U}_q^*$, we can find states $|k_q\rangle\in\mathcal{U}_q$ and $|l_q\rangle\in\mathcal{U}_q^*$ such that

$$\langle l_q|K^p|k_q\rangle=\delta_{p,n_q-1}. \tag{7·25}$$

Here a sign factor σ_q is unnecessary because $\mathcal{U}_q^*$ is a space different from $\mathcal{U}_q$. The submatrix of η in $\mathcal{V}(E)\oplus\mathcal{V}(E^*)$ is a direct sum of matrices

$$\begin{pmatrix}0 & \xi_q\\ \xi_q & 0\end{pmatrix}, \tag{7·26}$$

where ξ_q is an $n_q\times n_q$ matrix

$$\begin{pmatrix}0 & & 1\\ & \cdot^{\cdot^{1}} & \\ 1^{\cdot} & & 0\end{pmatrix}. \tag{7·27}$$

The matrix representation of H in $\mathcal{V}(E)\oplus\mathcal{V}(E^*)$ is a direct sum of matrices

$$\begin{pmatrix}\zeta_q(E) & 0\\ 0 & \zeta_q(E^*)\end{pmatrix}, \tag{7·28}$$

where $\zeta_q(E)$ is an $n_q\times n_q$ matrix having the form (7·18) (E is now complex). The product of (7·26) and (7·28) is hermitian as it should be. We note that the representation of H in $\mathcal{V}(E)\oplus\mathcal{V}(E^*)$ is also the Jordan form.

Summarizing the above results, we obtain the following theorem.

THEOREM For E_j real, $\mathcal{V}(E_j)$ is a direct sum of mutually orthogonal, cyclic invariant subspaces $\mathcal{W}_q$, in each of which we can choose a base such that η is of the form (7·16) and h (the matrix representation of H) is of the Jordan form (7·18). For E_j non-real, $\mathcal{V}(E_j)\oplus\mathcal{V}(E_j^*)$ is a direct sum of mutually orthogonal, invariant subspaces $\mathcal{U}_q\oplus\mathcal{U}_q^*$, where $\mathcal{U}_q\subset\mathcal{V}(E_j)$ and $\mathcal{U}_q^*\subset\mathcal{V}(E_j^*)$ with $\dim\mathcal{U}_q=\dim\mathcal{U}_q^*$. In $\mathcal{U}_q\oplus\mathcal{U}_q^*$, we can choose a base such that η and h are given by (7·26) and (7·28), respectively.*)

*) Since $\eta^{-1}=\eta$, the (orthogonal) projection operator to $\mathcal{W}_q$ (or $\mathcal{U}_q\oplus\mathcal{U}_q^*$) is given by $\sum_{p,p'}|p\rangle\eta_{pp'}\langle p'|$, where $\{|p\rangle\}$ denotes its base adopted above.

Combining this theorem with the primary decomposition theorem, we obtain the complete forms of η and h. Thus, as far as a *finite-dimensional* indefinite-metric space is concerned, its structure is known quite satisfactorily. The same applies also to a space which is a direct sum of an finite-dimensional indefinite-metric space and a Hilbert space, if both are invariant with respect to H.

Chapter 3
General Aspects of Indefinite-Metric Quantum Field Theory

§ 8. Topology

We consider an extension of the theory of a finite-dimensional indefinite-metric space formulated in Chapter 2 to the infinite-dimensional case. The axioms 1°∼10° stated in § 4 are unchanged except for 6°, in which N should now be regarded as infinite. This modification is, however, very essential. We should remember that almost all important results obtained in Chapter 2 are based on the finite dimensionality of the space. For example, in the infinite-dimensional case, in order to define a base, we have to consider a linear combination of an infinite number of states. To define it, it is necessary to introduce a concept of the limit. More precisely, the notion that two states are sufficiently *near* must be defined unambiguously. A space having such a relation is called a topological space. Thus, for the mathematical treatment of the indefinite-metric Hilbert space, we have to introduce topology into it. For the finite-dimensional space, its topology is *a priori* unique. For the infinite-dimensional space, however, there are many possibilities of introducing topology into it, and unfortunately it is extremely difficult to find the topology appropriate for indefinite-metric Hilbert space.

We first review the axioms of the Hilbert space $\mathcal{H}$. For $\mathcal{H}$, in addition to 1°∼5° and 7°∼9°, we have two properties (4·1) and (4·2). Hence the quantity

$$\| |k\rangle \| \equiv \langle k|k\rangle^{1/2} \tag{8·1}$$

defines a mathematical norm. Since to define $|n\rangle \to |k\rangle$ as $n\to\infty$ is equivalent to define $|n\rangle - |k\rangle \to 0$ as $n\to\infty$, it is sufficient to define $|n\rangle\to 0$. Because of (4·2), it is natural to define $|n\rangle \to 0$ by

$$\| |n\rangle \| \to 0. \tag{8·2}$$

It is also possible to define $|n\rangle\to 0$ without using norm, that is, we may say that $|n\rangle \to 0$ if

$$\langle l|n\rangle \to 0 \text{ for any } |l\rangle \in \mathcal{H} \tag{8·3}$$

and, in the ordinary case, if

$$\langle n|n\rangle \to 0. \tag{8·4}$$

This topology is called weak topology, and correspondingly the topology in the sense of (8·2) is called strong topology. Evidently, if $|n\rangle\to 0$ in strong topology, $|n\rangle \to 0$ in weak topology, but the converse is not true. In quantum field theory, weak topology is used only in the asymptotic condition.*)

In the Hilbert space, an additional axiom is completeness. In $\mathcal{H}$, if for any $\varepsilon > 0$ there exists a positive number N such that

$$\|(|n\rangle - |m\rangle)\| < \varepsilon \text{ for any } n, m > N, \tag{8·5}$$

then there exists $|k\rangle \in \mathcal{H}$ such that $|n\rangle \to |k\rangle$ as $n \to \infty$. [Furthermore, it is usually assumed that $\mathcal{H}$ is separable, that is, there is a countable subset $\mathcal{D}$ *dense* in $\mathcal{H}$ (i. e., the closure of $\mathcal{D}$ is $\mathcal{H}$).]

Now, we consider the problem of how to introduce the topology appropriate for the indefinite-metric Hilbert space $\mathcal{V}$.

(A) *Weak topology*

Since it is impossible to define a mathematical norm in $\mathcal{V}$ by (8·1), at first sight it looks natural to use weak topology in $\mathcal{V}$. Since by assumption, $\mathcal{V}$ is non-degenerate, (8·3) might look a satisfactory condition for defining $|n\rangle \to 0$. This is not the case, however, even in a Hilbert space $\mathcal{H}$. Suppose that $\{|n\rangle\}$ forms a complete orthonormal set in $\mathcal{H}$. Then any state $|l\rangle$ can be expanded into

$$|l\rangle = \sum_{n=0}^{\infty} a_n |n\rangle \tag{8·6}$$

with

$$\sum_{n=0}^{\infty} |a_n|^2 < \infty. \tag{8·7}$$

Then we have

$$\langle l|n\rangle = a_n^* \to 0, \tag{8·8}$$

but $\langle n|n\rangle = 1$ always. The unnatural statement $|n\rangle \to 0$ is excluded by (8·4).

The condition (8·4) becomes powerless in the indefinite-metric Hilbert space $\mathcal{V}$, because there are zero-norm states. Indeed, let $\{|n\rangle; n=0, 1, \cdots\}$ be an infinite sequence of linearly independent states in $\mathcal{V}$ such that

$$\langle n|m\rangle = (-1)^n \delta_{nm}. \tag{8·9}$$

We consider a sequence defined by

$$|\tilde{n}\rangle \equiv |2n\rangle + |2n+1\rangle \quad (n=0, 1, \cdots). \tag{8·10}$$

Then of course

$$\langle \tilde{n}|\tilde{n}\rangle = 0. \tag{8·11}$$

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*) In the asymptotic condition, (8·4) is excluded in order to remove oscillatory terms.

For any state $|l\rangle \in \mathcal{V}$ which can be expanded as (8·6) with (8·7), we have

$$\langle l|\tilde{n}\rangle = a_{2n}^* - a_{2n+1}^* \to 0. \qquad (8\cdot 12)$$

Thus in the sense of weak topology, we have

$$|\tilde{n}\rangle \to 0 \text{ as } n\to\infty, \qquad (8\cdot 13)$$

provided that $|l\rangle$ can be an arbitrary state in $\mathcal{V}$. If we do not require (8·7), then the above difficulty disappears, but in this case $\langle l|l\rangle$ becomes ambiguous, that is, it depends on the order of the summation in (8·6). Since the existence of a base (i.e., a complete set) is necessary for our purpose, it is hopeless to construct an appropriate indefinite-metric Hilbert space with weak topology.

(B) *η-formalism*

As mentioned in § 1, the η-formalism (unitary trick) is the old-fashioned way of treating the indefinite-metric Hilbert space. We consider a Hilbert space $\mathcal{H}$, which is in one-to-one correspondence with $\mathcal{V}$ in such a way that the linearity is preserved in this correspondence. Let $|\boldsymbol{k}\rangle$ and $|\boldsymbol{l}\rangle$ in $\mathcal{H}$ be the states corresponding to $|k\rangle$ and $|l\rangle$ in $\mathcal{V}$, respectively. The inner product in $\mathcal{V}$ is then expressed by

$$\langle k|l\rangle = \langle \boldsymbol{k}|\boldsymbol{\eta}|\boldsymbol{l}\rangle, \qquad (8\cdot 14)$$

where $\boldsymbol{\eta}$ is an *operator* in $\mathcal{H}$, which must be hermitian because (8·14) should equal

$$\langle l|k\rangle^* = \langle \boldsymbol{l}|\boldsymbol{\eta}|\boldsymbol{k}\rangle^* = \langle \boldsymbol{k}|\boldsymbol{\eta}^\dagger|\boldsymbol{l}\rangle. \qquad (8\cdot 15)$$

It is usual to suppose that there is a complete set of states $|\boldsymbol{n}\rangle$ in $\mathcal{H}$ such that for their corresponding states $|n\rangle$ in $\mathcal{V}$ we have

$$\langle n|m\rangle = \sigma_n \delta_{nm}. \quad (\sigma_n = \pm 1) \qquad (8\cdot 16)$$

The topology of $\mathcal{V}$ is defined by the strong topology of $\mathcal{H}$. An operator T on $\mathcal{V}$ corresponds to $\boldsymbol{T}$ on $\mathcal{H}$ in such a way that $T|l\rangle$ corresponds to $\boldsymbol{T}|\boldsymbol{l}\rangle$. Hence

$$\langle k|T|l\rangle = \langle \boldsymbol{k}|\boldsymbol{\eta}\boldsymbol{T}|\boldsymbol{l}\rangle. \qquad (8\cdot 17)$$

It is straightforward to show that $T^\dagger$ corresponds to $\boldsymbol{\eta}^{-1}\boldsymbol{T}^\dagger\boldsymbol{\eta}$.

It should be noted that the η-formalism is essentially based on a *particular* pseudo-orthonormal set $\{|n\rangle\}$ in $\mathcal{V}$, that is, we always have to describe the theory in terms of a particular base. This fact is very inconvenient if we wish to represent manifest-covariance properties of the theory in $\mathcal{V}$. Indeed, the use of the η-formalism in quantum electrodynamics is misleading,[S11] because the existence of η violates the manifest Lorentz covariance in the commutation relation between η and the electromagnetic field. Furthermore, the η-formalism is inconvenient for describing multipole ghosts.

(C) *Nevanlenna space**)

Instead of preferring a particular base in $\mathcal{V}$, we can construct a mathematically well-defined indefinite-metric Hilbert space $\mathcal{V}$, which was proposed by Nevanlenna, in the following way. A Nevanlenna space $\mathcal{V}$ is a direct sum of two mutually orthogonal subspaces $\mathcal{V}_+$ and $\mathcal{V}_-$, where $\mathcal{V}_+$ is a Hilbert space, and $\mathcal{V}_-$ is a modified Hilbert space such that $\langle k_-|k_-\rangle<0$ instead of >0 for any non-zero $|k_-\rangle\in\mathcal{V}_-$. For any $|k\rangle$, $|l\rangle\in\mathcal{V}$, we can uniquely write $|k\rangle=|k_+\rangle+|k_-\rangle$ and $|l\rangle=|l_+\rangle+|l_-\rangle$ with $|k_+\rangle$, $|l_+\rangle\in\mathcal{V}_+$ and $|k_-\rangle$, $|l_-\rangle\in\mathcal{V}_-$. The inner product is then given by

$$\langle k|l\rangle=\langle k_+|l_+\rangle+\langle k_-|l_-\rangle, \tag{8·18}$$

while a mathematical norm is defined by

$$\||k\rangle\|^2\equiv\langle k_+|k_+\rangle-\langle k_-|k_-\rangle\geq 0. \tag{8·19}$$

The decomposition of $\mathcal{V}$ into two subspaces is of course not unique. According to Ginzburg and Yokhvidov,**) however, the topology introduced above is independent of the decomposition.

The defect of this formalism is that the decomposition of $\mathcal{V}$ into $\mathcal{V}_+$ and $\mathcal{V}_-$ is, in general, not invariant under the transformations which leave the theory invariant.

(D) *Banach-space method*[N15]

Since the inner products between unequal states are unnecessary for defining the mathematical norm, in order to introduce strong topology into $\mathcal{V}$, it is more economical to consider a Banach space $\mathcal{B}$ than to do a Hilbert space $\mathcal{H}$, where a Banach space is a complete, normed vector space. For $\psi\in\mathcal{B}$, its norm is denoted by $\|\psi\|$, which has the following properties:

$$\begin{aligned}&\|a\psi\|=|a|\cdot\|\psi\|,\\&\|\psi+\phi\|\leq\|\psi\|+\|\phi\|,\\&\|\psi\|\geq 0,\\&\|\psi\|=0 \text{ if and only if } \psi=0.\end{aligned} \tag{8·20}$$

We suppose that there is a one-to-one correspondence between $\mathcal{V}$ and $\mathcal{B}$

*) For references, see Nagy's book.[N4]

**) The present author could not see the original paper.

such that linearity is preserved. The topology of $\mathcal{V}$ is introduced by that of $\mathcal{B}$, and therefore it is independent of the inner product in $\mathcal{V}$.

We consider linear functionals over $\mathcal{B}$; $\bar{\varphi}$ is called a linear functional if $\bar{\varphi}(\psi)$ for any $\psi\in\mathcal{B}$ is a complex number and if

$$\bar{\varphi}(a\psi_1+b\psi_2)=a\bar{\varphi}(\psi_1)+b\bar{\varphi}(\psi_2). \tag{8·21}$$

We can define a linear combination of linear functionals by

$$(a\bar{\varphi}_1+b\bar{\varphi}_2)(\psi)\equiv a^*\bar{\varphi}_1(\psi)+b^*\bar{\varphi}_2(\psi). \tag{8·22}$$

Furthermore, we define the mathematical norm of $\bar{\varphi}$ by

$$\|\bar{\varphi}\|\equiv\sup_{\|\psi\|\leq 1}|\bar{\varphi}(\psi)|, \tag{8·23}$$

where "sup" means supremum. Then the totality of the linear functionals over $\mathcal{B}$ spans a Banach space $\bar{\mathcal{B}}$, which is called a dual space of $\mathcal{B}$. The dual space, $\bar{\bar{\mathcal{B}}}$, of $\bar{\mathcal{B}}$ includes $\mathcal{B}$. In particular, if $\bar{\bar{\mathcal{B}}}=\mathcal{B}$, then $\mathcal{B}$ is called reflexive. We assume that $\mathcal{B}$ is reflexive.

It is convenient to rewrite $\bar{\varphi}(\psi)$ as $\bar{\varphi}\psi$. Given $\bar{\varphi}$ and an operator T, if there exists $\bar{\varphi}'$ such that $\bar{\varphi}(T\psi)=\bar{\varphi}'\psi$ for any $\psi\in\mathcal{B}$, then we may write $\bar{\varphi}'=\bar{\varphi}T$, and $\bar{\varphi}(T\psi)$ may be written as $\bar{\varphi}T\psi$ without confusion.

As is evident from the above construction, $\bar{\mathcal{B}}$ is essentially the space corresponding to bra-vectors. In this sense, we can represent the inner product in $\mathcal{V}$ by a sesquilinear form $\bar{\varphi}\psi$. Given $\psi\in\mathcal{B}$, we may define $\bar{\psi}\in\bar{\mathcal{B}}$ if both ψ and $\bar{\psi}$ correspond to the same state in $\mathcal{V}$. Of course, $\|\psi\|$ is independent of $\bar{\psi}\psi$.

The Banach-space method seems to be the most natural way for introducing topology into the indefinite-metric Hilbert space.

In subsequent sections, however, we do not take care of mathematical problems such as what topology is used, in what domain an operator considered is defined, etc.

§ 9. Problem of interpretation

The introduction of indefinite metric into quantum field theory causes a serious trouble in the physical interpretation of the state. We first review the fundamental postulates of the quantum theory in the ordinary framework.

(1) A state is represented by a normalized vector in a Hilbert space $\mathcal{H}$.

(2) An observable is represented by a hermitian operator on $\mathcal{H}$.

(3) The states $|k\rangle$ in which the measured value of an observable A is always λ are defined by an eigenvalue equation

$$A|k\rangle=\lambda|k\rangle. \tag{9·1}$$

(4) An arbitrary state in $\mathcal{H}$ can be represented as a linear combination of the simultaneous eigenstates, $|n\rangle$, of (a maximal set of) mutually commuting observables, where

$$\langle n|m\rangle=\delta_{nm}. \tag{9·2}$$

(5) In an arbitrary state $|k\rangle$, the probability of finding that the measured values of mutually commuting observables are all equal to the respective eigenvalues of $|n\rangle$ is given by

$$w_n=|\langle n|k\rangle|^2. \tag{9·3}$$

From Postulate (4), for any $|k\rangle$ we have

$$|k\rangle=\sum_n a_n|n\rangle. \tag{9·4}$$

Hence

$$\langle n|k\rangle=\sum_m a_m\langle n|m\rangle=a_n \tag{9·5}$$

with the aid of (9·2). Thus (9·4) becomes

$$|k\rangle=\sum_n|n\rangle\langle n|k\rangle. \tag{9·6}$$

Therefore we obtain the completeness condition

$$\sum_n|n\rangle\langle n|=1. \tag{9·7}$$

The total probability of finding that the measured values of mutually commuting observables are some c-numbers is given by

$$\begin{aligned}\sum_n w_n&=\sum_n|\langle n|k\rangle|^2=\sum_n\langle k|n\rangle\langle n|k\rangle\\&=\langle k|k\rangle\end{aligned} \tag{9·8}$$

because of Postulate (5) together with (9·7). Since a state is a normalized vector, (9·8) implies that

$$\sum_n w_n=1, \tag{9·9}$$

that is, the total probability is unity, as it should be.

There is also a postulate which prescribes the dynamics of a system.

(6) The time development of a system is described by

$$i(\partial/\partial t)|t\rangle=H|t\rangle, \tag{9·10}$$

provided that no observation is made. Here, $|t\rangle$ is the state of the system at time t, and H denotes the Hamiltonian of the system, which is a hermitian operator.

Because of (9·10), if $|t\rangle$ is normalized at a particular time, it is so at any time. Thus the total probability is always unity, that is, probability is conserved. This is the basis of the probabilistic interpretation of the state in quantum theory.

Now, we return to the indefinite-metric theory. In this case, (9·2) no longer holds. We have at best

$$\langle n|m\rangle=\sigma_n\delta_{nm}, \quad (\sigma_n=\pm 1) \tag{9·11}$$

so that

$$\sum_n |n\rangle\sigma_n\langle n|=1 \tag{9·12}$$

instead of (9·7). Hence (9·11) and Postulate (5) contradict the requirement that the total probability shall be unity. Therefore, Postulate (5) must be modified. There may be the following possibilities:

(a) We abolish the probabilistic interpretation. Unfortunately, we have no good idea of a substitute for it.

(b) We define w_n by

$$w_n=\frac{|\langle n|k\rangle|^2}{\sum_m |\langle m|k\rangle|^2} \tag{9·13}$$

instead of (9·3). Then of course the total probability is unity. But this definition is evidently quite *ad hoc*. Indeed, (9·13) is, in general, not invariant under the transformations which leave the theory invariant. The most crucial defect of (9·13) is that w_n depends not only on $|k\rangle$ and $|n\rangle$ but also on the choice of other eigenstates of a complete set.

(c) We define w_n by

$$w_n=\langle k|n\rangle\sigma_n\langle n|k\rangle \tag{9·14}$$

instead of (9·13). Then it is evident that (9·9) holds if $\langle k|k\rangle=1$.

Possibility (c) is the traditional way of the interpretation of the state in the indefinite-metric theory. Hereafter, we always adopt this definition of w_n. Since from (9·14) we see that w_n is real but it is not necessarily non-negative, we encounter the problem of negative probability. This is the origin of the name "ghost".

Since we know no adequate interpretation of negative probability, the usual way of avoiding this difficulty is to introduce a constraint (*or* a subsidiary condition). We postulate that only physically admissible states are those which satisfy the constraint; they are called "physical states"*) and span a subspace of $\mathcal{V}$. If all physical states have positive or zero norm, any probability concerning physical states is non-negative, and the total probability is unity as far as a normalized physical state is concerned. Then

*) A physical state used here does not mean a clothed (dressed) state.

we can retain Postulates (2)~(5) (however, we admit $\langle n|n\rangle=0$ in (9·2) for some $|n\rangle$) in the indefinite-metric theory if $\mathcal{H}$ is identified with the physical-state subspace of $\mathcal{V}$. Postulate (1) should be replaced by the postulate concerning physical states mentioned above. We note that $\mathcal{H}$ is, in general, degenerate. Therefore $\mathcal{V}$ is not generally decomposed into a direct sum of $\mathcal{H}$ and its *orthogonal* complement; if it does, the indefinite-metric theory considered is essentially equivalent to a local field theory having positive metric. It is supposed, of course, that any observable is extendable to an operator on $\mathcal{V}$, but it may not necessarily be hermitian in the whole space $\mathcal{V}$. We also note that the expectation value $\langle k|A|k\rangle$ of an operator A in a state $|k\rangle$ is physically meaningful only if $|k\rangle\in\mathcal{H}$.

Postulate (6) is left unchanged if $|t\rangle$ can be consistently restricted to $\mathcal{H}$, that is, if the constraint persists at all time. This condition is, however, very stringent. *It is unlikely that we can utilize indefinite metric to remove ultraviolet-divergence difficulty in such a way that the constraint is satisfied at all time, because no negative-norm states then can effectively contribute to physical processes.* In general, therefore, we should weaken the constraint in Postulate (6). This point will be discussed in the next section.

In quantum mechanics, we have several theories of observation. Though they are not quite satisfactory, we can meaningfully discuss the problem of observation at any *finite* time. In quantum field theory, however, the situation is much worse. It seems impossible to construct a theory of observation at finite time. This is because in quantum field theory there exist vacuum polarization and self-energy effects, which cannot be well defined unless we take into account all contributions between $t=-\infty$ and $t=+\infty$. In a finite time interval, we cannot decide whether a particle A emitted from a particle B is to be absorbed later by B or not. In quantum field theory, therefore, it is natural to postulate that observation can be made only in the asymptotic states ($t=\pm\infty$), in which all particles are mutually at very large distances. Then a question arises: If only the asymptotic states are of physical interest, then there would be no need for considering the Hamiltonian and field operators at finite time. This extreme standpoint is called the S-matrix theory, in which the S-matrix only is regarded as a meaningful quantity. We do not, however, adopt the S-matrix theory because of the following reasons:

(1) It is quite difficult to derive quantum mechanics from the S-matrix theory.

(2) The S-matrix theory cannot explain the reason for the great success of quantum electrodynamics. Furthermore, the classical Maxwell equations are completely foreign to the S-matrix theory.

(3) The treatment of the bound-state problem in the S-matrix theory is unsatisfactory.

Therefore, we adhere to quantum field theory throughout. Though we cannot observe a finite-time transition, we take the standpoint that it is meaningful to discuss a state at finite time. This standpoint is very crucial when we consider the indefinite-metric quantum field theory.

There are two ways of formulating the asymptotic states. The traditional method is the adiabatic hypothesis. The total Hamiltonian is supposed to approach adiabatically to the free Hamiltonian H_0 as $t\to\pm\infty$. More precisely, we replace the interaction Hamiltonian $H_1 \equiv H - H_0$ by $H_1 e^{-\varepsilon|t|}$ (or $H_1 e^{-\varepsilon^2 t^2}$ in the complex-ghost theory), and take the limit $\varepsilon\to+0$ after constructing the transition matrix between $t=-\infty$ and $t=+\infty$. This approach is suitable for the Hamiltonian formalism. The other method is based on the asymptotic condition. As $t\to\pm\infty$, field operators (or products of field operators) are supposed to approach to their asymptotic fields, which satisfy the commutation relations of free fields, in the sense of weak topology. This approach is suitable for discussing Heisenberg operators. We shall, however, adopt the adiabatic hypothesis throughout, because we work in the Hamiltonian formalism except in §11.

When we consider an indefinite-metric quantum field theory in which a constraint persists at all time, there is an important problem, which has hitherto been unnoticed. Should we apply the adiabatic hypothesis also to the constraint? In the conventional treatment, the answer seems to be "yes", though it seems to be never explicitly stated so. Recently, however, Haller and Landovitz[H2~H4),*)] have proposed a new formulation of quantum electrodynamics in which the answer to the above question is "no". It is not unreasonable that the constraint remains unchanged even for $t\to\pm\infty$, because the notion of the physical states should be independent of time. Under this postulate, Haller and Landovitz found that quantum electrodynamics in any gauge reduces to that in the Coulomb (radiation) gauge; for example, according to their conclusion, the renormalization constants Z_1 and Z_2 are gauge-independent.[T4)] Of course, in the conventional quantum electrodynamics, Z_1, Z_2 and most of off-the-mass-shell quantities are gauge-dependent. The difference between two formulations originates from the adiabatic hypothesis on the constraint. In the conventional formalism, the space $\mathscr{H}_0$ of the physical states at $t=\pm\infty$ is different from $\mathscr{H}$, the space of the physical states at finite time. This adiabatic change of the definition of the physical states induces a departure from the Coulomb-gauge formalism, that is, the possibility of various gauges is owing to the inequivalence between $\mathscr{H}_0$ and $\mathscr{H}$.

It may look unsatisfactory that the definition of the physical states at $t=\pm\infty$ is different from that at finite time, but we should note that an observer who can observe phenomena taking place at finite time cannot wait

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*) The description of their criticism of the Gupta-Bleuler constraint is misleading.[G9)] Their standpoint is properly stated in a very recent paper by Tomczak and Haller.[T4)]

for an infinite time interval, and conversely, an observer who can observe a transition from $t=-\infty$ to $t=+\infty$ cannot measure a finite time interval. Therefore, the existence of two kinds of physical states is not a logical inconsistency. We may say that quantum electrodynamics is a unified description of two theories: "infinite-time quantum electrodynamics" and "finite-time quantum electrodynamics", which correspond to an infinite-time observer and to a finite-time observer, respectively. The infinite-time observer can measure scattering cross sections, while the finite-time observer can observe bound states and the classical Maxwell field (as expectation values).

Finally, we discuss the question of causality. This problem crucially depends on the definition of causality. The microcausality is usually used as the same meaning as local commutativity, i.e., the property that any two field operators are commutative or anticommutative at spacelike separations. In this sense, the indefinite-metric theories usually considered are microcausal. If, however, the use of ghost fields is not allowed, we have to eliminate them. It is generally believed that if ghosts are eliminated, an indefinite-metric theory will always reduce to a non-local theory, in which microcausality is violated.

There are many possible definitions of macrocausality. In the weakest sense, it means a cluster property:[B6] If a system consists of two subsystems which are present at a very great distance, the S-matrix of the whole system is essentially equal to a product of the S-matrices of the two subsystems. Another definition of macrocausality is as follows: We can arbitrarily choose an initial state from a set *a priori* prescribed independently of dynamics, and then the corresponding final state is uniquely determined by dynamics, that is, the initial state can be specified without knowing the final state. This macrocausality is always satisfied if the S-matrix is derived from a linear dynamical equation like the Schrödinger equation. The indefinite-metric theories usually considered are macrocausal also in this sense.

§ 10. Unitarity of the physical S-matrix

Since, as pointed out in § 9, it seems too stringent to impose a constraint at all time for the purpose of constructing a divergence-free theory, in this section we consider only the infinite-time observer and require the following condition: *If the initial state is a physical state then the corresponding final state is also a physical state, and vice versa.* We call this condition "physical-state condition". Let S be the S-matrix and P be one of projection operators to $\mathcal{H}_0$ (the physical subspace at $t=\pm\infty$), that is, $P\mathcal{V}=\mathcal{H}_0$. Then the physical-state condition can be written as

$$(1-P^\dagger)SP=0,$$
$$P^\dagger S(1-P)=0, \qquad (10\cdot1)$$

that is,

$$SP = P^\dagger SP = P^\dagger S. \tag{10·2}$$

We assume that the semi-definite-metric space $\mathcal{H}_0$ is a direct sum of a Hilbert space $\mathcal{H}_0'$ and a space $\mathcal{O}$ which consists of zero-norm states alone and is orthogonal to $\mathcal{H}_0'$. Let P' be the *orthogonal* projection operator to $\mathcal{H}_0'$, namely, a projection operator such that $\mathcal{H}_0' = P'\mathcal{V}$ is orthogonal to $(1-P')\mathcal{V}$. Then, as is seen from $(1-P')^\dagger P' = P'^\dagger(1-P') = 0$, we have

$$P'^\dagger = P'. \tag{10·3}$$

Let P_0 be one of projection operators to $\mathcal{O}$ (note that no orthogonal projection to $\mathcal{O}$ exists). Then

$$P_0^\dagger P' = P'^\dagger P_0 = P_0^\dagger P_0 = 0. \tag{10·4}$$

It is convenient to set $P = P' + P_0$. Then (10·3) and (10·4) yield

$$P^\dagger P = P'. \tag{10·5}$$

Now, the S-matrix S is unitary in $\mathcal{V}$ if H is hermitian and if S is constructed as usual, but this fact does not imply the conservation of prob ability for physical states. The *physical S-matrix*, which is defined by

$$S_{\text{phys}} \equiv P^\dagger SP, \tag{10·6}$$

is not necessarily unitary in $\mathcal{H}_0$ or $\mathcal{H}_0'$. However, if S is unitary, i. e.,

$$S^\dagger S = SS^\dagger = 1, \tag{10·7}$$

and if the physical-state condition is satisfied, then, by sandwiching (10·7) with $P^\dagger$ and P and by using (10·2), (10·6) and (10·5), we find

$$S_{\text{phys}}^\dagger S_{\text{phys}} = S_{\text{phys}} S_{\text{phys}}^\dagger = P', \tag{10·8}$$

that is, S_{phys} *is unitary in the quotient space* $\mathcal{H}_0/\mathcal{O} (\simeq \mathcal{H}_0')$.

If H is not hermitian, the unitarity of S_{phys} is not guaranteed even if the physical-state condition is satisfied. Since the latter must be satisfied in order to avoid negative probability, the hermiticity of H is almost a necessary condition for the unitarity of S_{phys}, though of course we cannot exclude the possibility of fantastic exceptional examples in which H is non-trivially non-hermitian but S_{phys} is unitary.*) Hereafter, we always assume

*) The complex-ghost theory proposed by H. Yamamoto[Y1~Y4] has a non-hermitian Hamiltonian. His claim that S_{phys} is unitary is wrong unfortunately. His theory is interesting, however, as the only known example of a divergence-free, Lorentz-invariant quantum field theory which satisfies the physical-state condition. There is another interesting non-hermitian model, which was considered by Scarf and Umezawa[S1] and by Yokoyama[Y8] without being aware of its non-hermiticity. This model has a Yukawa-type interaction Lagrangian, but all fermion propagators reduce to boson ones.

that H is hermitian. Then we have only to check whether or not the physical-state condition is satisfied.

The physical-state condition is of course satisfied if there is a constraint which persists at all time. The most successful example is quantum electrodynamics (see §18). It is of course possible to construct various indefinite-metric theories which have a persisting constraint but are equivalent to positive-definite-metric local field theories (in this case, $\mathcal{O}$ is zero). On the other hand, we know two types of theories which have no persisting constraint but satisfy the physical-state condition. One is based on the energy conservation law. Because of the uncertainty principle, energy is not necessarily conserved at a finite time interval but it is strictly conserved at an infinite time interval. Therefore, if the physical states are characterized by energy values, then the physical-state condition is satisfied. The other example is called the Shimodaira model,[87] which is based on the mass-shell condition satisfied at the initial and final states. Suppose that a free ghost-field equation is

$$(\Box+m^2)\phi=0, \tag{10·9}$$

say. If the interaction Lagrangian involves ϕ only in the form of $(\Box+m^2)\phi$, then it is evident that no external lines of ϕ exist in any Feynman graph. Therefore, the physical-state condition is satisfied. This model is not welcome, however, because the ultraviolet-divergence problem becomes much worse.*)

From the above consideration, in order to construct a divergence-free theory by using the indefinite-metric device, it seems that the only possible way is the case of energy conservation. We analyze this possibility in detail.

We first consider a theory involving a simple ghost $|G\rangle$ ($\langle G|G\rangle<0$ and $(H-E)|G\rangle=0$ with E real). Since it is impossible to distinguish a many-particle state involving $|G\rangle$ from a physical many-particle state by using the eigenvalue equation of the Hamiltonian only, the existence of $|G\rangle$ generally violates the physical-state condition.

Next, we consider the complex-ghost case. As remarked in § 7, for E non-real, we have a pair of complex ghosts $|C\rangle$ and $|C^*\rangle$:

$$(H-E)|C\rangle=(H-E^*)|C^*\rangle=0,$$
$$\langle C|C\rangle=\langle C^*|C^*\rangle=0,\ \langle C^*|C\rangle\neq 0. \tag{10·10}$$

Since any many-particle state involving a complex ghost (either $|C\rangle$ or $|C^*\rangle$) has a non-real eigenvalue, it can clearly be distinguished from any physical many-particle state which has a real eigenvalue. In such a case, the

*) Here, of course, we confine ourselves to local interactions only. As for non-local interactions, see § 15.

physical-state condition is satisfied. Unfortunately, however, the above distinction is no longer possible if we consider a state involving *both* $|C\rangle$ *and* $|C^*\rangle$, because $E+E^*$ is real. Thus the complex-ghost theory looks hopeless. It is not the case, however, if we consider a *relativistic* theory of complex ghosts. We shall discuss this point in detail in §16.

Finally, we consider the multipole-ghost case. Since the complex multipole-ghost case is quite similar to the complex-ghost case, we suppose that E is real. Furthermore, for simplicity, we confine ourselves to the dipole-ghost case alone. This case is very delicate because the energy conservation law itself does not guarantee the physical-state condition.

Historically, Heisenberg[H7] first made the following consideration. If the theory considered contains a dipole ghost, then scattering-wave solutions of a time-independent Schrödinger equation asymptotically behave, apart from an incident wave, like either $ae^{\pm ikr}/r$ or $(a+br)e^{\pm ikr}/r$ $(b\neq 0)$, where $r\equiv|\boldsymbol{x}|$ and a, b, k are constants. Heisenberg called the set of the former-type wave functions "Hilbert space I" and the remainder "Hilbert space II" (the set of dipole-ghost scattering states). Since the states belonging to "Hilbert space II" are physically unreasonable, he postulated that one should adopt only the solutions belonging to "Hilbert space I". Then it is evident that no dipole ghosts contribute to the physical S-matrix; accordingly, the physical-state condition is to be satisfied.

Since Heisenberg's reasoning is rather heuristic, a more reasonable proof was presented by Ascoli and Minardi.[A3),A4)] They considered the problem in the time-dependent Schrödinger picture. Since the time-displacement operator is e^{iHt}, if the state considered is a superposition of the eigenstates of H at some particular time, this property persists at all time. Because a dipole ghost is not an eigenstate of H, it cannot be produced at any time. Thus the physical-state condition is to be satisfied.

The above reasoning is also unsatisfactory because the initial condition is usually given at $t=-\infty$ but not at finite time. The most satisfactory way of showing the unitarity of the physical S-matrix is due to Nagy.[N4)] Suppose that a complete set is formed by physical incoming-wave eigenstates $|m, \text{ in}\rangle$ of H, dipole ghosts $|l, D\rangle$ and their corresponding zero-norm eigenstates $|l, 0\rangle=(H-E_l)|l, D\rangle$, where $|l, 0\rangle$ is orthogonal to $|m, \text{ in}\rangle$ but not to $|l, D\rangle$. A physical outgoing-wave eigenstate $|n, \text{ out}\rangle$ can be expanded as

$$|n, \text{ out}\rangle=\sum_m a_{mn}|m, \text{ in}\rangle+\sum_l b_{ln}|l, 0\rangle+\sum_l c_{ln}|l, D\rangle. \qquad (10\cdot 11)$$

If $H-E_n$ acts on $(10\cdot 11)$, then we find

$$0=\sum_m a_{mn}(E_m-E_n)|m, \text{ in}\rangle+\sum_l b_{ln}(E_l-E_n)|l, 0\rangle$$
$$+\sum_l c_{ln}|l, 0\rangle+\sum_l c_{ln}(E_l-E_n)|l, D\rangle. \qquad (10\cdot 12)$$

From the linear independence of the states, we obtain

$$a_{mn}=0 \quad \text{for } E_m\neq E_n,$$
$$c_{ln}=b_{ln}=0 \text{ for } E_l\neq E_n. \tag{10·13}$$

Therefore, (10·12) reduces to

$$\sum_{E_l=E_n} c_{ln}|l, 0\rangle=0, \tag{10·14}$$

that is,

$$c_{ln}=0 \text{ for } E_l=E_n. \tag{10·15}$$

On substituting (10·13) and (10·15) in (10·11), we have

$$|n, \text{out}\rangle=\sum_{E_m=E_n} a_{mn}|m, \text{in}\rangle+\sum_{E_l=E_n} b_{ln}|l, 0\rangle. \tag{10·16}$$

From (10·16), we see that $S_{\text{phys}}=\{\langle m, \text{in}|n, \text{out}\rangle\}$ should be unitary.

Because of the above proofs, it was believed for a long time that a dipole-ghost field theory would give a unitary physical S-matrix, provided that we do not encounter negative-norm bound states. Some concrete examples,*) however, seemed to contradict the conclusion of this theorem. Nagy[N5] therefore constructed a modified Lee model in order to investigate this question more closely. In his model, the only source of negative probability is a dipole-ghost field, but the physical S-matrix can be explicitly shown to be *non-unitary*. Indeed, for scattering states, there is a complete set consisting of the eigenstates of H only. From this result, Nagy concluded that the above theorem concerning the dipole-ghost theory was valid but the dipole-ghost *situation* was "unstable".

The unitarity problem of the dipole-ghost theory was finally settled by Nakanishi,[N17] who showed that the above theorem was actually *wrong*. The point is that contrary to common belief, *dipole-ghost scattering states are expressible as superpositions of the eigenstates of H*. This curious phenomenon can take place because the spectrum of H is *continuous* but not discrete for scattering states. A dipole ghost, $|l, D\rangle$ satisfies

$$(H-E_l)|l, D\rangle=|l, 0\rangle, \tag{10·17}$$

and of course $|l, D\rangle$ is linearly independent of $|l, 0\rangle$. We can construct, however, another state satisfying the dipole-ghost equation (10·17) in terms of $|l, 0\rangle$:

$$|l, \widetilde{D}\rangle\equiv\int dE_m\delta'(E_l-E_m)|m, 0\rangle. \tag{10·18}$$

Indeed, with the aid of $x\delta'(x)=-\delta(x)$, we have

*) See Nakanishi[N13] and private communication to Nagy.[N5]

$$(H-E_l)|l,\tilde{D}\rangle=|l,0\rangle. \tag{10·19}$$

Therefore, a state

$$|l,\tilde{0}\rangle\equiv|l,D\rangle-|l,\tilde{D}\rangle \tag{10·20}$$

is an eigenstate of H and linearly independent of $|l,0\rangle$. We can form a complete set by using $|l,\tilde{0}\rangle$ instead of $|l,D\rangle$. In Nagy's modified Lee model, it is possible to construct $|l,D\rangle$ and $|l,\tilde{0}\rangle$ explicitly.

Heisenberg's reasoning is wrong because "Hilbert space I" is *not a closed vector space*, and hence a *matrix* cannot be defined within it. The incorrectness of the second proof is evident because a dipole ghost is a superposition of the eigenstates of H. The drawback of the third proof is the overlook of the possibility that b_{ln} can involve $\delta'(E_l-E_n)$ (hence (10·14) is wrong).

Thus, except for the relativistic complex-ghost theory, it seems hopeless to construct a divergence-free indefinite-metric quantum field theory which has a unitary physical S-matrix. Here we of course adhere the definition (10·6) of S_{phys}. If S_{phys} may *not* be a submatrix of S, that is, if we artificially define S_{phys}, then it is quite easy to construct a unitary physical S-matrix. Such a method of defining a physical S-matrix has no field-theoretical basis, and therefore we call it "artificial unitarization". To allow artificial unitarization is essentially to give up quantum field theory and adopt an S-matrix-theoretical point of view.

Since artificial unitarization is not a logical reasoning, there are indefinitely many ways of its realization. We classify them into three categories.

(A) *Non-linear function of S and P'*

A historically famous example of this type is due to Bogoliubov, Medvedev and Polivanov.[B5] They supposed that the initial state could have an adjustable non-physical component. If it is chosen to be equal to the non-physical component of the final state apart from a phase factor, we obtain

$$S_{\text{phys}}\equiv P'S[1+e^{i\theta}(1-P')S]^{-1}P', \tag{10·21}$$

where θ is a real number. It is not difficult to check that S_{phys} is unitary in $\mathcal{H}_0'$.

Another example of this type may be called the final-state normalization method. We assume that the operator $\tilde{S}\equiv P'SP'$ is invertible in $\mathcal{H}_0'$. Then $\tilde{S}\tilde{S}^\dagger$ is also invertible. Since $(\tilde{S}\tilde{S}^\dagger)^{-1}$ is hermitian, we can uniquely define $(\tilde{S}\tilde{S}^\dagger)^{-1/2}$ by using a unitary transformation. We then define

$$S_{\text{phys}}\equiv(\tilde{S}\tilde{S}^\dagger)^{-1/2}\tilde{S}. \tag{10·22}$$

It is straightforward to verify that (10·22) is unitary in $\mathcal{H}_0'$.

In the artificial unitarization of this type, since S_{phys} depends on S nonlinearly, the cluster property of $S_{\rm phys}$ is generally lost. Thus it badly violates macrocausality.

(B) *Adjustment of the absorptive part*

We define that the dispersive part of $S_{\rm phys}$ is equal to that of $P'SP'$, but the absorptive part of $S_{\rm phys}$ is determined in such a way that $S_{\rm phys}$ becomes unitary.*) This procedure is always possible to be carried out if it is worked out successively in perturbation theory.

The artificial unitarization of this type is essentially to neglect the absorptive part due to ghost production. A convenient prescription of doing this in the case of complex ghosts was proposed by Cutkosky, Landshoff, Olive and Porkinghorne.[C2)] The shadow-state theory of Sudarshan and his collaborators[S10),G2),N21),N22)] seems to be essentially of this type.

Some of drawbacks of the theories of this type are as follows: $S_{\rm phys}$ is not analytic in general so that it will violate the postulates of the *analytic* S-matrix theory,[C1)] and this method is applicable only to perturbation theory.

The analyticity can be maintained if the dispersive part is also successively modified in conformity with the modification of the same-order absorptive part. A celebrated example of this type is the massless Yang-Mills theory,[D2)] in which one introduces Feynman's fictitious quanta after constructing the S-matrix. Quite a similar situation is encountered also in the dual resonance model.[M1)]

(C) *Redefinition of physical states*

This method is based on the standpoint that the initial and final states should not be assigned more rigidly than they are assigned by good quantum numbers. If we can find a one-to-one mapping U from $\mathcal{V}$ onto $\mathcal{V}$ (an automorphism) such that good quantum numbers are preserved by U and if $P^\dagger U^\dagger SUP$ is unitary in $\mathcal{H}_0/\mathcal{O}$, we define

$$S_{\rm phys}\equiv P^\dagger U^\dagger SUP. \qquad (10\cdot 23)$$

The definition of this type was suggested by Schnitzer and Sudarshan[S2)] and recently used by Nagy.[N7)] Kita[K6)] independently made an analogous consideration in his non-local theory.

A defect of this method is that we cannot specify U unless we know S explicitly. In some solvable models, we can demonstrate the existence of U, but unfortunately it is not unique. There is no proper reason to choose as U a particular one among others.

§ 11. Consideration in the Heisenberg representation

It seems that there is no sufficiently comprehensive axiomatic field theory

*) This prescription should be applied only to *connected* Feynman graphs. If it is blindly applied to the old-fashioned perturbation theory, in which the contributions from disconnected Feynman graphs are not separated, the result badly violates macrocausality as in (A).

having indefinite metric. As pointed out in § 8, the trouble which we encounter is that we know no appropriate way of introducing topology into $\mathcal{V}$. In discussing the Heisenberg operators on $\mathcal{V}$, we can hope at best the Lehmann-Symanzik-Zimmermann (LSZ)[L11] level of mathematical rigor. In what follows, we examine how the LSZ-type postulates have to be modified in the indefinite-metric quantum field theory.

We first review the LSZ-type postulates in the positive-metric quantum field theory.

(1) The set of all states spans a Hilbert space $\mathcal{H}$. Fields $\phi_r(x)$, which correspond to the elementary particles labeled by r, are operators on $\mathcal{H}$ (more precisely, they are operator-valued distributions).

(2) Observables are (unbounded) hermitian operators on $\mathcal{H}$. Hence their eigenvalues are all real, and a complete set is formed by eigenstates.

(3) The theory is invariant under the Poincaré group and some other symmetry groups. In particular, the translational invariance implies the existence of the generators P_μ, which are hermitian and mutually commutative and form a Lorentz vector; furthermore,

$$[P_\mu, \phi_r(x)] = -i\partial_\mu \phi_r(x) \tag{11·1}$$

for any r.

(4) Local commutativity: $\phi_r(x)$ and $\phi_s(y)$ are commutative or anticommutative for $(x-y)^2<0$.

(5) Spectral condition: The eigenvalues of P_0 and those of P^2 are real and non-negative. Furthermore, the state whose eigenvalue of P_0 is zero is unique and is called the vacuum, which is the only Lorentz-invariant eigenstate. It is often assumed that P^2 has some particular spectrum consistent with the Lagrangian field theory.

(6) Asymptotic condition: Field operators $\phi_r(x)$ approach to $\phi_r^{\text{in}}(x)$ as $x_0 \to -\infty$ and to $\phi_r^{\text{out}}(x)$ as $x_0 \to +\infty$; $\phi_r^{\text{in}}(x)$ and $\phi_r^{\text{out}}(x)$ have the free-field properties.

In the indefinite-metric field theory, most of the above postulates have to be modified. We have already discussed how to modify Postulates (1) and (2) in § 9. Postulates (3) and (4) remain unmodified. Postulate (5) has to be weakened considerably, because the generalized eigenvalues of P_0 and P^2 are, in general, complex, and there may exist many non-invariant states whose eigenvalue of P_0 is zero (see § 17). Postulate (6) becomes impossible if there exist complex ghosts, because $\phi_r(x)$ may increase exponentially as $x_0 \to \pm\infty$.

Since, as seen above, it is quite difficult to discuss Heisenberg operators in the general framework of the indefinite-metric quantum field theory, we here exclude the existence of non-real eigenvalues,[*] that is, we assume that we encounter only simple ghosts and real multipole ghosts in addition to the normal states. In this case, Postulates (5) and (6) can remain essentially unmodified. Therefore, most results concerning the analyticity of Green's functions and scattering amplitudes can still be derived in the indefinite-metric theory. The only difference between the positive-metric case and our case is the properties of spectral functions. For example, as is well known, the spectral function of the one-particle Green's function is no longer positive definite if there exist simple ghosts. In the following, we discuss what happens about the spectral function of the one-particle Green's function if there exists a dipoleghost.

The one-particle Green's function is defined by

$$G(x) \equiv \langle \Omega | \mathrm{T}(\phi(x)\phi(0)) | \Omega \rangle, \tag{11·2}$$

where $|\Omega\rangle$ stands for the true vacuum, $\phi(x)$ is a Heisenberg operator, which is assumed to be hermitian scalar field for simplicity, and T is the chronological symbol. Suppose that there exists a dipole ghost $|p, D\rangle$ and its associated zero-norm state $|p, 0\rangle$ such that

$$\begin{aligned} &(P_\mu - p_\mu)|p, 0\rangle = 0, \\ &(P_\mu - p_\mu)|p, D\rangle = p_\mu |p, 0\rangle \end{aligned} \tag{11·3}$$

with

$$\begin{aligned} &\langle p, 0 | p, 0\rangle = \langle p, D | p, D\rangle = 0, \\ &\sigma^{-1} \equiv \langle p, D | p, 0\rangle = \langle p, 0 | p, D\rangle = \pm 1. \end{aligned} \tag{11·4}$$

Then the orthogonal projection operator $D(p)$ to the subspace spanned by those two states is given by

$$D(p) \equiv \sigma[\,|p, D\rangle\langle p, 0| + |p, 0\rangle\langle p, D|\,]. \tag{11·5}$$

From (11·1) we have

$$\phi(x) = e^{iPx}\phi(0)e^{-iPx}. \tag{11·6}$$

Furthermore, (11·3) is rewritten as

$$\begin{aligned} &e^{-iPx}|p, 0\rangle = e^{-ipx}|p, 0\rangle, \\ &e^{-iPx}|p, D\rangle = e^{-ipx}|p, D\rangle - ipxe^{-ipx}|p, 0\rangle. \end{aligned} \tag{11·7}$$

Let $G^{(+)}_{D(p)}(x)$ be the contribution from $|p, D\rangle$ and $|p, 0\rangle$ to $\langle \Omega | \phi(x)\phi(0) | \Omega \rangle$. With the aid of (11·6) and (11·7), we have

[*] It should be noted that even if this condition is satisfied for H_0, it is not necessarily satisfied for $H \equiv P_0$.

$$G^{(+)}_{D(p)}(x) \equiv \langle \Omega | \phi(x) D(p) \phi(0) | \Omega \rangle$$
$$= \sigma \{ [\langle \Omega | \phi(0) | p, D \rangle - ipx \langle \Omega | \phi(0) | p, 0 \rangle] \langle p, 0 | \phi(0) | \Omega \rangle$$
$$+ \langle \Omega | \phi(0) | p, 0 \rangle \langle p, D | \phi(0) | \Omega \rangle \} e^{-ipx}$$
$$= \sigma_1(p) e^{-ipx} + \sigma_2(p) p_\mu (\partial / \partial p_\mu) e^{-ipx}, \tag{11·8}$$

where

$$\sigma_1(p) \equiv 2\sigma \,\mathrm{Re}\, [\langle \Omega | \phi(0) | p, D \rangle \langle p, 0 | \phi(0) | \Omega \rangle],$$
$$\sigma_2(p) \equiv \sigma | \langle \Omega | \phi(0) | p, 0 \rangle |^2. \tag{11·9}$$

Because of Lorentz invariance, the Lorentz-transformed states of $|p, D\rangle$ and $|p, 0\rangle$ have to be taken into account. Suppose that $p^2 = m^2$ with $p_0 > 0$. Then the total contribution on the mass shell is

$$G^{(+)}_D(x) \equiv \int d^4p \theta(p_0) \delta(p^2 - m^2) G^{(+)}_{D(p)}(x)$$
$$= \int_0^\infty ds \ \rho_D(s) \Delta^{(+)}(x, s), \tag{11·10}$$

where

$$\Delta^{(+)}(x, s) \equiv \int d^4p \theta(p_0) \delta(p^2 - s) e^{-ipx},$$
$$\rho_D(s) \equiv [\tilde{\sigma}_1(m^2) - 2\tilde{\sigma}_2(m^2)] \delta(s - m^2) - 2m^2 \tilde{\sigma}_2(m^2) \delta'(s - m^2) \tag{11·11}$$

with $\sigma_j(p) \equiv \tilde{\sigma}_j(p^2)$ because of Lorentz invariance. As is well known, from (11·10) we have

$$G_D(x) = \int_0^\infty ds \ \rho_D(s) \Delta_F(x, s), \tag{11·12}$$

where $G_D(x)$ denotes the contribution to $G(x)$ from dipole ghosts. As seen in (11·11), the spectral function $\rho_D(s)$ contains δ'. Therefore if $s = m^2$ belongs to a discrete spectrum, it yields a double pole in $G(x)$. If it belongs to a continuous spectrum, however, we should integrate (11·11) over m^2, and hence $G(x)$ has a cut similar to the ordinary case. The only difference between the two cases is that the spectral function is a *measure* (in the mathematical sense) in the ordinary case, while it is a distribution which is not necessarily a measure in the dipole-ghost case.

Finally, we note that (11·3) is not necessarily satisfied in a Lorentz-invariant theory. A dipole-ghost state can appear as a particular component of a Lorentz vector (see § 18). In such a case, the dipole-ghost equations should be

$$(P^2 - p^2) | p, 0 \rangle = 0,$$
$$(P^2 - p^2) | p, D \rangle = p^2 | p, 0 \rangle. \tag{11·13}$$

Chapter 4

Examples of the Indefinite-Metric Quantum Field Theory

§ 12. Lee model

The Lee model with indefinite metric was first considered by Källén and Pauli.[K2] Heisenberg[H7] made use of the Lee model to demonstrate the possibility of the dipole-ghost situation. Since then, many authors investigated the indefinite-metric Lee model in detail and also discussed various modified models. It seems, however, that in the indefinite-metric Lee model the discussion of the eigenvalue problem is confined so far to the no-cutoff case. The consideration in this case depends on the method of renormalization. In discussing what situation we encounter in the eigenvalue problem, it will be better to consider a theory which contains no ambiguity related to the removal of ultraviolet divergences.

The Hamiltonian of the indefinite-metric *cutoff* Lee model is given by

$$H=-m_V\psi^\dagger\psi+m_N\phi^\dagger\phi+\int d\boldsymbol{p}\,\omega_{\boldsymbol{p}}\theta^\dagger(\boldsymbol{p})\theta(\boldsymbol{p})$$
$$+\psi^\dagger\phi\int d\boldsymbol{p}\,g(\boldsymbol{p})\theta(\boldsymbol{p})+\psi\phi^\dagger\int d\boldsymbol{p}\,g^*(\boldsymbol{p})\theta^\dagger(\boldsymbol{p}) \qquad (12\cdot1)$$

with*)

$$g(\boldsymbol{p})\equiv igG(\boldsymbol{p})/\sqrt{2\omega_{\boldsymbol{p}}}. \qquad (12\cdot2)$$

Here ψ, ϕ and $\theta(\boldsymbol{p})$ are field operators of V, N and θ, respectively, whose masses are m_V, m_N and m_θ, respectively; $\omega_{\boldsymbol{p}}\equiv\sqrt{m_\theta+\boldsymbol{p}^2}$ and g is a coupling constant; the form factor $G(\boldsymbol{p})$ is a (complex) smooth function such that $G(\boldsymbol{p})=O(|\boldsymbol{p}|^{-1-\varepsilon})$ with $\varepsilon>0$ as $|\boldsymbol{p}|\to\infty$.

The commutation relations are

$$\{\psi,\psi^\dagger\}=-1,$$
$$\{\phi,\phi^\dagger\}=+1,$$
$$[\theta(\boldsymbol{p}),\theta^\dagger(\boldsymbol{q})]=\delta(\boldsymbol{p}-\boldsymbol{q}), \qquad (12\cdot3)$$

and all other relevant commutators (or anticommutators) vanish. The vacuum $|0\rangle$ is defined by

$$\psi|0\rangle=\phi|0\rangle=\theta(\boldsymbol{p})|0\rangle=0,\ \langle 0|0\rangle=1. \qquad (12\cdot4)$$

The problem of our interest is to discuss the clothed V-particle states $|E\rangle$, which satisfy

*) The factor i in (12·2) is inessential for our discussion. It originates from the renormalization in the no-cutoff model.

$$H|E\rangle = E|E\rangle. \tag{12·5}$$

Because of the particle-number conservation laws, $|E\rangle$ can be expressed as

$$|E\rangle = [c\psi_V^\dagger + \phi^\dagger \int d\boldsymbol{p} f(\boldsymbol{p})\theta^\dagger(\boldsymbol{p})]\,|0\rangle, \tag{12·6}$$

where c and $f(\boldsymbol{p})$ are a constant and a function of $\boldsymbol{p}$, respectively. On substituting (12·6) in (12·5), we have

$$\begin{aligned}(m_V - E)c &= -\int d\boldsymbol{p}\, g(\boldsymbol{p}) f(\boldsymbol{p}),\\ (m_N + \omega_{\boldsymbol{p}} - E) f(\boldsymbol{p}) &= c g^*(\boldsymbol{p}).\end{aligned} \tag{12·7}$$

We solve (12·7) under the boundary condition of no incident wave. Non-trivial solutions of (12·7) exist if and only if E satisfies

$$h(E) \equiv E - m_V - \int d\boldsymbol{p} \frac{|g(\boldsymbol{p})|^2}{m_N + \omega_{\boldsymbol{p}} - E} = 0. \tag{12·8}$$

In the following, we investigate this equation in detail.

First, it is evident that (12·8) has no real roots for $E > E_0 \equiv m_N + m_\theta$. Furthermore, we see

$$h(-\infty) = -\infty, \quad |h(E_0)| < \infty, \tag{12·9}$$

$$h'(-\infty) = +1, \quad h'(E_0) = -\infty, \tag{12·10}$$

$$h''(E) < 0 \qquad \text{for } E \leq E_0. \tag{12·11}$$

Therefore, the zero point of

$$h'(E) = 1 - \int d\boldsymbol{p} \frac{|g(\boldsymbol{p})|^2}{(m_N + \omega_{\boldsymbol{p}} - E)^2} \tag{12·12}$$

is unique in the region $E \leq E_0$. We denote it by $E_{\max}$, that is,

$$\begin{aligned} &h'(E) > 0 \qquad \text{for } E < E_{\max},\\ &h'(E_{\max}) = 0,\\ &h'(E) < 0 \qquad \text{for } E_{\max} < E \leq E_0. \end{aligned} \tag{12·13}$$

We distinguish the following three cases.

(1) $h(E_{\max}) > 0$. In this case, (12·8) has *either* one real root $E = E_-$ ($< E_{\max}$) *or* two real roots $E = E_-$ and $E = E_+$ ($E_- < E_{\max} < E_+ \leq E_0$), depending on $h(E_0) > 0$ or ≤ 0.

(2) $h(E_{\max}) = 0$. In this case, (12·8) has only one real root $E = E_d$ ($= E_{\max}$), which is a double root. It is easy to show that

$$E_d > (1/2)(m_V + E_0), \qquad m_V < E_0. \tag{12·14}$$

(3) $h(E_{\max}) < 0$. In this case, (12·8) has no real roots.

Next, we investigate complex roots of (12·8). Let

$$E=E_1+iE_2, \qquad E_2\neq 0. \tag{12·15}$$

Then (12·8) is rewritten as

$$E_1-m_V-\int d\boldsymbol{p}\frac{|g(\boldsymbol{p})|^2(m_N+\omega_{\boldsymbol{p}}-E_1)}{(m_N+\omega_{\boldsymbol{p}}-E_1)^2+E_2^2}=0, \tag{12·16}$$

$$E_2[1-\int d\boldsymbol{p}\frac{|g(\boldsymbol{p})|^2}{(m_N+\omega_{\boldsymbol{p}}-E_1)^2+E_2^2}]=0. \tag{12·17}$$

In (12·17), because of $E_2\neq 0$, the quantity in the square bracket must vanish. We define a function

$$\varphi(E_1,\alpha)\equiv 1-\int d\boldsymbol{p}\frac{|g(\boldsymbol{p})|^2}{(m_N+\omega_{\boldsymbol{p}}-E_1)^2+\alpha} \tag{12·18}$$

in $\alpha\geq 0$. It has the following properties:

$$\begin{aligned}\varphi(E_1,0)&=h'(E_1) \quad \text{for } E_1<E_0,\\ &=-\infty \quad \text{for } E_1\geq E_0,\end{aligned} \tag{12·19}$$

$$\varphi(E_1,+\infty)=+1, \tag{12·20}$$

$$(\partial/\partial\alpha)\varphi(E_1,\alpha)>0. \tag{12·21}$$

Hence, because of (12·13), for E_1 fixed, $\varphi(E_1,\alpha)$ has no zero point if $E_1<E_{\max}(<E_0)$ and has only one zero point $\alpha=\alpha(E_1)$ if $E_1\geq E_{\max}$; in particular,

$$\alpha(E_{\max})=0. \tag{12.22}$$

Thus (12·17) with $E_2\neq 0$ is satisfied for $E_2^2=\alpha(E_1)$ with $E_1>E_{\max}$.

The next task is to analyze (12·16).*) We define a function

$$\chi(E_1)\equiv E_1-m_V-\int d\boldsymbol{p}\frac{|g(\boldsymbol{p})|^2(m_N+\omega_{\boldsymbol{p}}-E_1)}{(m_N+\omega_{\boldsymbol{p}}-E_1)^2+\alpha(E_1)} \tag{12·23}$$

in $E_1\geq E_{\max}$. It has the following properties:

$$\chi(E_{\max})=h(E_{\max}), \tag{12·24}$$

$$\chi(+\infty)=+\infty, \tag{12·25}$$

$$\chi'(E_1)>0. \tag{12·26}$$

Here (12·24) is evident from (12·22) and (12·8), and (12·25) is also obvious, but the proof of (12·26) is non-trivial.

For simplicity of notation, we set

*) This analysis is owing to Prof. H. Araki. The author is very grateful to him for his valuable comments on the problem of solving (12·8).

$$F(E_1, \boldsymbol{p}) \equiv (m_N + \omega_{\boldsymbol{p}} - E_1)^2 + \alpha(E_1). \tag{12·27}$$

Then

$$\chi'(E_1) = 1 + \int d\boldsymbol{p} \frac{|g(\boldsymbol{p})|^2}{F(E_1, \boldsymbol{p})} + (-2)\int d\boldsymbol{p} \frac{|g(\boldsymbol{p})|^2 (m_N + \omega_{\boldsymbol{p}} - E_1)^2}{[F(E_1, \boldsymbol{p})]^2}$$
$$+ \int d\boldsymbol{p} \frac{|g(\boldsymbol{p})|^2 (m_N + \omega_{\boldsymbol{p}} - E_1)}{[F(E_1, \boldsymbol{p})]^2} \alpha'(E_1). \tag{12·28}$$

Because of

$$\varphi(E_1, \alpha(E_1)) = 0, \tag{12·29}$$

the second term of (12·28) is equal to 1, and the third term equals

$$-2 + 2\int d\boldsymbol{p} \frac{|g(\boldsymbol{p})|^2 \alpha(E_1)}{[F(E_1, \boldsymbol{p})]^2}. \tag{12·30}$$

Therefore, the sum of the first three terms in (12·28) is positive. On differentiating (12·29) with respect to E_1, we have

$$\alpha'(E_1) = \frac{2\int d\boldsymbol{p} \dfrac{|g(\boldsymbol{p})|^2 (m_N + \omega_{\boldsymbol{p}} - E_1)}{[F(E_1, \boldsymbol{p})]^2}}{\int d\boldsymbol{p} \dfrac{|g(\boldsymbol{p})|^2}{[F(E_1, \boldsymbol{p})]^2}}. \tag{12·31}$$

On substituting (12·31) in (12·28), we find that the last term in (12·28) is also positive. Thus (12·26) has been established.

From (12·24)~(12·26), we see that $\chi(E_1)$ has no zero point if $h(E_{\max})>0$ and only one zero point if $h(E_{\max})\leq 0$. For $h(E_{\max})=0$, the zero point is $E_1=E_{\max}$, and hence $\alpha(E_1)=0$ because of (12·22). We therefore conclude that *non-real roots of* (12·8) *exist if and only if* $h(E_{\max})<0$ (i.e., Case (3)), and in this case we have exactly two roots $E=E_c$ and E_c^*, which are mutually complex conjugate.

Finally, we investigate the norm of $|E\rangle$. From (12·6) together with (12·3) and (12·4) and (12·7), we have

$$\langle E|E\rangle = -|c|^2 + \int d\boldsymbol{p} |f(\boldsymbol{p})|^2$$
$$= -|c|^2 \left[1 - \int d\boldsymbol{p} \frac{|g(\boldsymbol{p})|^2}{|m_N + \omega_{\boldsymbol{p}} - E|^2}\right]. \tag{12·32}$$

Therefore

$$\langle E|E\rangle = -|c|^2 h'(E) \qquad \text{for } E \text{ real,}$$
$$= -|c|^2 \operatorname{Im} h(E)/\operatorname{Im} E \text{ for } E \text{ non-real.} \tag{12·33}$$

For the three cases classified above, we have the following results.

(1) Since $h'(E_-)>0$ and $h'(E_+)<0$, from (12·33) we find that $|E_-\rangle$ has negative norm while $|E_+\rangle$ has positive norm. As $g\to 0$, $|E_-\rangle$ tends to $\psi_r^\dagger|0\rangle$, while $|E_+\rangle$ disappears; thus the latter is a bound state of N and θ.

(2) We obviously have $\langle E_d|E_d\rangle=0$. Hence there has to exist a dipole ghost $|E_d, D\rangle$ which satisfies

$$(H-E_d)|E_d,D\rangle=|E_d\rangle. \tag{12·34}$$

Let

$$|E_d, D\rangle=[\tilde{c}\psi_r^\dagger+\phi^\dagger\int d\boldsymbol{p}\tilde{f}(\boldsymbol{p})\theta^\dagger(\boldsymbol{p})]|0\rangle. \tag{12·35}$$

Then

$$(m_V-E_d)\tilde{c}=-\int d\boldsymbol{p}g(\boldsymbol{p})\tilde{f}(\boldsymbol{p})+c,$$

$$(m_N+\omega_{\boldsymbol{p}}-E_d)\tilde{f}(\boldsymbol{p})=\tilde{c}g^*(\boldsymbol{p})+f(\boldsymbol{p}). \tag{12·36}$$

By eliminating $\tilde{f}(\boldsymbol{p})$, we find

$$h(E_d)\tilde{c}+h'(E_d)c=0. \tag{12·37}$$

Since $h(E_d)=h'(E_d)=0$, (12·37) is identically satisfied. Thus (12·36) is solvable. Since

$$\tilde{f}(\boldsymbol{p})=\frac{\tilde{c}g^*(\boldsymbol{p})}{m_N+\omega_{\boldsymbol{p}}-E_d}+\frac{cg^*(\boldsymbol{p})}{(m_N+\omega_{\boldsymbol{p}}-E_d)^2}, \tag{12·38}$$

we find

$$\langle E_d|E_d, D\rangle=|c|^2\int d\boldsymbol{p}\frac{|g(\boldsymbol{p})|^2}{(m_N+\omega_{\boldsymbol{p}}-E_d)^3}>0. \tag{12·39}$$

The non-existence of a tripole ghost can also be confirmed by using (12·11).

(3) From (12·33) we have

$$\langle E_c|E_c\rangle=\langle E_c^*|E_c^*\rangle=0, \tag{12·40}$$

as it should be (see (7·3)), and

$$\langle E_c^*|E_c\rangle=-|c|^2h'(E_c)\neq 0. \tag{12·41}$$

We cannot say which of $|E_c\rangle$ and $|E_c^*\rangle$ is the elementary V-particle state. They are both particle-mixtures of elementary and composite particles.

§ 13. Froissart model

The Froissart model[F6)] is the simplest example of relativistic dipole-ghost fields. We consider two hermitian scalar fields A and B. The Lagrangian density is given by

$$\mathcal{L}=\partial^\mu A\partial_\mu B-m^2AB+\frac{1}{2}\lambda A^2. \quad (\lambda\neq 0) \tag{13·1}$$

Field equations are therefore

$$(\Box+m^2)A=0,$$
$$(\Box+m^2)B=\lambda A, \tag{13·2}$$

where $\Box\equiv\partial^\mu\partial_\mu$. Hence

$$(\Box+m^2)^2B=0. \tag{13·3}$$

The canonical quantization implies that

$$[A(x),\dot{B}(y)]_{x_0=y_0}=[B(x),\dot{A}(y)]_{x_0=y_0}=i\delta(\boldsymbol{x}-\boldsymbol{y}), \tag{13·4}$$

where a dot denotes $\partial/\partial y_0$, and all other equal-time commutators vanish. Let $\omega_{\boldsymbol{p}}\equiv\sqrt{m^2+\boldsymbol{p}^2}$; then, as usual, we can expand $A(x)$ as

$$A(x)=(2\pi)^{-3/2}\int d\boldsymbol{p}(2\omega_{\boldsymbol{p}})^{-1/2}[\alpha(\boldsymbol{p})\exp(-i\omega_{\boldsymbol{p}}x_0+i\boldsymbol{px})+\text{h.c.}], \tag{13·5}$$

where h. c. stands for the hermitian conjugate. Since $B(x)$ does not satisfy the Klein-Gordon equation, however, a similar expansion is impossible for $B(x)$. Therefore, *assuming* $m\neq 0$, we consider

$$B_D(x)\equiv B(x)+\frac{1}{2}\lambda m^{-2}(1+x^\mu\partial_\mu)A(x). \tag{13·6}$$

Since

$$(\Box+m^2)(x^\mu\partial_\mu A)=-2m^2A, \tag{13·7}$$

we have

$$(\Box+m^2)B_D(x)=0. \tag{13·8}$$

Hence it is possible to write

$$B_D(x)=(2\pi)^{-3/2}\int d\boldsymbol{p}(2\omega_{\boldsymbol{p}})^{-1/2}[\beta(\boldsymbol{p})\exp(-i\omega_{\boldsymbol{p}}x_0+i\boldsymbol{px})+\text{h. c.}]. \tag{13·9}$$

The equal-time commutators are transcribed as

$$[\alpha(\boldsymbol{p}),\beta^\dagger(\boldsymbol{q})]=[\beta(\boldsymbol{p}),\alpha^\dagger(\boldsymbol{q})]=\delta(\boldsymbol{p}-\boldsymbol{q}),$$
$$[\alpha(\boldsymbol{p}),\alpha^\dagger(\boldsymbol{q})]=[\beta(\boldsymbol{p}),\beta^\dagger(\boldsymbol{q})]=0, \text{ etc.} \tag{13·10}$$

The four-dimensional commutators are calculated from (13·5), (13·6), (13·9) and (13·10):

$$[A(x), A(y)]=0,$$
$$[A(x), B(y)]=i\Delta(x-y, m^2),$$
$$[B(x), B(y)]=i\lambda(-\partial/\partial m^2)\Delta(x-y, m^2), \qquad (13\cdot 11)$$

where

$$\Delta(x, m^2)\equiv -i(2\pi)^{-3}\int d^4p\varepsilon(p_0)\delta(p^2-m^2)e^{-ipx}. \qquad (13\cdot 12)$$

In deriving (13·11), we have used an identity*)

$$(\partial/\partial m^2)\Delta(x, m^2)=m^{-2}(1+(1/2)x^\mu\partial_\mu)\Delta(x, m^2). \qquad (13\cdot 13)$$

The generators P_μ of translations are given by

$$P_0=H=\int d\boldsymbol{p}\omega_{\boldsymbol{p}}N(\boldsymbol{p}),$$
$$P_l=\int d\boldsymbol{p}p_lN(\boldsymbol{p}), \qquad (13\cdot 14)$$

with

$$N(\boldsymbol{p})\equiv\alpha^\dagger(\boldsymbol{p})\beta(\boldsymbol{p})+\beta^\dagger(\boldsymbol{p})\alpha(\boldsymbol{p})+\frac{1}{2}\lambda m^{-2}\alpha^\dagger(\boldsymbol{p})\alpha(\boldsymbol{p}). \qquad (13\cdot 15)$$

The vacuum $|0\rangle$ is defined by

$$\alpha(\boldsymbol{p})|0\rangle=\beta(\boldsymbol{p})|0\rangle=0, \quad \langle 0|0\rangle=1. \qquad (13\cdot 16)$$

The one-particle state $\alpha^\dagger(\boldsymbol{p})|0\rangle$ and $\beta^\dagger(\boldsymbol{p})|0\rangle$ satisfy

$$(P_\mu-p_\mu)\alpha^\dagger(\boldsymbol{p})|0\rangle=0,$$
$$(P_\mu-p_\mu)\beta^\dagger(\boldsymbol{p})|0\rangle=\frac{1}{2}\lambda m^{-2}p_\mu\alpha^\dagger(\boldsymbol{p})|0\rangle, \qquad (13\cdot 17)$$

with $p_0\equiv\omega_{\boldsymbol{p}}$, and

$$\langle 0|\alpha(\boldsymbol{q})\alpha^\dagger(\boldsymbol{p})|0\rangle=\langle 0|\beta(\boldsymbol{q})\beta^\dagger(\boldsymbol{p})|0\rangle=0,$$
$$\langle 0|\alpha(\boldsymbol{q})\beta^\dagger(\boldsymbol{p})|0\rangle=\delta(\boldsymbol{p}-\boldsymbol{q}). \qquad (13\cdot 18)$$

Thus $\beta^\dagger(\boldsymbol{p})|0\rangle$ is a dipole ghost.

We emphasize that the above consideration *cannot* be applied to the case $m=0$. In order to define $B_D(x)$, we have to seek for an integro-differential operator K satisfying

$$\Box KA(x)=A(x), \qquad (13\cdot 19)$$

where

$$\Box A(x)=0. \qquad (13\cdot 20)$$

*) Unfortunately, this identity is erroneously written in Nagy's book.N4) The definition (13·6) is different from Froissart's oneF6) because of the correction of this error.

In the following, we prove the non-existence of a *Lorentz-invariant* K. [This proposition was inferred by Lukierski[L12)] on the basis of the $m\to 0$ limit of (13·7).]

Because of Lorentz invariance, K can be expressed as

$$K=F(x^2, x^\mu\partial_\mu, \Box). \tag{13·21}$$

On transferring $\Box$ to the right end, we can eliminate $\Box$ by using (13·20). Since $x^\mu\partial_\mu$ is dimensionless, the dimensional analysis shows that

$$K=x^2f(x^\mu\partial_\mu). \tag{13·22}$$

Since no property other than (13·20) and Lorentz scalarity is assumed for $A(x)$, (13·19) must hold for any invariant solution of (13·20). Hence we may substitute

$$D(x)\equiv\varDelta(x,0)=-(2\pi)^{-1}\varepsilon(x_0)\delta(x^2) \tag{13·23}$$

for $A(x)$. Since $D(x)$ is homogeneous, we find

$$KD(x)=f(-2)x^2D(x)=0. \tag{13·24}$$

Thus there is no invariant K satisfying (13·19).

It is of course easy to find a non-invariant K satisfying (13·19). For example,[L2)]

$$K=\frac{1}{2}\Delta^{-1}x_0\partial_0+F(\partial_\mu), \tag{13·25}$$

where Δ stands for the Laplacian $\sum_{l=1}^3\partial_l^2$, whose inverse can be defined unambiguously.*) To adopt (13·25) implies to violate manifest Lorentz invariance. Manifest Lorentz invariance is, however, the vital element of the Froissart model. Indeed, as shown recently by Nagy,[N7)] who translated the consideration on the dipole-ghost theory stated in §10 into an operator from, a *non-relativistic* version of the Froissart model is *equivalent* to a model consisting of a normal field and a simple-ghost field. For simplicity, we suppress the degrees of the freedom of spatial momentum. We start with a Hamiltonian

$$H=\int dE\cdot E[a^\dagger(E)a(E)-b^\dagger(E)b(E)] \tag{13·26}$$

with

$$[a(E), a^\dagger(E')]=-[b(E), b^\dagger(E')]=\delta(E-E'),$$
$$[a(E), b^\dagger(E')]=0, \text{ etc.} \tag{13·27}$$

*) In contrast with Δ^{-1}, $\Box^{-1}$ is ambiguous and violates the associative law $(D(x)=(\Box^{-1}\Box)\times D(x)\neq\Box^{-1}(\Box D(x))=0)$.

This model, of course, has no dipole ghost. If, however, with $k\neq 0$, we define

$$\alpha(E)\equiv k[a(E)-b(E)],$$
$$\beta(E)\equiv k^{-1}a(E)+k^{-1}E(d/dE)[a(E)-b(E)], \tag{13·28}$$

then

$$[\alpha(E),\beta^\dagger(E')]=[\beta(E),\alpha^\dagger(E')]=\delta(E-E'),$$
$$[\alpha(E),\alpha^\dagger(E')]=[\beta(E),\beta^\dagger(E')]=0,\ \text{etc.} \tag{13·29}$$

and

$$H=\int dE\cdot E[\alpha^\dagger(E)\beta(E)+\beta^\dagger(E)\alpha(E)+k^{-2}\alpha^\dagger(E)\alpha(E)]. \tag{13·30}$$

Evidently, (13·29) and (13·30) constitute a non-relativistic version of the Froissart model.

If manifest Lorentz covariance is required, however, then such a consideration as above becomes impossible. In the following, we show that *if* $m\neq 0$, the displacement operator P_μ, which is defined by (13·14) and (13·15) together with (13·10), cannot be reduced to P'_μ, where

$$P'_0\equiv\int d\boldsymbol{p}\omega_{\boldsymbol{p}}N'(\boldsymbol{p}),$$
$$P'_l\equiv\int d\boldsymbol{p}p_lN'(\boldsymbol{p}) \tag{13·31}$$

with

$$N'(\boldsymbol{p})\equiv a^\dagger(\boldsymbol{p})a(\boldsymbol{p})-b^\dagger(\boldsymbol{p})b(\boldsymbol{p}) \tag{13·32}$$

and

$$[a(\boldsymbol{p}),a^\dagger(\boldsymbol{q})]=-[b(\boldsymbol{p}),b^\dagger(\boldsymbol{q})]=\delta(\boldsymbol{p}-\boldsymbol{q}),$$
$$[a(\boldsymbol{p}),b^\dagger(\boldsymbol{q})]=0,\ \text{etc.} \tag{13·33}$$

Suppose that $P'_\mu=P_\mu$ when $\alpha(\boldsymbol{p})$ and $\beta(\boldsymbol{p})$ are certain linear combinations of $a(\boldsymbol{p})$ and $b(\boldsymbol{p})$. From $[\alpha,P'_\mu]=p_\mu\alpha$ and $[\alpha,\alpha^\dagger]=0$, we must have

$$\alpha(\boldsymbol{p})=k[a(\boldsymbol{p})-b(\boldsymbol{p})], \tag{13·34}$$

since $k(a+b)$ reduces to the above by redefining $-b$ as b. Then the relation $[\alpha,\beta^\dagger]=\delta$ implies that

$$\beta(\boldsymbol{p})=k^{-1}a(\boldsymbol{p})+D_{\boldsymbol{p}}[a(\boldsymbol{p})-b(\boldsymbol{p})], \tag{13·35}$$

where $D_{\boldsymbol{p}}$ is a differential operator. In order to have $[\beta,P'_\mu]=p_\mu(\beta+h\alpha)$ with $h\equiv\frac{1}{2}\lambda m^{-2}$, from (13·31) and (13·35), $D_{\boldsymbol{p}}$ must satisfy

$$[D_p, \omega_p] = hk\omega_p, \tag{13·36}$$

$$[D_p, p_l] = hkp_l. \tag{13·37}$$

From (13·37) we find

$$D_p = hk\boldsymbol{p}\partial/\partial\boldsymbol{p}, \tag{13·38}$$

but (13·38) does not satisfy (13·36) unless $m=0$. Thus only the $m=0$ case ($\lambda=\infty$ so that h be finite), which is not the genuine Froissart model as emphasized above, can be reduced to a simple-ghost model in a manifestly covariant way. ($[\beta, \beta^\dagger]=0$ is satisfied by setting $h=k^{-2}/3$.)

Finally, we note that there are some attempts to extend the Froissart model to the multipole-ghost case. Invariant solutions of the multiple Klein-Gordon equation

$$(\Box + m^2)^n \varphi = 0 \tag{13·39}$$

were investigated in detail by Bowman and Harris[B7] and later by Montaldi.[M4] Lukierski[L14),L15] constructed solutions of (13·39) in terms of the solutions of the Klein-Gordon equation by using translationally non-invariant coefficients. A natural extension of the Froissart model to the multipole-ghost case was made by Yokoyama and Kubo.[Y12]

§ 14. Multimass theory

The simplest covariant way of avoiding ultraviolet divergences in the Feynman integral is to introduce a Feynman cutoff.[F3] For example, as is well known, the photon propagator in the Feynman gauge is $1/(-p^2-i\varepsilon)$ apart from a factor $ig_{\mu\nu}$. If it is replaced by

$$\frac{1}{-p^2-i\varepsilon} \cdot \frac{\Lambda^2}{\Lambda^2-p^2-i\varepsilon}, \tag{14·1}$$

where Λ^2 is a very large quantity, all relevant Feynman integrals except for the second-order photon self-energy integral become convergent (apart from infrared divergences) in quantum electrodynamics. Since (14·1) is rewritten as

$$\frac{1}{-p^2-i\varepsilon} - \frac{1}{\Lambda^2-p^2-i\varepsilon}, \tag{14·2}$$

the above replacement is equivalent to the introduction of a ghost field having a mass Λ. This procedure, of course, violates the unitarity of the physical S-matrix, but in this section we do not take care of this problem.

The historically famous Pauli-Villars regulator method[P5] is of a similar line of thought. Though their original idea was formalistic, but as pointed out by Gupta,[G8] it is realized in terms of the indefinite-metric theory as follows. We suppose that a physical field $\phi_0(x)$ is always accompanied with a number of auxiliary fields $\phi_1(x), \cdots, \phi_n(x)$ (for simplicity, we assume that $\phi_0, \phi_1, \cdots, \phi_n$ are hermitian scalar fields). We consider a linear combination

$$\phi(x) \equiv \sum_{j=0}^{n} a_j \phi_j(x) \tag{14·3}$$

with a_j real and $a_0=1$. If

$$[\phi_j(x), \phi_k(y)] = i\sigma_j \delta_{jk} \Delta(x-y, m_j^2) \tag{14·4}$$

with $\sigma_0=1$ and $\sigma_j=\pm 1$ for $j=1, \cdots, n$, then

$$[\phi(x), \phi(y)] = i\sum_{j=0}^{n} \sigma_j a_j^2 \Delta(x-y, m_j^2). \tag{14·5}$$

We require the Pauli-Villars regularity conditions

$$\sum_{j=0}^{n} \sigma_j a_j^2 = 0,$$
$$\sum_{j=0}^{n} \sigma_j m_j^2 a_j^2 = 0, \text{ etc.} \tag{14·6}$$

which imply that $\phi(x)$ is much less singular than $\phi_0(x)$ on the light cone. Thus, given a conventional theory involving ϕ_0, if ϕ_0 is replaced by ϕ in the interaction Lagrangian, then we obtain a less singular propagator $\langle 0|\mathrm{T}(\phi\phi)|0\rangle$ instead of $\langle 0|\mathrm{T}(\phi_0\phi_0)|0\rangle$.

The above auxiliary-field method is closely related to the multimass theory, which was extensively investigated by Pais and Uhlenbeck[P1] without explicitly employing indefinite metric. The field equation in the multimass theory is of the form

$$F(\Box)\phi(x) = j(x), \tag{14·7}$$

where $F(s)$ is a polynomial in s and $j(x)$ denotes a source operator. The higher is the degree of $F(s)$, the less singular is the propagator of ϕ.

The free Lagrangian density of the multimass theory is given by

$$\mathcal{L}_0 = -\frac{1}{2}\phi F(\Box)\phi. \tag{14·8}$$

Because of the hermiticity of $\mathcal{L}_0$, the coefficients in $F(s)$ are all real. Assuming that $F(s)$ is monic (i.e., the highest-degree coefficient is one), we can write

$$F(s) = \prod_{j=0}^{n} (s-s_j), \tag{14·9}$$

where s_j may be complex, but then s_j^* also has to appear.

Hereafter we confine ourselves to the case in which all s_j are different from each other. Hence $F'(s_j) \neq 0$. From the well-known formula of partial fractions

$$\frac{1}{F(s)}=\sum_{j=0}^{n}\frac{1}{F'(s_j)(s-s_j)}\,, \qquad (14\cdot 10)$$

we obtain an identity

$$\sum_{j=0}^{n}\eta_j\prod_{k\neq j}(\Box-s_j)=1 \qquad (14\cdot 11)$$

with $\eta_j\equiv 1/F'(s_j)$. Therefore, on setting

$$\tilde{\phi}_j(x)\equiv[\prod_{k\neq j}(\Box-s_k)]\phi(x), \qquad (14\cdot 12)$$

we have

$$\phi(x)=\sum_{j=0}^{n}\eta_j\tilde{\phi}_j(x). \qquad (14\cdot 13)$$

By using (14·13) and (14·12), the free Lagrangian density (14·8) is rewritten as*)

$$\mathcal{L}_0=-\frac{1}{2}\sum_{j=0}^{n}\eta_j\tilde{\phi}_jF(\Box)\phi$$

$$=-\frac{1}{2}\sum_{j=0}^{n}\eta_j\tilde{\phi}_j(\Box-s_j)\tilde{\phi}_j. \qquad (14\cdot 14)$$

If $s_j=s_c$ is non-real, then as mentioned above, (14·14) also involves a field corresponding to s_c^*. On defining $\phi_c\equiv\sqrt{\eta_c}\tilde{\phi}_c$, we see that $\mathcal{L}_0$ involves $\mathcal{L}_c$, where

$$\mathcal{L}_c\equiv-\frac{1}{2}[\phi_c(\Box-s_c)\phi_c+\phi_c^\dagger(\Box-s_c^*)\phi_c^\dagger]. \qquad (14\cdot 15)$$

This Lagrangian density (after integrating by parts) is the starting point of the relativistic complex-ghost field theory (see § 16).

If all s_j are real, we may assume that $s_0>s_1>\cdots>s_n$ without loss of generality. Since $F(s)$ is a continuous function, the sign of $F'(s_j)$, namely, that of η_j, changes *alternately*. For simplicity, we assume that $\eta_0>0$. We can eliminate $|\eta_j|$ from (14·14) by setting $\phi_j=\sqrt{|\eta_j|}\tilde{\phi}_j$. Finally, we assume that $s_0<0$, and set $s_j=-m_j^2$. Then (14·14) and (14·13) become

$$\mathcal{L}_0=-\frac{1}{2}\sum_{j=0}^{n}\sigma_j\phi_j(\Box+m_j^2)\phi_j \qquad (14\cdot 16)$$

and

$$\phi(x)=\sum_{j=0}^{n}a_j\phi_j(x), \qquad (14\cdot 17)$$

respectively, where

$$\sigma_j\equiv(-1)^j,\quad a_j\equiv(-1)^j\sqrt{|\eta_j|}. \qquad (14\cdot 18)$$

Since $\sigma_j a_j^2=\eta_j$, the identity (14·10), i. e.,

$$\sum_{j=0}^{n}\eta_j(s+m_j^2)^{-1}=[F(s)]^{-1}, \qquad (14\cdot 19)$$

*) Pais and Uhlenbeck[P1] made a partial integration, but it is unnecessary in (14·14).

reproduces (14·6). [Expand $(s+m_j^2)^{-1}$ and compare the coefficients of s^{-k} $(k=1, 2, \cdots, n)$.]

As demonstrated above, the multimass theory realizes the auxiliary-field method in the most economical way. The former is more elegant than the latter because in the former the *ad hoc* introduction of the auxiliary fields into the *free Lagrangian* is avoided and the regularity conditions are satisfied automatically. A defect of the multimass theory is the use of higher derivatives in the Lagrangian.

Finally, we make some comments on the multimass version of quantum electrodynamics. If the electron field is associated with auxiliary fermion fields, then gauge invariance is violated, because fermion fields having different masses couple directly. In the photon self-energy part, Pauli and Villars[P5)] avoided this difficulty by introducing an artificial regularization, which could not be justified in terms of the indefinite-metric quantum field theory. Arons, Han and Sudarshan[A1)] formulated a divergence-free quantum electrodynamics by introducing two auxiliary fermion fields, but it was not gauge-invariant. A gauge-invariant, divergence-free quantum electrodynamics was first formulated by Lee and Wick.[L8)] Though their original theory is a complex-ghost theory (see §16), we may take all masses real if we forget about the unitarity problem. It does not violate gauge invariance to introduce an auxiliary vector field associated with the electromagnetic field. As noted at the beginning of this section, its introduction removes all ultraviolet divergences appearing in the S-matrix *except for* the second-order photon self-energy part. To remove its divergence, Lee and Wick introduced two auxiliary fermion fields, ψ_+ and ψ_-, having a purely imaginary charge. More precisely, the Lee-Wick additional Lagrangian density is given by

$$\mathcal{L}^{LW}=i\overline{\psi_-}(-i\gamma^\mu\partial_\mu+M-e\gamma^\mu A_\mu)\psi_+-i\overline{\psi_+}(-i\gamma^\mu\partial_\mu+M^*-e\gamma^\mu A_\mu)\psi_-, \tag{14·20}$$

where $\overline{\psi_\pm}\equiv\psi_\pm^\dagger\gamma_0$ and A_μ denotes the electromagnetic field; M may or may not be complex. The equal-time anticommutators are

$$\{\psi_\pm^\alpha(x), \bar{\psi}_\mp^\beta(y)\}_{x_0=y_0}=(\gamma_0)^{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{y}), \ \{\psi_\pm^\alpha(x), \bar{\psi}_\pm^\beta(y)\}_{x_0=y_0}=0. \tag{14·21}$$

As seen from (14·20), $\mathcal{L}^{LW}$ does not violate gauge invariance. Because of the purely imaginary charge, the second-order photon self-energy integrals due to these auxiliary fermion fields have the opposite sign to that of the second-order photon self-energy integrals due to the electron field and the muon field.*) Therefore, their ultraviolet-divergent parts exactly cancel out, because the coefficients of both quadratically and logarithmically divergent

*) If we do not take account of the muon field, the charge of the auxiliary fermion fields must be $ie/\sqrt{2}$. Thus the muon field has a *raison d'être* of avoiding an irrational magnitude of the charge of the auxiliary fields.

terms are independent of the loop mass.

The Lee-Wick method is quite interesting, but one should note that it is not a regularization method, but, so to speak, a cancellation method. The ultraviolet behavior is weakened only in the loop integrals which have $4n+2$ fermion lines ($n=0, 1, \cdots$). Furthermore, the Schwinger term[O1] cannot be removed by $\mathcal{L}^{LW}$. This is in contrast with the regularization method, in which the Schwinger term is completely swept away, as pointed out by Moffat.[M3] The appearance of the Schwinger term is a pathological feature of the present quantum field theory, and it is natural to remove it by introducing auxiliary fields. We note that the removal of the Schwinger term persists even if the masses of the auxiliary fields tend to infinity so that the unitarity of the physical S-matrix is not injured.

§ 15. Realization of intrinsic form factors

The multimass theory provides a method of regularizing the Feynman integral by modifying propagators. Another way of regularizing the Feynman integral is to introduce intrinsic form factors into vertices. Historically, this attempt is known as the non-local theory, or more precisely, the quantum field theory of non-local interaction.

One considers such an interaction Lagrangian density as

$$\mathcal{L}_I(x) = g\int d^4y\int d^4z F(x-y, x-z)\phi_1(x)\phi_2(y)\phi_3(z), \qquad (15\cdot1)$$

where $F(x, y)$ is a c-number function, called a form factor, hoping that certain good form factors could eliminate ultraviolet divergences.

The non-local theory, however, has two fatal drawbacks. First, in order to obtain a convergent Feynman integral in momentum space, the form factor should be a distribution containing the Feynman $-i\varepsilon$; otherwise the Feynman integral would become more singular because Dyson's power-counting theorem for ultraviolet divergences then would no longer hold. This requirement implies that $F(x, y)$ should be a *complex-valued* distribution, but that contradicts the hermiticity of $\int \mathcal{L}_I d^4x$. Second, since (15·1) contains action at a distance, the Tomonaga-Schwinger equation does not satisfy the integrability condition. Therefore, the Yang-Feldman in-out formalism was adopted to construct the S-matrix. Hayashi,[H6] however, pointed out that in this formalism an internal inconsistency appears in the order g^4. If the formalism is modified to remedy this difficulty, them macrocausality will be violated. Thus the non-local theory based on the non-local interaction Lagrangian seems to be quite unsatisfactory.*)

*) An attempt at a non-local theory without postulating the Lagrangian was made by Kita.[K4~K6] In his theory, the assumption that field operators are Lorentz-covariant is given up.

The purpose of this section is to demonstrate that it is possible on the basis of the indefinite-metric quantum field theory to construct a theory in which each vertex has a form factor of a Feynman integral-type. This theory is divergence-free and macrocausal, but its physical S-matrix is non-unitary, unfortunately. The theory which we propose here is a refinement of Yokoyama's theory. In 1961, Yokoyama[Y5)~Y7)] proposed a very interesting way of regularizing quantum field theory by introducing several auxiliary fields. He claimed the unitarity of his physical S-matrix. Munakata,[M5)] however, criticized his conclusion by constructing a modified Yokoyama model, whose physical S-matrix was explicitly non-unitary. The reason of this discrepancy was, contrary to Munakata's suggestion, Yokoyama's inadequate treatment of the mass-shell condition. If calculated correctly, one cannot forbid the appearance of ghosts in the final state.

Before entering into the main subject, we note that there is an exactly solvable model which satisfies all usual (Wightman) axioms of the relativistic quantum field theory *except for* the positive definiteness of metric. This model was proposed first by Glaser[G1)] in a brief comment and much later formulated independently by Bialynicki-Birula.[B3)] Let $\phi(x)$ $\tilde{\phi}(x)$ and be two scalar fields having an equal mass m. We consider a Lagrangian density

$$\mathcal{L}=\frac{1}{2}(\partial^{\mu}\phi\partial_{\mu}\phi-m^{2}\phi^{2})-\frac{1}{2}(\partial^{\mu}\tilde{\phi}\partial_{\mu}\tilde{\phi}-m^{2}\tilde{\phi}^{2})+F(\phi+\tilde{\phi}), \qquad (15\cdot2)$$

where $F(z)$ is an arbitrary real polynomial in z. The field equations are

$$(\Box+m^{2})\phi(x)=j(x),$$
$$(\Box+m^{2})\tilde{\phi}(x)=-j(x), \qquad (15\cdot3)$$

where

$$j\equiv F'(\phi+\tilde{\phi}). \qquad (15\cdot4)$$

From (15·3), we have

$$(\Box+m^{2})(\phi+\tilde{\phi})=0, \qquad (15\cdot5)$$

that is, $\phi+\tilde{\phi}$ is a free field. Therefore, we can write

$$\phi+\tilde{\phi}=\phi^{\text{in}}+\tilde{\phi}^{\text{in}}, \qquad (15\cdot6)$$

where

$$[\phi^{\text{in}}(x),\phi^{\text{in}}(y)]=i\Delta(x-y,m^{2}),$$
$$[\phi^{\text{in}}(x),\tilde{\phi}^{\text{in}}(y)]=0,$$
$$[\tilde{\phi}^{\text{in}}(x),\tilde{\phi}^{\text{in}}(y)]=-i\Delta(x-y,m^{2}). \qquad (15\cdot7)$$

We note that $j(x)$ is rewritten as $F'(\phi^{\text{in}}+\tilde{\phi}^{\text{in}})$. Hence the solution of (15·3) is given by

$$\phi(x)=\phi^{\text{in}}(x)+\int d^4y\Delta_R(x-y,m^2)j(y),$$

$$\tilde{\phi}(x)=\tilde{\phi}^{\text{in}}(x)-\int d^4y\Delta_R(x-y,m^2)j(y) \tag{15·8}$$

where

$$\Delta_R(x,m^2)\equiv\frac{1}{(2\pi)^4}\int d^4p\frac{ye^{-ipx}}{m^2+\boldsymbol{p}^2-(p_0+i\varepsilon)^2}. \tag{15·9}$$

In perturbation approach, propagators always appear in a pair of ϕ and $\tilde{\phi}$, and they cancel out exactly. Therefore, all Feynman graphs involving at least one internal line give no contribution. Thus the only contribution to $S-1$ arises from the Feynman graphs which have only one vertex in each connected component.

Now, Yokoyama's theory is based on the above exact cancellation of two equal-mass fields. In the following, we present Yokoyama-type models in a more transparent way.

We consider three physical fields $\phi_j(x)$ $(j=1,2,3)$ and six auxiliary fields $\varphi_j(x)$ and $\tilde{\varphi}_j(x)$ $(j=1,2,3)$; they are all hermitian and scalar. The Lagrangian density $\mathcal{L}=\mathcal{L}_0+\mathcal{L}_I^{(Y)}$ is given by

$$\mathcal{L}_0=\frac{1}{2}\sum_{j=1}^{3}(\partial^\mu\phi_j\partial_\mu\phi_j-m_j^2\phi_j^2)$$
$$+\frac{1}{2}\sum_{j=1}^{3}(\partial^\mu\varphi_j\partial_\mu\varphi_j-\mu_j^2\varphi_j^2)-\frac{1}{2}\sum_{j=1}^{3}(\partial^\mu\tilde{\varphi}_j\partial_\mu\tilde{\varphi}_j-\mu_j^2\tilde{\varphi}_j^2), \tag{15·10}$$

$$\mathcal{L}_I^{(Y)}=f\cdot(\varphi_1+\tilde{\varphi}_1)(\varphi_2+\tilde{\varphi}_2)(\varphi_3+\tilde{\varphi}_3)+\frac{1}{2}\sum_{j=1}^{3}f_j\cdot(\varphi_j-\tilde{\varphi}_j)\phi_j. \tag{15·11}$$

Without the last term of (15·11), the above model would be a trivial superposition of Glaser's model, that is, the φ_j propagator always cancels with the $\tilde{\varphi}_j$ propagator. Because of the presence of the last term of (15·11), it is possible to have a non-vanishing propagator

$$\langle 0|\mathrm{T}[(\varphi_j+\tilde{\varphi}_j)(\varphi_j-\tilde{\varphi}_j)]|0\rangle. \tag{15·12}$$

The physical fields ϕ_1, ϕ_2 and ϕ_3 can couple with each other only through the propagator (15·12). Each Feynman integral of this theory, therefore, is obtained from that of the theory having the Lagrangian density

$$\frac{1}{2}\sum_{j=1}^{3}(\partial^\mu\phi_j\partial_\mu\phi_j-m_j^2\phi_j^2)+g\phi_1\phi_2\phi_3 \tag{15·13}$$

by replacing g by

$$f\prod_{j=1}^{3}\frac{f_j}{\mu_j^2-p_j^2-i\varepsilon}, \tag{15·14}$$

p_j being the 4-momentum of the ϕ_j line $(p_1+p_2+p_3=0)$. Thus we have obtained a theory having form factor (15·14) at each vertex. If we wish to have a theory in which the form factor is given by the Feynman integral corresponding to a triangle graph instead of (15·14), we have only to adopt the interaction Lagrangian density

$$\mathcal{L}_I^{(\Delta)}=\frac{1}{2}g_1\phi_1(\varphi_2+\tilde{\varphi}_2)(\varphi_3-\tilde{\varphi}_3)+\frac{1}{2}g_2\phi_2(\varphi_3+\tilde{\varphi}_3)(\varphi_1-\tilde{\varphi}_1)$$

$$+\frac{1}{2}g_3\phi_3(\varphi_1+\tilde{\varphi}_1)(\varphi_2-\tilde{\varphi}_2) \qquad (15\cdot 15)$$

instead of $\mathcal{L}_I^{(Y)}$.

It is now an easy matter to construct a theory in which the form factor is exactly the Feynman integral corresponding to a given (connected) Feynman graph G. We prepare a physical field ϕ_k for each external line k of G and a pair of auxiliary fields φ_l and $\tilde{\varphi}_l$ for each internal line l of G, and construct the free Lagrangian density $\mathcal{L}_0$ in such a way that all ϕ_k and φ_l have positive norm and all $\tilde{\varphi}_l$ have negative norm:

$$\mathcal{L}_0=\frac{1}{2}\sum_k(\partial^\mu\phi_k\partial_\mu\phi_k-m_k^2\phi_k^2)$$

$$+\frac{1}{2}\sum_l(\partial^\mu\varphi_l\partial_\mu\varphi_l-\mu_l^2\varphi_l^2)-\frac{1}{2}\sum_l(\partial^\mu\tilde{\varphi}_l\partial_\mu\tilde{\varphi}_l-\mu_l^2\tilde{\varphi}_l^2). \qquad (15\cdot 16)$$

We arbitrarily assign an orientation to every internal line of G. For each vertex a of G, we denote the set of all external lines incident with a by $E[a]$, the set of all internal lines outgoing from a by $O[a]$ and the set of all internal lines ingoing to a by $I[a]$. We then define the interaction Lagrangian density by

$$\mathcal{L}_I^{(G)}=\sum_a\{g_a\prod_{k\in E[a]}\phi_k\cdot\prod_{l\in O[a]}\frac{1}{\sqrt{2}}(\varphi_l+\tilde{\varphi}_l)\cdot\prod_{l\in I[a]}\frac{1}{\sqrt{2}}(\varphi_l-\tilde{\varphi}_l)\}, \qquad (15\cdot 17)$$

where the summation goes over all vertices of G. In this way, we see that an arbitrary Feynman integral can be used as a form factor. We also note that any linear combination of Feynman integrals can be realized by taking the corresponding superposition of $\mathcal{L}_I^{(G)}$.

Finally, we point out that the above technique can also be used for generalizing Glaser's model. If we add

$$-\frac{1}{2}\sum_k(\partial^\mu\tilde{\phi}_k\partial_\mu\tilde{\phi}_k-m_k^2\tilde{\phi}_k^2) \qquad (15\cdot 18)$$

to (15·16) and if we replace ϕ_k in (15·17) by $\phi_k+\tilde{\phi}_k$, then we obtain a field theory in which G is the sole connected Feynman graph which contributes to the S-matrix.*) Likewise, we can construct a theory in which the

*) From this model follows an important conclusion that each Feynman integral has the analyticity and other properties which can be proved in the axiomatic field theory without using the positive-definiteness of metric.

S-matrix consists of a finite number of Feynman integrals.

§ 16. Relativistic complex-ghost field theory

As noted already (see §§11 and 10), there are two serious difficulties in formulating a complex-ghost quantum field theory. First, a complex-ghost field asymptotically increases exponentially,*) thus forbidding the asymptotic condition. Correspondingly, in the interaction picture, the infinite-time limit of the transition matrix $U(t, t')$ diverges exponentially. Second, since the energy of the state consisting of a complex ghost and its conjugate is real, it is very difficult to forbid its appearance in the final state, that is, the physical-state condition is usually violated. In the Lee model, the violation of the unitarity of the physical S-matrix was demonstrated in the $2N+3\theta$ sector by Ascoli and Minardi[A5)] and by Tanaka.[T3)]

Both difficulties stated above can be avoided if the masses of all complex ghosts contain a *negative* imaginary part. This standpoint is H. Yamamoto's complex ghost theory[Y1)~Y4)] (see also Tanaka's paper[T3)]), in which H is necessarily non-hermitian. In his theory, the total S-matrix is non-unitary and hence the physical S-matrix is also non-unitary in general unless we make artificial unitarization. In this section, therefore, we consider only the case in which H is hermitian.

The first difficulty of the complex-ghost theory was resolved by Lee and Wick.[L6),**)] They directly defined the physical S-matrix in terms of the scattering-wave eigenstates of H. Let $|E\rangle$ be an eigenstate of the free Hamiltonian H_0 belonging to a real eigenvalue E. Then the outgoing-wave and incoming-wave eigenstates of H are given by

$$|E,\ \text{out}\rangle \equiv [1-(H-E-i\varepsilon)^{-1}H_1]\,|E\rangle,$$
$$|E,\ \text{in}\rangle \equiv [1-(H-E+i\varepsilon)^{-1}H_1]\,|E\rangle, \tag{16·1}$$

where $H_1 \equiv H-H_0$ is the interaction Hamiltonian. The physical S-matrix is defined by

$$S_{\text{phys}} = \{\langle E', \text{in}\,|\,E, \text{out}\rangle\}. \tag{16·2}$$

It is straightforward to show that

$$\langle E', \text{in}\,|\,E, \text{out}\rangle = \langle E'|\,\{1-2\pi i\delta(E'-E)\,[H_1-H_1(H-E-i\varepsilon)^{-1}H_1]\}\,|E\rangle. \tag{16·3}$$

*) Pauli[P4)] was aware of this difficulty already in 1958.

**) There appeared a number of papers concerning the Lee-Wick complex-ghost theory. For example, see Lee and Wick,[L7)] Nachtman,[N1),N2)] Nagy,[N6)] Gleeson and Sudarshan,[G3)] Mehrotra and Patil[M2)] and Dobson, Jr.[D4)]

It is important to note that the $-i\varepsilon$ in $(H-E-i\varepsilon)^{-1}$ has no effect on the intermediate states having non-real energy. Lee and Wick showed that (16·2) can also be obtained from $U(t,t')$ (under the adiabatic hypothesis) if exponentially divergent terms are consistently separated before taking the infinite-time limit.

To see the implication of (16·2) explicitly, we consider a modified Lee model in which the free Hamiltonian H_0 contains complex-ghost fields ψ_V (having a mass M) and ψ_{V^*} (having a mass M^*), where Im $M>0$. The free propagators of V and V^* are

$$S_V(t)\equiv\frac{-1}{2\pi i}\int_\Gamma dE\frac{e^{-iEt}}{E-M},$$
$$S_{V^*}(t)\equiv\frac{-1}{2\pi i}\int_{\Gamma'} dE\frac{e^{-iEt}}{E-M^*}, \tag{16·4}$$

respectively, where the contours Γ and Γ' run from $E=-\infty$ to $E=+\infty$ in such a way that

$$S_V(t)=S_{V^*}(t)=0 \quad \text{for } t<0 \tag{16·5}$$

because of the absence of the antiparticles. Then the contours Γ and Γ' have to pass through *above* the pole of the integrand. Thus Γ' can be identified with the real axis R, but Γ must be a complex contour, which is equivalent (i. e., homologous) to $R-\delta(M)$, where $\delta(M)$ denotes a counterclockwise small circle enclosing $E=M$ (δ is called Leray's coboundary operator). The above results may symbolically be written as

$$\Gamma\sim R-\delta(M) \quad \text{in } C-\{M\},$$
$$\Gamma'\sim R \quad \text{in } C-\{M^*\}, \tag{16·6}$$

where C denotes the complex plane and $\{M\}$ indicates the set consisting of a point $E=M$ alone.

By using (16·6), (16·4) is rewritten as

$$S_V(t)=\frac{-1}{2\pi i}\int_R dE\frac{e^{-iEt}}{E-M}+e^{-iMt},$$
$$S_{V^*}(t)=\frac{-1}{2\pi i}\int_R dE\frac{e^{-iEt}}{E-M^*}. \tag{16·7}$$

The second term of $S_V(t)$ diverges exponentially as $t\to+\infty$. This divergence, for instance, appears in the lowest-order graph of the N-θ elastic scattering. The Lee-Wick prescription, in this case, is nothing but to drop the second term of $S_V(t)$. It is important, however, to note that the exponential term is dropped *only for the infinite time.* For intermediate states, the whole expression for $S_V(t)$ must be retained, that is, in any loop integral we should employ the complex contour Γ. For example, the second-order self-energy part having an intermediate state $V+V^*$ is given by

$$i\int_{\Gamma}\frac{dE'}{(E'-M)(E-E'-M^*)}, \tag{16·8}$$

where $\Gamma\sim R-\delta(M)$ but not $\Gamma\sim R$. If we replaced Γ by R, then we should have a Yamamoto-type theory.

Now, the second difficulty stated at the beginning of this section is manifest in (16·8); a pinch occurs when $E=M+M^*$, so that we have a real singularity due to a pair of complex ghosts. Evidently, this difficulty cannot be overcome as long as we consider a static or non-relativistic field theory. Lee,[L5)] however, made a very interesting observation: In a relativistic theory,[L4),L8)] the situation changes qualitatively. The relativitic energy of a complex ghost having a mass M and a spatial momentum $\boldsymbol{q}$ is given by

$$\omega_{\boldsymbol{q}}\equiv\sqrt{M^2+\boldsymbol{q}^2}. \tag{16·9}$$

Therefore, the total energy E of a complex-ghost pair state having a total spatial momentum $\boldsymbol{p}$ is given by

$$E=\omega_{\boldsymbol{p}}+\omega_{\boldsymbol{p}-\boldsymbol{q}}{}^*=\sqrt{M^2+\boldsymbol{q}^2}+\sqrt{M^{*2}+(\boldsymbol{p}-\boldsymbol{q})^2}, \tag{16·10}$$

where $\boldsymbol{p}$ and $\boldsymbol{q}$ are of course *real* 3-vectors. Then it is important to note that (16·10) is, in general, *not real* for $\boldsymbol{p}\neq 0$. Indeed, for $\boldsymbol{p}\neq 0$,*) when $\boldsymbol{q}$ runs over the whole three-dimensional Euclidean space, E sweeps a closed, simply-connected, *two-dimensional* region D in the complex E plane. The boundary ∂D of D intersects the real axis R at only one point $E=b$, where

$$b\equiv\omega_{\boldsymbol{p}/2}+\omega_{\boldsymbol{p}/2}^*=\mathrm{Re}\sqrt{4M^2+\boldsymbol{p}^2}. \tag{16·11}$$

Let

$$a\equiv\sqrt{(M+M^*)^2+\boldsymbol{p}^2}; \tag{16·12}$$

then we have

$$b<a. \tag{16·13}$$

Hence a half-line $L\equiv[a,+\infty)$ on R (this is the "expected cut") lies completely *inside* D.

The fact that the energy eigenvalues of complex-ghost-pair states occupy a two-dimensional region is a very important characteristic of the *relativistic* complex-ghost theory. The complex-ghost-pair states having a real eigenvalue also exist but are of *measure zero* in Lebesgue's sense. This property remains true for any state involving complex ghosts. Thus, because of

*) Hereafter we always consider this case. The $\boldsymbol{p}=0$ case should be regarded as the limit $\boldsymbol{p}\to 0$.

energy conservation, the physical-state condition (see §10) is satisfied if all real-energy-particle states are the physical states, and therefore the physical S-matrix is unitary. This conclusion is extremely remarkable.

To see the above situation more explicitly, we consider the Feynman integral of the second-order self-energy part involving a pair of complex ghosts. Analogously to (16·8) (a precise derivation of (16·14) will be given later), we have

$$I(p) \equiv i\int d\boldsymbol{q}\int_\Gamma dq_0 \left\{\frac{1}{(q_0^2-\omega_{\boldsymbol{q}}^2)\,[(p_0-q_0)^2-\omega_{\boldsymbol{p}-\boldsymbol{q}}^{*2}]} - (p_\mu = p_\mu^{(0)})\right\}, \tag{16·14}$$

where the contour Γ passes through below the left-hand poles $q_0 = -\omega_{\boldsymbol{q}}$ and $q_0 = p_0 - \omega_{\boldsymbol{p}-\boldsymbol{q}}^*$ and above the right-hand poles $q_0 = \omega_{\boldsymbol{q}}$ and $q_0 = p_0 + \omega_{\boldsymbol{p}-\boldsymbol{q}}^*$, that is, $\Gamma \sim R + \delta(-\omega_{\boldsymbol{q}}) - \delta(\omega_{\boldsymbol{q}})$. The subtraction term $(p_\mu = p_\mu^{(0)})$ is introduced in order to make (16·14) by itself finite; such a procedure is actually unnecessary in a finite theory in which the ultraviolet divergence always cancels out. It is straightforward to carry out the integration over q_0 of $I(p)$. After some manipulation, we have

$$I(p) = \frac{1}{2}\pi[F(p_0, \boldsymbol{p}) + F(-p_0, \boldsymbol{p})] \tag{16·15}$$

with

$$F(p_0, \boldsymbol{p}) \equiv \int d\boldsymbol{q}\left\{\frac{1}{\omega_{\boldsymbol{q}}\omega_{\boldsymbol{p}-\boldsymbol{q}}^*(p_0 - \omega_{\boldsymbol{q}} - \omega_{\boldsymbol{p}-\boldsymbol{q}}^*)} - (p_\mu = p_\mu^{(0)})\right\}. \tag{16·16}$$

It is evident from (16·16) that $F(p_0, \boldsymbol{p})$ is real for p_0 real. Furthermore, $F(p_0, \boldsymbol{p})$ is holomorphic in p_0 in the domain in which the relation

$$p_0 = \omega_{\boldsymbol{q}} + \omega_{\boldsymbol{p}-\boldsymbol{q}}^* \tag{16·17}$$

does not hold for any real value of $\boldsymbol{q}$, that is, $F(p_0, \boldsymbol{p})$ is holomorphic in p_0 outside D. For any point belonging to D, (16·17) holds only on a *one-dimensional* manifold in the $\boldsymbol{q}$ space. Hence, in (16·16), the contribution from a neighborhood of (16·17) is infinitesimal.[L5)] Accordingly, $F(p_0, \boldsymbol{p})$ is well defined and *continuous everywhere* in the p_0 plane. Thus $F(p_0, \boldsymbol{p})$ has no discontinuity on the real axis, that is, the absorptive part of $F(p_0, \boldsymbol{p})$ is identically zero; a complex-ghost pair cannot appear (more precisely, it can appear with *zero* probability) in the final state, as noted above.

The absence of the absorptive part is possible because the value of $F(p_0, \boldsymbol{p})$ in D is *not* equal to the analytic continuation of $F(p_0, \boldsymbol{p})$ from the outside of D (otherwise, $F(p_0, \boldsymbol{p})$ would be an entire function and therefore be identically zero because of Liouville's theorem). Indeed, the first derivative of $F(p_0, \boldsymbol{p})$ is *discontinuous* on the boundary ∂D.[N16)] In particular, $F(p_0, \boldsymbol{p})$ is not holomorphic at $p_0 = b$. It is important to note that the "expected cut" L is completely enclosed by the "non-analytic barrier" ∂D. We set $s=p_0^2-\boldsymbol{p}^2$ and correspondingly set

$$s_a \equiv a^2-\boldsymbol{p}^2=(M+M^*)^2,$$
$$s_b \equiv b^2-\boldsymbol{p}^2=(\mathrm{Re}\sqrt{4M^2+\boldsymbol{p}^2})^2-\boldsymbol{p}^2. \qquad (16\cdot18)$$

In contrast with $s=s_a$, the singular point $s=s_b$, is *not Lorentz-invariant*. Indeed, as $\boldsymbol{p}^2$ varies from $+\infty$ to 0, s_b moves from $2(M^2+M^{*2})$ to $(M+M^*)^2$. Since the location of a singularity of $F(p_0, \boldsymbol{p})$ is not Lorentz-invariant, $I(p)$ cannot be a Lorentz-invariant function. In this way, the Lorentz non-invariance of (16·14) was first proved by Nakanishi.[N16),*)] Later, Gleeson, Moore, Rechenberg and Sudarshan[G2)] explicitly calculated (16·14) for p_0 real and confirmed his conclusion. According to their result, $I(p)$ in $s_b<s<s_a$ is equal to a sum of a Lorentz-invariant function, which is the analytic continuation of $I(p)$ from $s<s_b$ to $s_b<s<s_a$, and a quantity $\Delta I(p)$, which is given by

$$\Delta I(p) \equiv -2\pi^3\left\{\frac{i(M^2-M^{*2})\,|\boldsymbol{p}|}{s(s+\boldsymbol{p}^2)^{1/2}}+\frac{1}{s}[s-(M-M^*)^2]^{1/2}[(M+M^*)^2-s]^{1/2}\right\}. \qquad (16\cdot19)$$

It is easy to check $\Delta I(b, \boldsymbol{p})=0$. It is noteworthy that the magnitude of the Lorentz-invariance violation in the *location* of the singularity $s=s_b$ is of second order in Im M, but, as is seen from (16·19), that in the amplitude itself is of *first order* in Im M.**) Summarizing the above consideration, we may say that the physical S-matrix of the relativistic complex-ghost field theory is unitary at the sacrifice of Lorentz invariance.***)

All the above consideration is crucially based on the assumption that all spatial momenta are real. Indeed, if they were also analytically continued, then $I(p)$ would become Lorentz-invariant but have the usual cut L on the real axis, violating the unitarity of the physical S-matrix. After Nakanishi's pointing-out of Lorentz non-invariance, Lee and Wick[L9)] became suspicious about taking spatial momenta real. [They gave up their original prescription and instead adopted an S-matrix-theoretical prescription proposed by Cutkosky, Landshoff, Olive and Polkinghorne.[C2)]] Their question to the real-spatial-momentum prescription is as follows: Under a Lorentz transformation, a

*) Before Nakanishi's work, the Lorentz non-invariance of the Lee-Wick theory was inferred, in a certain higher-order graph, by Cutkosky, Landshoff, Olive and Polkinghorne.[C2)] Unfortunately, however, what they discussed was the Lorentz-transformed Lee-Wick prescription but not the Lorentz transformation of the expression defined by the Lee-Wick prescription.

**) Compare this result with a claim made in Ref. C2).

***) In the Lee-Wick quantum electrodynamics,[L4),L8)] the complex-ghost self-energy graph itself does not exist because of gauge invariance, but we encounter it as a reduced graph of a fourth-order electron-positron scattering graph.

free complex-ghost wave function

$$\exp(-i\omega_{\boldsymbol{p}}x_0+i\boldsymbol{px}) \tag{16·20}$$

is transformed into a function having a non-real spatial momentum; then the Fourier transform to momentum space becomes impossible, and thus we cannot have the usual formulation of a field theory. Nakanishi[N18)] made an objection to their comment. To keep spatial momenta real is very essential in the framework of quantum field theory. In the commutator of an annihilation operator and a creation one, we encounter a three-dimensional Dirac δ-function, which is well defined in the sense of Schwartz's distribution[S5)] only if its argument, i. e., spatial momentum, is real. If one extended the definition of the δ-function to complex values of spatial momentum in the sense of analytic continuation (this is mathematically possible as described below), then the state-vector space $\mathcal{V}$ would no longer be a vector space in the usual sense. Furthermore, the Lee-Wick question on the difficulty of the Fourier transform is related to a c-number solution of the Klein-Gordon equation, but *it does not apply to the second-quantized field.* Indeed, the complex-ghost field operators with real spatial momenta can be shown to be a Lorentz scalar. In the following, we review Nakanishi's formulation[N20)] of the relativistic complex-ghost field theory, which is manifestly covariant at any *finite* time.

Let ϕ and $\phi^\dagger$ be spinless complex-ghost fields having M and M^*, respectively (Im $M>0$). As shown in § 14, the free Lagrangian density of complex-ghost fields is given by

$$\mathcal{L}_0=\frac{1}{2}(\partial^\mu\phi\partial_\mu\phi-M^2\phi^2+\partial^\mu\phi^\dagger\partial_\mu\phi^\dagger-M^{*2}\phi^{\dagger 2}). \tag{16·21}$$

The field operator $\phi(x)$ is expanded as

$$\phi(x)=\phi^{(+)}(x)+\phi^{(-)}(x) \tag{16·22}$$

with

$$\phi^{(+)}(x)\equiv(2\pi)^{-3/2}\int d\boldsymbol{p}(2\omega_{\boldsymbol{p}})^{-1/2}\alpha(\boldsymbol{p})\exp(i\boldsymbol{px}-i\omega_{\boldsymbol{p}}x_0),$$
$$\phi^{(-)}(x)\equiv(2\pi)^{-3/2}\int d\boldsymbol{p}(2\omega_{\boldsymbol{p}})^{-1/2}\beta^\dagger(\boldsymbol{p})\exp(-i\boldsymbol{px}+i\omega_{\boldsymbol{p}}x_0). \tag{16·23}$$

The canonical commutation relations imply that

$$[\alpha(\boldsymbol{p}),\beta^\dagger(\boldsymbol{q})]=[\beta(\boldsymbol{p}),\alpha^\dagger(\boldsymbol{q})]=\delta(\boldsymbol{p}-\boldsymbol{q}),$$
$$[\alpha(\boldsymbol{p}),\alpha^\dagger(\boldsymbol{q})]=[\beta(\boldsymbol{p}),\beta^\dagger(\boldsymbol{q})]=0,\ \text{etc.} \tag{16·24}$$

Therefore the four-dimensional commutation relations are

$$[\phi(x), \phi(y)] = i\Delta(x-y, M^2),$$
$$[\phi(x), \phi^\dagger(y)] = 0,$$
$$[\phi^\dagger(x), \phi^\dagger(y)] = i\Delta(x-y, M^{*2}), \tag{16·25}$$

where

$$\Delta(x, M^2) \equiv (2\pi)^{-3}\int d\boldsymbol{p}\omega_{\boldsymbol{p}}^{-1}\sin(\boldsymbol{p}\boldsymbol{x} - \omega_{\boldsymbol{p}}x_0). \tag{16·26}$$

It is easy to show that $\Delta(x, M^2)$ is Lorentz-invariant by using contour deformation in $\boldsymbol{p}$.

The generators P_μ and $M_{\mu\nu}$ of the Poincaré group can be explicitly constructed as

$$P_0 \equiv \int d\boldsymbol{p}\,[\omega_{\boldsymbol{p}}\beta^\dagger(\boldsymbol{p})\alpha(\boldsymbol{p}) + \omega_{\boldsymbol{p}}^*\alpha^\dagger(\boldsymbol{p})\beta(\boldsymbol{p})],$$
$$P_l \equiv \int d\boldsymbol{p}p_l\,[\beta^\dagger(\boldsymbol{p})\alpha(\boldsymbol{p}) + \alpha^\dagger(\boldsymbol{p})\beta(\boldsymbol{p})],$$
$$M_{0l} \equiv \frac{1}{2}i\int d\boldsymbol{p}\,[\omega_{\boldsymbol{p}}\beta^\dagger(\boldsymbol{p})\overleftrightarrow{\partial}_l\alpha(\boldsymbol{p}) + \omega_{\boldsymbol{p}}^*\alpha^\dagger(\boldsymbol{p})\overleftrightarrow{\partial}_l\beta(\boldsymbol{p})],$$
$$M_{kl} \equiv \frac{1}{2}i\int d\boldsymbol{p}\,[p_k\beta^\dagger(\boldsymbol{p})\overleftrightarrow{\partial}_l\alpha(\boldsymbol{p}) - p_l\beta^\dagger(\boldsymbol{p})\overleftrightarrow{\partial}_k\alpha(\boldsymbol{p})$$
$$+ p_k\alpha^\dagger(\boldsymbol{p})\overleftrightarrow{\partial}_l\beta(\boldsymbol{p}) - p_l\alpha^\dagger(\boldsymbol{p})\overleftrightarrow{\partial}_k\beta(\boldsymbol{p})], \tag{16·27}$$

where

$$A\overleftrightarrow{\partial}_l B \equiv (\partial A/\partial p_l)B - A(\partial B/\partial p_l). \tag{16·28}$$

By calculating $[\phi^{(\pm)}(x), M_{\mu\nu}]$, we can confirm that $\phi^{(\pm)}(x)$ is Lorentz-scalar, in spite of the fact that the spatial momentum $\boldsymbol{p}$ is real. This is because $M_{\mu\nu}$ is also written in terms of the operators of a real spatial momentum.

The vacuum $|0\rangle$ is defined by

$$\alpha(\boldsymbol{p})|0\rangle = \beta(\boldsymbol{p})|0\rangle = 0 \tag{16·29}$$

together with $\langle 0|0\rangle = 1$. Since (16·29) is equivalent to

$$\phi^{(+)}(x)|0\rangle = [\phi^{(-)}(x)]^\dagger|0\rangle = 0, \tag{16·30}$$

the vacuum is a Poincaré-invariant state. Thus both the Lagrangian and the quantization are Lorentz-invariant.

The one-particle states are expressed as wave-packet states such as

$$\int d^4x f(x)\,[\phi^{(+)}(x)]^\dagger|0\rangle, \tag{16·31}$$

where $f(x)$ is a function sufficiently localized in both space and time. The totality of the states (16·31) spans a Lorentz-invariant subspace. It is impotant to note that a Lorentz-transformed state of a pure plane-wave state $\alpha^\dagger(\boldsymbol{p})|0\rangle$ is *outside* our space $\mathcal{V}$, because such a "state" would be exponentially increasing in spatial directions. Thus we never need a Lorentz-transformed operator of $\alpha(\boldsymbol{p})$.

It is straightforward to calculate the two-point vacuum expectation values:

$$\begin{aligned}&\langle 0|\phi(x)\phi(y)|0\rangle=(2\pi)^{-3}\int d\boldsymbol{p}(2\omega_{\boldsymbol{p}})^{-1}\exp[i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})-i\omega_{\boldsymbol{p}}(x_0-y_0)],\\&\langle 0|\phi(x)\phi^\dagger(y)|0\rangle=0,\\&\langle 0|\phi^\dagger(x)\phi^\dagger(y)|0\rangle=(2\pi)^{-3}\int d\boldsymbol{p}(2\omega_{\boldsymbol{p}}^*)^{-1}\exp[i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})-i\omega_{\boldsymbol{p}}^*(x_0-y_0)].\end{aligned} \tag{16·32}$$

Since $\mathrm{Im}\,\omega_{\boldsymbol{p}}>0$, by Cauchy's theorem we have

$$\begin{aligned}&\theta(x_0-y_0)\langle 0|\phi(x)\phi(y)|0\rangle\\&\quad=i(2\pi)^{-4}\int d\boldsymbol{p}\int_{\Gamma_1}dp_0\frac{\exp[i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})-ip_0(x_0-y_0)]}{2\omega_{\boldsymbol{p}}(p_0-\omega_{\boldsymbol{p}})},\end{aligned} \tag{16·33}$$

where the p_0 contour Γ_1 runs from $-\infty$ to $+\infty$ but passes through *above* the pole $p_0=\omega_{\boldsymbol{p}}$, that is, $\Gamma_1\sim R-\delta(\omega_{\boldsymbol{p}})$. By interchanging x and y and transforming p_μ into $-p_\mu$, (16·33) becomes

$$\begin{aligned}&\theta(y_0-x_0)\langle 0|\phi(y)\phi(x)|0\rangle\\&\quad=-i(2\pi)^{-4}\int d\boldsymbol{p}\int_{\Gamma_2}dp_0\frac{\exp[i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})-ip_0(x_0-y_0)]}{2\omega_{\boldsymbol{p}}(p_0+\omega_{\boldsymbol{p}})},\end{aligned} \tag{16·34}$$

where Γ_2 runs from $-\infty$ to $+\infty$ but passes through *below* the pole $p_0=-\omega_{\boldsymbol{p}}$, that is, $\Gamma_2\sim R+\delta(-\omega_{\boldsymbol{p}})$. It is evident that $\Gamma_1\sim\Gamma$ in (16·33) and $\Gamma_2\sim\Gamma$ in (16·34), where

$$\Gamma\sim R-\delta(\omega_{\boldsymbol{p}})+\delta(-\omega_{\boldsymbol{p}}). \tag{16·35}$$

From (16·33) and (16·34), therefore, we obtain

$$\langle 0|\mathrm{T}[\phi(x)\phi(y)]|0\rangle=\Delta_F(x-y,M^2) \tag{16·36}$$

with

$$\Delta_F(x,M^2)\equiv-i(2\pi)^{-4}\int d\boldsymbol{p}\int_\Gamma dp_0\frac{e^{-ipx}}{M^2-p^2}, \tag{16·37}$$

where Γ is defined by (16·35). We note that $\Delta_F(x,M^2)$ is Lorentz-invariant, as it should be.

Likewise, we have

$$\begin{aligned}&\langle 0|\mathrm{T}[\phi(x)\phi^\dagger(y)]|0\rangle=0,\\&\langle 0|\mathrm{T}[\phi^\dagger(x)\phi^\dagger(y)]|0\rangle=\Delta_F(x-y,M^{*2}),\end{aligned} \tag{16·38}$$

where the p_0 contour Γ' of $\Delta_F(x, M^{*2})$ is homologous to R, that is, $\Delta_F(x, M^{*2})$ has an ordinary four-dimensional Fourier transform.

Now, we introduce interactions with other fields. Let H_0 and H_1 be a free Hamiltonian and an interaction Hamiltonian, respectively; H_0 includes not only complex-ghost fields but also physical fields. In the interaction picture, as is well known, the interaction Hamiltonian is given by

$$H_I(x_0) \equiv e^{iH_0x_0} H_1 e^{-iH_0x_0}. \tag{16·39}$$

As remarked in § 11, Dyson's S-matrix does not exist if we employ the usual exponential adiabatic factor $e^{-\varepsilon|x_0|}$. Hence we here employ the Gaussian adiabatic factor $e^{-\varepsilon^2 x_0^2}$, that is, we define

$$H_I^{\varepsilon}(x_0) \equiv H_I(x_0) e^{-\varepsilon^2 x_0^2}. \tag{16·40}$$

Then the transition matrix $U^{\varepsilon}(x_0, x_0')$ has the infinite-time limit. Therefore we obtain the S-matrix

$$S^{\varepsilon} \equiv \sum_{n=0}^{\infty} (1/n!)(-i)^n \int d^4x^{(1)} \cdots \int d^4x^{(n)} \mathrm{T}[\mathscr{H}_I^{\varepsilon}(x^{(1)}) \cdots \mathscr{H}_I^{\varepsilon}(x^{(n)})] \tag{16·41}$$

apart from the vacuum polarization factor, where

$$H_I^{\varepsilon}(x_0) \equiv \int d\boldsymbol{x} \mathscr{H}_I^{\varepsilon}(x). \tag{16·42}$$

Of course, S^{ε} is dependent on ε, and this ε-dependence violates the Lorentz invariance of S^{ε}. Hence, as usual, we should take the $\varepsilon \to 0$ limit, but then the exponential divergence due to complex ghosts reappears. To avoid this difficulty, we empoly the notion of a *finite part* of a divergent integral. It is quite common in modern mathematics to define a finite part of a divergent series or integral. The most reasonable definition of a finite part is based on the uniqueness of analytic continuation. For example, consider Euler's integral of the first kind:

$$\int_0^1 du\ u^{\alpha-1}(1-u)^{\beta-1}. \tag{16·43}$$

This integral is convergent only for $\mathrm{Re}\,\alpha > 0$ and $\mathrm{Re}\,\beta > 0$. If, however, we take Hadamard's finite part, which is essentially analytic continuation in α and β, then (16·43) equals $B(\alpha, \beta)$ everywhere.*) Hadamard's finite part is most reasonably defined in terms of Schwartz's distribution.[S5]

In order to define a finite part of $\lim_{\varepsilon\to 0} S^{\varepsilon}$, we have to make some extension of Schwartz's distribution. In calculating each term of $\lim_{\varepsilon\to 0} S^{\varepsilon}$, the difficulty, of course, arises from each integration over $x_0^{(j)}$. A typical x_0 integral which we encounter is

*) We should suppose that Hadamard's finite part is extensively used in the dual resonance model;[M1] otherwise the usual integral representation is almost useless.

$$\varphi^{\varepsilon}(p_0-q_0)\equiv(2\pi)^{-1}\int_{-\infty}^{+\infty}dx_0 e^{i(p_0-q_0)x_0}e^{-\varepsilon^2 x_0^2}. \tag{16·44}$$

If both p_0 and q_0 are real then $\lim_{\varepsilon\to 0}\varphi^{\varepsilon}(p_0-q_0)$ is identical with $\delta(p_0-q_0)$. As seen from (16·37), however, energy cannot always be restricted to real values. For instance, suppose that q_0 is complex; then evidently $\lim_{\varepsilon\to 0}\varphi^{\varepsilon}$ is exponentially divergent. Our problem is to extract a finite part from it unambiguously.

Let $f(p_0)$ be an arbitrary function holomorphic in a strip which includes the real axis $\mathrm{Im}p_0=0$ and a straight line $\mathrm{Im}p_0=\mathrm{Im}q_0$. If the test function $f(p_0)$ sufficiently damps at infinity, we can define

$$F^{\varepsilon}(q_0)\equiv\int_{-\infty}^{+\infty}dp_0\varphi^{\varepsilon}(p_0-q_0)f(p_0). \tag{16·45}$$

On substituting (16·44) in (16·45), we can interchange the order of the integrations, because the integral is absolutely and uniformly convergent for ε finite. We then have

$$F^{\varepsilon}(q_0)=(2\pi)^{-1}\int_{-\infty}^{+\infty}dx_0 e^{-\varepsilon^2 x_0^2}\int_{-\infty}^{+\infty}dp_0 e^{i(p_0-q_0)x_0}f(p_0). \tag{16·46}$$

Because of the assumed analyticity of $f(p_0)$, the p_0 contour can be deformed into a line $\mathrm{Im}p_0\equiv\mathrm{Im}q_0$. After the contour deformation, we again interchange the order of integrations. Since $\mathrm{Im}(p_0-q_0)=0$ now, we can take the $\varepsilon\to 0$ limit to obtain

$$\lim_{\varepsilon\to 0}F^{\varepsilon}(q_0)=f(q_0). \tag{16·47}$$

Comparing (16·47) with (16·45), we find that $\lim_{\varepsilon\to 0}\varphi^{\varepsilon}(p_0-q_0)$ has the same effect as a δ-function. Therefore, we call it a *complex δ-function* and denote it by $\delta_c(p_0-q_0)$. This concept is not new, and it is indeed a special case of the complex distribution introduced by Nakanishi[N8)] in 1958 for the purpose of constructing an exact eigenstate corresponding to an unstable particle.

By using the complex δ-function, it is now easy to write down any Feynman integral. At each vertex of a Feynman graph, we have a product of a three-dimensional δ-function and a complex δ-function instead of a four-dimensional δ-function. We need, however, to take no special care for the treatment of complex δ-functions. Everything goes as in the ordinary Feynman integral except for the point that $\Delta_F(x, M^2)$ has a complex contour Γ. Thus, for example, (16·14) (without the subtraction term) is a second-order self-energy Feynman integral. For any tree graph, we need no complex energies.

The above procedure is manifestly covariant because, according to the Hall-Wightman theorem,[H1)] Lorentz invariance cannot be injured by analytic continuation. On the other hand, we know that $I(p)$ is not Lorentz-invariant. The reason for this apparent dilemma is that the assumed analyticity of the test function $f(p_0)$ is not generally satisfied in the actual Feynman integral. Indeed, in

$$\lim_{\varepsilon\to 0}\int dx\int_{-\infty}^{+\infty} dx_0 e^{i(p-k-q)x} e^{-\varepsilon^2 x_0^2}(M^2-q^2)^{-1}(M^{*2}-k^2)^{-1}, \qquad (16\cdot 48)$$

$f(p_0-k_0)$ should be identified with $(M^{*2}-k^2)^{-1}$, which has a pole in the strip. The violation of invariance property in taking a finite part is not quite foreign to us; a similar situation is encountered also in the conventional quantum field theory. The renormalization procedure always preserves Lorentz invariance, but it generally violates scale invariance and the γ_5-phase invariance.

The above definition of the S-matrix should strictly be distinguished from an artificial definition of the S-matrix. The prescription of taking a finite part is of no *ad hoc* nature; it is unambiguous. The introduction of complex δ-functions is necessary if we define the total S-matrix satisfying the energy conservation, because there are complex-energy eigenstates. Furthermore, we can show[N20)] that the physical S-matrix, which is strictly a *submatrix* of the S-matrix defined here, coincides with the Lee-Wick physical S-matrix (16·2).

Finally, we again emphasize that the relativistic complex-ghost field theory is a non-trivial, divergence-free quantum field theory which is manifestly covariant at any finite time and whose physical S-matrix is unitary and macrocausal. So far, we know no other theory which has all these features. The Lorentz non-invariance of the physical S-matrix is not necessarily its demerit but it can be a merit.*) The well-known parity violation and time-reversal non-invariance could be regarded as a suggestion of the possible violation of Lorentz invariance. We also note that the relativistic complex-ghost field theory provides a convenient model for analyzing the possible experimental departure from quantum electrodynamics, since the usual Feynman-cutoff model violates unitarity.

§ 17. Quantization of a purely-imaginary-mass field

As is well known, the 4-momentum of any observed elementary or composite particle is timelike or lightlike. One might feel that the non-existence of the particles having a spacelike 4-momentum is not aesthetic. It might not be quite unreasonable, therefore, to suppose the existence of such particles. Indeed, in the theory of the Bethe-Salpeter equation,[N14)] it is widely known that the solutions of the homogeneous Bethe-Salpeter equation exist and are well-behaved even for any negative value of the invariant square of the bound-state 4-momentum. Recently, Schroer and Swieca[S4)] found

*) There are some experimental indications of the possible violation of Lorentz invariance. For example, the coexistence of the 3°K black-body radiation in the universe and the ultra-high energy cosmic-ray primaries (>10^{20} eV) is most naturally explained by the violation of Lorentz invariance at very high energies (H. Sato and T. Tati, Prog. Theor. Phys. **47** (1972), 1788).

a similar situation also in the relativistic Schrödinger equation for a particle moving in a strong external field. In this section, we discuss the field-theoretical treatment of the particles having a spacelike 4-momentum, namely, the problem of quantizing the field having a purely imaginary mass. We note that the purely-imaginary-mass field does not necessarily require the introduction of indefinite metric.

For simplicity, we consider a scalar field ϕ, which satisfies the purely-imaginary-mass Klein-Gordon equation

$$(\Box-\mu^2)\phi=0, \quad \mu>0. \tag{17·1}$$

The invariant c-number solutions of (17·1) can be obtained from those of the complex-mass Klein-Gordon equation

$$(\Box+M^2)\phi=0 \tag{17·2}$$

by means of analytic continuation in M^2. The two independent invariant solutions of (17·2) are

$$\Delta(x, M^2)\equiv-(2\pi)^{-3}\int d\boldsymbol{p}(M^2+\boldsymbol{p}^2)^{-1/2}e^{i\boldsymbol{px}}\sin[(M^2+\boldsymbol{p}^2)^{1/2}x_0],$$
$$\Delta^{(1)}(x, M^2)\equiv(2\pi)^{-3}\int d\boldsymbol{p}(M^2+\boldsymbol{p}^2)^{-1/2}e^{i\boldsymbol{px}}\cos[(M^2+\boldsymbol{p}^2)^{1/2}x_0]. \tag{17·3}$$

Since the energy $(M^2+\boldsymbol{p}^2)^{1/2}$ has a branch point at $M^2=-\boldsymbol{p}^2$, for $M^2=-\mu^2<-\boldsymbol{p}^2$ the sign of $(M^2+\boldsymbol{p}^2)^{1/2}$ depends on the path of analytic continuation. Since $\Delta(x, M^2)$ is independent of the sign of $(M^2+\boldsymbol{p}^2)^{1/2}$, its analytic continuation to $M^2=-\mu^2$ is unique, but $\Delta^{(1)}(x, M^2)$ is analytically continued in *two* inequivalent ways. Thus we obtain three*) independent invariant solutions of (17·1):

$$\Delta(x, -\mu^2)=-(2\pi)^{-3}\int d\boldsymbol{p}[\theta(\boldsymbol{p}^2-\mu^2)\omega_{\boldsymbol{p}}^{-1}e^{i\boldsymbol{px}}\sin\omega_{\boldsymbol{p}}x_0$$
$$+\theta(\mu^2-\boldsymbol{p}^2)\tilde{\omega}_{\boldsymbol{p}}^{-1}e^{i\boldsymbol{px}}\sinh\tilde{\omega}_{\boldsymbol{p}}x_0],$$
$$\mathrm{Re}\Delta^{(1)}(x, -\mu^2)=(2\pi)^{-3}\int d\boldsymbol{p}\theta(\boldsymbol{p}^2-\mu^2)\omega_{\boldsymbol{p}}^{-1}e^{i\boldsymbol{px}}\cos\omega_{\boldsymbol{p}}x_0,$$
$$\mathrm{Im}\Delta^{(1)}(x, -\mu^2)=(2\pi)^{-3}\int d\boldsymbol{p}\theta(\mu^2-\boldsymbol{p}^2)\tilde{\omega}_{\boldsymbol{p}}^{-1}e^{i\boldsymbol{px}}\cosh\tilde{\omega}_{\boldsymbol{p}}x_0, \tag{17·4}$$

where

$$\omega_{\boldsymbol{p}}\equiv\sqrt{\boldsymbol{p}^2-\mu^2} \quad \text{for} \quad \boldsymbol{p}^2\geq\mu^2,$$
$$\tilde{\omega}_{\boldsymbol{p}}\equiv\sqrt{\mu^2-\boldsymbol{p}^2} \quad \text{for} \quad \boldsymbol{p}^2\leq\mu^2. \tag{17·5}$$

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*) It seems that the third one of (17·4) has been unnoticed so far. The author is grateful to Dr. Kamoi and Prof. Kamefuchi for communication on thier paper.[K3)]

Among the above three invariant functions, only $\mathrm{Re}\Delta^{(1)}(x, -\mu^2)$, which we write as $\Delta_T(x, -\mu^2)$ for brevity, has a manifestly Lorentz-invariant expression

$$\Delta_T(x, -\mu^2) \equiv \mathrm{Re}\Delta^{(1)}(x, -\mu^2) = (2\pi)^{-3}\int d^4p\,\delta(p^2+\mu^2)e^{-ipx}. \tag{17·6}$$

It is important to note that two terms of $\Delta(x, -\mu^2)$ are not Lorentz-invariant separately. This is because $\varepsilon(p_0)\delta(p^2+\mu^2)$ is not Lorentz-invariant.

(A) *Tachyon theory*

In quantizing the field having a purely imaginary mass, Feinberg[F1] and Arons and Sudarshan[A2] employed $\Delta_T(x, -\mu^2)$ in the four-dimensional commutation relation. Since $\Delta_T(x, -\mu^2)$ does not vanish for $x^2<0$, those particles can travel faster than light velocity. Feinberg, who called them "tachyons", proposed to quantize the purely-imaginary-mass field $\phi(x)$ in the Fermi statistics by

$$\{\phi(x), \phi(y)\} = \Delta_T(x-y, -\mu^2), \tag{17·7}$$

because $\Delta_T(x, -\mu^2)$ is an even function of x. Since the creation of negative-energy particles has to be interpreted as the annihilation of positive-energy particles and since the sign of the energy p_0 is not Lorentz-invariant, creation and annihilation, and therefore the vacuum, cannot be Lorentz-invariant notions. In order to avoid this difficulty, Arons and Sudarshan[A2] considered a non-hermitian scalar field $\phi(x)$, which contains annihilation operators alone. Then, $\phi(x)$ can obey the Bose statistics. Indeed, we can set

$$[\phi(x), \phi^\dagger(y)] = \Delta_T(x-y, -\mu^2),$$
$$[\phi(x), \phi(y)] = 0, \tag{17·8}$$

that is, in terms of two real fields $\phi_1 \equiv (\phi+\phi^\dagger)/\sqrt{2}$ and $\phi_2 \equiv (\phi-\phi^\dagger)/\sqrt{2}i$, we set

$$[\phi_j(x), \phi_j(y)] = 0, \quad (j=1, 2)$$
$$[\phi_1(x), \phi_2(y)] = i\Delta_T(x-y, -\mu^2). \tag{17·9}$$

In order to avoid the appearance of negative-energy particles as physical particles, Arons and Sudarshan defined the S-matrix under the reinterpretation principle that negative-enegry particles in the final state (*or* in the initial state) should be regarded as positive-energy particles in the initial state (*or* in the final state). Thus macrocausality is evidently violated.

There appeared a number of papers*) concerning tachyons: theoretical and experimental, classical and quantum-theoretical, pro and con,**) etc.

*) For an extensive list of references, see Danburg et al.[D1]

**) See, for example, Kamoi and Kamefuchi.[K3]

The most serious difficulty of the tachyon theory is the acausal character of tachyons. If they existed and were observable, we could obtain the information about *future* events by using a fast-moving tachyon-receiver-emitter.

At any rate, the quantum field theory of tachyons is based on serious misunderstanding. The differential equation (17·1) does *not* mean that the particle travels faster than light velocity. As is well known in mathematics, the propagation character of (17·1), is the same as that of the positive-mass Klein-Gordon equation (this was first pointed out by Tanaka[T2)]). The use of $\Delta_T(x, -\mu^2)$ in the commutation relation is *not* a realization of the super-light velocity, because $\Delta_T(x, -\mu^2)$ does not vanish also at timelike distances (a classical tachyon cannot travel more slowly than light velocity) and because even for the positive-mass case, $\Delta^{(1)}(x, m^2)$ does not vanish at spacelike distances. Furthermore, the use of $\Delta_T(x, -\mu^2)$ contradicts the very motivation of introducing purely-imaginary-mass particles, because then the aesthetic uniformity among the three cases $m^2>0$, $=0$ and <0 is lost.

(B) *Non-tachyon theory*

A more reasonable quantization of the purely-imaginary-mass field was proposed by Tanaka[T2)] much earlier than Feinberg's work, and recently Schroer[S3)] independently developed a similar formulation. Both in Tanaka's theory and in Schroer's one, $\Delta(x, -\mu^2)$ is used in the four-dimensional commutation relation. Since $\Delta(x, -\mu^2)$ vanishes at spacelike distances, we encounter no acausality problem. What is of super-light velocity is the group velocity $d\omega_{\boldsymbol{p}}/d|\boldsymbol{p}|$.[T2)]

We consider a hermitian scalar field $\phi(x)$, whose free Lagrangian density is given by

$$\mathcal{L}_0 = \frac{1}{2}(\partial^\mu\phi\partial_\mu\phi + \mu^2\phi^2). \tag{17·10}$$

We assume that $\phi(x)$ is expanded as

$$\begin{aligned}\phi(x) = (2\pi)^{-3/2}\int d\boldsymbol{p}\,\{&\theta(\boldsymbol{p}^2-\mu^2)(2\omega_{\boldsymbol{p}})^{-1/2}[\alpha(\boldsymbol{p})\exp(i\boldsymbol{px}-i\omega_{\boldsymbol{p}}x_0)\\ &+\alpha^\dagger(\boldsymbol{p})\exp(-i\boldsymbol{px}+i\omega_{\boldsymbol{p}}x_0)] + \theta(\mu^2-\boldsymbol{p}^2)(2\tilde{\omega}_{\boldsymbol{p}})^{-1/2}\\ &\cdot[\beta(\boldsymbol{p})\exp(i\boldsymbol{px}+\tilde{\omega}_{\boldsymbol{p}}x_0)+\gamma(\boldsymbol{p})\exp(-i\boldsymbol{px}-\tilde{\omega}_{\boldsymbol{p}}x_0)]\}.\end{aligned} \tag{17·11}$$

It should be noted in (17·11) that the spatial momentum can take *any* real value in contrast with the tachyon theory; for $\boldsymbol{p}^2<\mu^2$ the energy becomes purely imaginary. The hermiticity of $\phi(x)$ implies that

$$\beta^\dagger(\boldsymbol{p}) = \beta(-\boldsymbol{p}), \quad \gamma^\dagger(\boldsymbol{p}) = \gamma(-\boldsymbol{p}). \tag{17·12}$$

The canonical commutation relations can be rewritten as

$$[\alpha(\boldsymbol{p}), \alpha^\dagger(\boldsymbol{q})] = \delta(\boldsymbol{p}-\boldsymbol{q}),$$
$$[\beta(\boldsymbol{p}), \gamma(\boldsymbol{q})] = -i\delta(\boldsymbol{p}-\boldsymbol{q}), \tag{17·13}$$

and zero commutators for any other pairs. From (17·11) and (17·13), we have

$$[\phi(x), \phi(y)] = i\Delta(x-y, -\mu^2), \tag{17·14}$$

as is expected.

The problem is how to define the vacuum $|0\rangle$. There is no question in setting $\alpha(\boldsymbol{p})|0\rangle=0$. From the idea of analytic continuation, it is expected to be good to set either $\beta(\boldsymbol{p})|0\rangle=0$ or $\gamma(\boldsymbol{p})|0\rangle=0$. If we assume the spatial-rotation invariance,*) however, from (17·13) and (17·12) we then have

$$0 = \langle 0|[\beta(\boldsymbol{p}), \gamma(\boldsymbol{q})]|0\rangle = -i\delta(\boldsymbol{p}-\boldsymbol{q})\langle 0|0\rangle, \tag{17·15}$$

that is, the vacuum has to have zero norm. This result is quite unsatisfactory.

As a natural way of avoiding the above trouble, we propose to introduce an operator

$$\xi(\boldsymbol{p}) \equiv (1/\sqrt{2})[e^{(1/4)\pi i}\beta(\boldsymbol{p}) + e^{-(1/4)\pi i}\gamma(-\boldsymbol{p})] \tag{17·16}$$

and define the vacuum $|0\rangle$ by

$$\alpha(\boldsymbol{p})|0\rangle = \xi(\boldsymbol{q})|0\rangle = 0. \tag{17·17}$$

From (17·13), we see that $\xi(\boldsymbol{p})$ satisfies

$$[\xi(\boldsymbol{p}), \xi(\boldsymbol{q})] = 0,$$
$$[\xi(\boldsymbol{p}), \xi^\dagger(\boldsymbol{q})] = \delta(\boldsymbol{p}-\boldsymbol{q}). \tag{17·18}$$

Therefore, the quantization for the part $\boldsymbol{p}^2<\mu^2$ is analogous to that for the normal part $\boldsymbol{p}^2>\mu^2$. We need no use of indefinite metric.

The generators of the Poincaré group are as follows:

$$P_0 = \int d\boldsymbol{p}\,\{\theta(\boldsymbol{p}^2-\mu^2)\omega_{\boldsymbol{p}}\alpha^\dagger(\boldsymbol{p})\alpha(\boldsymbol{p})$$
$$-\theta(\mu^2-\boldsymbol{p}^2)\tilde{\omega}_{\boldsymbol{p}}\frac{1}{2}[\xi(\boldsymbol{p})\xi(-\boldsymbol{p})+\xi^\dagger(\boldsymbol{p})\xi^\dagger(-\boldsymbol{p})]\},$$

$$P_l = \int d\boldsymbol{p}\, p_l[\theta(\boldsymbol{p}^2-\mu^2)\alpha^\dagger(\boldsymbol{p})\alpha(\boldsymbol{p}) + \theta(\mu^2-\boldsymbol{p}^2)\xi^\dagger(\boldsymbol{p})\xi(\boldsymbol{p})],$$

*) In Schroer's theory,[S3] a special direction is introduced to violate the spatial-rotation invariance.

$$M_{0l}=\frac{1}{2}i\int d\boldsymbol{p}\{\theta(\boldsymbol{p}^2-\mu^2)\omega_{\boldsymbol{p}}\alpha^\dagger(\boldsymbol{p})\overleftrightarrow{\partial_l}\alpha(\boldsymbol{p})$$
$$+\theta(\mu^2-\boldsymbol{p}^2)\tilde{\omega}_{\boldsymbol{p}}\cdot\frac{1}{2}[\xi(\boldsymbol{p})\overleftrightarrow{\partial_l}\xi(-\boldsymbol{p})-\xi^\dagger(\boldsymbol{p})\overleftrightarrow{\partial_l}\xi^\dagger(-\boldsymbol{p})]\},$$
$$M_{kl}=\frac{1}{2}i\int d\boldsymbol{p}\{\theta(\boldsymbol{p}^2-\mu^2)[p_k\alpha^\dagger(\boldsymbol{p})\overleftrightarrow{\partial_l}\alpha(\boldsymbol{p})-p_l\alpha^\dagger(\boldsymbol{p})\overleftrightarrow{\partial_k}\alpha(\boldsymbol{p})]$$
$$+\theta(\mu^2-\boldsymbol{p}^2)[p_k\xi^\dagger(\boldsymbol{p})\overleftrightarrow{\partial_l}\xi(\boldsymbol{p})-p_l\xi^\dagger(\boldsymbol{p})\overleftrightarrow{\partial_k}\xi(\boldsymbol{p})]\},\tag{17·19}$$

where $\overleftrightarrow{\partial_l}$ is defined by (16·28). The spatial generators P_l and M_{kl} are essentially the same as those in the positive-mass theory if $\xi(\boldsymbol{p})$ is regarded as the extension of $\alpha(\boldsymbol{p})$ to $\boldsymbol{p}^2<\mu^2$. However, P_0 and M_{0l} do not commute with the number operator. Therefore, $|0\rangle$ can be an eigenstate of neither P_0 nor M_{0l}, that is, the invariance under time translations and pure Lorentz transformations is *spontaneously* broken.

The above difficulty cannot be avoided by considering a non-hermitian field as in (17·8). Following Tanaka,[T2] suppose that

$$[\phi(x),\phi(y)]=0,$$
$$[\phi(x),\phi^\dagger(y)]=i\Delta(x-y,-\mu^2).\tag{17·20}$$

Let $\phi_1=(\phi+\phi^\dagger)/\sqrt{2}$ and $\phi_2=(\phi-\phi^\dagger)/\sqrt{2}i$; then (17·20) is equivalent to

$$[\phi_j(x),\phi_k(y)]=i\delta_{jk}\Delta(x-y,\,-\mu^2).\tag{17·21}$$

Thus the situation reduces to the hermitian case.

The only way out of the difficulty is to set

$$[\phi(x),\phi(y)]=i\Delta(x-y,\,-\mu^2),$$
$$[\phi(x),\phi^\dagger(y)]=0\tag{17·22}$$

instead of (17·20). This quantization is nothing but a special case of the relativistic complex-ghost field theory formulated in § 16. In this case, therefore, the purely-imaginary-mass particles are no longer observable at all, as is consistent with experimental data.[D1]

§ 18. Manifestly covariant quantization of the electromagnetic field

The most widely known example of the indefinite metric quantum field theory is the manifestly covariant quantum electrodynamics proposed by Gupta[G6] and completed by Bleuler.[B4],*) Since the photon mass is exactly zero,

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*) There are a number of papers concerning the Gupta-Bleuler theory and its extension. Some recent papers are those by Dürr and Rudolph,[D5],[D6] Brevik and Lautrup,[B8] Hayakawa and Yokoyama,[H5] Gomatam[G4] and Bertrand.[B2]

the longitudinal photon is not observable, but instead there is Coulomb interaction between charged particles. The positive-metric quantization of the electromagnetic field is possible if we add a non-local and non-covariant Coulomb interaction term in the Lagrangian; the S-matrix is then Lorentz invariant. This formalism is, however, not manifestly covariant. It is *impossible* to formulate a manifestly covariant quantum electrodynamics within the framework of the positive-metric theory,*) because the quanta of the Coulomb potential are not observable. This fact is closely related to the non-compactness of the little group of a lightlike vector.

In the Gupta-Bleuler theory, the longitudinal photon has positive norm, while the timelike photon has negative norm. The physical states are those in which longitudinal photons and timelike photons are superposed in an equal weight. The Lorentz condition $\partial^\mu A_\mu=0$ is not satisfied as an operator identity, but it is reproduced only as an expectation value in the physical state. Therefore, the photon's free propagator has no gradient term, that is, it is

$$\frac{ig_{\mu\nu}}{-p^2-i\varepsilon} \tag{18·1}$$

in momentum space.

As is well known, quantum electrodynamics is gauge invariant. The photon propagator is gauge-dependent; the gauge of (18·1) is called the Feynman gauge or the Fermi gauge. The photon propagator in more general covariant gauge is

$$\frac{ig_{\mu\nu}}{-p^2-i\varepsilon}+\lambda\frac{ip_\mu p_\nu}{(-p^2-i\varepsilon)^2}. \tag{18·2}$$

If $\lambda=0$, (18·2) reduces to (18·1). The case $\lambda=1$ is called the Landau-Khalatonikov gauge, or simply the Landau gauge. As will be seen later, the Landau gauge is most convenient; only in this gauge the electromagnetic field $A_\mu(x)$ satisfies the Lorentz condition as an operator identity. It is impossible to obtain the Landau-gauge quantum electrodynamics from the Feynman-gauge one in a manifestly covariant way by a q-number gauge transformation, because the photon propagator of the former contains a double pole as seen in (18·2) while (18·1) does not. In order to obtain (18·2) with $\lambda\neq0$, it is necessary to introduce a dipole-ghost field operator, as was pointed out by Nakanishi.[N11)]

A canonical formalism of the electromagnetic field in the general covariant gauge was formulated, within the framework of classical theory,

*) There are several papers which claim the success of constructing a manifestly covariant quantum electrodynamics without using indefinite metric. All of them are incorrect. Indeed, Strocchi[S8)] proved the non-existence of a Lorentz-vector operator $A_\mu(x)$ which reproduces the Maxwell equation (see also Wightman and Gårding[W1)]).

first by Utiyama[U2)] (much earlier than Nakanishi's work). The manifestly covariant quantum electrodynamics in the non-Feynman covariant gauge was proposed by Nakanishi[N11),N13)] and by Lautrup,[L2)] independently. Lautrup's theory was very thorough, but unfortunately he was not aware of the necessity of dipole-ghost quantization. Some modified versions of the Nakanishi-Lautrup formalism were presented by Lukierski,[L16)] by Goto and Obara[G5)] and by Yokoyama.[Y9),*)]

We first consider the manifestly covariant quantization of the free electromangetic field $A_\mu(x)$ in the general covariant gauge. For this purpose, it is necessary to introduce an auxiliary scalar field $B(x)$. Of course, both $A_\mu(x)$ and $B(x)$ are hermitian. The Lagrangian density is given by

$$\mathcal{L}_0 = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + B\partial^\mu A_\mu + \frac{1}{2}\alpha B^2, \tag{18·3}$$

where α is a real parameter and

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{18·4}$$

In (18·3), we may generalize the last term to an arbitrary function of B, but we do not consider such a general case. In contrast with a usual Bose field, B has a dimension of the mass *squared*. If $\alpha \neq 0$, then setting

$$C \equiv \partial^\mu A_\mu + \alpha B, \tag{18·5}$$

we can write

$$\mathcal{L}_0 = \mathcal{L}_0' + \frac{1}{2}\alpha^{-1}C^2 \tag{18·6}$$

with

$$\mathcal{L}_0' \equiv -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}\alpha^{-1}(\partial^\mu A_\mu)^2. \tag{18·7}$$

Hence we may start with $\mathcal{L}_0'$ by omitting C; thus we can eliminate B. It is impossible, however, to eliminate B in the case $\alpha = 0$, which is the most important Landau-gauge case. Therefore we use (18·3) throughout.

The field equations which follow from (18·3) are

$$\Box A_\mu - \partial_\mu \partial^\nu A_\nu - \partial_\mu B = 0, \tag{18·8}$$

$$\partial^\mu A_\mu + \alpha B = 0. \tag{18·9}$$

From (18·8), we have

$$\Box B = 0. \tag{18·10}$$

Hence B is a massless field. On eliminating $\partial^\nu A_\nu$ in (18·8) by means of

*) An extension to the weak gravitational field was made by Yokoyama and Kubo[Y11)] (and Yokoyama[Y10)]).

(18·9), we obtain

$$\Box A_\mu-(1-\alpha)\partial_\mu B=0. \tag{18·11}$$

From (18·10), therefore, we find

$$\Box^2 A_\mu=0, \tag{18·12}$$

that is, A_μ does not satisfy the d'Alembert equation but does the double d'Alembert equation.

Let π_μ be the canonical conjugate of A_μ, that is,

$$\begin{aligned}&\pi_l\equiv\delta\mathcal{L}_0/\delta\dot{A}_l=\dot{A}_l-\partial_l A_0,\\&\pi_0\equiv\delta\mathcal{L}_0/\delta\dot{A}_0=B,\end{aligned} \tag{18·13}$$

where a dot stands for differentiation with respect to x_0 as usual. We note that the canonical conjugate of B is A_0.

The Hamiltonian $H_0\equiv\int d\boldsymbol{x}\mathcal{H}_0(x)$ is defined by

$$\mathcal{H}_0\equiv\sum_{l=1}^{3}\pi_l\dot{A}_l+\pi_0\dot{A}_0-\mathcal{L}_0, \tag{18·14}$$

that is,

$$\mathcal{H}_0=\frac{1}{4}\sum_{k,l}F_{kl}^2+\frac{1}{2}\sum_l[(\dot{A}_l)^2-(\partial_l A_0)^2]+B\sum_l\partial_l A_l-\frac{1}{2}\alpha B^2. \tag{18·15}$$

The canonical commutation relations,

$$\begin{aligned}&[A_\mu(x),A_\nu(y)]_{x_0=y_0}=[\pi_\mu(x),\pi_\nu(y)]_{x_0=y_0}=0,\\&[A_\mu(x),\pi_\nu(y)]_{x_0=y_0}=i\delta_{\mu\nu}\delta(\boldsymbol{x}-\boldsymbol{y}),\end{aligned} \tag{18·16}$$

can be rewritten in terms of A_μ, $\dot{A}_l$ and B:

$$\begin{aligned}&[A_\mu(x),A_\nu(y)]_{x_0=y_0}=0,\\&[A_\mu(x),\dot{A}_l(y)]_{x_0=y_0}=i\delta_{\mu l}\delta(\boldsymbol{x}-\boldsymbol{y}),\\&[\dot{A}_k(x),\dot{A}_l(y)]_{x_0=y_0}=0,\\&[A_\mu(x),B(y)]_{x_0=y_0}=i\delta_{\mu 0}\delta(\boldsymbol{x}-\boldsymbol{y}),\\&[\dot{A}_k(x),B(y)]_{x_0=y_0}=i\partial_k^x\delta(\boldsymbol{x}-\boldsymbol{y}),\\&[B(x),B(y)]_{x_0=y_0}=0,\end{aligned} \tag{18·17}$$

where $\partial_\mu^x\equiv\partial/\partial x_\mu$. These relations involve neither $\dot{A}_0$ nor $\dot{B}$, but from the field equations (18·9) and (18·11), we obtain

$$\dot{A}_0=\sum_l\partial_l A_l-\alpha B, \tag{18·18}$$

$$\dot{B}=\sum_l\partial_l\dot{A}_l-\Delta A_0, \tag{18·19}$$

where Δ denotes the Laplacian. It is to be noted that (18·18) and (18·19) are equivalent to the Heisenberg equations $i\dot{A}_0=[A_0, H_0]$ and $i\dot{B}=[B_0, H_0]$, respectively.

From (18·19) and (18·17), we have

$$[\dot{B}(x), B(y)]_{x_0=y_0}=0, \tag{18·20}$$

We rewrite (18·10) as[L2)]

$$B(x)=\int d\boldsymbol{y} D(x-y)\overleftrightarrow{\partial_0^y}B(y), \tag{18·21}$$

where

$$f\overleftrightarrow{\partial_0^y}g\equiv(\partial f/\partial y_0)g-f\partial g/\partial y_0. \tag{18·22}$$

The validity of (18·21) is confirmed in the following way: Differentiate the right-hand side of (18·21) with respect to y_0 to see that it is independent of y_0, and then set $y_0=x_0$ to see that it is equal to $B(x)$. Since y_0 is arbitrary in (18·21), we can make use of (18·17) to compute the four-dimensional commutator. We find

$$[B(x), B(y)]=0. \tag{18·23}$$

Thus $B(x)$ is a zero-norm field. Analogously, we obtain

$$[A_\mu(x), B(y)]=-i\partial_\mu^x D(x-y). \tag{18·24}$$

In order to calculate $[A_\mu, A_\nu]$, we need a causal invariant solution of $\Box^2\varphi=0$ other than $D(x)$. It is defined by

$$\begin{aligned} E(x)&\equiv-(\partial/\partial m^2)\Delta(x, m^2)|_{m=0} \\ &=-i(2\pi)^{-3}\int d^4p\varepsilon(p_0)\delta'(p^2)e^{-ipx} \\ &=-(8\pi)^{-1}\varepsilon(x_0)\theta(x^2). \end{aligned} \tag{18·25}$$

As is easily seen, $E(x)$ has the following properties:

$$\Box E(x)=D(x), \tag{18·26}$$

$$\begin{aligned} &E(0, \boldsymbol{x})=\dot{E}(0, \boldsymbol{x})=\ddot{E}(0, \boldsymbol{x})=0, \\ &\dddot{E}(0, \boldsymbol{x})=-\delta(\boldsymbol{x}). \end{aligned} \tag{18·27}$$

By using $E(x)$, we rewrite (18·12) as

$$A_\mu(x)=\int d\boldsymbol{y} D(x-y)\overleftrightarrow{\partial_0^y}A_\mu(y)+\int d\boldsymbol{y} E(x-y)\overleftrightarrow{\partial_0^y}\Box A_\mu(y). \tag{18·28}$$

We can compute $[A_\mu, A_\nu]$ by means of (18·28) and (18·17):

$$[A_\mu(x), A_\nu(y)]=-ig_{\mu\nu}D(x-y)+i(1-\alpha)\partial_\mu^x\partial_\nu^x E(x-y). \tag{18·29}$$

In analogy with $D^{(\pm)}(x)$ (the positive and negative frequency parts of $D(x)$), we wish to define $E^{(\pm)}(x)$ by $-(\partial/\partial m^2)\Delta^{(\pm)}(x, m^2)|_{m=0}$, but the latter quantities are infrared divergent ($E(x)$ is not). To avoid this trouble, we define $E^{(\pm)}(x)$ by

$$E^{(\pm)}(x) \equiv \mp i(2\pi)^{-3}\int d^4p\,\theta(p_0)\delta'(p^2)(e^{-ipx}-1). \tag{18·30}$$

It is evident that

$$E(x) = E^{(+)}(x) + E^{(-)}(x), \tag{18·31}$$

$$[E^{(\pm)}(x)]^* = -E^{(\pm)}(-x) = E^{(\mp)}(x). \tag{18·32}$$

We can define $A_\mu^{(\pm)}(x)$ and $B^{(\pm)}(x)$ by replacing $D(x)$ and $E(x)$ by $D^{(\pm)}(x)$ and $E^{(\pm)}(x)$, respectively, in (18·28) and in (18·21). Then

$$A_\mu(x) = A_\mu^{(+)}(x) + A_\mu^{(-)}(x),$$

$$B(x) = B^{(+)}(x) + B^{(-)}(x), \tag{18·33}$$

$$(A_\mu^{(+)})^\dagger = A_\mu^{(-)},\ (B^{(+)})^\dagger = B^{(-)}. \tag{18·34}$$

Since $A_\mu^{(\pm)}$ and $B^{(\pm)}$ are translationally invariant and Lorentz-covariant, the vacuum $|0\rangle$ is consistently defined by

$$A_\mu^{(+)}(x)|0\rangle = 0,\ B^{(+)}(x)|0\rangle = 0 \tag{18·35}$$

with $\langle 0|0\rangle = 1$. Therefore

$$\langle 0|A_\mu(x)A_\nu(y)|0\rangle = -ig_{\mu\nu}D^{(+)}(x-y) + i(1-\alpha)\partial_\mu^x\partial_\nu^x E^{(+)}(x-y). \tag{18·36}$$

Since for an arbitrary function $f(x)$ we have

$$\theta(\pm x_0)\partial_0 f(x) = \partial_0[\theta(\pm x_0)f(x)] \mp \delta(x_0)f(x),$$

$$\theta(\pm x_0)\partial_0^2 f(x) = \partial_0^2[\theta(\pm x_0)f(x)] \mp \partial_0[\delta(x_0)f(x)] \mp \delta(x_0)\partial_0 f(x), \tag{18·37}$$

with the aid of (18·32), (18·31) and (18·27), we obtain

$$\langle 0|\mathrm{T}[A_\mu(x)A_\nu(y)]|0\rangle = -g_{\mu\nu}D_F(x-y) + (1-\alpha)\partial_\mu^x\partial_\nu^x E_F(x-y), \tag{18·38}$$

where*)

$$E_F(x) \equiv i\theta(x_0)E^{(+)}(x) - i\theta(-x_0)E^{(-)}(x)$$
$$= -i(2\pi)^{-4}\int d^4p[(-p^2-i\varepsilon)^{-2}e^{-ipx} + i\pi\delta'(p^2)], \tag{18·39}$$

*) Note that the second term of the integrand in (18·39) does not contribute to (18·38).

$$\Box E_F(x) = D_F(x). \tag{18·40}$$

The fourier transform of (18·38) coincides with (18·2) if we set $\lambda = 1 - \alpha$. Likewise, we have

$$\langle 0 | \mathrm{T}[A_\mu(x) B(y)] | 0 \rangle = -\partial_\mu^x D_F(x-y),$$
$$\langle 0 | \mathrm{T}[B(x) B(y)] | 0 \rangle = 0. \tag{18·41}$$

The physical states are defined by

$$B^{(+)}(x) | \text{phys} \rangle = 0. \tag{18·42}$$

Then it is evident from (18·11) and (18·9) that $\langle \text{phys} | A_\mu(x) | \text{phys} \rangle$ satisfies both the d'Alembert equation and the Lorentz condition.

Next, we consider the momentum representation of the field operators. Because of the existence of the second term in (18·29), we cannot directly write down the usual three-dimensional fourier transform of $A_\mu(x)$. Lautrup[L2] (see also Lukierski[L16]), therefore, took the following technique. Let

$$\Lambda(x) \equiv \frac{1}{2} \Delta^{-1} [x_0 \partial_0 B(x) - \frac{1}{2} B(x)]. \tag{18·43}$$

Then (cf. § 13) it satisfies

$$\Box \Lambda = B \tag{18·44}$$

because of (18·10). He defined

$$A_\mu^F \equiv A_\mu + (\alpha - 1) \partial_\mu \Lambda; \tag{18·45}$$

A_μ^F satisfies $\Box A_\mu^F = 0$ because of (18·11). With the aid of an identity

$$E(x) = \frac{1}{2} \Delta^{-1} [x_0 \partial_0 D(x) - D(x)], \tag{18·46}$$

it is straightforward to show that

$$[A_\mu^F(x), A_\nu^F(y)] = -i g_{\mu\nu} D(x-y). \tag{18·47}$$

Therefore, it is possible to write down the three-dimensional fourier transform of $A_\mu^F(x)$ just as in the Gupta-Bleuler theory. This formalism is, however, unsatisfactory because Λ is *not a Lorentz scalar* and therefore A_μ^F is not a Lorentz vector. As shown in §13, it is impossible to find an invariant solution of (18·44). Since manifest covariance is the principal motivation of the present formalism, its violation cannot be accepted. To avoid the use of dipole ghosts can be attained only at the sacrifice of manifest covariance (cf. §13), as long as $\alpha \neq 1$.

The four-dimensional fourier transform of $A_\mu(x)$, considered by Nakanishi,[N11] respects the manifest covariance of the theory. A defect of this method is that the Hamiltonian cannot be expressed in terms of momentum-space operators. Let

$$A_\mu(x)=(2\pi)^{-3/2}\int d^4p\theta(p_0)\,[a_\mu(p)e^{-ipx}+a_\mu^\dagger(p)e^{ipx}],$$
$$B(x)=(2\pi)^{-3/2}\int d^4p\theta(p_0)\,[b(p)e^{-ipx}+b^\dagger(p)e^{ipx}]. \qquad (18\cdot48)$$

The field equations (18·10), (18·9) and (18·11) are equivalent to

$$p^2b^\dagger(p)=0, \qquad (18\cdot49)$$
$$p^\mu a_\mu^\dagger(p)=i\alpha b^\dagger(p), \qquad (18\cdot50)$$
$$p^2a_\mu^\dagger(p)=-i(1-\alpha)p_\mu b^\dagger(p), \qquad (18\cdot51)$$

respectively. The four-dimensional commutation relations (18·29), (18·24) and (18·23) become

$$[a_\mu(p),a_\nu^\dagger(q)]=-\delta^4(p-q)\,[g_{\mu\nu}\delta(p^2)+(1-\alpha)p_\mu p_\nu\delta'(p^2)], \qquad (18\cdot52)$$
$$[a_\mu(p),b^\dagger(q)]=i\delta^4(p-q)p_\mu\delta(p^2), \qquad (18\cdot53)$$
$$[b(p),b^\dagger(q)]=0, \qquad (18\cdot54)$$

respectively. The definition (18·35) of the vacuum is rewritten as

$$a_\mu(p)\,|0\rangle=0, \quad b(p)\,|0\rangle=0. \quad (p_0>0) \qquad (18\cdot55)$$

There are four independent one-particle states having a 4-momentum p_μ ($p_0>0$) because of (18·50). We take a Lorentz frame in which $p_1=p_2=0$ (then $p_3\neq0$ on the mass shell). It is convenient to adopt the following four one-particle states:

$$|p,T_l\rangle\equiv a_l^\dagger(p)\,|0\rangle, \quad (l=1,2)$$
$$|p,L\rangle=a_3^\dagger(p)\,|0\rangle,$$
$$|p,S\rangle\equiv b^\dagger(p)\,|0\rangle. \qquad (18\cdot56)$$

The first two, $|p,T_l\rangle$ ($l=1,2$), are called transverse photons, the third, $|p,L\rangle$, is a longitudinal photon, and the fourth, $|p,S\rangle$, is a scalar photon. From (18·49)∼(18·55), we have

$$p^2|p,T_l\rangle=p^2|p,S\rangle=0,$$
$$p^2|p,L\rangle=-i(1-\alpha)p_3|p,S\rangle, \qquad (18\cdot57)$$

and

$$\langle p,T_k|q,T_l\rangle=\delta_{kl}\delta^4(p-q)\delta(p^2),$$
$$\langle p,L|q,L\rangle=\delta^4(p-q)\,[\delta(p^2)-(1-\alpha)p_3^2\delta'(p^2)],$$
$$\langle p,S|q,S\rangle=0,$$

$$\langle p, T_i | q, L\rangle = \langle p, T_i | q, S\rangle = 0,$$
$$\langle p, L | q, S\rangle = i\delta^4(p-q)p_3\delta(p^2). \tag{18·58}$$

Thus the transverse photons have positive norm and satisfy the d'Alembert equation, the longitudinal photon is a dipole ghost except for $\alpha=1$ and the scalar photon is the associated zero-norm state.

Now, we introduce the interaction Lagrangian density

$$\mathcal{L}_I = -j^\mu A_\mu, \tag{18·59}$$

where the current j_μ is assumed to be conserved:

$$\partial^\mu j_\mu = 0. \tag{18·60}$$

The Heisenberg operators $A_\mu(x)$ and $B(x)$ satisfy

$$\Box A_\mu - (1-\alpha)\partial_\mu B = j_\mu, \tag{18·61}$$
$$\partial^\mu A_\mu + \alpha B = 0, \tag{18·62}$$
$$\Box B = 0. \tag{18·63}$$

Because of (18·63), we can consistently define the positive-frequency part $B^{(+)}(x)$ of $B(x)$. The constraint

$$B^{(+)}(x)|\text{phys}\rangle = 0 \tag{18·64}$$

persists, and therefore the physical-state condition is satisfied. It should be noted that though $B(x)$ satisfies (18·63), it is not a free field because it does not commute with a matter field at lightlike distances.[L2)]

The S-matrix is constructed as usual. The only difference from the Gupta-Bleuler theory is of course the gauge of the photon propagator. Since $\mathcal{L}_I$ does not involve B, the propagators in (18·41) are unnecessary. The physical S-matrix is unitary; this result is owing to (18·60), but not owing to the use of dipole ghosts as discussed in §10.

As is well known, quantum electrodynamics is gauge-invariant. There is no problem for a c-number gauge transformation $A'_\mu = A_\mu + a\partial_\mu \Lambda_c$ and $B' = B$, where Λ_c is a c-number function satisfying $\Box \Lambda_c = 0$. In order to change the gauge of the photon propagator, it is necessary to consider a q-number gauge transformation.[L1)] If we employ an operator Λ satisfying (18·44), then the gauge parameter α is transformed into β by[L2)]

$$A'_\mu = A_\mu + (\alpha-\beta)\partial_\mu \Lambda, \qquad B' = B. \tag{18·65}$$

As emphasized above, however, Λ cannot be a Lorentz scalar. Therefore, *the gauge transformation (18·65) necessarily conflicts with manifest covariance.* Since for each value of α we have a different indefinite-metric Hilbert space $\mathcal{V}_\alpha$, the q-number gauge transformation which is compatible with manifest covariance could be realized only in a larger space $\sum_\alpha\oplus\mathcal{V}_\alpha$.

In this connection, Yokoyama's formalism[Y9)] is noteworthy because it has a scalar operator satisfying (18·44). His free Lagrangian density $\mathcal{L}_0$ consists of two parts $\mathcal{L}_0^{GB}$ and $\mathcal{L}_0^F$, where

$$\mathcal{L}_0^{GB}\equiv-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}-\frac{1}{2}(\partial^\mu A_\mu)^2, \tag{18·66}$$

$$\mathcal{L}_0^F\equiv\partial^\mu C\partial_\mu B-\frac{1}{2}\lambda B^2; \tag{18·67}$$

$\mathcal{L}_0^{GB}$ is equivalent to the free Lagrangian density in the Gupta-Bleuler theory and $\mathcal{L}_0^F$ is equivalent to the massless Froissart model (see § 13). Since

$$\Box C=-\lambda B,\quad \Box B=0, \tag{18·68}$$

$$[C(x),C(y)]=-i\lambda E(x-y), \tag{18·69}$$

if we define

$$A_\mu^{(\lambda)}\equiv A_\mu-\partial_\mu C, \tag{18·70}$$

then $A_\mu^{(\lambda)}$ is manifestly covariant and satisfies

$$[A_\mu^{(\lambda)}(x),A_\nu^{(\lambda)}(y)]=-ig_{\mu\nu}D(x-y)+i\lambda\partial_\mu^x\partial_\nu^x E(x-y). \tag{18·71}$$

Since we do not have a relation like (18·9), however, $\partial^\mu A_\mu^{(\lambda)}$ cannot be expressed in terms of B only. In particular, for the Landau gauge $\lambda=1$, the Lorentz condition is not satisfied as an operator identity; we have to introduce a constraint to reproduce the Lorentz condition. Thus the genuine Landau-gauge theory cannot be described by Yokoyama's formalism. This point is crucial when we consider the Heisenberg operators.

Finally, we discuss the renormalization of $A_\mu(x)$. An important feature of the electromagnetic field is that, except in the Landau gauge, the renormalized field A_μ^r has a gauge *different* from the unrenormalized one, as was observed first by Källén.[K1),H5)]

For simplicity, we consider the Gupta-Bleuler theory. Let $|\Omega\rangle$ be the true vacuum. Then from Lorentz invariance, local commutativity and $\partial^\mu\Box A_\mu^r=0$, we have a spectral representation

$$\begin{aligned}\langle\Omega|\,[A_\mu^r(x),A_\nu^r(y)]\,|\Omega\rangle\\ =-i\int_0^\infty ds\{[\delta(s)+\sigma(s)]g_{\mu\nu}+[s^{-1}\sigma(s)-K\delta(s)]\partial_\mu^x\partial_\nu^x\}\Delta(x-y,s)\end{aligned} \tag{18·72}$$

with $\sigma(s)\geq0$ and $K\equiv\int_0^\infty ds\sigma(s)/s$ (from an equal-time commutator). Therefore

$$\begin{aligned}\langle\Omega|\,[\partial^\mu A_\mu^r(x),A_\nu^r(y)]\,|\Omega\rangle&=-i\partial_\nu^x D(x-y)\\ &=\langle\Omega|\,[\partial^\mu A_\mu(x),A_\nu(y)]\,|\Omega\rangle.\end{aligned} \tag{18·73}$$

Accordingly, the simple-minded renormalization $A_\mu^r = Z_3^{-1/2} A_\mu$ contradicts (18·73). We should define

$$A_\mu = Z_3^{1/2} A_\mu^r - (1-Z_3)\partial_\mu \Lambda, \qquad (18\cdot 74)$$

where

$$Z_3^{-1} = 1 + \int_0^\infty ds\,\sigma(s), \qquad (18\cdot 75)$$

$$\Box \Lambda = -\partial^\mu A_\mu, \qquad (18\cdot 76)$$

so that

$$\partial^\mu A_\mu = Z_3^{-1/2} \partial^\mu A_\mu^r. \qquad (18\cdot 77)$$

As emphasized several times, however, there is no Lorentz scalar Λ satisfying (18·76) in the Gupta-Bleuler theory. Thus we conclude that *in the Gupta-Bleuler theory it is impossible to carry out the renormalization of the field operators in a manifestly covariant way.* The same is true also in any other covariant gauge except for the Landau gauge. *A manifestly covariant renormalization is possible only in the Landau gauge* (for detail, see § 19).

§ 19. Massive vector field

A massive vector field is usually quantized in a positive-metric Hilbert space. This is possible because the little group of a timelike vector is compact. For simplicity, we consider a hermitian vector field U_μ, whose free Lagrangian density is given by

$$\mathcal{L}_0 = -\frac{1}{4} V^{\mu\nu} V_{\mu\nu} + \frac{1}{2} m^2 U^\mu U_\mu \qquad (19\cdot 1)$$

with $m > 0$, where

$$V_{\mu\nu} \equiv \partial_\mu U_\nu - \partial_\nu U_\mu. \qquad (19\cdot 2)$$

The field equation, which is called the Proca equation, is

$$\Box U_\mu - \partial_\mu \partial^\nu U_\nu + m^2 U_\mu = 0, \qquad (19\cdot 3)$$

that is,

$$\partial^\mu U_\mu = 0, \quad (\Box + m^2) U_\mu = 0. \qquad (19\cdot 4)$$

Thus the Lorentz condition is identically satisfied. The four-dimensional commutation relation is

$$[U_\mu(x), U_\nu(y)] = -i(g_{\mu\nu} + m^{-2}\partial_\mu^x \partial_\nu^x) \Delta(x-y, m^2). \qquad (19\cdot 5)$$

There is another widely-known formalism, called the Stueckelberg formalism,[S9] which is a generalization of the electromagnetic field in the Feynman gauge. We consider a hermitian vector field A_μ and a hermitian scalar field B. They are supposed to have an equal mass.*) The free Lagrangian density is

$$\mathcal{L}_0=\frac{1}{2}[(\partial^\mu A^\nu)(\partial_\mu A_\nu)-m^2A^\mu A_\mu+(\partial^\mu B)(\partial_\mu B)-m^2B^2]. \qquad (19\cdot6)$$

Hence the field equations are

$$(\Box+m^2)A_\mu=0,$$
$$(\Box+m^2)B=0. \qquad (19\cdot7)$$

The four-dimensional commutation relations are

$$[A_\mu(x), A_\nu(y)]=-ig_{\mu\nu}\Delta(x-y, m^2),$$
$$[A_\mu(x), B(y)]=0,$$
$$[B(x), B(y)]=i\Delta(x-y, m^2). \qquad (19\cdot8)$$

Since $A_0(x)$ satisfies an abnormal commutation relation, we have to employ an indefinite-metric Hilbert space, as was remarked by Gupta.[G7] The vector field of physical interest is defined by

$$U_\mu\equiv A_\mu+m^{-1}\partial_\mu B. \qquad (19\cdot9)$$

It satisfies

$$(\Box+m^2)U_\mu=0, \qquad (19\cdot10)$$
$$[U_\mu(x), U_\nu(y)]=-i(g_{\mu\nu}+m^{-2}\partial_\mu^x\partial_\nu^x)\Delta(x-y, m^2). \qquad (19\cdot11)$$

In spite of the right-hand side of (19·11), $U_\mu(x)$ does not satisfy the Lorentz condition. Hence one has to introduce a constraint

$$(\partial^\mu U_\mu)^{(+)}|\text{phys}\rangle=0, \qquad (19\cdot12)$$

that is,

$$[(\partial^\mu A_\mu)^{(+)}-mB^{(+)}]|\text{phys}\rangle=0. \qquad (19\cdot13)$$

We note that as $m\to0$ (19·13) tends to Gupta's constraint in quantum electrodynamics. The above formulation is invariant under the gauge transformation

$$A'_\mu=A_\mu+\partial_\mu\Lambda_c, \qquad B'=B-m\Lambda_c \qquad (19\cdot14)$$

with $(\Box+m^2)\Lambda_c=0$. The two extra degrees of freedom introduced in (19·6) are suppressed by (19·12) and (19·14). The interaction Lagrangian

*) An extension to the unequal-mass case was made by Fujii and Kamefuchi.[F7]

density is given by $-U^\mu j_\mu$. If the current j_μ is conserved, then in spite of its appearance the *hermitian* vector field theory is known to be renormalizable,[U1)] because $B(x)$ can be formally eliminated in the S-matrix by a unitary transformation.

We now point out some defects of the above-mentioned conventional formalisms of the massive vector field.

(1) If one correctly calculates the Feynman propagator, one does not obtain

$$\langle 0|\mathrm{T}[U_\mu(x)U_\nu(y)]|0\rangle = -(g_{\mu\nu}+m^{-2}\partial_\mu^x\partial_\nu^x)\Delta_F(x-y, m^2). \quad (19\cdot15)$$

Since $\partial_0^x \equiv \partial/\partial x_0$ does not commute with $\theta(\pm x_0 \mp y_0)$, there appears an additional non-covariant term $-im^{-2}\delta_{\mu 0}\delta_{\nu 0}\delta^4(x)$, as is seen by using (18·37).

(2) Even if the non-covariant term is neglected, the Feynman propagator does not satisfy the Lorentz condition. Indeed, in momentum space, one has

$$p^\mu(-g_{\mu\nu}+m^{-2}p_\mu p_\nu)\frac{-i}{m^2-p^2-i\varepsilon} = im^{-2}p_\nu \neq 0. \quad (19\cdot16)$$

This fact implies that angular momentum may not be conserved in virtual states.*)

(3) The $m\to 0$ limit of U_μ does not exist. Therefore, we cannot discuss the massive vector field and the massless one in a unified way. In the past, this feature caused much inconvenience. For example, Johnson's proof[J1)] of the proposition that the vanishing of the bare mass m necessarily implies the vanishing of the physical mass was devoid of its foundation, as was criticized by Schwinger.[S6)] Furthermore, in quantum electrodynamics, one usually introduces a fictitious photon mass λ *only into the S-matrix* in order to avoid infrared divergences. Hence the non-perturbational treatments of the infrared-divergence problem could not, logically consistently, be compared with its Feynman-integral approach.

(4) The renormalizability is not manifest, and the *off-the-mass-shell* quantities cannot be renormalized. Furthermore, it seems to be unclear how the bare mass m involved in the gradient term in (19·15) is renormalized to the physical mass, when we take the picture that the physical mass consists of the bare mass and the radiative mass.

All the above defects can be dissolved by constructing a vector field theory in which the Feynman propagator becomes

$$i\frac{g_{\mu\nu}+p_\mu p_\nu(-p^2-i\varepsilon)^{-1}}{m^2-p^2-i\varepsilon} \quad (19\cdot17)$$

in momentum space. Since (19·17) is rewritten as

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*) If so, in the electromagnetic structure of the nucleon, the virtual photon could be converted not only into vector mesons but also into any kind of tensor mesons.

$$i\frac{g_{\mu\nu}-m^{-2}p_\mu p_\nu}{m^2-p^2-i\varepsilon}+i\frac{m^{-2}p_\mu p_\nu}{-p^2-i\varepsilon}, \tag{19·18}$$

we see that a massless scalar field of negative norm should be present. The vector field theory having the above propagator is very naturally obtained by extending the Landau-gauge quantum electrodynamics, as was recently pointed out by Nakanishi.[N19),*)]

The Lagrangian density is given by

$$\mathcal{L}_0=-\frac{1}{4}V^{\mu\nu}V_{\mu\nu}+\frac{1}{2}m^2U^\mu U_\mu+B\partial^\mu U_\mu, \tag{19·19}$$

where $V_{\mu\nu}$ is defined by (19·2). Field equations are

$$\partial^\mu U_\mu=0, \tag{19·20}$$

$$(\Box+m^2)U_\mu-\partial_\mu B=0. \tag{19·21}$$

From (19·21) together with (19·20), we have

$$\Box B=0; \tag{19·22}$$

therefore (19·21) implies that

$$\Box(\Box+m^2)U_\mu=0. \tag{19·23}$$

Since the canonical commutation relations are independent of the mass term in (19·19), the equal-time commutation relations are the same as (18·17) if we replace A_μ by U_μ.

As in § 18, we rewrite (19·23) into an integral form:

$$U_\mu(x)=m^{-2}\left[\int d\boldsymbol{y}D(x-y)\overleftrightarrow{\partial}_0^y(\Box+m^2)U_\mu(y)\right.$$
$$\left.-\int d\boldsymbol{y}\Delta(x-y,m^2)\overleftrightarrow{\partial}_0^y\Box U_\mu(y)\right]. \tag{19·24}$$

We note that the $m\to 0$ limit of (19·24) precisely reduces to (18·28) if we replace U_μ by A_μ. On rewriting (19·24) as

$$U_\mu(x)=\int d\boldsymbol{y}\Delta(x-y,m^2)\overleftrightarrow{\partial}_0^y U_\mu(y)$$
$$-m^{-2}\int d\boldsymbol{y}[\Delta(x-y,m^2)-D(x-y)]\overleftrightarrow{\partial}_0^y[\partial_\mu B(y)] \tag{19·25}$$

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*) There are several theories which have some resemblances to, but are different from, Nakanishi's formalism. A massive scalar field of negative norm was introduced in the ξ-limiting process of Lee and Yang[L10)] and in the massive electrodynamics of Feldman and Matthews[F2)] for different purposes. Veltman[V1)] transformed the conventional propagator into (19·17) by constructing an extra Lagrangian in the massive Yang-Mills theory. Fradkin and Tyutin[F4)] considered the Yang-Mills version of the Lagrangian density (19·19) in their functional-integral formalism.

Note added: Very recently, P. Ghose and A. Das [Nucl. Phys. **B41** (1972), 299] have proposed a massive vector field theory quite similar to the one presented in the text, but in their theory the massless limit is not well defined.

with the aid of (19·21), we can easily calculate the four-dimensional commutation relation, as done in § 18. we find

$$[U_\mu(x),\ U_\nu(y)] = -i(g_{\mu\nu} + m^{-2}\partial_\mu^x\partial_\nu^x)\Delta(x-y, m^2)$$
$$+ im^{-2}\partial_\mu^x\partial_\nu^x D(x-y). \qquad (19\cdot26)$$

As $m\to 0$, the right-hand side of (19·26) tends to (18·29) with $\alpha=0$. Likewise

$$[U_\mu(x), B(y)] = -i\partial_\mu^x D(x-y), \qquad (19\cdot27)$$

$$[B(x), B(y)] = -im^2 D(x-y). \qquad (19\cdot28)$$

In contrast with the $m=0$ case, the three-dimensional fourier transform of $U_\mu(x)$ exists for $m\neq 0$. Let

$$p_\mu \equiv (\omega_{\boldsymbol{p}}, \boldsymbol{p}) \text{ with } \omega_{\boldsymbol{p}} = \sqrt{m^2+\boldsymbol{p}^2},$$
$$\tilde{p}_\mu \equiv (|\boldsymbol{p}|, \boldsymbol{p}) \text{ (hence } \tilde{p}^2=0). \qquad (19\cdot29)$$

Then

$$U_\mu(x) = m^{-2}\partial_\mu B(x) + (2\pi)^{-3/2}\int d\boldsymbol{p}(2\omega_{\boldsymbol{p}})^{-1/2}[\alpha_\mu(\boldsymbol{p})e^{-ipx} + \text{h.c.}],$$
$$B(x) = (2\pi)^{-3/2} m\int d\boldsymbol{p}(2|\boldsymbol{p}|)^{-1/2}[\beta(\boldsymbol{p})e^{-i\tilde{p}x} + \text{h.c.}] \qquad (19\cdot30)$$

for $m\neq 0$ only, where

$$p^\mu\alpha_\mu(\boldsymbol{p}) = 0 \qquad (19\cdot31)$$

and

$$[\alpha_\mu(\boldsymbol{p}), \alpha_\nu^\dagger(\boldsymbol{q})] = -(g_{\mu\nu} - m^{-2}p_\mu p_\nu)\delta(\boldsymbol{p}-\boldsymbol{q}),$$
$$[\alpha_\mu(\boldsymbol{p}), \beta^\dagger(\boldsymbol{q})] = 0,$$
$$[\beta(\boldsymbol{p}), \beta^\dagger(\boldsymbol{q})] = -\delta^3(\boldsymbol{p}-\boldsymbol{q}). \qquad (19\cdot32)$$

The vacuum $|0\rangle$ is defined by

$$\alpha_\mu(\boldsymbol{p})|0\rangle = 0, \quad \beta(\boldsymbol{p})|0\rangle = 0 \qquad (19\cdot33)$$

with $\langle 0|0\rangle = 1$, that is,

$$U_\mu^{(+)}(x)|0\rangle = 0, \quad B^{(+)}(x)|0\rangle = 0. \qquad (19\cdot34)$$

The constraint for physical states is

$$\beta(\boldsymbol{p})|\text{phys}\rangle = 0, \qquad (19\cdot35)$$

that is,

$$B^{(+)}(x)|\text{phys}\rangle = 0. \qquad (19\cdot36)$$

Though (19·33) and (19·35) are not well defined for $m=0$, (19·34) and (19·36) have the same form as in the case of the Landau-gauge electromagnetic field.

Now, we make an important observation. For $m\neq 0$, $B(x)$ is a field of negative norm. Hence the constraint (19·36) implies that there are no B particles in the physical state. For $m=0$, however, since $B(x)$ satisfies (18·23), $B^{(+)}(x)$ commutes with $[B^{(+)}(y)]^\dagger=B^{(-)}(y)$. Therefore the B particles *can* be present in the physical state; instead, (18·42) forbids the existence of the dipole ghosts, i.e., the longitudinal photons. Since the B particles are not observable because of zero norm, there remain only two degrees of the observable freedom in the $m=0$ case. Thus, in the present formalism, the well-known reduction of the degrees of the observable freedom as $m\to 0$ is embodied in quite an elegant way.

The Feynman propagator,

$$\Delta_{F,\mu\nu}(x-y, m^2)\equiv\langle 0|\mathrm{T}[U_\mu(x)U_\nu(x)]|0\rangle, \tag{19·37}$$

can easily be calculated. Since the non-covariant term owing to $\partial_\mu\partial_\nu\Delta^{(\pm)}$ is canceled by the one owing to $\partial_\mu\partial_\nu D^{(\pm)}$, we exactly obtain

$$\Delta_{F,\mu\nu}(x, m^2)=-(g_{\mu\nu}+m^{-2}\partial_\mu\partial_\nu)\Delta_F(x, m^2)+m^{-2}\partial_\mu\partial_\nu D_F(x). \tag{19·38}$$

The fourier tranform of (19·38) precisely reproduces (19·18), i.e., (19·17). Likewise,

$$\langle 0|\mathrm{T}[U_\mu(x)B(y)]|0\rangle=-\partial_\mu^x D_F(x-y),$$
$$\langle 0|\mathrm{T}[B(x)B(y)]|0\rangle=-m^2 D_F(x-y). \tag{19·39}$$

The interaction Lagrangian density is given by

$$\mathcal{L}_I=-j^\mu U_\mu, \tag{19·40}$$

where we assume that

$$\partial^\mu j_\mu=0 \tag{19·41}$$

and that j_μ explicitly involves neither U_μ nor B. The field equations for Heisenberg operators are

$$\partial^\mu U_\mu=0, \tag{19·42}$$
$$(\Box+m^2)U_\mu-\partial_\mu B=j_\mu. \tag{19·43}$$

Because of (19·41),(19·22) still holds for the Heisenberg operator $B(x)$. Hence the constraint (19·36) persists, and therefore the physical-state condition is satisfied. Without (19·41), we cannot prove that S_{phys} is unitary. Therefore, if we wish to extend the present formalism to a non-hermitian vector field, in order to secure the current conservation law, we should consider the massive Yang-Mills field. Unfortunately, in this case, the B field can no longer satisfy the d'Alembert equation, because it carries isospin 1, and therefore contributes to the isospin current. Thus, in order to have a unitary physical S-matrix, we have to confine ourselves to the hermitian vector field alone.

If (19·40) is of Yukawa type, the above theory is manifestly renormalizable as is seen by the simple power-counting. It should be noted that the renormalization is applicable not only to the on-the-mass-shell quantities but also to the off-the-mass-shell ones.

We discuss the vacuum expectation value of the four-dimensional commutator of the Heisenberg operator $U_\mu(x)$. The $m=0$ case should always be understood as the limit $m\to 0$. From Lorentz covariance, local commutativity and the Lorentz condition (19·42), we have a spectral representation

$$\langle\Omega|[U_\mu(x),\ U_\nu(y)]|\Omega\rangle$$
$$=-i\int_0^\infty ds\,\rho(s)\,(g_{\mu\nu}+s^{-1}\partial_\mu^x\partial_\nu^x)\,\Delta(x-y,s)+im^{-2}h\partial_\mu^x\partial_\nu^x D(x-y). \tag{19·44}$$

In order to determine a parameter h, we first prove the following relations:

$$[B(x),\ B(y)]=-im^2 D(x-y), \tag{19·45}$$

$$[B(x),\ j_\mu(y)]=0. \tag{19·46}$$

Since, by assumption, j_μ involves neither U_μ nor B, the operators U_l, $\dot{U}_l$, U_0 and B will commute with j_μ at the equal time, apart from some possible pathological terms.*) Hence by using

$$B(x)=\int d\boldsymbol{z}D(x-z)\overset{\leftrightarrow}{\partial_0^z}B(z) \tag{19·47}$$

together with

$$\dot{B}=\sum_l\partial_l\dot{U}_l-(\Delta-m^2)\,U_0-j_0, \tag{19·48}$$

we obtain (19·45) and (19·46).

On rewriting (19·43) and using (19·45) and (19·46), we have

$$(\Box^x+m^2)(\Box^y+m^2)\langle\Omega|[U_\mu(x),\ U_\nu(y)]|\Omega\rangle$$
$$=\langle\Omega|[j_\mu(x),j_\nu(y)]|\Omega\rangle+im^2\partial_\mu^x\partial_\nu^x D(x-y). \tag{19·49}$$

*) In the calculation of $[B(x),j_l(y)]$, we encounter the Schwinger term $[j_0,j_l]$ and the "seagull" term $\sum_k[\partial_k\dot{U}_k,j_l]$.[91] They will cancel with each other, because the left-hand side of (19·46) should be a Lorentz vector and its zeroth component vanishes. At any rate, we suppose that all pathological terms like the Schwinger term is removed by introducing auxiliary fields having an infinite mass (see § 14).

Since from (19·46) we have

$$[B^{(+)}(x), j_\mu(y)]=0, \tag{19·50}$$

$B^{(+)}(x)j_\mu(y)|\text{phys}\rangle$ vanishes, that is, $j_\mu(y)|\text{phys}\rangle$ is a physical state. Since $|\Omega\rangle$ is a physical state, therefore, $j_\mu(y)|\Omega\rangle$ is so. Thus, in the spectral representation of $\langle\Omega|[j_\mu(x), j_\nu(y)]|\Omega\rangle$, all relevant intermediate states are physical states. Hence, because of Lorentz covariance, local commutativity and (19·41), we have

$$\langle\Omega|[j_\mu(x), j_\nu(y)]|\Omega\rangle$$
$$=-i\int_0^\infty ds\tilde{\rho}(s)(g_{\mu\nu}+s^{-1}\partial_\mu^x\partial_\nu^x)\Delta(x-y, s) \tag{19·51}$$

with

$$\tilde{\rho}(s)\geq 0 \quad \text{for} \quad 0\leq s<\infty. \tag{19·52}$$

By comparing (19·44) with (19·49) together with (19·51), we find

$$h=1, \tag{19·53}$$

$$(s-m^2)^2\rho(s)=\tilde{\rho}(s). \tag{19·54}$$

We note that (19·53) is based on the assumption that j_μ involves neither U_μ nor B; otherwise h can be different from unity. From (19·54) and (19·52), $\rho(s)$ is seen to be positive definite (except possibly at $s=m^2$).

On substituting (19·53) in (19·44) and calculating the equal-time commutator $[U_k, \dot{U}_l]$, we have

$$i\delta_{kl}\delta(\boldsymbol{x}-\boldsymbol{y})=-i\int_0^\infty ds\rho(s)(-\delta_{kl}+s^{-1}\partial_k^x\partial_l^x)\delta(\boldsymbol{x}-\boldsymbol{y})$$
$$+im^{-2}\partial_k^x\partial_l^x\delta(\boldsymbol{x}-\boldsymbol{y}). \tag{19·55}$$

Therefore

$$\int_0^\infty ds\rho(s)=1, \tag{19·56}$$

$$m^{-2}=\int_0^\infty ds\rho(s)/s. \tag{19·57}$$

In the present formalism, we can always take the $m\to 0$ limit smoothly. From (19·57), as $m\to 0$ the right-hand integral must diverge, but because of (19·56) and $\rho(s)\geq 0$ for $s\neq m^2$ a divergent contribution can arise only from the lower limit of the integration. Accordingly, for $m=0$, there is no $a>0$ such that $\rho(s)=0$ for $s<a$, that is, the P^2 spectrum of the physical states extends to zero. In other words, there must exist at least one kind of physical particles whose *physical mass* is exactly zero. As is well established experimentally, however, there are no massless physical particles which interact strongly. Hence no baryon-number gauge field,

which necessarily has $m=0$, can exist in hadron physics.

Hereafter, we consider the case in which $\rho(s)$ has only one point-spectrum. Since $\rho(s)$ is a measure unless $m=0$, its point-spectrum must be a δ-function. Hence, for $m\neq 0$ we can write

$$\rho(s)=Z\delta(s-m_{\text{phys}}^2)+Z\sigma(s)\theta(s-b), \tag{19·58}$$

where m_{phys} and Z are the physical mass of U_μ and the wave-function renormalization constant, respectively. Because of the stability of the vector meson (if unstable, no point-spectrum can be existent), we have

$$b\geq m_{\text{phys}}^2. \tag{19·59}$$

On substituting (19·58) in (19·56) and in (19·57), we obtain

$$Z^{-1}=1+\int_b^\infty ds\sigma(s)>1, \tag{19·60}$$

$$\frac{1}{m^2}=\frac{Z}{m_{\text{phys}}^2}+Z\int_b^\infty ds\frac{\sigma(s)}{s}. \tag{19·61}$$

The above two relations are exactly the ones obtained by Johnson[J1)] in the framework of the conventional (positive-metric) vector field theory. In Johnson's case, one cannot take the limit $m\rightarrow 0$ in (19·61), but we can now take this limit safely. Hence we see that $m\rightarrow 0$ implies $m_{\text{phys}}\rightarrow 0$.

The substitution of (19·58) and (19·61) in (19·44) together with (19·53) yields

$$\begin{aligned}Z^{-1}&\langle\Omega|\,[U_\mu(x),\,U_\nu(y)]\,|\Omega\rangle\\&=-i(g_{\mu\nu}+m_{\text{phys}}^{-2}\partial_\mu^x\partial_\nu^x)\Delta(x-y,\,m_{\text{phys}}^2)+im_{\text{phys}}^{-2}\partial_\mu^x\partial_\nu^x D(x-y)\\&+\int_b^\infty ds\sigma(s)\,[-i(g_{\mu\nu}+s^{-1}\partial_\mu^x\partial_\nu^x)\Delta(x-y,\,s)+is^{-1}\partial_\mu^x\partial_\nu^x D(x-y)].\end{aligned} \tag{19·62}$$

We consider the $m\rightarrow 0$ limit, i.e., the $m_{\text{phys}}\rightarrow 0$ limit of (19·62). On writing $\lim_{m\rightarrow 0}U_\mu(x)=A_\mu(x)$, $\lim_{m\rightarrow 0}Z=Z_3$ and $\lim_{m\rightarrow 0}\sigma(s)=\pi(s)$, we have

$$\begin{aligned}Z_3^{-1}&\langle\Omega|\,[A_\mu(x),\,A_\nu(y)]\,|\Omega\rangle\\&=-i[g_{\mu\nu}D(x-y)-\partial_\mu^x\partial_\nu^x E(x-y)]\\&+\int_{+0}^\infty ds\pi(s)\,[-i(g_{\mu\nu}+s^{-1}\partial_\mu^x\partial_\nu^x)\Delta(x-y,s)+is^{-1}\partial_\mu^x\partial_\nu^x D(x-y)].\end{aligned} \tag{19·63}$$

This formula is nothing but the vacuum expectation value of the commutator of the Landau-gauge electromagnetic field.

Finally, we discuss the renormalization procedure. As suggested by (19·62), the renormalized field $U_\mu^r(x)$ is related to the unrenormalized one $U_\mu(x)$ through

$$U_\mu(x)=Z^{1/2}U_\mu^r(x). \tag{19·64}$$

Since $B(x)$ is not observable, there seems to be no definite criterion for choosing a particular wave-function renormalization. From the consideration in the case of the electromagnetic field in the general covariant gauge,[I.2] it seems natural to set (cf. (18·77))

$$B(x)=Z^{-1/2}B^r(x). \tag{19·65}$$

Then we have

$$\langle\Omega|\,[U_\mu^r(x),B^r(y)]\,|\Omega\rangle=-i\partial_\mu^x D(x-y), \tag{19·66}$$

$$[B^r(x),B^r(y)]=-im_{\text{phys}}^2[1+m_{\text{phys}}^2\int_b^\infty ds\sigma(s)/s]^{-1}D(x-y), \tag{19·67}$$

and

$$(\Box+m_{\text{phys}}^2)U_\mu^r-\partial_\mu B^r=J_\mu, \tag{19·68}$$

where

$$J_\mu\equiv Z^{-1/2}j_\mu+\delta m^2\cdot U_\mu^r+(Z^{-1}-1)\partial_\mu B^r \tag{19·69}$$

with $\delta m^2\equiv m_{\text{phys}}^2-m^2$. The renormalization in the Landau-gauge quantum electrodynamics is obtained as the $m_{\text{phys}}^2\to 0$ limit of the above. Unlike the other covariant gauges, we encounter no trouble in the Landau-gauge case. Thus the Landau-gauge quantum electrodynamics is more satisfactory than the Gupta-Bleuler theory.

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